Due by start of class time on Fri 2. Submit it in \LaTeX{} by email to Yimu Yin (yimuy@andrew.cmu.edu)

(1) Let $G$ be a group and $X$ be a set. An action of $G$ on $X$ is transitive if and only if $X$ forms a single orbit, equivalently for all $x, y \in X$ there is $g \in G$ such that $gx = y$.

Suppose that we are given actions of some group $G$ on sets $X$ and $X'$. The actions are equivalent iff there is a bijection $\alpha : X \to X'$ such that $\alpha(gx) = g\alpha(x)$ for all $g, x$.

(a) Let $G$ act transitively on $X$, let $x \in X$ and let $H = \text{Stab}(x)$. Let $X'$ be the set of left cosets of $H$. Show that the action of $G$ on $X$ is equivalent to the action of $G$ on $X'$ by left multiplication.

(b) Let $G$ act transitively on $X$. Show that if $x, y \in X$ then the stabiliser subgroups $\text{Stab}(x)$ and $\text{Stab}(y)$ are conjugate.

(c) Let $G$ act transitively on $X$, and let $H \leq G$ be such that $H$ also acts transitively on $X$. Show that for any $x \in X$, $G = H \text{Stab}(x)$.

(d) Let $G$ act transitively on $X$ and define an action of $G$ on $X^2$ by $g(x, y) = (gx, gy)$. Show that if $|X| > 1$ then the action of $G$ on $X^2$ is not transitive.

(2) Let $G$ act transitively on $X$ and let $F$ be a function with domain $X$. We say that the function is $G$-invariant iff $F(x) = F(y) \implies F(gx) = F(gy)$ for all $x, y \in X$. The action is said to be primitive if and only if every $G$-invariant $F$ is either constant or 1-1.

For any group $G$ a subgroup $H$ is maximal iff $H < G$, and there is no subgroup $K$ with $G < K < H$ (that is to say $H$ is maximal among proper subgroups of $G$).

Let $G$ act transitively on $X$ with $|X| > 1$, and let $x \in X$. Show that the action is primitive iff $\text{Stab}(x)$ is a maximal subgroup of $G$.

Hint: what can you say about $\{g : F(gx) = F(x)\}$?

(3) Let $G$ be finite, let $H \leq G$ and let $p$ be a prime dividing $|H|$. Let $H < G$ and let $p$ be a prime dividing $|H|$. Let $P$ be any Sylow $p$-subgroup of $H$. Then $G = HN_G(P)$.

(a) Show that if $P$ is any Sylow $p$-subgroup of $H$, then $G = HN_G(P)$.

Hint: an earlier question may help.

(b) Show that if $Q$ is a Sylow $p$-subgroup of $G$ then $Q \cap H$ is a Sylow $p$-subgroup of $H$.

(4) Recall that $G$ is simple iff the only normal subgroups of $G$ are $\{e\}$ and $G$. Let $G$ be a finite simple group, let $H < G$ and consider the action of $G$ on the set $X$ of left cosets of $H$ by multiplication from the left. Show that this action is an injective map from $G$ to $\text{Sym}(X)$, and deduce that $|G| \leq [G : H]!$.

Hint: HW1.

(5) Let $G$ be a simple group of order 60, and for $p = 2, 3, 5$ let $n_p$ be the number of Sylow $p$-subgroups. Find $n_p$. 

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(6) (You only \textit{have} to do this one if you know about categories. Anyone who took Math Studies or Commutative Algebra with me will be considered to know about categories.)

Recall that if \( A \) and \( B \) are objects then an \textit{isomorphism} from \( A \) to \( B \) is \( f : A \rightarrow B \) such that for some (necessarily unique) \( g : B \rightarrow A \) we have \( fg = id_B \) and \( gf = id_A \). We write \( g = f^{-1} \) in this case.

If \( \mathcal{C} \) is a category and \( X \) is an object of \( \mathcal{C} \), then \( Aut(X) \) is the set of all isomorphisms from \( X \) to \( X \).

(a) Prove that \( Aut(X) \) forms a group under composition.

(b) Prove that if \( \alpha : A \rightarrow B \) is an isomorphism in \( \mathcal{C} \), then \( \beta \mapsto \alpha \beta \alpha^{-1} \) is an isomorphism from \( Aut(A) \) to \( Aut(B) \).