HW 1

Due by start of class time on Fri 26. Please see instructions on the web page for how to type the homework in \LaTeX. Submit it by email to me.

You may wish to download the \LaTeX code for this homework from the web page and use it as a template for your solutions.

(1) Let $K, L \triangleleft G$ with $K \cap L = \{e\}$.
   (a) Show that if $k \in K$, $l \in L$ then $kl = lk$.
   Hint: $klk^{-1}l^{-1}$.
   (b) Show that $KL$ is isomorphic to the product group $K \times L$.

(2) Let $H \leq G$, $N \triangleleft G$, $G = HN$, $H \cap N = \{e\}$. As we saw in class every element of $G$ is $hn$ for unique $h \in H$, $n \in N$. Let $\psi$ be the HM from $H$ to $\text{Aut}(N)$ given by $\psi(h)(n) = n^h$.
   (a) Show that $h_1n_1h_2n_2 = (h_1h_2)(\psi(h_2^{-1})n_1)n_2$ where $h_1 \in H$, $n_1 \in N$.
   (b) Let $h \in H$ and $n \in N$. Express $(hn)^{-1}$ as a product $h'n'$ where $h' \in H$ and $n' \in N$.

(3) Let $H,N$ be arbitrary groups and let $\psi : H \to \text{Aut}(N)$ be some HM. We define a binary operation on the set $G$ of ordered pairs $(h,n)$ with $h \in H$, $n \in N$ by $(h_1,n_1)(h_2,n_2) = (h_1 \times H h_2, \psi(h_2^{-1})(n_1) \times N n_2)$. It can be shown that this makes $G$ into a group (you need not verify this).
   (a) Show that $(e_h,e_N)$ is the identity of $G$, and find a formula for the inverse of $(h,n)$.
   (b) What is $(h,e_N) \times_G (e_H,n)$?
   (c) What is the conjugate of $(e, n)$ by $(h, e)$?
   (d) Let $\bar{H} = H \times \{e_N\}$ and $\bar{N} = \{e_H\} \times N$. Show that $\bar{H} \leq G$, $\bar{N} \triangleleft G$,
   $\bar{H} \cong H$, $\bar{N} \cong N$, $G = \bar{H}\bar{N}$, $\bar{H} \cap \bar{N} = \{e_G\}$.
   (e) What is the relationship between this question and question 2?

(4) Let $G$ be a group and $H \leq G$. Let $X$ be the set of left cosets of $H$ in $G$ and define an action of $G$ on $X$ by $g(aH) = (ga)H$.
   (a) What is the stabiliser of the point $H$ in the set $X$?
   (b) Considering the action as a HM from $G$ to $\Sigma_X$, show that the kernel is the largest normal subgroup of $G$ which is contained in $H$.

(5) Let $D = \langle d \rangle$ be a cyclic group of order 7.
   (a) Describe the automorphism group $\text{Aut}(D)$. Hint: an automorphism is specified once you know where it takes the generator $d$.
   (b) Let $C = \langle c \rangle$ be cyclic of order 3. Describe all HMs from $C$ to $\text{Aut}(D)$.

(6) Let $G$ be a finite group of order 21.
   (a) Use Sylow’s theorem to show that $G$ has a cyclic subgroup $C$ of order 3 and a normal cyclic subgroup $D$ of order 7.
   (b) Show that $C \cap D = \{1\}$ and $G = CD$.
   (c) Show that if $C$ is normal if and only if $G$ is cyclic of order 21.
(d) Describe (up to isomorphism) the other possibilities for the structure of $G$. Hint: some previous Qs may help!

(7) Let $G$ be a finite group and $p$ a prime dividing the order of $G$. Show that the set of elements whose order is a power of $p$ is the union of the Sylow $p$-subgroups of $G$.

(8) Find for each of the following groups all the Sylow $p$-subgroups (for the relevant primes $p$), and their normalisers: $S_3$, $A_4$, $S_4$, $A_5$. 