Due by midnight on Sat May 12. You may not collaborate but may consult any printed or online source (please credit these sources). Please email your completed final to me (not Yimu).

(1) Let $K = \mathbb{Q}(\sqrt{2}, e^{2\pi i/5})$.

(a) Find $[K : \mathbb{Q}]$.

(b) Show that $K$ is a Galois extension of $\mathbb{Q}$.

(c) Describe the Galois group $G = \Gamma(K/\mathbb{Q})$.

(d) Give an explicit isomorphism between $G$ and a subgroup $H$ of $S_5$.

(e) Find all the subgroups of $G$ and all the fields intermediate between $\mathbb{Q}$ and $K$. Which ones are Galois extensions of $\mathbb{Q}$? Identify $K \cap \mathbb{R}$.

(f) Find all the roots of unity in $K$.

(2) Let $\mathbb{F}$ be a finite degree extension of $\mathbb{E}$, and assume $\mathbb{F}$ is a Galois extension of $\mathbb{E}$. Say that $\mathbb{F}_1$ is an “abelian extension of $\mathbb{E}$” if $\mathbb{F}_1$ is a subfield of $\mathbb{F}$ containing $\mathbb{E}$, $\mathbb{F}_1$ is a Galois extension of $\mathbb{E}$ and $\Gamma(\mathbb{F}_1/\mathbb{E})$ is abelian. Show that there is an abelian extension $\mathbb{F}_{\text{max}}$ of $\mathbb{E}$ which contains all the abelian extensions.

(3) Let $n > 1$ and let $p$ be an odd prime. Let $f = x^n + x + p$. Show that all complex roots of $f$ have absolute value strictly greater than 1. Show further that $f$ is irreducible in $\mathbb{Q}[x]$.

(4) Let $p$ be an odd prime and let $\zeta = e^{2\pi i/p}$, $\alpha = \zeta + \zeta^{-1}$. Show that $\mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$ and find its degree over $\mathbb{Q}$. For $p = 7$ find the minimal polynomial of $\alpha$ over $\mathbb{Q}$.

(5) Let $k$ be a field, $V$ a vector space over $k$ and $R$ the endomorphism ring of $V$. Let $W$ be a vector space over $k$ and and $M = \text{Hom}_k(W,V)$ be the group of all $k$-linear maps from $W$ to $V$ under pointwise addition. Show that $rm = r \circ m$ makes $M$ into a left $R$-module. If $W$ is a direct sum of finitely many copies of $V$, show that $M$ is a free $R$-module. Show by example that this is not true for all $W$.

(6) Let $\zeta = e^{2\pi i/13}$ and $F = \mathbb{Q}(\zeta)$. Show that there is a unique subfield $E$ of $F$ with $[E : \mathbb{Q}] = 4$, and find $\theta \in F$ such that $E = \mathbb{Q}(\theta)$.

(7) Let $k$ be a field and let $R$ be the polynomial ring $k[x]$. Show that the quotient of the polynomial ring $R[y]$ by the principal ideal generated by $1 - xy$ is a PID.