INDEXED SQUARES

JAMES CUMMINGS AND ERNEST SCHIMMERLING

ABSTRACT. We study some combinatorial principles intermediate between square and weak square. We construct models which distinguish various square principles, and show that a strengthened form of weak square holds in the Prikry model. Jensen proved that a large cardinal property slightly stronger than 1-extendibility is incompatible with square; we prove this is close to optimal by showing that 1-extendibility is compatible with square.

1. INTRODUCTION

Several lines of research motivate the results in this paper. What the lines have in common is Jensen's celebrated combinatorial principle $\square_\kappa$, which is pronounced "square kappa".

Definition 1.1 (Jensen [12]). *Let κ be an infinite cardinal. A sequence $\langle C_\alpha\colon \alpha < \kappa^+\rangle$ is called a $\square_\kappa$-*sequence *iff whenever β is a limit ordinal and $\kappa < \beta < \kappa^+$,*

(1) *C_β is a closed unbounded subset of β,*
(2) *C_β has order type at most κ, and*
(3) *if α is a limit point of C_β, then $C_\alpha = C_\beta \cap \alpha$.*

We say that $\square_\kappa$ holds *iff there exists a $\square_\kappa$-sequence.*

If β is a limit ordinal between κ and κ^+, then clearly there exists a closed unbounded $C \subseteq \beta$ with $\mathrm{ot}(C) \leq \kappa$. What gives $\square_\kappa$ strength is the last clause in the definition, which is commonly referred to as *coherence*. Although $\square_\kappa$ is a technical principle, it has turned out

First author partially supported by NSF grants DMS-9703945 and DMS-0070549.

Second author partially supported by NSF Grants DMS-9305990, DMS-9712580, DMS-9996280 and DMS-0088948.

to be one of the most important links between diverse parts of set theory.

A major theme in set theory is the tension between "compactness" and "incompactness". Examples of the kind of compactness phenomena we have in mind include stationary reflection (qv), the tree property, Shelah's singular compactness theorem, or Silver's theorem that GCH does not first fail at a singular cardinal of uncountable cofinality; some examples of incompactness are non-reflecting stationary sets, Aronszajn trees, or Magidor's theorem that GCH can fail first at $\aleph_\omega$. As we see shortly square and related principles are generally on the incompactness side, and can be used as a measure of the extent of compactness in the universe of set theory.

The following results are due to Jensen, and indicate some of the power of square principles. Let κ be a singular strong limit cardinal; then $\square_\kappa$ implies that

(1) there exists a special κ^+-Aronszajn tree,
(2) under GCH, there exists a κ^+-Souslin tree, and
(3) under GCH, for every structure $\mathfrak{A}$ of type $(\aleph_1, \aleph_0)$ there exists a structure $\mathfrak{B}$ of type (κ^+, κ) such that $\mathfrak{A} \equiv \mathfrak{B}$.

We recall that a stationary subset S of a regular cardinal λ is said to *reflect* if there is $\alpha < \lambda$ of uncountable cofinality such that $S \cap \alpha$ is stationary. Solovay showed that if $\square_\kappa$ holds then every stationary subset of κ^+ contains a non-reflecting stationary set. He also showed that if κ is κ^+-strongly compact then every stationary subset of the set $\kappa^+ \cap \mathrm{cof}(< \kappa)$ reflects, so that $\square_\kappa$ fails. This is a typical example of the kind of tension between compactness and incompactness which we discussed above.

Another point of departure for the work in this paper was the results of Cummings, Foreman and Magidor [5] on the relationship between square principles, stationary reflection and PCF theory. That paper uses ideas from PCF theory (in particular the concept of a "very good scale" as discussed in section 2) to clarify the relationship between squares and reflection. Several of our results are directly motivated by the results of [5].

It is natural to ask when $\square_\kappa$ holds. Jensen showed that in L the principle $\square_\kappa$ holds for all cardinals κ. On the other hand, the consistency strength of the failure of $\square_\kappa$ is strictly greater than that of

ZFC. For example, the theory ZFC + $\square_{\aleph_1}$ fails is equiconsistent with ZFC + there exists a Mahlo cardinal; the lower bound is by Jensen and uses L, while the upper bound is a forcing argument by Solovay that uses the Levy algebra. More generally, the failure of $\square_\kappa$ with κ regular is well-understood. There is a serious gap, however, in our understanding the case in which κ is singular: roughly, there are upper bounds of about one supercompact cardinal, and lower bounds of many Woodin cardinals. This leads us to the second point of departure for [5] and us too, namely inner model theory.

The main technique for proving lower bounds on the failure of $\square_\kappa$ for κ singular involves generalizations of Jensen's theorem that $\square_\kappa$ holds in L for all infinite cardinals κ, and Jensen's Covering Theorem [6], which implies that if $0^\#$ does not exist then $(\kappa^+)^L = \kappa^+$ for all V-singular cardinals κ. An immediate consequence of these results is that if $0^\#$ does not exist and κ is a singular cardinal, then $\square_\kappa$ holds. We note that this helps to explain a well-known phenomenon in combinatorial set theory; if κ is singular then it is typically hard to build models without "incompact" objects of size κ^+, such as κ^+-Aronszajn trees [18] or non-reflecting stationary sets [16].

However, this sort of covering fails for L if $0^\#$ exists, so other models must be used. The kind of transitive proper class models to which Jensen's theorems have been generalized are known as *core models.* In his early attempts in this direction, Schimmerling introduced the following hierarchy of weakenings of $\square_\kappa$.

Definition 1.2 (Schimmerling [20]). *Let κ be an infinite cardinal and let λ be a cardinal such that $1 < \lambda < \kappa^+$. A sequence of sets $\langle \mathcal{C}_\alpha \colon \alpha < \kappa^+\rangle$ is called a* $\square_\kappa^{<\lambda}$-sequence *iff whenever β is a limit ordinal and $\kappa < \beta < \kappa^+$, then*

(1) *$1 \le |\mathcal{C}_\beta| < \lambda$ and*
(2) *for all $C \in \mathcal{C}_\beta$,*
    (a) *C is a closed unbounded subset of β,*
    (b) *C has order type at most κ, and*
    (c) *if α is a limit point of C, then $C \cap \alpha \in \mathcal{C}_\alpha$.*

We say that $\square_\kappa^{<\lambda}$ holds iff there exists a $\square_\kappa^{<\lambda}$-sequence.

We write $\square_\kappa^\lambda$ for $\square_\kappa^{<\lambda^+}$. Clearly, $\square_\kappa^1$ is equivalent to $\square_\kappa$. The principle $\square_\kappa^\kappa$ appears in the literature as $\square_\kappa^*$ or "weak square kappa";

Jensen isolated $\square_\kappa^*$ roughly at the same time as $\square_\kappa$, and showed that it holds if and only if there is a special κ^+-Aronszajn tree. It is worth noticing that if $\kappa^{<\kappa} = \kappa$ then by an easy argument $\square_\kappa^*$ holds, so that $\square_\kappa^*$ is of greatest interest when κ is singular.

As an example of how the $\square_\kappa^\lambda$ hierarchy has been used, we review some results about forcing axioms and square principles. The Proper Forcing Axiom (PFA) and Martin's Maximum (MM) are strong versions of Martin's Axiom (MA) which are consistent relative to a supercompact cardinal, and whose exact consistency strength is still unknown.

Todorcevic [25] proved that under PFA, $\square_\kappa$ fails for all $\kappa \geq \aleph_1$. Hence, by the results of Jensen mentioned earlier, PFA implies that $0^\#$ exists. Magidor observed that Todorcevic's proof actually shows that PFA implies the failure of $\square_\kappa^{\aleph_1}$ for all $\kappa \geq \aleph_1$. Schimmerling used Magidor's observation together with the Mitchell-Schimmerling-Steel Covering Theorem to prove that one Woodin cardinal is a lower bound on the large cardinal consistency strength of PFA (the absolute square principles (qv) from Section 3 played a role in the original version of this argument).

We can use the $\square_\kappa^\lambda$ hierarchy to measure the combinatorial strength of a proposition P, by computing the least λ such that P is consistent with $\square_\kappa^{<\lambda}$. Magidor showed that PFA + $\square_\kappa^{\aleph_2}$ for all $\kappa \geq \aleph_2$ is consistent relative to a supercompact cardinal, while by contrast MM is incompatible with $\square_\kappa^*$ for any singular κ of cofinality ω; this is one measure of the gap in strength between these two axioms. In a similar vein, in [5] the $\square_\kappa^\lambda$ hierarchy is used to calibrate the strength of various stationary reflection principles (see the discussion in Section 2). An interesting open problem is to determine whether GCH + $\square_{\aleph_\omega}^\omega$ suffices to construct an $\aleph_{\omega+1}$-Souslin tree; a straightforward adaptation of Jensen's arguments shows that GCH + $\square_{\aleph_\omega}^{<\omega}$ is sufficient.

As the authors were writing this paper, the story of $\square_\kappa$ took an interesting turn. Schimmerling and Zeman proved that in all core models[1] if κ is not a *subcompact* cardinal, then $\square_\kappa$ holds. Subcompactness

[1]By *core model* we mean any proper class model of the form $L[\vec{E}]$ where $\vec{E}$ is a coherent sequence of extenders, subject to certain fine structural conditions. See [22] for a full explanation.

is a new large cardinal property that was introduced by Jensen; subcompactness follows from supercompactness. Recall the fact due to Solovay that was mentioned earlier: if κ is κ^+-strongly compact, then $\square_\kappa$ fails. Jensen observed that subcompactness suffices in Solovay's theorem and hence the converse to the Schimmerling-Zeman theorem holds. Therefore in the relevant core models $\square_\kappa$ holds iff κ is not subcompact. The results in the last section of our paper were inspired by these new developments.

We end this introduction with a summary of our results. The sections can be read independently, though some of them use definitions from earlier sections.

- In [5] it is shown that certain forms of stationary reflection are consistent with forms of $\square^\lambda_\kappa$. In Section 2 we show that these results are optimal in the sense that λ cannot be decreased. This is an example of the sort of calibration of combinatorial strength discussed above.
- In Section 3, we define and study "indexed square" principles stronger than $\square^\lambda_\kappa$. The main new idea is that the members of $\mathcal{C}_\nu$ are given indices up to λ and we explore various forms of coherence along the indices. These combinatorial principles were originally motivated by core model theory. The main result is Theorem 3.1 which identifies a square principle with some strong upwards absoluteness properties.
- In Section 4 we show that if $\mathbb{P}$ is the Prikry forcing notion associated to a normal measure over κ, then $\square^{\aleph_0}_\kappa$ holds in $V^{\mathbb{P}}$. The proof uses ideas from the results about absolute squares in Section 3 but is self-contained; the main technical point is that given an inaccessible κ we can construct a "good matrix", which is (roughly speaking) a form of $\square^*_\kappa$-sequence with additional coherence properties. We note that by results from [5], if κ is κ^+-supercompact then $\square^{<\aleph_0}_\kappa$ fails in $V^{\mathbb{P}}$.
- Jensen showed in unpublished work [11] that $\square_{\aleph_1}$ is strictly stronger than $\square^2_{\aleph_1}$. In Section 5 we use similar methods to compare these principles with some of the simplest of their "indexed" counterparts from Section 3. The upshot is that the indexed $\square^\lambda_\kappa$-hierarchy is interleaved with the original one.

- The results in Section 6 were obtained by the first author after he learned of the Schimmerling-Zeman result mentioned a few paragraphs above. Subcompactness is a natural strengthening of 1-extendibility, and so one would expect κ being 1-extendible to be consistent with $\square_\kappa$. Actually we prove something stronger but more technical to state: there is a transitive set W and a predicate $\vec{C}$ on W such that $(W, \in, \vec{C})$ is a model of $\mathsf{ZFC}_{\vec{C}}$ + $\mathsf{there\ is\ a\ 1\text{-}extendible\ cardinal}$ + $\vec{C}$ $\mathsf{is\ a\ global\ square\ sequence}$. Here $\mathsf{ZFC}_{\vec{C}}$ is the version of ZFC written in the language of set theory expanded by a predicate symbol for $\vec{C}$, and "global square" is a class version of $\square$ (introduced by Jensen) which implies that $\square_\lambda$ holds for all λ; this result leaves open whether we can have a 1-extendible cardinal and a *definable* global square sequence.

2. Stationary reflection

In this section we make two observations on the relationship between the $\square^\lambda_\kappa$ principles and stationary reflection. The following result was observed by Schimmerling and independently by Foreman and Magidor.

Theorem 2.1. *Assume that $\kappa^{<\lambda} = \kappa$ and $\square^{<\lambda}_\kappa$ holds. Let $T \subseteq \kappa^+$ be stationary. Then there exists $S \subseteq T$ such that S is stationary and S does not reflect at any $\nu < \kappa^+$ with $\mathrm{cf}(\nu) \geq \lambda$.*

Proof. Let $\vec{\mathcal{C}}$ witness $\square^{<\lambda}_\kappa$. Define $F(\nu) = \{\mathrm{ot}(C)\colon C \in \mathcal{C}_\nu\}$ for all $\nu \in T$, and find $S \subseteq T$ stationary such that F is constant on S with value A. Assume S reflects to ν, and choose $C \in \mathcal{C}_\nu$. Now the function which takes $\mu \in \lim(C) \cap S$ to $\mathrm{ot}(C \cap \mu)$ is an injection from $\lim(C) \cap S$ to A. Since $\lim(C) \cap S$ is unbounded (indeed stationary) in ν, we have

$$\mathrm{cf}(\nu) \leq |\lim(C) \cap S| \leq |A| < \lambda.$$

$\square$

In particular $\square^{<\omega}_\kappa$ implies that every stationary subset of κ^+ has a non-reflecting stationary subset.

Cummings, Foreman and Magidor [5] have made a systematic study of the connection between $\square_\kappa^\lambda$ and other combinatorial principles. To set our second result in context we quote some theorems from [5].

Fact 2.2 ([5])**.** *Let κ be singular, and let $\square_\kappa^\lambda$ hold for some $\lambda < \kappa$. Then for every stationary $T \subseteq \kappa^+$ there exists $\langle S_i \colon i < \mathrm{cf}(\kappa) \rangle$ such that each S_i is a stationary subset of T, and there is no $\nu < \kappa^+$ such that $\mathrm{cf}(\nu) > \mathrm{cf}(\kappa)$ and all the S_i reflect at ν.*

The proof of Fact 2.2 falls into two parts; in the first part a principle VGS_κ ("Very Good Scale at κ") is derived from $\square_\kappa^\lambda$, and in the second part the conclusion of Fact 2.2 is derived from VGS_κ. One theme of [5] is that certain constructions of incompact objects from square principles can be done using very good scales: Fact 2.2 is an example of this. The next result shows that $\square_\kappa^*$ is not powerful enough to imply the conclusion of Fact 2.2.

Fact 2.3 ([5])**.** *If the existence of infinitely many supercompact cardinals is consistent, then it is consistent that*

(1) $\square_{\aleph_\omega}^*$ *holds.*
(2) *For all m, n with $1 \leq m \leq n < \omega$, if $\langle S_i \colon i < \aleph_m \rangle$ is a sequence of stationary subsets of $\{\alpha < \aleph_{\omega+1} \colon \mathrm{cf}(\alpha) < \aleph_m\}$ then there exists $\nu < \aleph_{\omega+1}$ such that $\mathrm{cf}(\nu) = \aleph_n$ and all the S_i reflect at ν.*

On the other hand, Fact 2.2 cannot in general be strengthened to rule out simultaneous reflection of fewer than $\mathrm{cf}(\kappa)$ many sets.

Fact 2.4 ([5])**.** *If the existence of infinitely many supercompact cardinals is consistent, then it is consistent that*

(1) $\square_{\aleph_\omega}^\omega$ *holds.*
(2) *For every $n < \omega$, if $\langle S_i \colon i < n \rangle$ is a sequence of stationary subsets of $\aleph_{\omega+1}$ then there exists $M < \omega$ such that for all m with $M \leq m < \omega$ there exists $\nu < \aleph_{\omega+1}$ such that $\mathrm{cf}(\nu) = \aleph_m$ and all the S_i reflect at ν.*

The second theorem of this section shows that Fact 2.4 is close to optimal, in that $\square_{\aleph_\omega}^{<\aleph_\omega}$ is incompatible with the conclusion. Before proving it we need a technical lemma (an easy generalisation of a well-known fact about $\square_\kappa$).

Lemma 2.5. *Let κ be singular. If $\square_\kappa^{<\lambda}$ holds then there exists a $\square_\kappa^{<\lambda}$-sequence $\langle \mathcal{D}_\alpha \colon \alpha < \kappa^+ \rangle$, with the additional property that all the clubs in $\bigcup_\alpha \mathcal{D}_\alpha$ are of order type less than κ.*

Proof. Fix $D \subseteq \kappa$ such that D is closed and unbounded and $\mathrm{ot}(D) = \mathrm{cf}(\kappa)$. Given C a club subset of some $\alpha < \kappa^+$ with $\mathrm{ot}(C) \leq \kappa$ we define $C^* \subseteq C$:

- If $\mathrm{ot}(C) \in \lim(D)$ or $\mathrm{ot}(C) = \kappa$ then
$$C^* = \{\delta \in C \colon \mathrm{ot}(C \cap \delta) \in D\}.$$
- If $\mathrm{ot}(C) \notin \lim(D)$, then
$$C^* = \{\delta \in C \colon \mathrm{ot}(C \cap \delta) > \max(\mathrm{ot}(C) \cap \lim(D))\}.$$

It is easy to check that C^* is club in $\sup(C)$, and that if $\gamma \in \lim(C^*)$ then $C^* \cap \gamma = (C \cap \gamma)^*$. Given a $\square_\kappa^{<\lambda}$-sequence $\langle \mathcal{C}_\alpha \colon \alpha < \kappa^+ \rangle$, we set $\mathcal{D}_\alpha = \{C^* \colon C \in \mathcal{C}_\alpha\}$. $\square$

Theorem 2.6. *Assume that κ is a singular strong limit cardinal, and let T be a stationary subset of κ^+. Suppose that $\square_\kappa^{<\kappa}$ holds. Then there is a sequence $\langle S_i \colon i < \mathrm{cf}(\kappa) \rangle$ of stationary subsets of κ and a cardinal $\mu < \kappa$ such that*

1. *$S_i \subseteq T \cap \mathrm{cof}(< \mu)$.*
2. *If $\nu < \kappa^+$ is any point with $\mathrm{cf}(\nu) \geq \mu$, then $\langle S_i \colon i < \mathrm{cf}(\kappa) \rangle$ does not reflect simultaneously to ν.*

Proof. We fix a $\square_\kappa^{<\kappa}$ sequence $\mathcal{C}$, and we assume (as we may by Lemma 2.5) that all the club sets appearing in $\mathcal{C}$ have order type less than κ. We also fix an increasing sequence $\langle \kappa_i \colon i < \mathrm{cf}(\kappa) \rangle$ of regular cardinals cofinal in κ, and a stationary $T \subseteq \kappa^+$. Let T' be a stationary subset of T on which the functions $\nu \mapsto |\mathcal{C}_\nu|$ and $\nu \mapsto \mathrm{cf}(\nu)$ are constant. Let $i < \mathrm{cf}(\kappa)$ be large enough so that both of these constant values are less than κ_i. For each $\nu < \kappa^+$ and $j < \mathrm{cf}(\kappa)$, the set
$$\{\mathrm{ot}(C) \colon C \in \mathcal{C}_\nu\} \cap \kappa_j$$
is an element of V_κ. Since κ is a strong limit cardinal, by Fodor's lemma there are sequences $\langle S_j \colon j < \mathrm{cf}(\kappa) \rangle$ and $\langle A_j \colon j < \mathrm{cf}(\kappa) \rangle$ such that for every $j < \mathrm{cf}(\kappa)$, S_j is a stationary subset of T' and for every $\nu \in S_j$,
$$A_j = \{\mathrm{ot}(C) \colon C \in \mathcal{C}_\nu\} \cap \kappa_j \ .$$

Suppose that $\langle S_j \colon j < \mathrm{cf}(\kappa)\rangle$ reflects simultaneously to an ordinal $\nu < \kappa^+$. Let C be any element of $\mathcal{C}_\nu$; by our assumptions on $\mathcal{C}$ we know that $\mathrm{ot}(C) < \kappa$ and so we may choose $j < \mathrm{cf}(\kappa)$ large enough that $\kappa_j > \mathrm{ot}(C)$. Then $\lim(C) \cap S_j$ has fewer than κ_i many elements. This is because, if $\mu_0 < \mu_1$ are both elements of $\lim(C) \cap S_j$, then $\mathrm{ot}(C \cap \mu_0)$ and $\mathrm{ot}(C \cap \mu_1)$ are distinct elements of A_j, but $|A_j| < \kappa_i$. Therefore $\mathrm{cf}(\nu) < \kappa_i$. □

3. Indexed square principles

In this section, we introduce several weak square principles that were distilled from the core model combinatorics of [20] and [21]. Perhaps the most interesting of these principles is Slick-$\square^*_\kappa$. Rather than give the definition of Slick-$\square^*_\kappa$ here, at the start of this section, we will lead up to it in steps. However, to give the reader an idea of the goal, let us go ahead and state a corollary to what we are about to do.

Theorem 3.1. *Let $V \subseteq W$ be transitive models of* ZFC *and κ be a limit cardinal in W. Suppose that $(\kappa^+)^V = (\kappa^+)^W$ and* Slick-$\square^*_\kappa$ *holds in V. Then both* Slick-$\square^*_\kappa$ *and $\square^\lambda_\kappa$ hold in W where $\lambda = \mathrm{cf}^W(\kappa)$.*

Combinatorics similar to the proof of this theorem were used to obtain lower bounds on the consistency strength of PFA in [20]. Theorem 3.1 was the inspiration for the result in the next section that Prikry forcing at a measurable cardinal κ adds a $\square^{\aleph_0}_\kappa$ sequence.

Whacky-$\square^*_\kappa$ is another principle that we will define later in this section. On the surface, Whacky-$\square^*_\kappa$ seems like a slight improvement of $\square^{<\kappa}_\kappa$, but there are cases in which it does the work of $\square^{<\aleph_0}_\kappa$. Again, we will state a result well in advance of giving the definitions. Combinatorics similar to the proof of Theorem 3.2 were used to obtain consistency strength lower bounds on stationary reflection in [20].

Theorem 3.2. *Suppose that κ is a strong limit cardinal and that the principle* Whacky-$\square^*_\kappa$ *holds. Then every stationary subset of κ^+ has a non-reflecting stationary subset.*

To orient the reader we will fill in the missing definitions to make sense of the series of implications

$$\square^{<\kappa}_\kappa \Longleftarrow \text{Index-}\square^{<\kappa}_\kappa \Longleftarrow \text{Card-index-}\square^{<\kappa}_\kappa \Longleftarrow \text{Whacky-}\square^*_\kappa$$

from left to right, after which we will define Slick-$\square^*_\kappa$.

Our first definition is that of Index-$\square^{<\lambda}_\kappa$, which strengthens $\square^{<\lambda}_\kappa$ by assigning ordinal indices from $\kappa+1$ to each club in each $\mathcal{C}_\nu$, and demanding that initial segments of a club with a given index all get the same index.

Definition 3.3. *The pair* (A, C) *is said to* witness that Index-$\square^{<\lambda}_\kappa$ holds *iff* A *and* C *are functions such that if* ν *is a limit ordinal and* $\kappa < \nu < \kappa^+$, *then*

1. $A(\nu)$ *is a non-empty subset of* $\kappa + 1$ *of cardinality* $< \lambda$, *and*
2. *if* $\alpha \in A(\nu)$, *then*
   - (a) $C(\nu, \alpha)$ *is a club subset of* ν,
   - (b) $\mathrm{ot}(C(\nu, \alpha)) \leq \kappa$, *and*
   - (c) *if* $\mu \in \lim(C(\nu, \alpha))$, *then* $\alpha \in A(\mu)$ *and* $C(\nu, \alpha) \cap \mu = C(\mu, \alpha)$.

As one would expect, Index-$\square^{\lambda}_\kappa$ means Index-$\square^{<\lambda^+}_\kappa$. It might seem more natural to require $A(\nu) \subseteq \kappa$ in Definition 3.3; the value of allowing κ as an index will be remarked on after Definition 3.4. In Section 5, we will prove that indexing gives something new, namely that Index-$\square^2_{\aleph_1}$ is strictly between $\square_{\aleph_1}$ and $\square^2_{\aleph_1}$ in its strength. However, for the rest of this section, most of our results will be about the case in which κ is a limit cardinal.

The point of our next principle, Card-index-$\square^{<\lambda}_\kappa$, is to link information about the order type of a given club to its index. Of course, several ways of doing this are possible. Considerations tied to the core model combinatorics of [20] led to our choice of "α^+" here, in a way that we will not make precise.

Definition 3.4. *We say that* (A, C) witnesses that Card-index-$\square^{<\lambda}_\kappa$ holds *iff* (A, C) *witnesses that* Index-$\square^{<\lambda}_\kappa$ *holds and for each limit ordinal* ν *between* κ *and* κ^+,

1. *either* $A(\nu) \subseteq \kappa$ *and* $|C(\nu, \alpha)| \leq \alpha^+$ *for all* $\alpha \in A(\nu)$,
2. *or* $A(\nu) = \{\kappa\}$.

Here are some remarks on the definition.

- If κ is a successor cardinal and the principle Index-$\square^{<\lambda}_\kappa$ holds, then the principle Card-index-$\square^{<\lambda+1}_\kappa$ holds.

- The only point of allowing κ to be an index (rather than requiring $A(\nu) \subseteq \kappa$ in all cases) is to get a consistent principle when κ is an inaccessible cardinal. Note that if κ is an inaccessible cardinal and (A, C) is a witness that Card-index-$\square_\kappa^\lambda$ holds, then $A(\nu) = \{\kappa\}$ whenever $\mathrm{cf}(\nu) = \kappa$. Also note that if $\mu \in \lim(C(\nu, \kappa))$, then $A(\mu) = \{\kappa\}$.
- Let κ be a singular cardinal and $\lambda = \mathrm{cf}(\kappa)$. Suppose that there exists a witness that Card-index-$\square_\kappa^\lambda$ holds. Then there exists a witness (A, C) that Card-index-$\square_\kappa^\lambda$ holds such that $A(\nu) \subseteq \kappa$ for all ν. The construction is as follows. First arrange that $\mathrm{ot}(C(\nu, \kappa)) < \kappa$ whenever $A(\nu) = \{\kappa\}$ as in the proof of Lemma 2.5. Then fix a sequence $\langle \kappa_i \colon i < \lambda \rangle$ that is increasing and unbounded in κ. If $A(\nu) \subseteq \kappa$, then let $B(\nu) = A(\nu)$ and $D(\nu, \alpha) = C(\nu, \alpha)$. If $A(\nu) = \{\kappa\}$, then let
$$B(\nu) = \{\kappa_i < \kappa \colon \mathrm{ot}(C(\nu, \kappa)) \leq \kappa_i^+\}$$
and $D(\nu, \kappa_i) = C(\nu, \kappa)$ for $\kappa_i \in B(\nu)$. Then (B, D) is a witness that Card-index-$\square_\kappa^\lambda$ holds and $B(\nu) \subseteq \kappa$ for all ν.

We will prove Theorem 3.2 after defining the principle Whacky-$\square_\kappa^*$, which strengthens Card-index-$\square_\kappa^{<\kappa}$ by requiring, roughly, that the set of indices for a given ordinal ν be bounded in κ.

Definition 3.5. *We say that (A, C) is a* witness that Whacky-$\square_\kappa^*$ holds *iff (A, C) is a witness that* Card-index-$\square_\kappa^{<\kappa}$ *holds and for every limit ordinal ν between κ and κ^+, if $A(\nu) \subseteq \kappa$, then* $\sup(A(\nu)) < \kappa$.

We remark that if κ is a regular cardinal, then Whacky-$\square_\kappa^*$ holds iff Card-index-$\square_\kappa^{<\kappa}$ holds.

Proof of Theorem 3.2. Say κ is a strong limit cardinal and (A, C) is a witness that Whacky-$\square_\kappa^*$ holds. Let $S \subseteq \kappa^+$ be a stationary set of limit ordinals.

First suppose that there exists a set $S' \subseteq S$ that is stationary in κ such that $A(\nu) = \{\kappa\}$ for all $\nu \in S'$. By Fodor's lemma, there exists an ordinal $\tau \leq \kappa$ and a stationary set $S'' \subseteq S'$ such that $\mathrm{ot}(C(\nu, \kappa)) = \tau$ for all $\nu \in S''$. Consider an arbitrary ν. Let $\alpha \in A(\nu)$. Suppose that $\mu \in S'' \cap \lim(C(\nu, \alpha))$. Then $\alpha = \kappa$, $A(\mu) = \{\kappa\}$ and
$$\mathrm{ot}(C(\nu, \kappa) \cap \mu) = \mathrm{ot}(C(\mu, \kappa)) = \tau.$$
But this can hold for at most one μ, so S'' does not reflect to ν.

Thus, without loss of generality $A(\nu) \subseteq \kappa$ for all $\nu \in S$. Then

$$\sup(A(\nu)) < \kappa$$

and

$$\begin{aligned}\sup\left(\{\mathrm{ot}(C(\nu,\alpha))\colon \alpha \in A(\nu)\}\right) &\leq \sup\left(\left\{|C(\nu,\alpha)|^{+}\colon \alpha \in A(\nu)\right\}\right)\\ &\leq \sup\left(\left\{\alpha^{++}\colon \alpha \in A(\nu)\right\}\right)\\ &< \kappa\end{aligned}$$

for all $\nu \in S$. Since κ is a strong limit cardinal,

$$\langle \mathrm{ot}(C(\nu,\alpha))\colon \alpha \in A(\nu)\rangle \in H_\kappa$$

for all $\nu \in S$. By Fodor's lemma, there is a stationary $S' \subseteq S$, a set $B \in H_\kappa$ and a sequence of ordinals $\tau = \langle \tau_\alpha\colon \alpha \in B\rangle$ such that

$$A(\nu) = B$$

and

$$\langle \mathrm{ot}(C(\nu,\alpha))\colon \alpha \in A(\nu)\rangle = \tau$$

for all $\nu \in S'$. Consider an arbitrary ν. Let $\alpha \in A(\nu)$. Suppose that $\mu \in S' \cap \lim(C(\nu,\alpha))$. Then $\alpha \in B = A(\mu)$ and

$$\mathrm{ot}(C(\nu,\alpha) \cap \mu) = \mathrm{ot}(C(\mu,\alpha)) = \tau_\alpha.$$

This can hold for at most one μ, therefore S' does not reflect to ν. $\square$

Definition 3.6. *We say that (A, C) is a* witness that Slick-$\square^*_\kappa$ holds *iff A and C are functions such that for all limit ν with $\kappa < \nu < \kappa^+$,*

(1) *either*
  (a) *$A(\nu)$ is a non-empty closed subset of κ, and*
  (b) *if $\alpha \in A(\nu)$, then*
    (i) *$C(\nu,\alpha)$ is a club subset of ν,*
    (ii) *$\mathrm{ot}(C(\nu,\alpha)) \leq \kappa$ and $|C(\nu,\alpha)| \leq \alpha^+$,*
    (iii) *if $\mu \in \lim(C(\nu,\alpha))$, then $\alpha \in A(\mu)$ and*

$$C(\nu,\alpha) \cap \mu \subseteq C(\mu,\alpha),$$

and
    (iv) *if $\alpha < \beta$ and $\beta \in A(\nu)$, then $C(\nu,\alpha) \subseteq C(\nu,\beta)$,*

(2) *or else*
  (a) *$A(\nu) = \{\kappa\}$,*
  (b) *$C(\nu,\kappa)$ is a club subset of ν,*
  (c) *$\mathrm{ot}(C(\nu,\kappa)) \leq \kappa$, and*

(d) *if* $\mu \in \lim(C(\nu, \kappa))$, *then* $A(\mu) = \{\kappa\}$ *and*

$$C(\nu, \kappa) \cap \mu = C(\mu, \kappa).$$

A witness to Slick-$\square^*_\kappa$ need not be a witness to any of the other indexed square principles that we have defined, not even $\square^*_\kappa$, because of the weaker coherence condition ("$\subseteq$" instead of "$=$") in clause 1(b)(iii). In fact, the proof of Theorem 3.1 shows that if Slick-$\square^*_\kappa$ holds, then there exists a witness with the stronger form of coherence. On the other hand, we have added clause 1(b)(iv), which is a different kind of coherence across indices. Another new feature is the requirement, clause 1(a), that $A(\nu)$ be closed; note that $A(\nu)$ is not required to be bounded or have small cardinality.

Let us also remark that if κ is a singular cardinal, then the second possibility in Definition 3.6 is not needed in the sense that Slick-$\square^*_\kappa$ holds iff there is a witness (A, C) that Slick-$\square^*_\kappa$ holds such that $A(\nu) \subseteq \kappa$ for all ν. The reason is just like that given in the third remark after Definition 3.4.

Theorem 3.1 follows immediately from the following two lemmas, the first of which is obvious.

Lemma 3.7. *Let* $V \subseteq W$ *be transitive models of* ZFC *and* κ *be a cardinal in* W. *Suppose that* $(\kappa^+)^V = (\kappa^+)^W$ *and*

$$V \models (A, C) \text{ is a witness that Slick-}\square^*_\kappa \text{ holds.}$$

Then

$$W \models (A, C) \text{ is a witness that Slick-}\square^*_\kappa \text{ holds.}$$

Lemma 3.8. *Suppose that* κ *is a limit cardinal,* $\lambda = \mathrm{cf}(\kappa)$, *and there exists a witness that* Slick-$\square^*_\kappa$ *holds. Then there exists a witness* (A, C) *that* Slick-$\square^*_\kappa$ *holds such that if* $A(\nu) \subseteq \kappa$, *then*

(1) $|A(\nu)| \leq \lambda$, *and*

(2) *if* $\mu \in \lim(C(\nu, \alpha))$, *then* $\alpha \in A(\mu)$ *and* $C(\nu, \alpha) \cap \mu = C(\mu, \alpha)$.

In particular, (A, C) *is also a witness that* Card-index-$\square^\lambda_\kappa$ *holds.*

Proof. Let $\langle \kappa_i \colon i < \lambda \rangle$ be a strictly increasing continuous sequence of cardinals that is unbounded in κ. Let (A, C) witness that Slick-$\square^*_\kappa$ holds. From this data, we will define a pair $(A^{\mathrm{fat}}, C^{\mathrm{fat}})$ satisfying the requirements of the lemma. The intuitive idea is to recursively "fatten up" each $C(\nu, \alpha)$ for $\alpha < \kappa$.

If $A(\nu) = \{\kappa\}$, then let $A^{\mathrm{fat}}(\nu) = \{\kappa\}$ and $C^{\mathrm{fat}}(\nu, \kappa) = C^{\mathrm{fat}}(\nu, \kappa)$. There is nothing to check in this case.

For the rest of the proof, we turn to the only other case, namely $A(\nu) \subseteq \kappa$. Define

$$A^{\mathrm{fat}}(\nu) = \{\kappa_i \colon A(\nu) \cap (\kappa_i + 1) \neq \emptyset\}.$$

Clearly $A^{\mathrm{fat}}(\nu)$ is a club subset of κ of cardinality at most λ. If $\kappa_i \in A^{\mathrm{fat}}(\nu)$, then define an ordinal

$$\alpha(\nu, i) = \sup \Big(A(\nu) \cap (\kappa_i + 1) \Big).$$

Since $A(\nu)$ is closed,

$$\alpha(\nu, i) \in A(\nu) \cap (\kappa_i + 1)$$

whenever $\kappa_i \in A^{\mathrm{fat}}(\nu)$.

Claim 3.9. *If $\kappa_i \in A^{\mathrm{fat}}(\nu)$ and $\mu \in \lim(C(\nu, \alpha(\nu, i))$, then*

$$\alpha(\nu, i) \in A(\mu) \cap (\kappa_i + 1),$$
$$\kappa_i \in A^{\mathrm{fat}}(\mu),$$
$$\alpha(\nu, i) \leq \alpha(\mu, i)$$

and

$$C(\nu, \alpha(\nu, i)) \cap \mu \subseteq C(\mu, \alpha(\nu, i)) \subseteq C(\mu, \alpha(\mu, i)).$$

Proof. This is immediate from the definitions given before the statement of Claim 3.9 with clause 1(b)(iii) and clause 1(b)(iv) of Definition 3.6. $\square$

By recursion on ν, define for $\kappa_i \in A^{\mathrm{fat}}(\nu)$,

$$C^{\mathrm{fat}}(\nu, \kappa_i) = C(\nu, \alpha(\nu, i)) \cup \bigcup \left\{ C^{\mathrm{fat}}(\mu, \kappa_i) \colon \mu \in \lim \left(C(\nu, \alpha(\nu, i)) \right) \right\}.$$

This definition makes sense since if $\mu \in \lim(C(\nu, \alpha(\nu, i)))$, then $\mu < \nu$ and, by Claim 3.9, $\kappa_i \in A^{\mathrm{fat}}(\mu)$.

The next claim shows that $(A^{\mathrm{fat}}, C^{\mathrm{fat}})$ satisfies clause 1(b)(ii) of Definition 3.6.

Claim 3.10. *If $\kappa_i \in A^{\mathrm{fat}}(\nu)$, then $|C^{\mathrm{fat}}(\nu, \kappa_i)| \leq \kappa_i^+$.*

Proof. By induction on ν, we see that $C^{\mathrm{fat}}(\nu, \kappa_i)$ is the union of at most $(\alpha(\nu, i))^+$-many sets, each of cardinality at most κ_i^+. Since $\alpha(\nu, i) \leq \kappa_i$, we are done. $\square$

The following claim is another step towards seeing that $(A^{\text{fat}}, C^{\text{fat}})$ satisfies clause 2 of Lemma 3.8; the full verification of this will be given in Claim 3.14.

Claim 3.11. *If $\kappa_i \in A^{\text{fat}}(\nu)$ and $\mu \in \lim\left(C(\nu, \alpha(\nu, i))\right)$, then*

$$\kappa_i \in A^{\text{fat}}(\mu)$$

and

$$C^{\text{fat}}(\nu, \kappa_i) \cap \mu = C^{\text{fat}}(\mu, \kappa_i).$$

Proof. The part about $\kappa_i \in A^{\text{fat}}(\mu)$ was already proved in Claim 3.9. We prove the other part by induction on ν. Assume that Claim 3.11 holds for all $\nu' < \nu$ and that $\mu \in \lim\left(C(\nu, \alpha(\nu, i))\right)$.

First suppose that μ is the largest limit point of $C(\nu, \alpha(\nu, i))$. Consider an arbitrary $\mu' < \mu$ such that

$$\mu' \in \lim(C(\nu, \alpha(\nu, i))).$$

By Claim 3.9,

$$C(\nu, \alpha(\nu, i)) \cap \mu \subseteq C(\mu, \alpha(\mu, i)).$$

In particular,

$$\mu' \in \lim(C(\mu, \alpha(\mu, i))).$$

By the induction hypothesis

$$C^{\text{fat}}(\mu, \kappa_i) \cap \mu' = C^{\text{fat}}(\mu', \kappa_i).$$

By the arbitrariness of μ' and the definition of $C^{\text{fat}}(\nu, \kappa_i)$,

$$C^{\text{fat}}(\nu, \kappa_i) = C(\nu, \alpha(\nu, i)) \cup C^{\text{fat}}(\mu, \kappa_i).$$

But

$$C(\nu, \alpha(\nu, i)) \cap \mu \subseteq C(\mu, \alpha(\mu, i)) \subseteq C^{\text{fat}}(\mu, \kappa_i)$$

by Claim 3.9 and the definition of $C^{\text{fat}}(\mu, \kappa_i)$. Thus

$$C^{\text{fat}}(\nu, \kappa_i) = C^{\text{fat}}(\mu, \kappa_i) \cup \left(C(\nu, \alpha(\nu, i)) - \mu\right).$$

It follows from the last equation that Claim 3.11 holds in the first case.

Second suppose that $C(\nu, \alpha(\nu, i))$ has no largest limit point. Consider an arbitrary $\nu' > \mu$ such that $\nu' \in \lim\left(C(\nu, \alpha(\nu, i))\right)$. By Claim 3.9 applied to ν' and ν,

$$C(\nu, \alpha(\nu, i)) \cap \nu' \subseteq C(\nu', \alpha(\nu', i)).$$

In particular,

$$\mu \in \lim\left(C(\nu', \alpha(\nu', i))\right).$$

By the induction hypothesis,

$$C^{\mathrm{fat}}(\nu', \kappa_i) \cap \mu = C^{\mathrm{fat}}(\mu, \kappa_i).$$

By the arbitrariness of ν' and the definition of $C^{\mathrm{fat}}(\nu, \kappa_i)$,

$$C^{\mathrm{fat}}(\nu, \kappa_i) \cap \mu = (C(\nu, \alpha(\nu, i)) \cap \mu) \cup C^{\mathrm{fat}}(\mu, \kappa_i).$$

But

$$C(\nu, \alpha(\nu, i)) \cap \mu \subseteq C(\mu, \alpha(\mu, i)) \subseteq C^{\mathrm{fat}}(\mu, \kappa_i)$$

by Claim 3.9 and the definition of $C^{\mathrm{fat}}(\mu, \kappa_i)$. Thus Claim 3.11 follows in the second case too. □

Claim 3.12. *Suppose that $\kappa_i \in A^{\mathrm{fat}}(\nu)$ and*

$$\mu = \sup\Big(\lim\left(C(\nu, \alpha(\nu, i))\right)\Big).$$

Then

$$C^{\mathrm{fat}}(\nu, \kappa_i) = \begin{cases} C^{\mathrm{fat}}(\mu, \kappa_i) \cup (C(\nu, \alpha(\nu, i)) - \mu) & \text{if } \mu < \nu \\ \bigcup\left\{C^{\mathrm{fat}}(\nu', \kappa_i) \colon \nu' \in \lim\left(C(\nu, \alpha(\nu, i))\right)\right\} & \text{if } \mu = \nu \end{cases}$$

Proof. The characterization follows by induction on ν from the definition of $(A^{\mathrm{fat}}, C^{\mathrm{fat}})$ and Claim 3.11. □

Claim 3.13. *If $\kappa_i \in A^{\mathrm{fat}}(\nu)$, then $C^{\mathrm{fat}}(\nu, \kappa_i)$ is club in ν.*

Proof. By induction on ν. Assume that Claim 3.13 holds for all $\nu' < \nu$ and that $\mu \in \lim\left(C^{\mathrm{fat}}(\nu, \kappa_i)\right)$. We will show that $\mu \in C^{\mathrm{fat}}(\nu, \kappa_i)$.

First suppose that there is no limit point of $C(\nu, \alpha(\nu, i))$ strictly greater than μ. By Claim 3.12, μ must be the largest limit point of $C(\nu, \alpha(\nu, i))$. So

$$\mu \in C(\nu, \alpha(\nu, i)) \subseteq C^{\mathrm{fat}}(\nu, \kappa_i).$$

On the other hand, if $\nu' > \mu$ and $\nu' \in \lim\left(C(\nu, \alpha(\nu, i))\right)$, then by the induction hypothesis and Claim 3.11,

$$\begin{aligned} \mu \in \lim\left(C^{\mathrm{fat}}(\nu, \kappa_i) \cap \nu'\right) &= \lim\left(C^{\mathrm{fat}}(\nu', \kappa_i)\right) \\ &\subseteq C^{\mathrm{fat}}(\nu', \kappa_i) = C^{\mathrm{fat}}(\nu, \kappa_i) \cap \nu'. \end{aligned}$$

□

Claim 3.14. *If $\kappa_i \in A^{\mathrm{fat}}(\nu)$ and $\mu \in \lim\left(C^{\mathrm{fat}}(\nu, \kappa_i)\right)$, then*

$$\kappa_i \in A^{\mathrm{fat}}(\mu)$$

and

$$C^{\mathrm{fat}}(\nu, \kappa_i) \cap \mu = C^{\mathrm{fat}}(\mu, \kappa_i).$$

Proof. By induction on ν. Assume that Claim 3.14 holds for all $\nu' < \nu$ and that $\mu \in \lim\left(C^{\mathrm{fat}}(\nu, \kappa_i)\right)$. If $\mu \in \lim\left(C(\nu, \alpha(\nu, i))\right)$, then we are done by Claim 3.11. So we may assume that there is a $\nu' > \mu$ such that $\nu' \in \lim\left(C(\nu, \alpha(\nu, i))\right)$. Then

$$C^{\mathrm{fat}}(\nu, \kappa_i) \cap \nu' = C^{\mathrm{fat}}(\nu', \kappa_i)$$

by Claim 3.11, so $\mu \in \lim\left(C^{\mathrm{fat}}(\nu', \kappa_i)\right)$. By the induction hypothesis, $C^{\mathrm{fat}}(\nu', \kappa_i) \cap \mu = C^{\mathrm{fat}}(\mu, \kappa_i)$. Putting the equations together, we are done. □

The next result implies that $(A^{\mathrm{fat}}, C^{\mathrm{fat}})$ satisfies clause 1(b)(iv) of Definition 3.6.

Claim 3.15. *If $\kappa_i \in A^{\mathrm{fat}}(\nu)$ and $i < j$, then $\kappa_j \in A^{\mathrm{fat}}(\nu)$ and*

$$C^{\mathrm{fat}}(\nu, \kappa_i) \subseteq C^{\mathrm{fat}}(\nu, \kappa_j).$$

Proof. Obvious from the corresponding assumption on (A, C) and the definition of $(A^{\mathrm{fat}}, C^{\mathrm{fat}})$. □

From the claims above, it is immediate that $(A^{\mathrm{fat}}, C^{\mathrm{fat}})$ satisfies the requirements of Lemma 3.8.

□

4. Prikry forcing, good matrices and weak square

It is proved in [17] that after forcing with Prikry forcing at a measurable cardinal κ the weak square principle $\square^*_\kappa$ holds. In this section we strengthen this result, showing that if κ is measurable in V and W is a Prikry extension of V then $\square^\omega_\kappa$ holds in W. In general we can not hope to improve this; by Theorem 2.1 and the following result, doing Prikry forcing at a sufficiently large cardinal κ will make $\square^{<\omega}_\kappa$ fail in the generic extension.

Fact 4.1 ([5]). *If κ is κ^+-supercompact, $\mathbb{P}$ is Prikry forcing defined from some normal measure on κ, and $S = \{\alpha < \kappa^+ \colon \mathrm{cf}(\alpha) < \kappa\}$ then*

$$V^{\mathbb{P}} \models \text{“finite sets of stationary subsets of } S \text{ reflect simultaneously”}.$$

A note on history: Originally we had a false proof of Theorem 4.2 based on Theorem 3.1 and an incorrect version of Lemma 4.4. Matt Foreman pointed out that we could get the conclusion more directly from the first version of Lemma 4.4. We then discovered and fixed the problem in Lemma 4.4, retaining Foreman's direct way of drawing the desired conclusion.

Theorem 4.2. *Let κ be measurable in V. Let U be a normal measure on κ and let $\mathbb{P}_U$ be the Prikry forcing defined from U. If W is a generic extension of V by $\mathbb{P}_U$ then $\square^{\omega}_{\kappa}$ holds in W.*

Proof. The key idea is to do most of the work in V. We will build in V an object called a “good matrix”, and then working in W we will read off the required $\square^{\omega}_{\kappa}$-sequence. It is helpful to think of the construction of a good matrix as a refinement of the (very easy) construction of a $\square^{*}_{\kappa}$-sequence for κ inaccessible.

The proof will be structured as follows: we will start by defining a good matrix, will show how to use one to build a $\square^{\omega}_{\kappa}$-sequence (hopefully motivating the definition) and will finish by constructing one.

Let λ be a regular cardinal with $\lambda > \aleph_1$. We will say that a set $A \subseteq \lambda$ is a club* subset of λ if and only if there is C club in λ such that $\{\alpha \in C \colon \mathrm{cf}(\alpha) > \omega\} \subseteq A$. It is easy to see that the collection of club* subsets of λ is a normal filter on λ, and that any unbounded subset of λ which is closed under uncountable suprema is club*.

We claim that every club* subset A of κ has measure one for the normal measure U. To see this let C be club in κ such that $\{\alpha \in C \colon \mathrm{cf}(\alpha) > \omega\} \subseteq A$, and let $j \colon V \longrightarrow M$ be the ultrapower map associated with U. Since $j(C) \cap \kappa = C$ and $j(C)$ is closed we see that $\kappa \in j(C)$, and since ${}^{\kappa}M \subseteq M$ we see that $M \models \mathrm{cf}(\kappa) > \omega$; it follows by the elementarity of j that $\kappa \in j(A)$, and so by the normality of U that $A \in U$.

It is a well-known fact about Prikry forcing that any $\mathbb{P}_U$-generic ω-sequence is eventually contained in any set in U. In particular we see that a $\mathbb{P}_U$-generic ω-sequence is eventually contained in any club* subset of κ from the ground model.

Definition 4.3. *Let λ be an inaccessible cardinal, and let*

$$S = \{\alpha < \lambda^+ \colon \mathrm{cf}(\alpha) < \lambda\}.$$

A good matrix for λ *is an array of sets*

$$\langle C(\alpha, i) \colon \alpha \in S, i \in X_\alpha \rangle$$

such that

(1) *$C(\alpha, i)$ is club in α.*
(2) *X_α is a club* subset of λ.*
(3) $\mathrm{ot}(C(\alpha, i)) < \lambda$.
(4) *If $i \in X_\alpha$ and $\beta \in \lim(C(\alpha, i))$ then $i \in X_\beta$ and $C(\alpha, i) \cap \beta = C(\beta, i)$.*
(5) *If $i, j \in X_\alpha$ and $i < j$ then $C(\alpha, i) \subseteq C(\alpha, j)$.*
(6) *If $\alpha, \beta \in S$ with $\beta < \alpha$ then $\beta \in \lim(C(\alpha, i))$ for some $i \in X_\alpha$ (and thus for all larger $i \in X_\alpha$ by the preceding clause).*

We now show how to finish the proof of Theorem 4.2, given the existence of a good matrix for κ. Let $\langle C(\alpha, i) \colon \alpha \in S, i \in X_\alpha \rangle$ be such a matrix. Let $\langle \kappa_i \colon i < \omega \rangle$ be a Prikry sequence generic for the forcing $\mathbb{P}_U$. As we showed above, for every α the club* set X_α contains a final segment of $\langle \kappa_i \colon i < \omega \rangle$.

We define our $\square^\omega_\kappa$-sequence $\langle \mathcal{D}_\alpha \colon \alpha < \kappa^+, \lim(\alpha) \rangle$. Let $\alpha < \kappa^+$ be a limit ordinal. We distinguish two cases.

Case I. $\alpha \in S$. Let $\mathcal{D}_\alpha = \{C(\alpha, \kappa_j) \colon \kappa_j \in X_\alpha\}$.

Case II. $\alpha \notin S$, so that $V \models \mathrm{cf}(\alpha) = \kappa$ and $W \models \mathrm{cf}(\alpha) = \mathrm{cf}(\kappa) = \omega$. Choose C_α to be any set which is club in α with $\mathrm{ot}(C_\alpha) = \omega$, and then set $\mathcal{D}_\alpha = \{C_\alpha\}$.

We need to verify that we have defined a $\square^\omega_\kappa$-sequence. It is clear that $|\mathcal{D}_\alpha| \leq \omega$ and $\mathcal{D}_\alpha$ is a family of clubs each with order type less than κ. To finish, suppose that $C \in \mathcal{D}_\alpha$ and $\beta \in \lim(C)$. Clearly $\alpha \in S$, because otherwise $C = C_\alpha$ and C_α has no limit points. So $C = C(\alpha, \kappa_j)$ for some j with $\kappa_j \in X_\alpha$. By the properties of a good matrix $\kappa_j \in X_\beta$ and $C(\beta, \kappa_j) = C \cap \beta$, so that $C \cap \beta \in \mathcal{D}_\beta$.

This shows that $\langle \mathcal{D}_\alpha \colon \alpha < \kappa^+ \rangle$ is a $\square^\omega_\kappa$-sequence, so $\square^\omega_\kappa$ holds in W and we are done once we have shown the following Lemma.

Lemma 4.4. *If λ is inaccessible there is a good matrix.*

Proof. We construct a good matrix by induction on $\alpha \in S$.

Case 1: $\alpha = \omega$. We set $X_\omega = \lambda$ and $C(\omega, i) = \omega$ for all i.

Case 2: $\alpha = \beta + \omega$ for some limit ordinal β with $\mathrm{cf}(\beta) < \lambda$ (that is to say $\beta \in S$). We set $X_\alpha = X_\beta$ and $C(\alpha, i) = C(\beta, i) \cup [\beta, \alpha)$ for all $i \in X_\alpha$.

Clearly $C(\alpha, i)$ is club in α. By definition $X_\alpha = X_\beta$, and so X_α is club*. Since $\alpha = \beta + \omega$, $\mathrm{ot}(C(\alpha, i)) = \mathrm{ot}(C(\beta, i)) + \omega$ and so $\mathrm{ot}(C(\alpha, i)) < \lambda$.

If $i \in X_\alpha$ and $\gamma \in \lim C(\alpha, i)$ then either $\gamma \in \lim C(\beta, i)$ or $\gamma = \beta$. In the former case we have by induction that $i \in X_\gamma$ and $C(\gamma, i) = C(\beta, i) \cap \gamma$, in the latter that $i \in X_\beta = X_\gamma$ and $C(\gamma, i) = C(\beta, i)$: in either case $C(\alpha, i) \cap \gamma = C(\gamma, i)$.

If $i, j \in X_\alpha$ with $i < j$ then by induction $C(\beta, i) \subseteq C(\beta, j)$, so that $C(\alpha, i) \subseteq C(\alpha, j)$. Finally if $\gamma \in S \cap \alpha$ then either $\gamma \in S \cap \beta$ or $\gamma = \beta$: if $\gamma \in S \cap \beta$ then by induction $\gamma \in \lim(C(\beta, i))$ for some i and then $\gamma \in \lim(C(\alpha, i))$ for the same i, while if $\gamma = \beta$ then $\gamma \in \lim(C(\alpha, i))$ for every $i \in X_\alpha$.

Case 3: $\mathrm{cf}(\alpha) = \omega$ and α is a limit of limit ordinals. We choose $\langle \alpha_m : m < \omega \rangle$ an increasing sequence of ordinals in S which is cofinal in α. We set

$$X_\alpha = \{i < \lambda : \forall m < \omega \; i \in X_{\alpha_m} \wedge \forall m < n < \omega \; \alpha_m \in \lim(C(\alpha_n, i))\}.$$

X_α is a club* set because it is a final segment of $\bigcap_j X_{\alpha_j}$.

We observe that if $i \in X_\alpha$ then $C(\alpha_m, i) = C(\alpha_n, i) \cap \alpha_m$ for all $m < n < \omega$. We now set $C(\alpha, i) = \bigcup_m C(\alpha_m, i)$ for all $i \in X_\alpha$.

$C(\alpha, i)$ is club in α because every initial segment is an initial segment of $C(\alpha_m, i)$ for some m. A similar argument shows that $\mathrm{ot}(C(\alpha, i)) < \lambda$. If $\beta \in \lim(C(\alpha, i))$ then $\beta \in \lim(C(\alpha_m, i))$ for some m, and by induction $i \in X_\beta$ and $C(\beta, i) = C(\alpha_m, i) \cap \beta = C(\alpha, i) \cap \beta$.

If $i, j \in X_\alpha$ with $i < j$ then by induction $C(\alpha_m, i) \subseteq C(\alpha_m, j)$ for all $m < \omega$, so that $C(\alpha, i) \subseteq C(\alpha, j)$. Finally if $\beta \in S \cap \alpha$ then $\beta \in S \cap \alpha_m$ for some m, and so by induction $\beta \in \lim(C(\alpha_m, i))$ for all large $i \in X_{\alpha_m}$; it follows that $\beta \in \lim(C(\alpha, i))$ for any large enough $i \in X_\alpha$.

Case 4: $\omega < \mathrm{cf}(\alpha) < \lambda$. Let $\mathrm{cf}(\alpha) = \rho$ say. As in Case 3 we fix $\langle \alpha_m \colon m < \rho \rangle$ an increasing and continuous sequence of members of S which is cofinal in α. We define

$$Y_\alpha = \{i < \lambda \colon \forall m < \rho \ \ i \in X_{\alpha_m} \text{ and } \forall m < n < \rho \ \ \alpha_m \in \lim C(\alpha_n, i)\}.$$

Note that Y_α depends on the choice of the sequence $\langle \alpha_m \colon m < \rho \rangle$ used in its definition. Exactly as in Case 3 Y_α is a club* set, and if $i \in Y_\alpha$ then $C(\alpha_m, i) = C(\alpha_n, i) \cap \alpha_m$ for all $m < n < \rho$.

Unfortunately Y_α will not quite do as a candidate for X_α because its dependence on the choice of $\langle \alpha_m \colon m < \rho \rangle$ would cause a problem in Case 5. We choose X_α in a more canonical way and make it as large as possible. To be more precise we let

$$X_\alpha = \{i < \lambda \colon \exists E \text{ club in } \alpha \ \forall \gamma \in \lim(E)(i \in X_\gamma \wedge E \cap \gamma = C(\gamma, i))\}.$$

If $i \in Y_\alpha$ and we let $E = \bigcup_m C(\alpha_m, i)$ then it is easy to check that E witnesses $i \in X_\alpha$, so that $Y_\alpha \subseteq X_\alpha$.

Suppose that $i \in X_\alpha$ and E, E' are both clubs in α witnessing this. Then $E \cap E'$ is club in α and

$$E = \bigcup_{\gamma \in \lim(E \cap E')} C(\gamma, i) = E'.$$

For each $i \in X_\alpha$, we now define $C(\alpha, i)$ to be the *unique* E which is club in α and is such that $\forall \gamma \in \lim(E)\ E \cap \gamma = C(\gamma, i)$. Notice that if $i \in Y_\alpha$ then automatically $C(\alpha, i) = \bigcup_m C(\alpha_m, i)$.

Since every initial segment of $C(\alpha, i)$ is an initial segment of $C(\gamma, i)$ for some $\gamma < \alpha$, $\mathrm{ot}(C(\alpha, i)) < \lambda$. If $\beta \in \lim(C(\alpha, i))$ then $\beta \in \lim(C(\gamma, i))$ for some $\gamma \in \lim(C(\alpha, i))$, and we have by induction that $i \in X_\beta$ and $C(\beta, i) = C(\gamma, i) \cap \beta = C(\alpha, i) \cap \beta$.

Let $i, j \in X_\alpha$ with $i < j$. Let $C(\alpha, i) = E$ and $C(\alpha, j) = F$. Then

$$E = \bigcup_{\gamma \in \lim(E \cap F)} C(\gamma, i) \subseteq \bigcup_{\gamma \in \lim(E \cap F)} C(\gamma, j) = F,$$

that is to say that $C(\alpha, i) \subseteq C(\alpha, j)$. Finally we may argue as in Case 3 that $S \cap \alpha \subseteq \bigcup_{i \in Y_\alpha} \lim C(\alpha, i)$, which suffices since $Y_\alpha \subseteq X_\alpha$.

Case 5: $\alpha = \beta + \omega$ where $\mathrm{cf}(\beta) = \lambda$. We fix $\langle \beta_i \colon i < \lambda \rangle$ an increasing and continuous sequence of members of S which is cofinal in β. Let

$$Z = \{i < \lambda \colon \forall j < i \ \ i \in X_{\beta j} \text{ and } \forall j < k < i \ \ \beta_j \in \lim(C(\beta_k, i))\}.$$

We claim that Z is club* in λ. To see this first observe that if $D = \{i < \lambda\colon \forall j < i \;\; i \in X_{\beta_j}\}$ then D is a diagonal intersection of sets in the club* filter, and since that filter is normal D is a club* set. Define $f\colon [\lambda]^2 \longrightarrow \lambda$ by setting $f(j,k)$ equal to the least $i \in X_{\beta_k}$ with $\beta_j \in \lim C(\beta_k, i)$, and let C be the club set of $i < \lambda$ which are closed under f. If $i \in D \cap C$ then

(1) Since $i \in D$, $i \in X_{\beta_j}$ for all $j < i$.
(2) If $j, k < i$ then since $i \in C$ we have $f(j,k) < i$, and by definition $f(j,k) \in X_{\beta_k}$ and $\beta_j \in \lim(C(\beta_k, f(j,k)))$. Since $i \in D$ we also have $i \in X_{\beta_k}$, and so by the properties of a good matrix $C(\beta_k, f(j,k)) \subseteq C(\beta_k, i)$ and so $\beta_j \in \lim(C(\beta_k, i))$.

It follows that $D \cap C \subseteq Z$, and so Z is a club* set.

We let $X_\alpha = \{i \in D \cap C\colon \mathrm{cf}(i) > \omega\}$. Let $i \in X_\alpha$ and consider the construction at level β_i; since $\mathrm{cf}(i) > \omega$ and the sequence $\langle \beta_j\colon j < \lambda\rangle$ is continuous, $\mathrm{cf}(\beta_i) = \mathrm{cf}(i) > \omega$ and the relevant clause of the definition is Case 4.

If we let $E = \bigcup_{j<i} C(\beta_j, i)$ then the fact that $i \in Z$ and the coherence properties of the good matrix imply that $\forall \gamma \in \lim(E)\; E \cap \gamma = C(\gamma, i)$, so that by the definition of X_{β_i} and $C(\beta_i, i)$ from Case 4 $i \in X_{\beta_i}$ and $C(\beta_i, i) = \bigcup_{j<i} C(\beta_j, i)$.

We define

$$C(\alpha, i) = C(\beta_i, i) \cup \{\beta_i\} \cup [\beta, \alpha).$$

Clearly $C(\alpha, i)$ is club in α, and $\mathrm{ot}(C(\alpha, i)) = \mathrm{ot}(C(\beta_i, i)) + \omega < \lambda$. If $\gamma \in \lim C(\alpha, i)$ then either $\gamma \in \lim C(\beta_i, i)$ or $\gamma = \beta_i$, and in either case it is easy to see that $i \in X_\gamma$ and $C(\gamma, i) = C(\beta_i, i) \cap \gamma = C(\alpha, i) \cap \gamma$.

Let $i, j \in X_\alpha$ with $i < j$. By induction

$$C(\beta_i, i) = \bigcup_{k<i} C(\beta_k, i) \subseteq \bigcup_{k<i} C(\beta_k, j) \subseteq \bigcup_{k<j} C(\beta_k, j) = C(\beta_j, j).$$

Since $C(\beta_j, j)$ is club in β_j and $C(\beta_i, i)$ is cofinal in β_i, it follows that $\beta_i \in C(\beta_j, j)$. Therefore by definition $C(\alpha, i) \subseteq C(\alpha, j)$.

Finally let $\gamma \in S \cap \alpha$, and observe that since $\beta \notin S$ we have $S \cap \alpha = S \cap \beta$. Find i such that $\gamma < \beta_i$, and then $j \in X_\alpha$ such that $i < j$ and $\gamma \in \lim C(\beta_i, j)$. Since $C(\beta_j, j) = \bigcup_{k<j} C(\beta_k, j)$, $\gamma \in \lim C(\beta_j, j)$.

This concludes the proof of Lemma 4.4. $\square$

The construction of a good matrix for κ in Lemma 4.4 concludes the proof of Theorem 4.2. $\square$

It is natural to ask what happens when the cofinality of κ is changed to some value other than ω, for example by Radin forcing. Apter and Cummings [1] studied this question and used the ideas of Theorem 4.2 and Fact 4.1 to show

Fact 4.5. *Let* GCH *hold and let κ be a κ^{+5}-supercompact cardinal. Then there exists a forcing poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$*

(1) *κ is κ^{+5}-supercompact.*
(2) *For every singular cardinal $\lambda < \kappa$*
   (a) *There exists $S \subseteq \lambda^+$ stationary such that any family of size less than $\mathrm{cf}(\lambda)$ of stationary subsets of S reflects simultaneously to a point of cofinality μ for unboundedly many $\mu < \lambda$.*
   (b) *The combinatorial principle $\square_\lambda^{\mathrm{cf}(\lambda)}$ holds.*

We also note a connection with some work of Gitik, Dzamonja and Shelah. Strengthening a result of Gitik [10], Dzamonja and Shelah [7] showed some results on "outside guessing of clubs" which have the following corollary:

Fact 4.6. *Let $V \subseteq W$ be inner models of* ZFC *and let* GCH *hold in V. Suppose that κ is a W-cardinal such that $\kappa_V^+ = \kappa_W^+$, $W \models \mathrm{cf}(\kappa) = \omega$ and $V \models \kappa$ is inaccessible. Then there is in W an ω-sequence which is cofinal in κ and is eventually contained in every club* subset of κ from V.*

It follows that Theorem 4.2 can be generalised to a wider class of extensions.

5. Distinguishing squares

Jensen showed in unpublished work [11] that $\square_{\aleph_1}$ is strictly stronger than $\square_{\aleph_1}^2$. His methods can be used to distinguish the principles $\square_\kappa^\lambda$ for a fixed regular κ, and similar results can be proved [5] for κ singular. In this section we use methods similar to those of [11] to show where the simplest indexed versions of weak square principles fit in.

Theorem 5.1. *Let κ be Mahlo. Then*

(1) *There is a forcing extension in which* $\kappa = \aleph_2$, $\square^2_{\aleph_1}$ *holds and Index-*$\square^2_{\aleph_1}$ *fails.*
(2) *There is a forcing extension in which* $\kappa = \aleph_2$, *Index-*$\square^2_{\aleph_1}$ *holds and* $\square_{\aleph_1}$ *fails.*

Proof. We will prove the first claim of the theorem in some detail, and then indicate how to modify the proof to give the second claim.

Let δ be inaccessible. We begin by describing a countably closed forcing $\mathbb{P}_\delta$ which will collapse δ to be $\aleph_2$ and at the same time will add a $\square^2_{\aleph_1}$-sequence. The sequence we add will have the special property that at points of uncountable cofinality it only gives a single club set.

$p \in \mathbb{P}_\delta$ iff p is a function such that

(1) $\mathrm{dom}(p)$ is a countable set of limit ordinals less than δ.
(2) If $\mathrm{cf}(\alpha) = \omega$ and $\alpha \in \mathrm{dom}(p)$ then $1 \leq |p(\alpha)| \leq 2$ and each set in $p(\alpha)$ is a club subset of α with countable order type.
(3) If $\mathrm{cf}(\alpha) > \omega$ then $p(\alpha) = \{C\}$ where C is a closed bounded subset of α with countable order type, and the largest point of C is greater than $\sup(\mathrm{dom}(p) \cap \alpha)$.
(4) If $\alpha \in \mathrm{dom}(p)$, $C \in p(\alpha)$ and $\beta \in \lim(C)$, then $\beta \in \mathrm{dom}(p)$ and $C \cap \beta \in p(\beta)$.

If $p, q \in \mathbb{P}_\delta$ then $p \leq q$ iff

(1) $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$.
(2) For all $\alpha \in \mathrm{dom}(q)$
    (a) If $\mathrm{cf}(\alpha) = \omega$ then $p(\alpha) = q(\alpha)$.
    (b) If $\mathrm{cf}(\alpha) > \omega$, $p(\alpha) = \{C\}$ and $q(\alpha) = \{D\}$ then $D = C \cap (\max(D) + 1)$.

Lemma 5.2. *Let* δ *be inaccessible. Then*

- $\mathbb{P}_\delta$ *is* δ*-c.c. and countably closed.*
- $\mathbb{P}_\delta$ *collapses* δ *to* $\aleph_2$ *and adds a* $\square^2_{\aleph_1}$*-sequence.*

Proof. This is routine. The only slightly delicate point comes in checking that $\mathbb{P}_\delta$ is countably closed. Let $\langle p_n \colon n < \omega \rangle$ be a decreasing sequence of conditions, and let $\alpha \in \bigcup_n \mathrm{dom}(p_n)$ be an ordinal such that $\mathrm{cf}(\alpha) > \omega$ and the value of $p_n(\alpha)$ does not eventually stabilise for large n. The third clause in the definition of a condition implies that $\max p_n(\alpha) > \sup(\mathrm{dom}(p_n) \cap \alpha)$, so that if $\beta = \sup_n \max p_n(\alpha)$ then $\beta \notin \bigcup_n \mathrm{dom}(p_n)$ and we are at liberty to define a lower bound p_ω for $\langle p_n \colon n < \omega \rangle$ with $p_\omega(\beta) = \{\bigcup_n p_n(\alpha)\}$. $\square$

Now we suppose that γ, δ are inaccessible with $\gamma < \delta$. We will show that $\mathbb{P}_\delta$ can be viewed as a three step iteration $\mathbb{P}_\gamma * \mathbb{T} * \mathbb{Q}$, where $\mathbb{T}$ adds a suitable club at γ and $\mathbb{Q}$ adds suitable clubs in the interval (γ, δ). Conditions in $\mathbb{T}$ and $\mathbb{Q}$ are countable sets of ordinals, and so since $\mathbb{P}_\delta$ is countably closed we will have $\mathbb{T} \subseteq V$ and $\mathbb{Q} \subseteq V$ (though of course these posets will not be members of V).

Definition 5.3. *Let γ, δ be inaccessible with $\gamma < \delta$.*

(1) *If $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \colon \alpha < \gamma \rangle$ is the sequence added by $\mathbb{P}_\gamma$, then $\mathbb{T}$ is the poset in $V[\vec{\mathcal{C}}]$ defined as follows.*
   - (a) *$t \in T$ iff t is a countable, closed and bounded subset of γ such that $\forall \alpha \in \lim(t)$ $t \cap \alpha \in \mathcal{C}_\alpha$.*
   - (b) *If $t, t' \in \mathbb{T}$ then $t \leq t'$ iff $t = t' \cap (\max(t) + 1)$.*

(2) *If $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \colon \alpha < \gamma \rangle$ is the sequence added by $\mathbb{P}_\gamma$ then $\mathbb{Q}$ is the poset in $V[\vec{\mathcal{C}}]$ defined as follows:*
   - (a) *$q \in \mathbb{Q}$ iff q is a function such that*
     - (i) *$\mathrm{dom}(q)$ is a countable set of limit ordinals in the interval (γ, δ).*
     - (ii) *If $\mathrm{cf}(\alpha) = \omega$ and $\alpha \in \mathrm{dom}(q)$ then $1 \leq |q(\alpha)| \leq 2$ and each set in $q(\alpha)$ is a club subset of α with countable order type.*
     - (iii) *If $\mathrm{cf}(\alpha) > \omega$ then $q(\alpha) = \{C\}$ where C is a closed bounded subset of α, C has countable order type, and $\max(C) > \sup(\mathrm{dom}(q) \cap \alpha)$.*
     - (iv) *If $\alpha \in \mathrm{dom}(q)$, $C \in q(\alpha)$ and $\beta \in \lim(C)$ then*
       - (A) *If $\beta > \gamma$, then $\beta \in \mathrm{dom}(q)$ and $C \cap \beta \in q(\beta)$.*
       - (B) *If $\beta < \gamma$, then $C \cap \beta \in \mathcal{C}_\beta$.*
   - (b) *If $q, q' \in \mathbb{Q}$ then $q' \leq q$ iff*
     - (i) *$\mathrm{dom}(q) \subseteq \mathrm{dom}(q')$.*
     - (ii) *For all $\alpha \in \mathrm{dom}(q)$*
       - (A) *If $\mathrm{cf}(\alpha) = \omega$ then $q'(\alpha) = q(\alpha)$.*
       - (B) *If $\mathrm{cf}(\alpha) > \omega$, $q(\alpha) = \{C\}$ and $q'(\alpha) = \{D\}$ then $D = C \cap (\max(D) + 1)$.*

Remark: We can define $\mathbb{Q}$ in $V^{\mathbb{P}_\gamma}$ because γ can not be a limit point of any club in $\mathcal{C}_\alpha$ for $\gamma < \alpha < \delta$.

Lemma 5.4. *Let γ, δ be inaccessible cardinals with $\gamma < \delta$. Then there is an isomorphism between a dense subset of $\mathbb{P}_\delta$ and a dense subset of $\mathbb{P}_\gamma * \mathbb{T} * \mathbb{Q}$.*

Moreover, $\mathbb{Q}$ is countably closed in $V^{\mathbb{P}_\gamma * \mathbb{T}}$.

Proof. Let $D_0 = \{p \in \mathbb{P}_\delta\colon \gamma \in \mathrm{dom}(p)\}$ and $D_1 = \{q\colon \exists p \in D_0\ q = (p \restriction \gamma, p(\gamma), p \restriction (\gamma, \delta))\}$.

It is easy to see that D_0 is dense in $\mathbb{P}_\delta$, $D_1 \subseteq \mathbb{P}_\gamma * \mathbb{T} * \mathbb{Q}$, and the map $\phi\colon p \longmapsto (p \restriction \gamma, p(\gamma), p \restriction (\gamma, \delta))$ is an isomorphism between D_0 and D_1. In fact we wrote the definitions of $\mathbb{T}$ and $\mathbb{Q}$ to make this true.

It remains to be seen that D_1 is dense in $\mathbb{P}_\gamma * \mathbb{T} * \mathbb{Q}$. To see this let $(p, \dot{t}, \dot{q})$ be an arbitrary condition in $\mathbb{P}_\gamma * \mathbb{T} * \mathbb{Q}$. Since $\mathbb{Q} \subseteq V$ we may find $(p_1, \dot{t}_1) \leq (p, \dot{t})$ and q such that $(p_1, \dot{t}_1) \Vdash \dot{q} = \check{q}$ and then $p_2 \leq p_1$ and t_1 such that $p_2 \Vdash \dot{t}_1 = \check{t}_1$. By construction $(p_2, t_1, q) \in P_\gamma * \mathbb{T} * \mathbb{Q}$ and $(p_2, t_1, q) \leq (p, \dot{t}, \dot{q})$.

Now $p_2 \Vdash \check{t}_1 \in \mathbb{T}$ and $(p_2, t_1) \Vdash \check{q} \in \mathbb{Q}$. It is routine to check that if we define $p^* = p_2 \cup \{(\gamma, t_1)\} \cup q$ then $p^* \in \mathbb{P}_\delta$ and $\phi(p^*) = (p_2, t_1, q)$.

The proof that $\mathbb{Q}$ is countably closed in $V^{\mathbb{P}_\gamma * \mathbb{T}}$ is just like the proof that $\mathbb{P}_\delta$ is countably closed in V. □

We will be done once we have proved the following result.

Claim 5.5. *If κ is Mahlo then Index-$\square^2_{\aleph_1}$ fails in $V^{\mathbb{P}_\kappa}$.*

Proof. Suppose not. For simplicity we assume that the empty condition forces that the principle holds, say

$$\Vdash_{\mathbb{P}_\kappa} \text{“}(\dot{A}, \dot{C}) \text{ witnesses Index-}\square^2_{\aleph_1}.\text{”}$$

By the κ-c.c. for $\mathbb{P}_\kappa$ and the Mahloness of κ we may find $\delta < \kappa$ such that δ is inaccessible and $(\dot{A} \restriction \delta, \dot{C} \restriction \delta \times \aleph_1)$ is a name in $V^{\mathbb{P}_\delta}$. This implies that

$$\Vdash_{\mathbb{P}_\delta} \text{“}(\dot{A} \restriction \delta, \dot{C} \restriction \delta \times \aleph_1) \text{ witnesses Index-}\square^2_{\aleph_1}.\text{”}$$

We now identify $\mathbb{P}_\kappa$ with $\mathbb{P}_\delta * \mathbb{T} * \mathbb{Q}$ where $\mathbb{T}$, $\mathbb{Q}$ are defined as in Lemma 5.4. Fix a condition (p, t, q) which forces that $\alpha \in A(\delta)$ for some $\alpha < \aleph_1$. If $\dot{D} = \dot{C}(\delta, \alpha)$ then (p, t, q) forces that $\dot{D}$ is club in δ, $\mathrm{ot}(\dot{D}) = \aleph_1$ and $\forall \gamma \in \lim(\dot{D})\ \dot{D} \cap \gamma = \dot{C}(\gamma, \alpha)$.

The object $\dot{D}$ cannot exist in the generic extension by $\mathbb{P}_\delta$, so we claim that we may find conditions (p', t_0, q_0) and (p', t_1, q_1) both extending (p, t, q) and an ordinal $\zeta < \delta$ such that

$$\begin{array}{rcl} (p', t_0, q_0) & \Vdash & \check{\zeta} \in \dot{D} \\ (p', t_1, q_1) & \Vdash & \check{\zeta} \notin \dot{D} \end{array}$$

If this were not so then we would have

$$\forall p' \leq p \ \forall \zeta, t_0, q_0, t_1, q_1 \ (p', t_0, q_0) \Vdash \check{\zeta} \in \dot{D} \iff (p', t_1, q_1) \Vdash \check{\zeta} \in \dot{D},$$

which would imply that below (p, t, q) the name $\dot{D}$ was equivalent to a $\mathbb{P}_\delta$-name.

We build sequences $\langle p_n \colon 1 \leq n < \omega \rangle$, $\langle t_0^{2n+1} \colon n < \omega \rangle$, $\langle t_1^{2n+2} \colon n < \omega \rangle$, $\langle q_0^{2n+1} \colon n < \omega \rangle$, $\langle q_1^{2n+2} \colon n < \omega \rangle$, and $\langle \zeta_n \colon 1 \leq n < \omega \rangle$ such that

(1) $p_n \in \mathbb{P}_\delta$, $p_1 \leq p'$ and $\langle p_n \colon 1 \leq n < \omega \rangle$ is decreasing.
(2) $(p_1, t_0^1, q_0^1) \leq (p', t_0, q_0)$ and $\langle (p_{2n+1}, t_0^{2n+1}, q_0^{2n+1}) \colon n < \omega \rangle$ is decreasing.
(3) $(p_2, t_1^2, q_1^2) \leq (p', t_1, q_1)$ and $\langle (p_{2n+2}, t_1^{2n+2}, q_1^{2n+2}) \colon n < \omega \rangle$ is decreasing.
(4) $\langle \zeta_n \colon 1 \leq n < \omega \rangle$ is an increasing sequence of ordinals such that
  (a) $\zeta_1 > \max\{\max(t_0), \max(t_1), \zeta\}$.
  (b) $\zeta_{2n+1} < \max\{\max(t_0^{2n+1}), \sup \operatorname{dom}(p_{2n+1})\} < \zeta_{2n+2}$.
  (c) $\zeta_{2n+2} < \max\{\max(t_1^{2n+2}), \sup \operatorname{dom}(p_{2n+2})\} < \zeta_{2n+3}$.
  (d) $(p_{2n+1}, t_0^{2n+1}, q_0^{2n+1}) \Vdash \check{\zeta}_{2n+1} \in \dot{D}$.
  (e) $(p_{2n+2}, t_1^{2n+2}, q_1^{2n+2}) \Vdash \check{\zeta}_{2n+2} \in \dot{D}$.

Let $p_\omega \in \mathbb{P}_\delta$ be a lower bound for the sequence $\langle p_n \colon 1 \leq n < \omega \rangle$. Since $\mathbb{Q}$ is countably closed in $V^{\mathbb{P}_\delta}$ we may find q_0^* and q_1^* such that (p_ω, q_0^*) is a lower bound for $\langle (p_{2n+1}, q_0^{2n+1}) \colon n < \omega \rangle$ and (p_ω, q_1^*) is a lower bound for $\langle (p_{2n+2}, q_1^{2n+2}) \colon n < \omega \rangle$. Now define

$$\begin{array}{rcl} \zeta^* & = & \sup_n \zeta_n, \\ t_0^* & = & \bigcup_n t_0^{2n+1} \cup \{\zeta^*\}, \\ t_1^* & = & \bigcup_n t_1^{2n+2} \cup \{\zeta^*\}, \\ p^* & = & p_\omega \cup \{(\zeta^*, \{t_0^*, t_1^*\})\}. \end{array}$$

It is routine to check that (p^*, t_0^*, q_0^*) and (p^*, t_1^*, q_1^*) are both conditions in $\mathbb{P}_\delta * \mathbb{T} * \mathbb{Q}$.

The conditions (p^*, t_0^*, q_0^*) and (p^*, t_1^*, q_1^*) both force ζ^* to be a limit point of $\dot{D}$, so

$$\begin{array}{lcl} (p^*, t_0^*, q_0^*) & \Vdash & \check{\zeta} \in \dot{C}(\zeta^*, \alpha) \\ (p^*, t_1^*, q_1^*) & \Vdash & \check{\zeta} \notin \dot{C}(\zeta^*, \alpha) \end{array}$$

This is absurd because $\dot{C}(\zeta^*, \alpha)$ is a name in $V^{\mathbb{P}_\delta}$, so that the preceding equations imply $p^* \Vdash \check{\zeta} \in \dot{C}(\zeta^*, \alpha)$ and $p^* \Vdash \check{\zeta} \notin \dot{C}(\zeta^*, \alpha)$. $\square$

This concludes the proof of the first claim of Theorem 5.1.

For the second claim, we start by defining a poset $\mathbb{P}^*_\delta$ which is designed to add an Index-$\square^2_{\aleph_1}$-sequence while collapsing an inaccessible δ to become $\aleph_2$. This sequence will have the special properties that it only gives one club set at limit ordinals of cofinality greater than ω, and that the only indices which are used are 0 and 1.

$p \in \mathbb{P}^*_\delta$ iff p is a pair (a, c) where

(1) a is a function with $\mathrm{dom}(a)$ a countable set of limit ordinals less than δ.
(2) For every $\nu \in \mathrm{dom}(a)$, $a(\nu)$ is a nonempty subset of $\{0, 1\}$. If $\mathrm{cf}(\nu) > \omega$ then $|a(\nu)| = 1$.
(3) c is a function with domain $\{(\nu, \alpha)\colon \nu \in \mathrm{dom}(a), \alpha \in a(\nu)\}$.
(4) If $\mathrm{cf}(\nu) = \omega$ and $(\nu, \alpha) \in \mathrm{dom}(c)$ then $c(\nu, \alpha)$ is a club subset of ν with countable order type.
(5) If $\mathrm{cf}(\nu) > \omega$ and $(\nu, \alpha) \in \mathrm{dom}(c)$ then $c(\nu, \alpha)$ is a closed bounded subset of ν with countable order type, with the additionla property that $\max(c(\nu, \alpha)) > \sup(\mathrm{dom}(a) \cap \nu)$.
(6) If $(\nu, \alpha) \in \mathrm{dom}(c)$ and $\beta \in \lim(c(\nu, \alpha))$ then $(\nu, \beta) \in \mathrm{dom}(c)$ and $c(\nu, \beta) = \beta \cap c(\nu, \alpha)$.

Conditions in $\mathbb{P}^*_\delta$ are ordered as follows: $(a_1, c_1) \leq (a_0, c_0)$ iff

(1) $\mathrm{dom}(a_0) \subseteq \mathrm{dom}(a_1)$.
(2) For all $\nu \in \mathrm{dom}(a_0)$
   (a) $a_0(\nu) = a_1(\nu)$.
   (b) For all $\alpha \in a_0(\nu)$, $c_0(\nu, \alpha) = c_1(\nu, \alpha)$.

As before it is easy to see that

(1) $\mathbb{P}^*_\delta$ is countably closed and δ-c.c.
(2) $\mathbb{P}^*_\delta$ collapses δ to $\aleph_2$ and adds (A, C) witnessing Index-$\Box^2_{\aleph_1}$.

$\mathbb{P}^*_\delta$ is susceptible to a factor analysis very similar to that which we gave for $\mathbb{P}_\delta$ above. The main difference is that we need two versions of $\mathbb{T}$ and $\mathbb{Q}$, reflecting the fact that at γ we must decide whether to put a club set with index 0 or a club set with index 1.

Lemma 5.6. *Let γ, δ be inaccessible with $\gamma < \delta$. There exist posets $\mathbb{T}^0, \mathbb{T}^1, \mathbb{Q}^0, \mathbb{Q}^1 \in V^{\mathbb{P}_\gamma}$ such that*

(1) *If $p = (a, c) \in \mathbb{P}^*_\delta$ and $(\gamma, 0) \in \mathrm{dom}(c)$ then $\mathbb{P}^*_\delta/p$ is isomorphic to a dense subset of*

$$\mathbb{P}^*_\gamma/(a \restriction \gamma, c \restriction \gamma \times 2) \times \mathbb{T}^0/c(\gamma, 0) \times \mathbb{Q}^0/(a \restriction (\gamma, \delta), c \restriction (\gamma, \delta) \times 2).$$

(2) *If $p = (a, c) \in \mathbb{P}^*_\delta$ and $(\gamma, 1) \in \mathrm{dom}(c)$ then $\mathbb{P}^*_\delta/p$ is isomorphic to a dense subset of*

$$\mathbb{P}^*_\gamma/(a \restriction \gamma, c \restriction \gamma \times 2) \times \mathbb{T}^1/c(\gamma, 1) \times \mathbb{Q}^1/(a \restriction (\gamma, \delta), c \restriction (\gamma, \delta) \times 2).$$

(3) *$\mathbb{Q}^j$ is countably closed in $V^{\mathbb{P}^*_\gamma * \mathbb{T}^j}$.*

Proof. The definitions and proofs are like those of Definition 5.3 and Lemma 5.4. $\Box$

Claim 5.7. *If κ is Mahlo then $\Box_{\aleph_1}$ fails in $V^{\mathbb{P}^*_\kappa}$.*

Proof. Suppose that

$$\Vdash_{\mathbb{P}^*_\kappa} \text{“}\langle \dot{D}_\alpha \colon \alpha < \kappa \rangle \text{ is a } \Box_{\aleph_1}\text{-sequence”}$$

By the κ-c.c. for $\mathbb{P}^*_\kappa$ and the Mahloness of κ we may find $\delta < \kappa$ such that δ is inaccessible and $\langle \dot{D}_\alpha \colon \alpha < \delta \rangle$ is a name in $V^{\mathbb{P}^*_\delta}$. This implies that

$$\Vdash_{\mathbb{P}^*_\delta} \text{“}\langle \dot{D}_\alpha \colon \alpha < \delta \rangle \text{ is a } \Box_{\aleph_1}\text{-sequence”}$$

We now consider the $\mathbb{P}^*_\kappa$-name $\dot{D} = \dot{D}_\delta$. We claim that we may find conditions $(p, t^0, q^0) \in \mathbb{P}^*_\delta * \mathbb{T}^0 * \mathbb{Q}^0$ and $(p, t^1, q^1) \in \mathbb{P}^*_\delta * \mathbb{T}^1 * \mathbb{Q}^1$ together with an ordinal $\zeta < \delta$ such that either

$$\begin{array}{lcl} (p, t^0, q^0) & \Vdash & \check{\zeta} \in \dot{D} \\ (p, t^1, q^1) & \Vdash & \check{\zeta} \notin \dot{D} \end{array}$$

or

$$(p, t^0, q^0) \quad \Vdash \quad \check{\zeta} \notin \dot{D}$$

$$(p, t^1, q^1) \Vdash \check{\zeta} \in \dot{D}$$

To see this we first find conditions (p^0, u^0_0, r^0_0) and (p^0, u^0_1, r^0_1) from $\mathbb{P}_\delta * \mathbb{T}^0 * \mathbb{Q}^0$ and an ordinal ζ such that

$$\begin{array}{rcl} (p^0, u^0_0, r^0_0) & \Vdash & \check{\zeta} \in \dot{D}, \\ (p^0, u^0_1, r^0_1) & \Vdash & \check{\zeta} \notin \dot{D}. \end{array}$$

This is possible because $\dot{D}$ names a set which is not in $V^{\mathbb{P}_\delta}$. We now find (p, u^1, r^1) in $\mathbb{P}_\delta * \mathbb{T}^1 * \mathbb{Q}^1$ such that $p \leq p^0$ and (p, u^1, r^1) decides the statement "$\zeta \in \dot{D}$", and then choose t^i and q^i accordingly.

We build sequences $\langle p_n \colon 1 \leq n < \omega \rangle$, $\langle t^{2n+1}_0 \colon n < \omega \rangle$, $\langle t^{2n+2}_1 \colon n < \omega \rangle$, $\langle q^{2n+1}_0 \colon n < \omega \rangle$, $\langle q^{2n+2}_1 \colon n < \omega \rangle$, and $\langle \zeta_n \colon 1 \leq n < \omega \rangle$ such that

1. $p_n \in \mathbb{P}_\delta$, $p_1 \leq p'$ and $\langle p_n \colon 1 \leq n < \omega \rangle$ is decreasing.
2. $(p_{2n+1}, t^{2n+1}_0, q^{2n+1}_0) \in \mathbb{P}_\delta * \mathbb{T}^0 * \mathbb{Q}^0$, $(p_1, t^1_0, q^1_0) \leq (p, t^0, q^0)$, and $\langle (p_{2n+1}, t^{2n+1}_0, q^{2n+1}_0) \colon n < \omega \rangle$ is decreasing.
3. $(p_{2n+2}, t^{2n+2}_1, q^{2n+2}_1) \in \mathbb{P}_\delta * \mathbb{T}^1 * \mathbb{Q}^1$, $(p_2, t^2_1, q^2_1) \leq (p, t^1, q^1)$, and $\langle (p_{2n+2}, t^{2n+1}_1, q^{2n+2}_1) \colon n < \omega \rangle$ is decreasing.
4. $\langle \zeta_n \colon 1 \leq n < \omega \rangle$ is an increasing sequence of ordinals such that
   (a) $\zeta_1 > \max\{\max(t^0), \max(t^1), \zeta\}$.
   (b) $\zeta_{2n+1} < \max\{\max(t^{2n+1}_0), \sup \mathrm{dom}(p_{2n+1})\} < \zeta_{2n+2}$.
   (c) $\zeta_{2n+2} < \max\{\max(t^{2n+2}_1), \sup \mathrm{dom}(p_{2n+2})\} < \zeta_{2n+3}$.
   (d) $(p_{2n+1}, t^{2n+1}_0, q^{2n+1}_0) \Vdash \check{\zeta}_{2n+1} \in \dot{D}$.
   (e) $(p_{2n+2}, t^{2n+2}_1, q^{2n+2}_1) \Vdash \check{\zeta}_{2n+2} \in \dot{D}$.

Let $\zeta^* = \bigcup_n \zeta_n$, and let p_ω be a lower bound for $\langle p_n \colon 1 \leq n < \omega \rangle$. Define

$$\begin{array}{rcl} u^0 & = & \bigcup_n t^0_{2n+1} \cup \{\zeta^*\}, \\ u^1 & = & \bigcup_n t^1_{2n+2} \cup \{\zeta^*\}, \\ p^* & = & p_\omega \cup \{((\zeta^*, 0), u^0), ((\zeta^*, 1), u^1).\} \end{array}$$

Using the countable closure of the $\mathbb{Q}^j$, we find r^0 and r^1 such that (p^*, u^0, r^0) is a lower bound for $\langle (p_{2n+1}, t^{2n+1}_0, q^{2n+1}_0) \colon n < \omega \rangle$, and (p^*, u^1, r^1) is a lower bound for $\langle (p_{2n+2}, t^{2n+2}_1, q^{2n+2}_1) \colon n < \omega \rangle$.

The conditions (p^*, u^0, r^0) and (p^*, u^1, r^1) both force ζ^* to be a limit point of $\dot{D}$, so

$$\begin{array}{rcl} (p^*, u^0, r^0) & \Vdash & \check{\zeta} \in \dot{D}_{\zeta^*} \\ (p^*, u^1, r^1) & \Vdash & \check{\zeta} \notin \dot{D}_{\zeta^*} \end{array}$$

But $\dot{D}_{\zeta^*}$ is a $V^{\mathbb{P}_\delta}$-name so $p^* \Vdash \check{\zeta} \in \dot{D}_{\zeta^*}$ and $p^* \Vdash \check{\zeta} \notin \dot{D}_{\zeta^*}$. This is a contradiction. □

This concludes the proof of Theorem 5.1. □

6. Global square and 1-extendible cardinals

In this section we investigate the question of how strong a large cardinal axiom has to be before it becomes incompatible with the existence of square sequences. We start by recalling the definition of a 1-extendible cardinal.

Definition 6.1. *κ is* 1-extendible *iff there exist a cardinal $\lambda > \kappa$ and π: $H_{\kappa^+} \longrightarrow H_{\lambda^+}$ an elementary embedding with* $\mathrm{crit}(\pi) = \kappa$ *and $\pi(\kappa) = \lambda$.*

For more information about extendible cardinals see Kanamori's book[14]. We note that if γ is a cardinal then γ is definable in H_{γ^+} as the largest cardinal, so that the demand that $\pi(\kappa) = \lambda$ in the definition of 1-extendibility is superfluous; it follows from the elementarity of the map π.

Jensen [13] introduced a strengthening of 1-extendibility called *quasicompactness*. For expository purposes we will also define an intermediate notion 1-*extendible in A*.

Definition 6.2. *Let κ be a cardinal.*

(1) *For $A \subseteq H_{\kappa^+}$, κ is* 1-extendible in A *iff there exist a cardinal $\lambda > \kappa$, a set $B \subseteq H_{\lambda^+}$ and an elementary embedding π from $(H_{\kappa^+}, \in, A)$ to $(H_{\lambda^+}, \in, B)$, such that $\pi(\kappa) = \lambda$ and the critical point of π is κ.*
(2) *κ is* quasicompact *iff κ is 1-extendible in A for all $A \subseteq H_{\kappa^+}$.*

Jensen showed that if $\vec{C} = \langle C_\alpha \colon \alpha < \kappa^+ \rangle$ is such that $C_\alpha \subseteq \alpha$ for all α, and κ is 1-extendible in $\vec{C}$ then $\vec{C}$ is not a $\square_\kappa$-sequence. In particular if κ is quasicompact then $\square_\kappa$ fails. Reflecting on this proof Jensen introduced the notion of *subcompactness*.

Definition 6.3. *Let κ be a cardinal. κ is* subcompact *iff for all $A \subseteq H_{\kappa^+}$ there exist a cardinal $\alpha < \kappa$, a set $a \subseteq H_{\alpha^+}$ and an elementary embedding π from $(H_{\alpha^+}, \in, a)$ to $(H_{\kappa^+}, \in, A)$, such that* $\mathrm{crit}(\pi) = \alpha$ *and $\pi(\alpha) = \kappa$.*

Jensen's argument shows that if κ is subcompact then $\square_\kappa$ fails. We note that a subcompact cardinal need not be measurable. In fact if κ is measurable and subcompact and U is any normal measure on κ then it is routine to check that κ is subcompact in $Ult(V, U)$, so that there are many subcompact cardinals below κ.

At this point a few words about the inner model program are in order. The goals of the program are to construct canonical "L-like" inner models for large cardinal axioms, and to analyse the internal structure of these models and their relation to V. This analysis can be used to obtain lower bounds on consistency strength for combinatorial statements. We refer the reader to the survey papers [23] and [15] and the books [19] and [24] for more information.

The inner models which are studied in the inner model program have the form $L[\vec{E}]$, where $\vec{E}$ is a sequence of extenders which is subject to certain fine-structural conditions; we will refer to models of this standard type as "$L[\vec{E}]$ models". It is anticipated that all large cardinal axioms below the level of supercompactness can hold in $L[\vec{E}]$-models, but currently this has only been proved up to slightly beyond the level of a measurable limit of Woodin cardinals.

Schimmerling and Zeman have shown that in any $L[\vec{E}]$-model, if there are no subcompact cardinals then $\square_\lambda$ holds for all λ. From the discussion in the previous paragraph, this shows that $\square_\lambda$ holds for every λ is consistent with large cardinals up to slightly beyond a measurable limit of Woodin cardinals. It should eventually be possible to show that $\square_\lambda$ holds for every λ is consistent with the existence of a 1-extendible cardinal by constructing a suitable $L[\vec{E}]$-model; in this section we will use forcing to prove this consistency result. Actually we prove something slightly stronger but more technical to state, which needs a preliminary definition.

Definition 6.4. $\langle C_\alpha \colon \alpha \in ON, \mathrm{cf}(\alpha) < \alpha \rangle$ *is a* global $\square$-sequence *iff*

(1) *For every singular ordinal* α, C_α *is club in* α *with* $\mathrm{ot}(C_\alpha) < \alpha$.
(2) *If* $\mathrm{cf}(\alpha) < \alpha$ *and* $\beta \in \lim(C_\alpha)$, *then* $\mathrm{cf}(\beta) < \beta$ *and* $C_\beta = C_\alpha \cap \beta$.

Jensen proved that if $V = L$ there is a global square sequence, and that if a global square sequence exists then $\square_\kappa$ holds for all κ. We can now state the result of this section precisely.

Theorem 6.5. *Let* GCH *hold, let* κ *be* 1*-extendible as witnessed by* $\pi\colon H_{\kappa^+} \longrightarrow H_{\lambda^+}$, *and let* δ *be inaccessible with* $\delta > \lambda$. *Then in some generic extension there is a transitive set* W *and a predicate* $\vec{C}$ *on* W *such that* $(W, \in, \vec{C})$ *is a model of* $\mathsf{ZFC}_{\vec{C}}$ + κ is 1-extendible + $\vec{C}$ is a global square sequence.

The rest of this section will be devoted to a proof of this theorem. Before starting the proof a few remarks are in order:

(1) Doug Burke [3] showed that the existence of a superstrong cardinal is consistent with $\square_\lambda$ holds for every λ.
(2) At first sight the most natural procedure for showing that a 1-extendible cardinal is consistent with global square would be to start with a model with some large cardinal κ, use class forcing to add a global square sequence and then argue that the resulting structure is a model of set theory in which κ is 1-extendible. We were unable to make this scenario work without assuming some additional reflection properties for the class of ordinals, which amounted to assuming that the universe has the form V_δ for δ inaccessible; we therefore decided to eliminate the complications of class forcing and build a transitive set model of our desired hypothesis by set forcing.
(3) It is easy to see that if κ is 1-extendible then κ is 1-extendible in A for every definable A, so that there can be no $\square_\kappa$-sequence which is definable in H_{κ^+}. While we are on the subject of definability we note that in Theorem 6.5 the sequence $\vec{C}$ is not definable in W, so our theorem leaves open whether a *definable* global square sequence is consistent with the existence of a 1-extendible cardinal.
(4) Jensen showed that if $\square_\kappa$ holds for all κ and a weak form of global square holds on singular cardinals, then global square holds. Zeman showed that the weak form of global square

holds in all $L[\vec{E}]$ models. Combining these results with the Schimmerling-Zeman result, we see that global square holds in $L[\vec{E}]$ if $L[\vec{E}]$ has no subcompact cardinals. See [22].

The following definition is not standard usage but is convenient here.

Definition 6.6. *Let η be an ordinal. A $GS(\eta)$-sequence is a sequence $\langle C_\alpha \colon \alpha < \eta, \mathrm{cf}(\alpha) < \alpha \rangle$ where*

(1) *For every singular ordinal $\alpha < \eta$, C_α is a club subset of α with $\mathrm{ot}(C_\alpha) < \alpha$.*
(2) *If $\mathrm{cf}(\alpha) < \alpha$ and $\beta \in \lim(C_\alpha)$, then $\mathrm{cf}(\beta) < \beta$ and $C_\beta = C_\alpha \cap \beta$.*

Intuitively a $GS(\eta)$-sequence is a potential initial segment of a global square sequence.

We now state our large cardinal hypothesis, which will be in effect for the rest of this section:

Hypothesis: GCH holds and there are regular cardinals $\kappa < \lambda < \delta$ such that

(1) There exists $j \colon H_{\kappa^+} \longrightarrow H_{\lambda^+}$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and j is elementary (that is to say j witnesses that κ is 1-extendible).
(2) δ is inaccessible.

Our plan for proving Theorem 6.5 is as follows: we will build a two-step generic extension $V[G][g]$ such that

(1) δ is inaccessible in $V[G][g]$.
(2) $V_\delta^{V[G]} = V_\delta^{V[G][g]}$ (we denote this model by $V_\delta[G]$ below).
(3) $V_\delta[G] \models$ "κ is 1-extendible".
(4) In $V[G][g]$ there is a sequence $\vec{C} = \langle C_\alpha \colon \alpha < \delta, \mathrm{cf}(\alpha) < \alpha \rangle$ such that
    (a) $\mathrm{ot}(C_\alpha) < \alpha$, and $\forall \beta \in \lim(C_\alpha)\ C_\beta = C_\alpha \cap \beta$.
    (b) $(V_\delta[G], \in, \vec{C})$ is a model of $\mathsf{ZFC}_{\vec{C}}$.

Before giving the details of the construction we discuss a couple of distinctive features. We note that very similar issues arise (and are discussed in more detail) in a paper by Cummings, Dzamonja and Shelah [4].

The construction is a "Reverse Easton" iteration of the same general type as those discussed in Baumgartner's survey [2]. It is common in Reverse Easton iterations for the forcing being done at stage γ to be γ-closed, but in our situation we will only assume that it is $< \gamma$-strategically closed. We recall the definition of strategic closure.

Definition 6.7 (Foreman [9])**.** *Let γ be a cardinal. A poset $\mathbb{P}$ is $< \gamma$-strategically closed if and only if for every ordinal $\zeta < \gamma$ player II wins the following two-player game of perfect information. Players I and II collaborate to build a decreasing chain $\langle p_\alpha \colon 0 < \alpha \rangle$ in $\mathbb{P}$ with player I playing at odd α and player II at even α (including all limit stages). Player II wins if play proceeds for ζ many moves, that is to say p_α is defined for all $\alpha < \zeta$.*

Replacing closure by strategic closure necessitates a few changes in the standard Reverse Easton arguments. We outline these changes at the relevant points below.

In our iteration, at each regular γ we will force with a poset $\mathbb{Q}_\gamma$ which adds a $GS(\gamma)$-sequence by approximation via initial segments. A potential problem with this strategy is that *a priori* there may not be enough conditions in $\mathbb{Q}_\gamma$, in fact what we need (see Claim 6.11 for the details) is that $GS(\alpha)$-sequences already exist for all ordinals $\alpha < \gamma$; we will arrange this using the fact that we already forced with $\mathbb{Q}_\mu$ for all regular $\mu < \gamma$ and the following sequence of technical lemmas.

Lemma 6.8. *Let ν be an infinite cardinal. If there exists a $GS(\nu)$-sequence, then there exists a $GS(\eta)$-sequence for every $\eta < \nu^+$.*

Proof. Let $\langle C_\alpha \colon \alpha < \nu, \mathrm{cf}(\alpha) < \alpha \rangle$ be a $GS(\nu)$-sequence. We prove the existence of a $GS(\eta + 1)$-sequence by induction on limit η in the interval $[\nu, \nu^+)$.

Case 1: $\eta = \nu$. If ν is regular there is nothing to do, so we assume that ν is singular. Choose $\langle \nu_i \colon i < \mathrm{cf}(\nu) \rangle$ increasing, continuous and cofinal in ν with $\nu_0 = 0$ and $\mathrm{cf}(\nu) < \nu_1$. Define for singular ordinals $\alpha \leq \nu$

$$D_\alpha = \begin{cases} \{\nu_i \colon i < j\} & \alpha = \nu_j,\ j \text{ limit} \\ \{\nu_i \colon i < \mathrm{cf}(\nu)\} & \alpha = \nu \\ C_\alpha \setminus (\nu_i + 1) & \nu_i < \alpha \leq \nu_{i+1} \end{cases}$$

Case 2: $\eta = \gamma + \omega$, γ limit. Let $\langle D_\alpha\colon \alpha \leq \gamma, \mathrm{cf}(\alpha) < \alpha\rangle$ be a $GS(\gamma + 1)$-sequence. We may extend this to be a $GS(\eta + 1)$-sequence by defining $D_\eta = \{\gamma + n\colon n < \omega\}$.

Case 3: $\nu < \eta < \nu^+$, η a limit of limit ordinals. Let $\mathrm{cf}(\eta) = \mu$, where necessarily $\mu \leq \nu$. Choose $\langle \eta_i\colon i < \mu\rangle$ increasing, continuous and cofinal in η, in such a way that

(1) $\eta_0 = 0$.
(2) η_{i+1} is a singular limit ordinal for all i.
(3) $\eta_1 > \nu$.

Fix $\langle C^{i+1}_\alpha\colon \alpha \leq \eta_{i+1}, \mathrm{cf}(\alpha) < \alpha\rangle$ a $GS(\eta_{i+1} + 1)$-sequence for each $i < \mu$.

Define for singular ordinals $\alpha \leq \eta$

$$D_\alpha = \begin{cases} \{\eta_i\colon i < j\} & \alpha = \eta_j,\ j \text{ limit} \\ \{\eta_i\colon i < \mu\} & \alpha = \eta \\ C^{i+1}_\alpha \setminus (\eta_i + 1) & \eta_i < \alpha \leq \eta_{i+1} \end{cases}$$

□

Lemma 6.9. *If ν is a singular cardinal and there is a $GS(\mu)$-sequence for every regular $\mu < \nu$, then there is a $GS(\nu)$-sequence.*

Proof. Like Case 3 in Lemma 6.8. □

Lemma 6.10. *Let γ be a cardinal and suppose that for every regular cardinal $\mu < \gamma$ there is a $GS(\mu)$-sequence. Then for every ordinal $\alpha < \gamma$ there is a $GS(\alpha)$-sequence.*

Proof. If γ is a limit cardinal then there are unboundedly many regular cardinals less than γ, and the result is clear. So suppose $\gamma = \mu^+$ for some cardinal μ. If μ is regular then there is a $GS(\mu)$-sequence by assumption, if μ is singular then there is a $GS(\mu)$-sequence by Lemma 6.9. In either case, by Lemma 6.8 there is a $GS(\alpha)$-sequence for every $\alpha < \mu^+ = \gamma$. □

We can now describe our iterated forcing construction. Given a regular cardinal γ we define a poset $\mathbb{Q}_\gamma$. $p \in \mathbb{Q}_\gamma$ if and only if $p = \langle C_\alpha\colon \mathrm{cf}(\alpha) < \alpha, \alpha \leq \beta\rangle$ where

(1) β is a singular limit ordinal less than γ.
(2) p is a $GS(\beta + 1)$-sequence.

If $p = \langle C_\alpha \colon \mathrm{cf}(\alpha) < \alpha, \alpha \leq \beta \rangle$ and $q = \langle D_\alpha \colon \mathrm{cf}(\alpha) < \alpha, \alpha \leq \beta^* \rangle$ are in $\mathbb{Q}_\gamma$ then $p \leq q$ iff $\beta \geq \beta^*$ and $C_\alpha = D_\alpha$ for all $\alpha \leq \beta^*$. We note that by GCH $\mathbb{Q}_\gamma$ is a poset of size at most γ, and so trivially has the γ^+-chain condition.

Claim 6.11. *If there is a $GS(\alpha)$-sequence for every $\alpha < \gamma$ then forcing with $\mathbb{Q}_\gamma$ adds a $GS(\gamma)$-sequence.*

Proof. We need to check that for every $\zeta < \gamma$ the set of $GS(\zeta + 1)$-sequences is dense. Let $p = \langle C_\alpha \colon \mathrm{cf}(\alpha) < \alpha, \alpha \leq \beta \rangle$ be a $GS(\beta + 1)$ sequence for some singular ordinal β, and let ζ be a singular ordinal with $\beta < \zeta < \gamma$. Let $q = \langle D_\alpha \colon \mathrm{cf}(\alpha) < \alpha, \alpha \leq \zeta \rangle$ be a $GS(\zeta + 1)$-sequence.

We define E_α for singular α with $\alpha \leq \zeta$ by letting $E_\alpha = C_\alpha$ for $\alpha \leq \beta$ and $E_\alpha = D_\alpha \setminus (\beta + 1)$ for $\beta < \zeta \leq \zeta$. It is routine to check that if $r = \langle E_\alpha \colon \mathrm{cf}(\alpha) < \alpha, \alpha \leq \zeta \rangle$ then r is a $GS(\zeta + 1)$-sequence extending p. $\square$

We define $\mathbb{P}_{\delta+1}$ to be the Reverse Easton iteration of $\mathbb{Q}_\gamma$ for regular $\gamma \leq \delta$. To be a little more explicit we define sequences $\langle \mathbb{P}_\alpha \colon \alpha \leq \delta + 1 \rangle$ and $\langle \dot{\mathbb{Q}}_\alpha \colon \alpha \leq \delta \rangle$ inductively by

(1) $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the version of $\mathbb{Q}_\alpha$ computed by $V^{\mathbb{P}_\alpha}$, if α is regular in $V^{\mathbb{P}_\alpha}$ (which will turn out to be the case for every regular α, see claim 6.12). Otherwise $\dot{\mathbb{Q}}_\alpha$ names the trivial forcing.
(2) $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.
(3) For $\lambda \leq \delta$ a limit ordinal, $\mathbb{P}_\lambda$ is the direct limit of $\langle \mathbb{P}_\alpha \colon \alpha < \lambda \rangle$ for λ inaccessible, and the inverse limit otherwise.

Claim 6.12. *Let γ be regular. Then in $V^{\mathbb{P}_\gamma}$*

1_γ. *For every condition $p \in \mathbb{Q}_\gamma$ and every $\zeta < \gamma$ there is a condition $q \in \mathbb{Q}_\gamma$ with $q \leq p$ and* $\max(\mathrm{dom}(q)) \geq \zeta$.
2_γ. *$\mathbb{Q}_\gamma$ is $< \gamma$-strategically closed.*
3_γ. *Cardinals and cofinalities are preserved.*

Proof. We proceed by induction. Assume that we have 1_δ, 2_δ and 3_δ for regular $\delta < \gamma$. We start by outlining the argument that $\mathbb{P}_\gamma$ preserves all cardinals and cofinalities.

Given a cardinal $\mu < \gamma$, we factor $\mathbb{P}_\gamma$ in the standard way as $\mathbb{P}_\mu * \mathbb{Q}_\mu * \mathbb{R}$ and note that μ^+ is the first point at which the iteration

$\mathbb{R}$ does non-trivial forcing. The arguments of [2], suitably adapted for strategically closed forcing, gives us that $\mathbb{R}$ is $< \mu^+$-strategically closed in $V^{\mathbb{P}_{\mu+1}}$. The usual counting arguments give us that $\mathbb{P}_{\mu+1}$ is always μ^+-c.c. A suitable adaptation of Easton's arguments from [8] shows that all μ-sequences from $V^{\mathbb{P}_\gamma}$ must lie in $V^{\mathbb{P}_{\mu+1}}$, and arguments exactly like those of [8] then show that all cardinals and cofinalities are preserved in $V^{\mathbb{P}_\gamma}$.

It follows by the induction hypothesis and Claim 6.11 that in $V^{\mathbb{P}_\gamma}$ we will have a $GS(\delta)$-sequence for every regular δ with $\delta < \gamma$. By Lemma 6.10 there is a $GS(\alpha)$-sequence for every ordinal $\alpha < \gamma$. By Lemma 6.11 again 1_γ holds in $V^{\mathbb{P}_\gamma}$.

Recall the strategic closure game from Definition 6.7. We describe a winning strategy for player Even in the game of length $\alpha + 1$ played on $\mathbb{Q}_\gamma$, where $\alpha < \gamma$ is a limit ordinal. Let $p_\beta = \langle E_\zeta \colon \mathrm{cf}(\zeta) < \zeta, \zeta \leq \gamma_\beta \rangle$ be the condition which is played at stage β, where player Even's strategy will guarantee that $\langle \gamma_\beta \colon \beta \leq \alpha \rangle$ is continuous.

Case 1: $\beta = 2$**:** Even plays a condition $p_2 \leq p_1$ with $\gamma_2 > \alpha$. Notice that for all limit $\beta \leq \alpha$ we will have $\mathrm{cf}(\gamma_\beta) = \mathrm{cf}(\beta) \leq \alpha < \gamma_2 < \gamma_\beta$.

Case 2: $\beta = \beta_0 + 2$**,** $\beta_0 > 0$ **even:** Even sets $\gamma_\beta = \gamma_{\beta_0+1} + \omega$ and $E_{\gamma_\beta} = \{\gamma_{\beta_0+1} + n \colon n < \omega\}$.

Case 3: $\lim(\beta)$**:** Even sets $\gamma_\beta = \sup_{\bar\beta < \beta} \gamma_{\bar\beta}$ and $E_{\gamma_\beta} = \{\gamma_{\bar\beta} \colon \bar\beta < \beta\}$. This is a legal move because

(1) If $\zeta \in \lim(E_{\gamma_\beta})$ then $\zeta = \gamma_{\bar\beta}$ for $\bar\beta$ limit, and so

$$E_\zeta = \{\gamma_\eta \colon \eta < \bar\beta\} = E_{\gamma_\beta} \cap \zeta.$$

(2) $\mathrm{ot}(E_{\gamma_\beta}) = \beta \leq \alpha < \gamma_2 < \gamma_\beta$.

It is routine to check that this is a winning strategy, concluding the proof of Claim 6.12. □

We will choose G to be some $\mathbb{P}_\delta$-generic filter subject to a certain technical condition; if $\vec{C}^\kappa$ and $\vec{C}^\lambda$ are the sequences added at stages κ and λ, then we choose G so that $\vec{C}^\lambda \restriction \kappa = \vec{C}^\kappa$. This is possible because κ is regular in $V[G_\lambda]$, so that $\vec{C}^\kappa$ can be extended [2] to a condition M in $\mathbb{Q}_\lambda$ and we may force below M to get $\vec{C}^\lambda$ as desired.

The reason for doing this is explained in detail in Claim 6.15 below; in the jargon of large cardinal theorists M is a "master condition" in $\mathbb{Q}_\lambda$, which is to say that forcing below M at stage λ will guarantee that we can lift our original embedding j to an elementary embedding $j\colon H_{\kappa^+}[G_\kappa * g_\kappa] \longrightarrow H_{\lambda^+}[G_\lambda * g_\lambda]$. For more about master conditions see the section on Reverse Easton forcing in Baumgartner's survey paper [2].

By Lemma 6.12, $V[G][g]$ has the same cardinals and cofinalities as V and δ is inaccessible in $V[G][g]$. This implies that every set of rank less than δ is coded by a bounded subset of δ, so $V_\delta^{V[G]} = V_\delta^{V[G][g]}$; to save the reader from a plague of superscripts we denote this set by $V_\delta[G]$ in what follows.

Let $\vec{C}$ be the generic $GS(\delta)$-sequence added by g. The structure $(V_\delta[G], \in, \vec{C})$ is a model of $\mathsf{ZFC}_{\vec{C}}$ because $V[G][g]$ is a model of ZFC, δ is inaccessible in $V[G][g]$ and $\vec{C} \in V[G][g]$.

To finish, we show that in $V_\delta[G]$ there is an elementary embedding from $H_{\kappa^+}^{V_\delta[G]}$ to $H_{\lambda^+}^{V_\delta[G]}$. It will then follow that κ is 1-extendible in $V_\delta[G]$. The elements of H_{κ^+} are coded by subsets of κ and no subsets of κ are added past stage κ of the iteration so $H_{\kappa^+}^{V_\delta[G]} = H_{\kappa^+}^{V[G_{\kappa+1}]} = H_{\kappa^+}^{V_\delta[G_{\kappa+1}]}$ Similarly we see that $H_{\lambda^+}^{V_\delta[G]} = H_{\lambda^+}^{V[G_{\lambda+1}]} = H_{\lambda^+}^{V_\delta[G_{\lambda+1}]}$.

The intuition behind the rest of the proof is that we want to treat $H_{\kappa^+}^{V_\delta[G_{\kappa+1}]}$ as a generic extension of H_{κ^+} by $\mathbb{P}_{\kappa+1}$, and then apply the techniques of Reverse Easton forcing to lift the embedding j. The argument requires a little care because H_{κ^+} is not a model of ZFC.

Since $|\mathbb{P}_{\kappa+1}| = \kappa$, every element of $H_{\kappa^+}^{V_\delta[G_{\kappa+1}]}$ has the form $\dot{\tau}^{G_{\kappa+1}}$ for some $\mathbb{P}_{\kappa+1}$-name $\dot{\tau} \in H_{\kappa^+}$. What is more $\mathbb{P}_{\kappa+1} \in H_{\kappa^+}$. A tedious but routine argument now shows that for any formula ϕ there exists a formula ϕ^* such that for any $\mathbb{P}_{\kappa+1}$-name $\dot{\tau}$ and condition $p \in \mathbb{P}_{\kappa+1}$

$$p \Vdash^V_{\mathbb{P}_{\kappa+1}} \text{“}H_{\kappa^+}^{V_\delta[\dot{G}_{\kappa+1}]} \models \phi(\dot{\tau}^{\dot{G}_{\kappa+1}})\text{”} \iff H_{\kappa^+} \models \phi^*(p, \mathbb{P}_{\kappa+1}, \dot{\tau}).$$

[2]Define M to agree with $\vec{C}^\kappa$ up to κ, to have $\kappa + \omega$ as the largest point in its domain and to associate $\{\kappa + n \colon n < \omega\}$ to $\kappa + \omega$.

Abusing notation slightly we write $p \Vdash^{H^+_\kappa}_{\mathbb{P}_{\kappa+1}} \phi(\dot\tau)$ for this relation, where the key point is that the relation is definable in H_{κ^+}. An exactly similar analysis works for $H^{V_\delta[G_\kappa]}_{\kappa^+}$ and we write $p \Vdash^{H^+_\kappa}_{\mathbb{P}_\kappa} \phi(\dot\tau)$ as an abbreviation for the indigestible $p \Vdash^V_{\mathbb{P}_\kappa}$ "$H^{V_\delta[\dot G_\kappa]}_{\kappa^+} \models \phi(\dot\tau_{\dot G_\kappa})$". In line with our intuitive remarks above we further abuse notation and write $H_{\kappa^+}[G_\kappa]$ for $H^{V_\delta[G_\kappa]}_{\kappa^+}$, $H_{\kappa^+}[G_{\kappa+1}]$ for $H^{V_\delta[G_{\kappa+1}]}_{\kappa^+}$, $H_{\lambda^+}[G_\lambda]$ for $H^{V_\delta[G_\lambda]}_{\lambda^+}$, and $H_{\lambda^+}[G_{\lambda+1}]$ for $H^{V_\delta[G_{\lambda+1}]}_{\lambda^+}$.

As usual, the problem is to lift the embedding j. We break up $G_{\lambda+1}$ as $G_\kappa * g_\kappa * H * g_\lambda$, where g_κ is the generic object added at κ, H the generic object added between κ and λ, and g_λ the generic object added at λ. Here $G_\lambda = G_\kappa * g_\kappa * H$ will be the generic object for $\mathbb{P}_\lambda$.

Claim 6.13. *j"$G_\kappa \subseteq G_\lambda$.*

Proof. Let $p \in G_\kappa$, then since we did a Reverse Easton iteration we know that the support of p is some ordinal α with $\alpha < \kappa$. Now $\mathrm{crit}(j) = \kappa$, so the support of $j(p)$ is also α and $p \restriction \alpha = j(p) \restriction \alpha$. So clearly we have $j(p) \in G_\kappa * g_\kappa * H = G_\lambda$, as desired. □

We now attempt to extend the embedding j to the larger domain $H_{\kappa^+}[G_\kappa]$ by defining $j(\dot\tau^{G_\kappa}) = j(\dot\tau)^{G_\lambda}$ for all $\dot\tau \in H_{\kappa^+}$.

Claim 6.14. *This definition gives a well-defined elementary embedding $j\colon H_{\kappa^+}[G_\kappa] \longrightarrow H_{\lambda^+}[G_\lambda]$ which extends our original map $j\colon H_{\kappa^+} \longrightarrow H_{\lambda^+}$.*

Proof. Suppose that $\dot\tau^{G_\kappa} = \dot\sigma^{G_\kappa}$. By the truth lemma there is $p \in G_\kappa$ such that $p \Vdash^{H_{\kappa^+}}_{\mathbb{P}_\kappa} \dot\tau = \dot\sigma$. This is a first-order statement in H_{κ^+} and so since j is elementary $j(p) \Vdash^{H_{\lambda^+}}_{\mathbb{P}_\lambda} j(\dot\tau) = j(\dot\sigma)$. $j(p) \in G_\lambda$ and so $j(\dot\tau)^{G_\lambda} = j(\dot\sigma)^{G_\lambda}$.

The proofs that the map we have defined is elementary and extends the original map are very similar. □

Claim 6.15. *j"$g_\kappa \subseteq g_\lambda$.*

Proof. Let $p \in g_\kappa$. Then p is in an initial segment of $\vec{C}^\kappa$, and $j(p) = p$. Since we chose $\vec{C}^\lambda$ to extend $\vec{C}^\kappa$, $j(p) \in g_\lambda$. □

By the same method as in Claim 6.14 we may further extend j to get an elementary embedding $j\colon H_{\kappa^+}[G_\kappa * g_\kappa] \longrightarrow H_{\lambda^+}[G_\lambda * g_\lambda]$. Since

$H_{\kappa^+}[G_\kappa * g_\kappa] = H_{\kappa^+}^{V_\delta[G]}$ and $H_{\lambda^+}[G_\lambda * g_\lambda] = H_{\lambda^+}^{V_\delta[G]}$, we have shown that κ is 1-extendible in $V_\delta[G]$. This concludes the proof of Theorem 6.5.

References

[1] A. Apter and J. Cummings. A global version of a theorem of Ben-David and Magidor. *Annals of Pure and Applied Logic*, 102(3):199–222, 2000.

[2] J. Baumgartner. Iterated forcing. In A. R. D. Mathias, editor, *Surveys in Set Theory*, pages 1–59. Cambridge University Press, Cambridge, 1983.

[3] D. Burke. Generic embeddings and the failure of box. *Proceedings of the American Mathematical Society*, 123(9):2867–2871, 1995.

[4] J. Cummings, M. Džamonja, and S. Shelah. A consistency result on weak reflection. *Fund. Math.*, 148(1):91–100, 1995.

[5] J. Cummings, M. Foreman, and M. Magidor. Squares, scales and stationary reflection. *Journal of Mathematical Logic*, 1(1):35–99, 2001.

[6] K. Devlin and R. Jensen. Marginalia to a theorem of Silver. In *Logic Colloquium, Kiel, 1974*, pages 115–142, Berlin, 1975. Springer-Verlag.

[7] M. Džamonja and S. Shelah. On squares, outside guessing of clubs and $I_{<f}[\lambda]$. *Fundamenta Mathematicae*, 148(2):165–198, 1995.

[8] W. Easton. Powers of regular cardinals. *Annals of Mathematical Logic*, 1:139–178, 1970.

[9] M. Foreman. Games played on boolean algebras. *Journal of Symbolic Logic*, 48:714–723, 1983.

[10] M. Gitik. Some results on the nonstationary ideal. II. *Israel Journal of Mathematics*, 99:175–188, 1997.

[11] R. Jensen. Some remarks on $\square$ below zero-pistol. Circulated notes.

[12] R. Jensen. The fine structure of the constructible hierarchy. *Annals of Mathematical Logic*, 4:229–308, 1972.

[13] R. Jensen. Circulated notes, 1998.

[14] A. Kanamori. *The higher infinite*. Springer-Verlag, Berlin, 1994. Large cardinals in set theory from their beginnings.

[15] B. Löwe and J. Steel. An introduction to core model theory. In *Sets and proofs (Leeds, 1997)*, pages 103–157. Cambridge Univ. Press, Cambridge, 1999.

[16] M. Magidor. Reflecting stationary sets. *Journal of Symbolic Logic*, 47:755–771, 1982.

[17] M. Magidor and S. Ben-David. The weak $\square^*$ is really weaker than the full $\square$. *Journal of Symbolic Logic*, 51:1029–1033, 1986.

[18] M. Magidor and S. Shelah. The tree property at successors of singular cardinals. *Archive for Mathematical Logic*, 35(5-6):385–404, 1996.

[19] W. Mitchell and J. Steel. *Fine structure and iteration trees*, volume 3 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1994.

[20] E. Schimmerling. Combinatorial principles in the core model for one Woodin cardinal. *Annals of Pure and Applied Logic*, 74:153–201, 1995.

[21] E. Schimmerling. A finite family weak square principle. *Journal of Symbolic Logic*, 64(3):1087–1110, 1999.

[22] E. Schimmerling and M. Zeman. Square in core models. *Bulletin of Symbolic Logic*, 7(3):305–314, 2001.

[23] J. Steel. An outline of inner model theory. To appear in the Handbook of Set Theory.

[24] J. Steel. *The core model iterability problem*, volume 8 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1996.

[25] S. Todorčević. A note on the Proper Forcing Axiom. In J. E. Baumgartner, D. A. Martin, and S. Shelah, editors, *Axiomatic Set Theory*, pages 209–218. American Mathematical Society, Providence RI, 1983.

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213-3890, USA

E-mail address: `jcumming@andrew.cmu.edu`

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213-3890, USA

E-mail address: `eschimme@andrew.cmu.edu`