Singular cardinal problems

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(Cantor) $2^\kappa > \kappa$.

(König) $\kappa^{\mathrm{cf}(\kappa)} > \kappa$, so in particular $\mathrm{cf}(2^\lambda) > \lambda$ for all λ.

(Easton) If GCH holds and $F : REG \longrightarrow CARD$ is a class function such that $\kappa < \lambda \Longrightarrow F(\kappa) \leq F(\lambda)$, $\mathrm{cf}(F(\kappa)) > \kappa$ then there is a cardinal and cofinality preserving class generic extension in which $2^\kappa = F(\kappa)$ for every regular κ.

(Scott) If U is a normal measure on κ, $\mu < \kappa$ and $\{\alpha < \kappa : 2^\alpha \leq \alpha^{+\mu}\} \in U$ then $2^\kappa \leq \kappa^{+\mu}$. In particular GCH does not first fail at κ.

(Silver) If κ is singular strong limit of uncountable cofinality, $\mu < \kappa$ and $\{\alpha < \kappa : 2^\alpha \leq \alpha^{+\mu}\}$ is stationary then $2^\kappa \leq \kappa^{+\mu}$. In particular GCH does not first fail at κ.

(Galvin and Hajnal) If κ is singular strong limit of uncountable cofinality, and $\kappa = \aleph_\eta$ for some $\eta < \kappa$ (that is to say κ is not a cardinal fixed point) then $2^\kappa < \aleph_{(2^{|\eta|})^+}$.

The Singular Cardinals Hypothesis (SCH): For all singular λ

$$\lambda^{\mathrm{cf}(\lambda)} = \max\{\lambda^+, 2^{\mathrm{cf}(\lambda)}\}.$$

Fact: if SCH holds then cardinal arithmetic is determined by the continuum function on the regulars and the cofinality function, and roughly speaking λ^μ has the least value possible. To be more precise if SCH holds then

1. Let κ be singular. If the continuum function is eventually constant below κ then $2^\kappa = 2^{<\kappa}$, otherwise $2^\kappa = (2^{<\kappa})^+$.

2. If $\lambda \leq 2^\mu$, $\lambda^\mu = 2^\mu$. If $2^\mu < \lambda$ then $\lambda^\mu = \lambda$ if $\mu < \mathrm{cf}(\lambda)$, and $\lambda^\mu = \lambda^+$ if $\mu \geq \mathrm{cf}(\lambda)$.

(Solovay) If κ is strongly compact and $\lambda > \kappa$ is singular, strong limit then $2^\lambda = \lambda^+$.

The following are equiconsistent:

1. SCH fails.
2. GCH first fails at $\aleph_\omega$.
3. GCH fails at a measurable
4. There exists κ with $o(\kappa) = \kappa^{++}$.

Outline of the lectures.

1) How close can we come to proving SCH? Pcf theory (Shelah)

2) Upper bounds for the failure of SCH. Forcing (Gitik, Magidor, Woodin).

3) Lower bounds for the failure of SCH. Core models (Dodd, Gitik, Jensen, Mitchell)

4) Other combinatorics at singulars. Scales, squares, reflection (Cummings, Foreman, Magidor, Shelah).

Let $\mathbb{P}$ be a poset.

1. $A \subseteq \mathbb{P}$ is *cofinal* iff $\forall p \in \mathbb{P}\ \exists q \in A\ p \leq q$.

2. $\mathrm{cf}(\mathbb{P}) = \min\{|A| : A$ is cofinal in $\mathbb{P}\}$.

3. $\langle p_i : i < \lambda \rangle$ is a *scale in* $\mathbb{P}$ iff $\forall i < j\ p_i < p_j$, $\{p_i : i < \lambda\}$ is cofinal and $\lambda = \mathrm{cf}(\lambda)$.

4. $\mathrm{tcf}(\mathbb{P}) = \lambda$ iff there is a scale of length λ in $\mathbb{P}$.

5. $\mathbb{P}$ is λ-*directed* iff for all $A \subseteq \mathbb{P}$ with $|A| < \lambda$ there is $q \in A$ such that $\forall p \in A\ p \leq q$.

6. Let $\vec{p} = \langle p_i : i < \lambda \rangle$ be increasing. q is a *bound* for $\vec{p}$ iff $\forall i\ p_i \leq q$. q is an *least upper bound (lub)* iff $q \leq r$ for all bounds r.

Let I be an ideal on a set X, let F be a filter on X.

1. If $f, g \in {}^X ON$ then

$$\begin{array}{rcl} f <_I g & \iff & \{x : f(x) \geq g(x)\} \in I \\ f \leq_I g & \iff & \{x : f(x) > g(x)\} \in I \\ f <_F g & \iff & \{x : f(x) < g(x)\} \in F \\ f \leq_F g & \iff & \{x : f(x) \leq g(x)\} \in F \end{array}$$

   We study $({}^X ON, \leq_I)$.

   Remark: "$f <_I g$" is (in general) stronger than "$f \leq_I g$ and $f \neq_I g$".

2. Let $\vec{f} = \langle f_i : i < \lambda \rangle$ be a sequence such that $\forall i < j \; f_i <_I f_j$. Then f is an *exact upper bound (eub)* for $\vec{f}$ iff f is a bound for $\vec{f}$ and

$$\forall g <_I f \; \exists i < \lambda \; g <_I f_i.$$

Fact: an eub is an lub.

Proof: let f be an eub for $\vec{f}$ which is not an lub. Let g be a bound such that $A = \{x : g(x) < f(x)\} \in I^+$. Let h be such that $h \restriction A = g \restriction A$ and $h \restriction A^c = 0$. Then $h <_I f$, so $h <_I f_i$ for some i and g is not a bound. Contradiction.

The converse is false in general.

Example: let I be the NS ideal on $\aleph_1$, and let $f_\alpha(i) = \alpha$ for $\alpha < \aleph_1$. Then $f = \mathrm{id}$ is an lub but not an eub for $\langle f_\alpha : \alpha < \aleph_1 \rangle$.

Proof: f is an lub by Fodor. If we write $\aleph_1 = \bigcup_{\beta < \aleph_1} S_\beta$ with S_β stationary, and define $g(i)$ to be the least β such that $i \in S_\beta$ if $\beta < i$ and 0 otherwise, then $g <_I f$ but g is above each f_α for $\alpha < \aleph_1$ on a positive set.

Convention: A is usually a set of regular cardinals with $|A|^+ < \min(A)$ and no largest element. D is usually an ultrafilter on A.

Remark: for any singular μ we may choose such an A with $\sup(A) = \mu$. If $\mu < \aleph_\mu$ we may in addition choose A to be an interval of regular cardinals.

$$\mathrm{pcf}(A) = \{\lambda : \exists D\ \mathrm{cf}(\prod A/D) = \lambda\}.$$

$$J_{<\lambda}(A) = \{B : \forall D \ni B\ \mathrm{cf}(\prod A/D) < \lambda\}.$$

Remark: clearly $|\mathrm{pcf}(A)| \leq 2^{2^{|A|}}$, $\sup \mathrm{pcf}(A) \leq |\prod A|$.

Some basic facts.

1. $J_{<\lambda}$ is an ideal.

2. $\prod A/J_{<\lambda}$ is λ-directed.

3. $\mathrm{cf}(\prod A/D) < \lambda$ iff $D \cap J_{<\lambda} \neq \emptyset$.

4. $\lambda \in \mathrm{pcf}(A)$ iff $J_{<\lambda^+} \setminus J_{<\lambda} \neq \emptyset$.

5. For λ a limit cardinal, $J_{<\lambda} = \bigcup_{\mu<\lambda} J_{<\mu}$.

6. $\mathrm{pcf}(A)$ has a maximal element, namely the least λ where $J_{<\lambda^+} = P(A)$.

7. $|\mathrm{pcf}(A)| \leq 2^{|A|}$.

Fact: $\prod A/J_{<\lambda}$ is λ-directed.

Proof: let $J = J_{<\lambda}$, $\kappa = |A|$. Reduce to the case where $|A|^+ < \mu = \mathrm{cf}(\mu) < \lambda$ and we need an upper bound for $\langle f_\alpha : \alpha < \mu \rangle$ which is $<_J$-increasing. Suppose for a contradiction that no bound exists and build a pointwise strictly increasing sequence $\langle g_\beta : \beta < \kappa^+ \rangle$ of elements of $\prod A$ as follows.

g_β is not an upper bound so $\{i : f_\alpha(i) > g_\beta(i)\} \in J^+$ for all large α, and we can find D such that $\mathrm{cf}(\prod A/D) \geq \lambda$ and $g_\beta <_D f_\alpha$ for all large α. Choose $g_{\beta+1}$ so that $f_\alpha <_D g_{\beta+1}$ for all α; for all large α there is i such that

$$g_\beta(i) < f_\alpha(i) < g_{\beta+1}(i).$$

Find α so large that this holds for every $\beta < \kappa^+$, and then find i which works for two values of β. Contradiction since the g_β are pointwise increasing.

More basic facts.

1. If $B \in J_{<\lambda^+} \setminus J_{<\lambda}$, then $\mathrm{tcf}(\prod B / J_{<\lambda}) = \lambda$.

2. $J_{<\lambda^+}$ is generated from $J_{<\lambda}$ by a single set: that is to say there is a set B such that $J_{<\lambda^+} = J_{<\lambda} + B = \{C : C \setminus B \in J_{<\lambda}\}$.

3. Let $\langle B_\lambda : \lambda \in \mathrm{pcf}(A) \rangle$ be such that B_λ generates $J_{<\lambda^+}$ over $J_{<\lambda}$, and for each λ fix $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ a scale in $\prod B_\lambda / J_{<\lambda}$. Then for any D we have

$$\mathrm{cf}(\prod A/D) = \min\{\mu : B_\mu \in D\},$$

and if $\mathrm{cf}(\prod A/D) = \lambda$ then $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ is a scale in $\prod B_\lambda / D$.

Shelah's trichotomy theorem.

Theorem: let I be an ideal on κ, let $\langle f_\alpha : \alpha < \lambda\rangle$ be a $<_I$-increasing sequence from ${}^\kappa ON$ with $\lambda = \mathrm{cf}(\lambda) > \kappa^+$. Then one of the following holds:

(The Good case) There exists an eub f for $\vec{f}$.

(The Bad case) There exists an ultrafilter D on κ with $D \cap I = \emptyset$, and a sequence of sets of ordinals $\langle S_i : i \leq \kappa\rangle$ with $|S_i| \leq \kappa$, such that

$$\forall\alpha < \lambda\ \exists h \in \prod_i S_i\ \exists\beta < \lambda\ f_\alpha \leq_D h <_D f_\beta.$$

(The Ugly Case) For some $g \in {}^\kappa ON$ the sequence of sets $\{i : f_\alpha(i) < g(i)\}$ is not eventually constant modulo I.

Remark : if I is prime then the Ugly case cannot occur, and the only D which can occur in the Bad case is the dual of I.

Remark: If $\mathrm{cf}(\lambda) > 2^\kappa$, easy counting arguments show we are in the good case.

Proof : We assume that we are not in the Good, Bad or Ugly case and work towards contradiction.

Step One: show that since we are not in Ugly case, an lub for $\vec{f}$ is automatically an eub.

Proof: let f be an lub which fails to be an eub. Let $g <_I f$ be such that

$$A_\alpha = \{i : g(i) > f_\alpha(i)\} \in I^+$$

for all α. As we are not in Ugly case the sequence of A_α eventually stabilises; fix A which is equal to A_α mod I for all large α. Define h so that $h \restriction A = g \restriction A$ and $h \restriction A^c = f \restriction A^c$, then h is a bound and below f on a positive set. Contradiction!

Step Two: show there is $\langle g_\beta : \beta < \kappa^+ \rangle$ a strictly $\leq_I$-decreasing sequence such that each g_β is a bound for $\vec{f}$.

Proof: Build $\vec{g}$ inductively. By Step One g_β is not an lub, so we may choose $g_{\beta+1}$ suitably. At limit β define

$$h^\beta_\alpha(i) = \min(\{g_\delta(i) : \delta < \beta\} \setminus f_\alpha(i)).$$

We claim that h^β_α stabilises mod I for large α, so that we may continue by choosing g_β as a representative of the stable class.

Suppose h^β_α does not stabilise; this means that for a fixed α, $A_{\alpha\gamma} = \{i : h^\beta_\alpha(i) < f_\gamma(i)\} \in I^+$ for all large γ. As we are not in Ugly case we may choose A_α such that $A_\alpha =_I A_{\alpha\gamma}$ for all large γ. The A_α are positive and decreasing mod I, so choose D such that $D \cap I = \emptyset$ and $A_\alpha \in D$ for all α. By construction

$$\forall\alpha\, \exists\gamma\; f_\alpha \leq_D h^\beta_\alpha <_D f_\gamma$$

so setting $S_i = \{g_\delta(i) : \delta < \beta\}$ we are in Bad case. Contradiction.

Step Three: no sequence as constructed in Step Two can exist.

Proof: let

$$h_\alpha(i) = \min(\{g_\delta(i) : \delta < \kappa^+\} \setminus f_\alpha(i)).$$

For all α there exists limit $\beta < \kappa^+$ such that $h_\alpha = h_\alpha^\beta$. Fix β such that $h_\alpha = h_\alpha^\beta$ for unboundedly many α, then choose α such that $h_\alpha = h_\alpha^\beta =_I g_\beta$. This is a contradiction since by construction $h_\alpha \leq_I g_{\beta+1}$.

This concludes the proof.

Remark: if Bad and Ugly fail then an eub f exists. f will have the property that $\mathrm{cf}(f(i)) > \kappa$ for almost all i, otherwise we would be Bad.

Definition: Let $\langle f_\alpha : \alpha < \lambda \rangle$ be $<_I$-increasing. β is a *good point for $\vec{f}$* iff $\mathrm{cf}(\beta) > \kappa$ and there is a pointwise strictly increasing sequence $\langle h_\gamma : \gamma < \mathrm{cf}(\beta) \rangle$ such that $\vec{h}$ is "cofinally interleaved" with $\vec{f} \restriction \beta$ mod I; that is $\forall \alpha < \beta \, \exists \gamma < \mathrm{cf}(\beta) \; f_\alpha <_I h_\gamma$ and $\forall \gamma < \mathrm{cf}(\beta) \, \exists \alpha < \beta \; h_\gamma <_I f_\alpha$.

Fact: if $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is $<_I$-increasing, $\lambda = \mathrm{cf}(\lambda) > \kappa^+$, and the set of good points is stationary in λ then there exists an eub for $\vec{f}$.

Proof : it is enough to show that we are not in Bad or Ugly case. Suppose D and $\vec{S}$ witness that we are in Bad case. Let

$$C = \{\gamma : \forall \alpha < \gamma \, \exists h \in \prod_i S_i \, \exists \beta < \gamma \; f_\alpha \leq_D h <_D f_\beta\}.$$

Let $\beta \in \lim(C)$ be a good point, as witnessed by $\langle h_\gamma : \gamma < \mathrm{cf}(\beta) \rangle$. Find $\langle \gamma_\delta : \delta < \mathrm{cf}(\beta) \rangle$ increasing and $H_\delta \in \prod S_i$ such that $h_{\gamma_\delta} \leq_D H_\delta <_D h_{\gamma_{\delta+1}}$.

Find $i < \kappa$ such that $h_{\gamma_\delta}(i) \leq H_\delta(i) < h_{\gamma_{\delta+1}}(i)$ for an unbounded set X of $\delta < \mathrm{cf}(\beta)$. Since $\langle h_\gamma : \gamma < \mathrm{cf}(\beta) \rangle$ is pointwise strictly increasing the values $H_\delta(i)$ for $\delta \in X$ are distinct, a contradiction since $\mathrm{cf}(\beta) > \kappa$ but $|S_i| \leq \kappa$.

The proof for Ugly is similar.

Theorem: Let δ, λ be regular with $\delta^+ < \lambda$. Then there exists $S \subseteq \lambda \cap \mathrm{cof}(\delta)$ stationary and $\langle S_\gamma : \gamma < \lambda \rangle$ such that

1. $\mathrm{ot}(S_\gamma) \leq \delta$, $S_\gamma \subseteq \gamma$.
2. $\forall \beta \in S_\gamma \; S_\beta = S_\gamma \cap \beta$.
3. If $\gamma \in S$, $\sup(S_\gamma) = \gamma$.

Definition: if S and $\vec{S}$ are as above then an $<_I$-increasing sequence $\langle f_\alpha : \alpha < \lambda \rangle$ is *docile* iff

1. $\forall \alpha \in S_\gamma \; \forall i < \kappa \; f_\alpha(i) < f_\gamma(i)$.
2. If $\gamma \in S$ then $f_\gamma(i) = \sup_{\alpha \in S_\gamma} f_\alpha(i)$.

Key point: if $\delta > \kappa$ then an docile $\vec{f}$ has a stationary set of good points, hence an eub.

In a typical application of docility we would have $\kappa = |A|$, $\delta = |A|^+$, $\lambda \in \mathrm{pcf}(A)$. Here is such an application.

Theorem: if $B \in J_{<\lambda^+} \setminus J_{<\lambda}$, $\mathrm{tcf}(\prod B / J_{<\lambda}) = \lambda$.

Proof: $I = J_{<\lambda} \cup \{C : \mathrm{tcf}(\prod C / J_{<\lambda}) = \lambda\}$ is an ideal. If $B \notin I$ choose D such that $B \in D$ and $D \cap I = \emptyset$, and observe $\mathrm{cf}(\prod B/D) = \lambda$. Build $\langle f_\alpha : \alpha < \lambda \rangle$ a sequence in $\prod B$ which is docile, increasing mod $J_{<\lambda}$ and cofinal mod D. Let g be an eub where WLOG $g(b) \leq b$ for all $b \in B$, and let $C = \{b : g(b) = b\}$.

$C \in I$, and so $B \setminus C \in D$, so modulo D we have $g \in \prod B$. Find α with $g <_D f_\alpha$. $f_\alpha <_I g$, contradiction!

A combinatorial lemma.

Lemma: let $\langle f_\beta : \beta < \kappa^+ \rangle$ be pointwise increasing and define $C_{\beta\gamma} = \{i : f_\beta(i) < f_\gamma(i)\}$. Then there is a club $E \subseteq \kappa^+$ such that $C_{\beta,\gamma}$ is constant on $[E]^2$.

Proof: $C_{\beta\gamma}$ is constant for all large γ, say $C_{\beta\gamma} = D_\beta$. D_β is constant for all large β, say $D_\beta = D$. Now let

$$E = \{\zeta : D_\zeta = D, \forall \eta < \zeta \; C_{\eta\zeta} = D_\eta\}.$$

Fact: $J_{<\lambda^+}$ is generated from $J_{<\lambda}$ by a single set.

Proof: Suppose not. Let $\kappa = |A|$, reduce to case where $\lambda = \text{cf}(\lambda) > \kappa^{++}$. Fix stationary $S \subseteq \lambda \cap \text{cof}(\kappa^+)$ and $\langle S_\alpha : \alpha < \lambda \rangle$ as above. Let $J = J_{<\lambda}$, $K = J_{<\lambda^+}$.

We will build a matrix of functions

$$\langle f_\alpha^\beta : \beta < \kappa^+, \alpha < \lambda \rangle$$

from $\prod A$ such that $\langle f_\alpha^\beta : \alpha < \lambda \rangle$ is docile and $<_J$-increasing, and $\langle f_\alpha^\beta : \beta < \kappa^+ \rangle$ is pointwise increasing. Let h^β be an eub for $\langle f_\alpha^\beta : \alpha < \lambda \rangle$, where we may assume $h^\beta(a) \leq a$ for all a; let $S_\beta = \{a : h^\beta(a) = a\}$, then it is easy to see that $S_\beta \in K$. S_β does not generate K over J.

We find $C \in K$ such that $C \setminus S_\beta \notin J$, and then choose D_β such that $C \in D_\beta$, $S_\beta \notin D_\beta$, $\mathrm{cf}(\prod A/D_\beta) = \lambda$. Now we choose $\langle f^{\beta+1}_\alpha : \alpha < \lambda\rangle$ cofinal mod D_β where $\forall\zeta\,\forall\eta\, f^\beta_\zeta <_{D_\beta} f^{\beta+1}_\eta$.

If $\beta < \gamma$, $E(\beta,\gamma) = \{\eta : \forall\zeta < \eta\; f^{\gamma+1}_\zeta <_{D_\beta} f^{\beta+1}_\eta\}$. If $\eta \in S \cap E(\beta,\gamma)$ then $f^{\beta+1}_\eta =_{D_\beta} f^{\gamma+1}_\eta$. To see this observe that by obedience there is $\zeta < \eta$ such that

$$\{i : f^{\beta+1}_\eta(i) < f^{\gamma+1}_\eta(i)\} = \{i : f^{\beta+1}_\eta(i) < f^{\gamma+1}_\zeta(i)\},$$

which is a set of D_β-measure zero.

Now choose $\eta \in S \cap (\bigcap_{\beta<\gamma<\kappa^+} E(\beta,\gamma))$, and define $C_{\beta\gamma} = \{i : f^{\beta+1}_\eta(i) < f^{\gamma+1}_\eta(i)\}$. Find $\beta_1 < \beta_2 < \beta_3$ such that $C_{\beta_1\beta_2} = C_{\beta_2\beta_3} = C_{\beta_1\beta_3}$. This is impossible because by construction

$$f^{\beta_1+1}_\eta <_{D_{\beta_2}} f^{\beta_2+1}_\eta =_{D_{\beta_2}} f^{\beta_3+1}_\eta.$$

A technical lemma.

Lemma: Suppose $\sup A < \mu = \text{cf}(\mu) < \nu$. Let J be an ideal on A such that $\prod A/J$ is ν-directed, and $A \cap \gamma \in J$ for all $\gamma < \sup A$. Then there is a $<_J$-increasing sequence $\vec{f} = \langle f_\alpha : \alpha < \mu \rangle$ such that

1. $\vec{f}$ has an eub $g \in \prod A$.

2. For all $\gamma < \sup A$, $\{a : \text{cf}(g(a)) < \gamma\} \in J$.

Proof: Fix $\langle \mathcal{C}_\alpha : \alpha < \mu \rangle$ such that

1. $\mathcal{C}_\alpha$ is a family of clubs in α, $|\mathcal{C}_\alpha| \leq \mu$.

2. There is $C \in \mathcal{C}_\alpha$ with $\text{ot}(C) = \text{cf}(\alpha)$.

3. $\forall D \in \mathcal{C}_\alpha \, \forall \beta \in \lim(D) \; D \cap \beta \in \mathcal{C}_\beta$.

At every limit stage α, compute for each $E \in \mathcal{C}_\alpha$ with $\mathrm{ot}(E) < \sup A$ the function $g^\alpha_E : a \longmapsto \sup_{\gamma \in E} f_\gamma(a)$. Choose f_α to dominate all such g^α_E.

Claim 1: Let $\sigma < \sup A$. There cannot exist an ultrafilter D with $D \cap J = \emptyset$, and a sequence of sets of ordinals $\langle S_i : i \leq \kappa \rangle$ with $|S_i| \leq \sigma$, such that

$$\forall \alpha < \mu \; \exists h \in \prod_i S_i \; \exists \beta < \mu \; f_\alpha \leq_D h <_D f_\beta.$$

Proof: let E be the club of $\gamma < \mu$ such that $\forall \alpha < \mu \; \exists h \in \prod_i S_i \; \exists \beta < \mu \; f_\alpha \leq_D h <_D f_\beta$. Let $\beta \in \lim(E) \cap \mathrm{cof}(\sigma^+)$ and let $F \in \mathcal{C}_\beta$. Choose $\langle \gamma_j : j < \sigma^+ \rangle$ increasing and $h_j \in \prod_i S_i$ such that $\gamma_j \in E \cap \lim(F)$ and $f_{\gamma_j} \leq_D h_j <_D f_{\gamma_{j+1}}$. By construction $f_{\gamma_j} >_D g^{\gamma_j}_{F \cap \gamma_j}$, so choose a_j such that

$$g^{\gamma_j}_{F \cap \gamma_j}(a_j) < f_{\gamma_j}(a_j) \leq_D h_j(a_j) <_D f_{\gamma_{j+1}}(a_j).$$

Find X unbounded such that $\forall j \in X \; a_j = a$, then by construction $\{h_j(a) : j \in X\}$ has size σ^+, contradiction.

Claim 2: We are not in the Ugly case.

Proof: Similar to Claim 1.

Conclusion: Claim 1 implies we are not in the Bad case, so by trichotomy there is a suitable g. Claim 1 implies that the cofinality of $g(a)$ tends to $\sup A$ modulo J.

Let λ be a singular cardinal, $\kappa = \mathsf{cf}(\lambda)$.

Definition (non standard): $\mathsf{PP}(\lambda)$ is the set of regular μ such that for some uf D on κ and some sequence $\langle \lambda_i : i < \kappa \rangle$ of regular cardinals we have $\lim_D \lambda_i = \lambda$, $\mu = \mathsf{cf}(\prod_i \lambda_i / D)$.

Definition: $\mathsf{pp}(\lambda)$ is the sup of $\mathsf{PP}(\lambda)$.

Theorem: $\mathsf{PP}(\lambda)$ is an interval of regular cardinals.

Proof: let $\lambda < \mu < \nu \in \mathsf{PP}(\lambda)$, say $\nu = \mathsf{cf}(\prod_i \lambda_i / D)$. Using the technical lemma, find $\langle f_\alpha : \alpha < \mu \rangle$ in $\prod_i \lambda_i$ increasing mod D with an eub g such that $\lim_D \mathsf{cf}(g(i)) = \lambda$. Now $\mathsf{cf}(\prod_i \mathsf{cf}(g(i))/D) = \mu$.

Theorem: if A is an interval of regular cardinals then so is $\mathsf{pcf}(A)$.

Proof: similar.

Convention: θ is some very large regular cardinal, and $<_\theta$ is a fixed well ordering of H_θ. We will form substructures of $(H_\theta, \in, <_\theta)$. If $X \subseteq H_\theta$ and Hull(X) is the set of points definable from parameters in X, then Hull(X) is the least substructure containing the set X.

IA chains: A chain of substructures $\vec{X}$ is *internally approachable (IA)* iff $\vec{X}$ is increasing and continuous, and $\vec{X} \restriction (\beta + 1) \in X_{\beta+1}$ for all β. It is easy to see that $\alpha \subseteq X_\alpha$ and that $\alpha < \beta \Longrightarrow X_\alpha \in X_\beta$.

Characteristic function of a structure: if B is a set of regular cardinals and $|X| < \min(B)$ then we may define $\chi_X \in \prod B$ by $\chi_X(b) = \sup(X \cap b)$ for all $b \in B$.

Fact: Let $\vec{X}$ be an IA chain, let $X = X_\alpha$ where $\mathrm{cf}(\alpha) > \omega$. Suppose $B \subseteq REG$ with $|X| < \min(B)$. Suppose $B \subseteq X_0$. Then for all $b \in B$, $X \cap b$ contains a club in $\chi_X(b)$.

Proof: $X_i, b \in X_{i+1}$ and $|X_i| < b$, so $\chi_{X_i}(b) \in X_{i+1} \cap b$. So $\langle \chi_{X_i}(b) : i < \alpha \rangle$ is continuous, increasing and cofinal in $\chi_X(b)$.

Fact: Suppose also that B is an interval of regular cardinals, $\min(B) = |X|^+$ and $|X| \subseteq X$. If $Z \subseteq X$ is such that Z is unbounded in $\chi_X(b)$ for all $b \in B$, then $\mathrm{Hull}(Z) \cap \sup B = X \cap \sup B$.

Proof: Show by induction on $b \in Z$ that

$$\mathrm{Hull}(Z) \cap b = X \cap b.$$

Step: let $\alpha \in Z \cap b^+$ and fix $f \in Z$ such that $f : \alpha \simeq b$. Then $f : \mathrm{Hull}(Z) \cap \alpha \simeq \mathrm{Hull}(Z) \cap b = X \cap b$, so $\mathrm{Hull}(Z) \cap \alpha = X \cap \alpha$. As Z is unbounded in $X \cap b^+$, we are done.

Fact: In the situation above we can reconstruct $X \cap \sup B$ from χ_X.

Proof: Intersect $\mathrm{Hull}(Z) \cap \sup B$ for all the $Z \subseteq H_\theta$ such that, for all $b \in B$, $Z \cap \chi_X(b)$ contains a club in $\chi_X(b)$.

1. For any such Z we have that $\mathrm{Hull}(Z) \cap \sup B \supseteq \mathrm{Hull}(Z \cap X) \cap \sup B = X \cap \sup B$.

2. Setting $Z = X$ gives $\mathrm{Hull}(Z) \cap \sup B = X \cap \sup B$.

So the intersection is precisely $X \cap \sup B$.

Smoothing: we may as well assume that for every limit β of cofinality $|A|^+$ we have $f^\lambda_\beta(i) = \min\{\sup_{\gamma \in C} f^\lambda_\gamma(i) : C \text{ club in } \beta\}$. Note we can fix C club in β such that for any club $D \subseteq C$ we have $f^\lambda_\beta = \sup_{\gamma \in D} f^\lambda_\gamma$.

Theorem: Let $A \subseteq REG$ be an interval of regular cardinals, where $\min(A) = \delta^+$ for some δ and $\max\{|A|^+, |\mathrm{pcf}(A)|\} < \min(A)$, Fix a sequence of generators $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ such that $B_{\max \mathrm{pcf}(A)} = A$. Fix $\langle f^\lambda_\alpha : \alpha < \lambda, \lambda \in \mathrm{pcf}(A)\rangle$ such that $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$ is cofinal in $\prod B_\lambda / J_{<\lambda}$.

Let $\langle X_\alpha : \alpha \leq |A|^+\rangle$ be an IA chain such that $A, \vec{B}, \vec{f} \in X_0$, $\delta \subseteq X_0$, $|X_\alpha| = \delta$ for all α. Let χ_α be the characteristic function of X_α, $X = X_{|A|^+}$, $\chi = \chi_{|A|^+}$. Then χ can be obtained by taking a pointwise sup of finitely many of the f^λ_α.

Proof: let $\lambda \in \text{pcf}(A)$, $\mu = \sup(X \cap \lambda)$. We claim

1) $\forall a \in B_\lambda\ f^\lambda_\mu(a) \leq \chi(a)$

2) $\{a \in B_\lambda : f^\lambda_\mu(a) < \chi(a)\} \in J_{<\lambda}$.

For the first claim, observe that by smoothing and the construction of $\vec{X}$ there is $D \subseteq X \cap \mu$ such that D is club in μ and $f^\lambda_\mu = \sup_{\gamma \in D} f^\lambda_\gamma$. Each $f^\lambda_\gamma \in X$, so f^λ_μ is pointwise dominated by χ.

For the second claim, choose $\alpha < |A|^+$ such that $\{a \in B_\lambda : f^\lambda_\mu(a) < \chi(a)\} = \{a \in B_\lambda : f^\lambda_\mu(a) < \chi_\alpha(a)\}$ Now $\chi_\alpha \in X$ so that by elementarity $\chi_\alpha <_{J<\lambda} f^\lambda_\mu$.

We now define inductively a decreasing sequence of points $\lambda_i \in \text{pcf}(A)$, along with $F_i \in J_{<\lambda_i}$. We let $\mu_i = \sup(X \cap \lambda_i)$ for all i.

$\lambda_0 = \max \mathrm{pcf}(A)$, $F_0 = \{a : f^{\lambda_0}_{\mu_0}(a) < \chi(a)\}$. If F_i is not empty then let λ_{i+1} be such that $F_i \in J_{<\lambda^+_{i+1}} \setminus J_{<\lambda_{i+1}}$, or to put it another way $\lambda_{i+1} = \max \mathrm{pcf}(F_i)$. $F_i \setminus B_{\lambda_{i+1}} \in J_{<\lambda_{i+1}}$ as the $\vec{B}$ are generators.

$$F_{i+1} = \{a \in F_i : a \notin B_{\lambda_{i+1}} \vee f^{\lambda_{i+1}}_{\mu_{i+1}}(a) < \chi(a)\}.$$

Eventually we reach n such that $F_n = \emptyset$. By construction, for every $a \in A$ there is $i < n$ such that $f^{\lambda_i}_{\mu_i}(a) = \chi(a)$. Since χ dominates each $f^{\lambda_i}_{\mu_i}$ pointwise, we have that χ is the pointwise supremum of $\{f^{\lambda_i}_{\mu_i} : i < n\}$.

Remark: these ideas are also relevant to Shelah's "Strong covering" theorems.

A sample application: bounding $\aleph_\omega^{\aleph_0}$.

Theorem: If $2^{\aleph_0} < \aleph_\omega$ then $\aleph_\omega^{\aleph_0} = \mathrm{pp}(\aleph_\omega)$.

Proof: Let $A = \{\aleph_k : k < \omega, 2^{\aleph_0} < \aleph_k\}$. Then $\mathrm{pp}(\aleph_\omega) = \max \mathrm{pcf}(A)$, and clearly $\max \mathrm{pcf}(A) \leq \prod A = \aleph_\omega^{\aleph_0}$.

Notice that $\max\{|A|^+, |\mathrm{pcf}(A)|\} < \min(A) = (2^{\aleph_0})^+$. If $a \in [\aleph_\omega]^{\aleph_0}$ then we can build an IA chain $\langle X_\alpha : \alpha \leq 2^{\aleph_0}\rangle$ such that $a \cup 2^{\aleph_0} \subseteq X_0$ and each structure has size $2^{\aleph_0}$.

There are at most $\mathrm{pp}(\aleph_\omega)$ possibilities for the characteristic function of $X_{2^{\aleph_0}}$, so there are at most $\mathrm{pp}(\aleph_\omega)$ possibilities for $X_{2^{\aleph_0}} \cap \aleph_\omega$. This means there are at most $\mathrm{pp}(\aleph_\omega) \times (2^{\aleph_0})^{\aleph_0} = \mathrm{pp}(\aleph_\omega)$ possibilities for a.

Remark: $\mathrm{pp}(\aleph_\omega) = \max \mathrm{pcf}(A) < \aleph_{(2^{\aleph_0})^+}$.

Generalisation: if A is an interval of regular cardinals, $\min(A)^{|A|} < \sup(A)$ then $\sup(A)^{|A|} = \max \mathrm{pcf}(A)$.

Let μ be a singular strong limit cardinal which is not a cardinal fixed point, say $\mu = \aleph_\eta$ for some $\eta < \mu$. Let $A = \mu \cap REG$. Then $2^\mu = \max \mathrm{pcf}(A)$.

Since $\mathrm{pcf}(A)$ is an interval of regular cardinals and $|\mathrm{pcf}(A)| \leq 2^{|A|} = 2^{|\eta|}$, $2^\mu < \aleph_{(2^{|\eta|})^+}$. This generalises the Galvin-Hajnal result.

Theorem: if λ is singular with $\kappa = \mathrm{cf}(\lambda)$ and J^{bd}_κ is the bounded ideal on κ then there is $\langle \lambda_i : i < \kappa \rangle$ increasing and cofinal in λ such that $\mathrm{tcf}(\prod_i \lambda_i / J^{\mathrm{bd}}_\kappa) = \lambda^+$.

Proof: as $\mathrm{PP}(\lambda)$ is an interval we may choose $A \subseteq \lambda$ with $\mathrm{ot}(A) = \kappa$ such that $\lambda^+ \in \mathrm{pcf}(A)$. Let $B \in J_{<\lambda^{++}} \setminus J_{<\lambda^+}$, then $\mathrm{tcf}(\prod B / J_{<\lambda^+}) = \lambda^+$. Now it is easy to see that $J_{<\lambda^+} = J_{<\lambda}$ and every element of $J_{<\lambda}$ is a bounded subset of A, so if $\langle \lambda_i : i < \kappa \rangle$ enumerates B then $\mathrm{tcf}(\prod_i \lambda_i / J^{\mathrm{bd}}_\kappa) = \lambda^+$ as required.

Theorem: $|\mathrm{pcf}(A)| < \min(A) \Rightarrow \mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$.

Proof: Let $A^* = \mathrm{pcf}(A)$, where clearly $A^* \subseteq \mathrm{pcf}(A^*)$. For each $\lambda \in A^*$ choose D_λ with $\mathrm{cf}(\prod A/D_\lambda) = \lambda$, and then $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ cofinal in $\mathrm{cf}(\prod A/D_\lambda)$. Now let $\mu \in \mathrm{pcf}(A^*)$, and let $\langle g_\beta : \beta < \mu \rangle$ be cofinal in $\prod A^*/D$ for some D.

Define $D^* = \{X \subseteq A : \{\lambda : X \in D_\lambda\} \in D\}$, and let $h_\beta(a) = \sup_{\lambda \in A^*} f^\lambda_{g_\beta(\lambda)}(a)$.

Claim: if $h \in \prod A$ then $h <_{D^*} h_\beta$ for all large β.

Proof: fix $g \in \prod A^*$ such that $h <_{D_\lambda} f^\lambda_{g(\lambda)}$ for all λ, and then β such that $g <_D g_\beta$. For each of the D-many λ such that $g(\lambda) < g_\beta(\lambda)$, we have $h <_{D_\lambda} h_\beta$, and so by definition $h <_{D^*} h_\beta$.

Using the claim we can thin out $\langle h_\beta : \beta < \mu \rangle$ to a μ-sequence which is increasing and cofinal in $\prod(A/D^*)$. So $\mu \in \mathrm{pcf}(A) = A^*$, and we have proved that $A^* = \mathrm{pcf}(A^*)$.

Theorem: let λ be singular with $\kappa = \mathrm{cf}(\lambda) > \omega$. Let $\langle \lambda_i : i < \kappa \rangle$ be an increasing and continuous sequence of singular cardinals with limit λ. Let $\mu_i = \lambda_i^+$, $\mu = \lambda^+$. If $A = \{\mu_i : i < \kappa\}$ and B_μ is such that $J_{<\mu^+} = J_{<\mu} + B_\mu$ then $\{i : \mu_i \in B_\mu\}$ contains a club. In particular $\mu \in \mathrm{pcf}(A)$ and $\mu = \mathrm{tcf}(\prod_{i \in C} \mu_i / J_\kappa^{\mathrm{bd}})$ for some club C.

Proof: Let $B \in J \iff \{i : \mu_i \in B \setminus B_\mu\} \in NS_\kappa$. Suppose $\{i : \mu_i \notin B_\mu\}$ is stationary, so that J is not trivial.

Since $J_{<\mu^+} \subseteq J$, $\prod A/J$ is μ^+-directed. We may therefore apply the technical lemma to build a $<_J$-increasing $\langle f_\alpha : \alpha < \mu \rangle$, which has an eub $g \in \prod A$ such that $\mathrm{cf}(g(\mu_i))$ tends to λ mod J. Now $\mathrm{cf}(g(\mu_i)) < \lambda_i$ for all i, so by Fodor there is $j < \kappa$ such that $\mathrm{cf}(g(\mu_i)) < \lambda_j$ for stationarily many i with $\mu_i \notin B_\mu$; this contradicts the stated property of g.

Localisation.

Theorem: if $|\mathrm{pcf}(A)| < \min(A)$ (so in particular $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$) then for all $B \subseteq \mathrm{pcf}(A)$ and all $\lambda \in \mathrm{pcf}(B)$ there exists $B_0 \in [B]^{|A|}$ such that $\lambda \in \mathrm{pcf}(B_0)$.

Proof: Does anyone know a proof of this that would fit on two slides?

Remark: if $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} > \aleph_{\omega_1}$ then there are sets $\langle A_\alpha : \alpha < \aleph_1 \rangle$ with $A_\alpha \subseteq REG$ such that $\alpha < \beta \Rightarrow \sup(A_\alpha) < \min(A_\beta)$ and $\aleph_{\omega_1+1} \in \mathrm{pcf}(A_\alpha)$ for all α.

Roughly speaking this is what makes the problem of forcing "$\aleph_\omega$ is strong limit and $2^{\aleph_\omega} > \aleph_{\omega_1}$" so hard. It is also crucial in the proof that this statement implies inner models for substantial large cardinals.

Club guessing:

Theorem: if κ, λ are regular and uncountable with $\kappa^+ < \lambda$ then there exists $\langle S_\alpha : \alpha \in \lambda \cap \text{cof}(\kappa) \rangle$ such that

1. S_α is club in α, $\text{ot}(C_\alpha) = \kappa$.

2. For every E club in λ, $\{\alpha : S_\alpha \subseteq E \cap \alpha\}$ is stationary.

Proof: Start with any choice of S_α. Every time you see a bad club E, replace S_α by $S_\alpha \cap E$. Repeat for κ^+ steps and argue that for some α, S_α shrank κ^+ times.

Sample bound on pp.

Theorem: $\mathrm{pp}(\aleph_\omega) < \aleph_{\omega_4}$.

Proof: If not we can manufacture a topology τ on $\aleph_4 + 1$ such that for any $Y \neq \emptyset$, γ.

1. $\mathrm{cl}(Y)$ has a maximal element.

2. $\forall x \in \mathrm{cl}(Y)\ \exists Y_0 \in [Y]^{\aleph_0}\ x \in \mathrm{cl}(Y_0)$.

3. If $\mathrm{cf}(\gamma) > \omega$ then there is C club in γ such that $\gamma = \max(\mathrm{cl}(C))$

We claim no such topology τ can exist. To see this fix $\langle S_\alpha : \alpha \in \aleph_3 \cap \mathrm{cof}(\aleph_1)\rangle$ which guesses clubs in $\aleph_3$. Build an IA chain $\langle X_\alpha : \alpha \leq \aleph_3\rangle$ of structures of size $\aleph_3$ such that $\aleph_3 \cup \{\tau, \vec{S}\} \in X_0$. If $\gamma_\alpha = X_\alpha \cap \aleph_4$ then $\langle \gamma_\alpha : \alpha \leq \aleph_3\rangle$ is increasing and continuous; let $\gamma = \gamma_{\aleph_3}$.

By the hypotheses on τ, fix $D \subseteq \gamma$ club with $\gamma = \max \mathrm{cl}(D)$. Fix α such that $S_\alpha \subseteq \{\beta : \gamma_\beta \in D\}$, and let $S^* = \{\gamma_\beta : \beta \in S_\alpha\}$. Now let $\delta = \max \mathrm{cl}(S^*)$, so that $\gamma_\alpha \leq \delta < \aleph_4$. As $\mathrm{cf}(\alpha) = \aleph_1$ we may find $\bar{\alpha} < \alpha$ such that $\delta = \max \mathrm{cl}\{\gamma_\beta : \beta \in S_\alpha \cap \bar{\alpha}\}$.

Now $S_\alpha \in M_0$ and $\langle \gamma_\beta : \beta < \bar{\alpha}\rangle \in M_{\bar{\alpha}+1}$, so that $\delta \in M_{\bar{\alpha}+1} \cap \aleph_4 = \gamma_{\bar{\alpha}+1}$. Contradiction!

Remark: a similar method shows that if λ is singular, $\lambda = \aleph_{\alpha+\delta}$ and $\delta < \aleph_\alpha$ then $\mathrm{pp}(\lambda) < \aleph_{\alpha+|\delta|^{+4}}$.

Remark: there is a theory of "pcf structures".

More interesting facts:

$\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) = \mathrm{pp}(\aleph_\omega)$.

$\mathrm{cf}(\prod A, <) = \max \mathrm{pcf}(A)$.

If $|\mathrm{pcf}(A)| < \min(A)$ then we can choose generators $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ such that $\forall \lambda \in B_\mu\ B_\lambda \subseteq B_\mu$ and $\mathrm{pcf}(B_\lambda) = B_\lambda$.

Analogues of Silver's theorem and the Galvin-Hajnal theorem hold for pp.

If $\delta < \aleph_4$ and $\mathrm{cf}(\delta) = \omega$ then $\mathrm{pp}(\aleph_\delta) < \aleph_{\omega_4}$.

If λ is the least counterexample to SCH then $\lambda > 2^{\aleph_0}$, $\mathrm{cf}(\lambda) = \aleph_0$, $\mu^{\aleph_0} \leq \mu^+$ for $2^{\aleph_0} < \mu < \lambda$, and there exists $\langle \lambda_n : n < \omega\rangle$ increasing and cofinal in λ such that $\mathrm{tcf}(\prod_n \lambda_n / J^{\mathrm{bd}}_\omega) = \lambda^{++}$.

Some hypotheses proposed by Shelah:

STRONG: for all singular λ, $\mathrm{pp}(\lambda) = \lambda^+$.

MEDIUM: $|\mathrm{pcf}(A)| = |A|$.

WEAK: For singular λ

$\{\mu < \lambda : \mathrm{pp}(\mu) \leq \lambda, \mathrm{cf}(\mu) = \omega\}$ is countable.

$\{\mu < \lambda : \mathrm{pp}(\mu) \leq \lambda, \mathrm{cf}(\mu) > \omega\}$ is finite.

Review of large cardinal notions:

κ is measurable iff there exists a κ-complete non-principal ultrafilter on κ iff there is $j : V \longrightarrow M$ such that crit$(j) = \kappa$ and ${}^{\kappa}M \subseteq M$.

The Mitchell order: if U, V are normal measures on κ then $U < V$ iff U is in the ultrapower by V. $<$ is a well founded partial ordering of height at most $(2^{\kappa})^{+}$. $o(\kappa)$ is the height of $<$.

κ is λ-strong iff there is $j : V \longrightarrow M$ such that $j(\kappa) > \lambda$, $V_{\lambda} \subseteq M$ and ${}^{\kappa}M \subseteq M$.

Remark: if κ is $(\kappa + 2)$-strong then $o(\alpha) = (2^{\alpha})^{+}$ for many $\alpha < \kappa$.

κ is λ-supercompact iff there is $j : V \longrightarrow M$ such that $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$.

Some forcing results relevant to SCH:

(Silver: supercompact) κ supercompact with $2^\kappa > \kappa^+$.

(Prikry) A measurable κ can be made to have cofinality ω by a cardinal preserving forcing.

(Magidor) A cardinal of Mitchell order at least $\lambda = \mathrm{cf}(\lambda) < \kappa$ can be made to have cofinality λ in a cardinal preserving extension.

SCH can fail at large singular strong limit cardinals.

(Magidor: supercompact) Supercompact Prikry forcing.

(Magidor: supercompact) $\aleph_\omega$ strong limit, $2^{\aleph_\omega} = \aleph_{\omega+2}$.

(Magidor: huge) $2^{\aleph_n} = \aleph_{n+1}$ for all n, $2^{\aleph_\omega} = \aleph_{\omega+2}$.

(Shelah: supercompact) $\aleph_\omega$ strong limit and $2^{\aleph_\omega} = \aleph_{\alpha+1}$, for $\alpha < \aleph_1$.

(Magidor: huge) $2^{\aleph_n} = \aleph_{n+1}$ for all n and $2^{\aleph_\omega} = \aleph_{\alpha+1}$, for $\alpha < \aleph_1$.

(Shelah: supercompact) μ the least cardinal fixed point of order ω, μ strong limit with 2^μ arbitrarily large.

(Radin) Adding a club of V-regulars to large cardinal κ, preserving some large cardinal properties of κ.

(Foreman and Woodin: supercompact) Supercompact Radin forcing.

(Foreman and Woodin: supercompact) GCH fails everywhere, in fact 2^κ weakly inaccessible for all κ. For fixed $n < \omega$, can build the model to contain many λ which are $\beth_n(\lambda)$-supercompact.

(Woodin: supercompact) $2^\kappa = \kappa^{+n}$ for all κ.

Woodin: reduction of hypotheses to level of hypermeasurability.

$(*)$ GCH + there is $j : V \longrightarrow M$ and $f : \kappa \longrightarrow \kappa$ such that crit$(j) = \kappa$, $j(f)(\kappa) = \kappa^{++}$ and ${}^{\kappa}M \subseteq M$.

Remark: $(*)$ follows from GCH + "κ is $(\kappa+2)$-strong", and if j witnesses the strength we may take $f(\alpha) = \alpha^{++}$.

(Woodin: $(*)$) κ measurable with $2^\kappa = \kappa^{++}$.

(Woodin: $(*)$) $2^{\aleph_n} = \aleph_{n+1}$ for all n, $2^{\aleph_\omega} = \aleph_{\omega+2}$.

(Cummings: strong) GCH holds at every successor, fails at every limit.

(Gitik) $(*)$ can be forced starting from $o(\kappa) = \kappa^{++}$.

The paradigm shift: up to now powerset of κ was blown up keeping κ large, and then κ was made singular. Problem: GCH will fail at many points on the Prikry sequence, and collapsing to restore GCH tends to collapse at κ also.

(Gitik-Magidor: strong) adding many cofinal ω-sequences to κ without adding bounded subsets.

Example: from κ which is $(\kappa+\omega_1)$-strong, $2^{\aleph_n} = \aleph_{n+1}$ for all n and $2^{\aleph_\omega} = \aleph_{\alpha+1}$, for $\alpha < \aleph_1$.

(Segal:strong) Gitik-Magidor for uncountable cofinalities.

(Gitik and Merimovich:strong) $\aleph_\omega$ strong limit, $2^{\aleph_\omega} = \aleph_{\omega+m}$ for $m < \omega$, complete freedom below $\aleph_\omega$.

$o(\kappa) = \kappa^{++}$ is an upper bound for the consistency strength of the failure of SCH. More: we can build a model where $\aleph_\omega$ is strong limit (or even GCH holds below $\aleph_\omega$) and $2^{\aleph_\omega} = \aleph_{\omega+2}$ starting from this hypothesis.

Two routes are available:

1. Gitik showed that forcing over a suitable model with $o(\kappa) = \kappa^{++}$ we can produce a model of $(*)$. Work of Woodin then gives models as required. We also get a model where GCH fails at a measurable.

2. Gitik showed that forcing over a suitable model with $o(\kappa) = \kappa^{++}$ gives a suitable ground model for a version of the Gitik-Magidor construction, which will produce a model where GCH holds up to $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$.

Prikry forcing: if κ is measurable, can define a κ^+-c.c. forcing which adds no bounded subsets of κ and makes cf$(\kappa) = \omega$.

U normal measure on κ. Conditions in $\mathbb{P}_U$ are (s, A) where $a \in [\kappa]^{<\omega}$ and $A \in U$. Intuition: s is an initial segment of the ω-sequence and the rest is inside A. Accordingly $(s, A) \subseteq (t, B)$ iff t extends s, $A \subseteq B$ and $s \setminus t \subseteq B$. $(s, A) \leq^* (t, B)$ iff $s = t$ and $A \subseteq B$.

Some attractive properties:

Strongly κ^+-c.c. as a union of κ filters.

(Prikry) For every sentence ϕ and condition p there is $q \leq^* p$ deciding ϕ. $\leq^*$ is κ-closed (as U is κ-complete), so we add no bounded subsets of κ.

(Mathias) $\vec{x}$ is generic iff every $A \in U$ contains a tail of $\vec{x}$.

(Kunen) Generic sequence can be obtained by iterating ultrapowers.

Tree version: V a κ-complete ultrafilter on κ. Conditions are trees $T \subseteq {}^{<\omega}\kappa$, which have some finite stem and V-large branching above the stem. $T \leq^* U$ iff $T \subseteq U$ and they have the same stem.

More generally: there is a family V_s of ultrafilters, and the tree has V_s-large branching at every s above the stem.

Abstractly: $(\mathbb{P}, \leq, \leq^*)$ is "Prikry-like" if $\leq^*$ is stronger than $\leq$, and all questions about the generic extension by $(\mathbb{P}, \leq)$ can be decided by strengthening in $\leq^*$.

(Magidor) Iterated Prikry forcing: finite support on s-parts, full support on A-parts.

Gitik's iteration technique. A is a set of inaccessibles. For $\gamma \in A$ have a $\mathbb{P}_\gamma$-name for a Prikry-like $(\mathbb{Q}_\gamma, \leq_\gamma, \leq^*_\gamma)$, which is forced to have size less than $\min(A \setminus (\gamma+1))$.

$\mathbb{P}_\alpha$ is the set of α-sequences p such that $\mathrm{supp}(p)$ is an *Easton* subset of A (bounded in every inaccessible), $p \restriction \gamma \in \mathbb{P}_\gamma$ and $p \restriction \gamma \Vdash p(\gamma) \in \mathbb{Q}_\gamma$ for all $\gamma \in \mathrm{supp}(p)$.

$p \leq q$ iff $\mathrm{supp}(q) \subseteq \mathrm{supp}(p)$, $p \restriction \gamma \Vdash p(\gamma) \leq q(\gamma)$ for all $\gamma \in \mathrm{dom}(q)$, $p \restriction \gamma \Vdash p(\gamma) \leq^* q(\gamma)$ for all but finitely many $\gamma \in \mathrm{dom}(q)$.

$p \leq^* q$ iff $\mathrm{supp}(q) \subseteq \mathrm{supp}(p)$, $p \restriction \gamma \Vdash p(\gamma) \leq^* q(\gamma)$ for all $\gamma \in \mathrm{dom}(q)$.

Iteration? $\mathbb{P}_{\gamma+1} \simeq \mathbb{P}_\gamma * \mathbb{Q}_\gamma$ and $p \longmapsto p \restriction \beta$ is a projection from $\mathbb{P}_\alpha$ to $\mathbb{P}_\beta$.

Theorem (Gitik): $(\mathbb{P}_\alpha, \leq, \leq^*)$ is Prikry-like.

Gitik: from $o(\kappa) = \kappa^{++}$ to $(*)$.

By work of Mitchell, we may assume that our ground model has nice L-like properties (GCH, squares) and there is $\langle U_\alpha : \alpha < \kappa^{++} \rangle$ increasing in the Mitchell ordering.

Aim: force to get a model with $\langle \bar{U}_\alpha : \alpha < \kappa^{++} \rangle$ a sequence of ultrafilters on κ which is increasing in the Rudin-Keisler ordering. Also we want $\langle \pi_{\alpha\beta} : \alpha < \beta < \kappa^{++} \rangle$ such that $X \in \bar{U}_\alpha \iff \pi^{-1}_{\alpha\beta}[X] \in \bar{U}_\beta$ and $\pi_{\alpha\gamma} =_{\bar{U}_\gamma} \pi_{\alpha\beta} \circ \pi_{\beta\gamma}$.

Now if $j_\alpha : V \longrightarrow M_\alpha = Ult(V, \bar{U}_\alpha)$ is the ultrapower map, then $\pi_{\alpha\beta}$ induces a map from M_α to M_β by $\pi^*_{\alpha\beta} : [f]_{\bar{U}_\alpha} \longmapsto [f \circ \pi_{\alpha\beta}]_{\bar{U}_\beta}$. These maps commute with the j_α and with each other, that is $j_\beta = j_\alpha \circ \pi^*_{\alpha\beta}$ and $\pi^*_{\alpha\gamma} = \pi^*_{\beta\gamma} \circ \pi^*_{\alpha\beta}$.

Given this we can form a limit ultrapower $j : V \longrightarrow M$ such that $j(\kappa) > \kappa^{++}$ and ${}^\kappa M \subseteq M$, and it is then possible to force the existence of a function f such that $j(f)(\kappa) = \kappa^{++}$.

A motivating example (Mitchell).

Assume GCH. Let κ be minimal with $o(\kappa) = 2$, $A = \{\alpha < \kappa : o(\alpha) = 1\}$. Fix $\langle U_\alpha : \alpha \in A\rangle$ measures of order zero on $\alpha \in A$. Let U, V be measures on κ where U has order zero, V has order 1 and $U = [\alpha \longmapsto U_\alpha]_V$. Notice that U_α and U concentrate on A^c, V concentrates on A.

Iterate Prikry forcing at $\alpha \in A$. At stage α observe that $j_{U_\alpha}(\mathbb{P}_\alpha)$ is a Gitik iteration whose support does not contain α. Use GCH and the Prikry lemma to build (canonically) a decreasing $\langle q_\gamma : \gamma < \alpha^+\rangle$ in $j(\mathbb{P}_\alpha)/G_\alpha$, such that for every $\dot{A}$ a $\mathbb{P}_\alpha$-name for a subset of α there is $r \in G_\alpha$ and $\gamma < \alpha^+$ such that $r \frown q_\gamma$ decides $\alpha \in j_{U_\alpha}(\dot{A})$.

$$\bar{U}_\alpha = \{\dot{A}^{G_\alpha} : \exists r \in G_\alpha\, \exists \gamma\, r \frown q_\gamma \Vdash \alpha \in j_{U_\alpha}(\dot{A})\}.$$

$\mathbb{Q}_\alpha$ is Prikry forcing defined from the normal measure $\bar{U}_\alpha$.

In $V[G_\kappa]$, extend U to $\bar{U}$ in the same way. Now consider $j_V(\mathbb{P}_\kappa)$: this is a Gitik iteration of Prikry forcing where the support does contain κ, and in fact the measure used at stage κ in this iteration is $\bar{U}$.

Building a suitable sequence $\langle q^*_\gamma : \gamma < \kappa^+ \rangle$ we define in $V[G_\kappa]$ $\dot{A}^{G_\kappa} \in \bar{V}$ iff

$$\exists r \in G_\kappa \, \exists B \in \bar{U} \; r \frown (\langle\rangle, B) \frown q^*_\gamma \Vdash \kappa \in j_V(\dot{A}).$$

$\bar{U}$ is normal and concentrates on A^c. $\bar{V}$ concentrates on A, and so is not normal since A consists of cofinality ω ordinals. Define π_{01} with domain A by $\pi_{01}(\delta) = \min(b_\delta)$, then we claim that π_{01} projects $\bar{V}$ to $\bar{U}$.

If not then find $E \in \bar{U}$, $F \in \bar{V}$ such that $\pi_{01}[F] \cap E = \emptyset$. Let $F = \dot{F}^{G_\kappa}$, $E = \dot{E}^{G_\kappa}$ and find $r \in G_\kappa$, $E^* \in \bar{U}$ and γ such that $r \Vdash \pi_{01}[\dot{F}] \cap \dot{E} = \emptyset$ and $r \frown (\langle\rangle, E^*) \frown q^*_\gamma \Vdash \kappa \in j_V(\dot{F})$. Now the condition $r \frown (\langle\rangle, E \cap E^*) \frown q^*_\gamma$ forces that $\min(b_\kappa) \in E$, contradiction.

Woodin: from $(*)$ to a measurable cardinal where GCH fails. Fix $j : V \longrightarrow M$ and f such that crit$(j) = \kappa$, ${}^{\kappa}M \subseteq M$ and $j(f)(\kappa) = \kappa^{++}$.

Stage 1: Iterate $Coll(\alpha^+, < f(\alpha))$ for $\alpha < \kappa$ and then do $Coll(\kappa^+, < \kappa^{++})$. Show that in the extension we may lift j and get a situation in which $(*)$ holds and $\kappa^{++} = \kappa_M^{++}$, so that we may take $f(\alpha) = \alpha^{++}$.

Stage 2: Iterate $Add(\alpha, \alpha^{++})$ for $\alpha \leq \kappa$, and argue that j can be lifted so that the measurability of κ is preserved. Less closure than in Silver's argument from a supercompact, so harder to build generic filters and master conditions: these problems are resolved by forcing, transferring and rearranging of generic filters.

Remark: if you just want failure of GCH at a measurable, you can start with $(*)$ and iterate $Add(\alpha, f(\alpha))$ for $\alpha < \kappa$ followed by $Add(\kappa, \kappa^{++})$. In this two stage approach you get a bonus; in the final model we can arrange

$(**)$: κ is measurable, $2^\kappa = \kappa^{++}$ and for some normal measure U on κ there exists F which is generic over $M = Ult(V, U)$ for the forcing $Coll(\kappa_M^{+5}, < j_U(\kappa))_M$.

Woodin: given a model of $(**)$ we can force using "Prikry forcing with interleaved collapses" to make $\kappa = \aleph_\omega$ while preserving κ^+. Magidor did supercompact Prikry forcing with interleaved collapses and then went to a certain inner model, analysis of this inner model done by Foreman and Woodin for the GCH fails everywhere result motivated the construction from $(**)$.

How to prove the Prikry lemma for Prikry forcing. U normal, $U = \{X : \kappa \in j_U(X)\}$. $\mathbb{P} = \mathbb{P}_U$.

Diagonal intersection: Given a family of conditions (s, A_s) let

$$A = \{\delta : \forall s \ \mathrm{max}(s) < \delta \Longrightarrow \delta \in A_s\}.$$

Then $A \in U$ and $(s, A) \leq (s, A_s)$ for all s.

Now fix a statement ϕ.

Stage 1: find A such that if (s, B) decides ϕ for any B, then (s, A) decides ϕ.

Stage 2: Consider $(s \frown \kappa, j_U(A))$ and $j_U(\phi)$. Find $A_s \in U$ such that all $(s \frown \delta, A)$ with $\delta \in A_s$ behave the same way wrt ϕ (all force ϕ, all force $\neg\phi$ or all fail to decide ϕ). Take a diagonal intersection to get A^*.

Stage 3: Given (s, A^*) take an extension of minimal length which decides ϕ. Wlog extension is (t, A^*). If $t \neq s$ then $t = t_0 \frown \delta$ for some $\delta \in A_{t_0}$, but now by construction (t_0, A^*) decides ϕ.

(**): κ is measurable, $2^\kappa = \kappa^{++}$ and for some normal measure U on κ there exists F which is generic over $M = Ult(V, U)$ for the forcing $Coll(\kappa_M^{+5}, < j_U(\kappa))_M$.

Define generalised Prikry forcing $\mathbb{P}$ as follows: conditions have the form

$$p = (\kappa_0, p_0, \kappa_1, p_1, \ldots \kappa_{n-1}, p_{n-1}, A, H)$$

where $\kappa_0 < \ldots < \kappa_{n-1} < \kappa$, dom$(H) = A \in U$, $p_j \in Coll(\kappa_j^{+5}, < \kappa_{j+1})$, $H(\delta) \in Coll(\delta^{+5}, < \kappa)$, $[H]_U \in F$.

A typical extension is

$$q = (\kappa_0, p_0^*, \kappa_1, p_1^*, \ldots \kappa_{m-1}, p_{m-1}^*, A^*, H^*)$$

where $p_i^* \leq p_i$ for $i < n$, $A^* \subseteq A$, $H^*(\delta) \leq H(\delta)$ for $\delta \in A^*$.

$q \leq^* p$ iff $m = n$.

Theorem: $(\mathbb{P}, \leq, \leq^*)$ is Prikry-like and κ^+-c.c.

Corollary: in the extension by $(\mathbb{P}, \leq)$ we have $\kappa = \kappa_0^{+\omega}$, κ strong limit, $2^\kappa = \kappa^{++}$. Now we can do $Coll(\aleph_0, \kappa_0)$ to get $\kappa = \aleph_\omega$.

Proof of theorem: Conditions have form $p = x \frown (A, H)$ where $x \in V_\kappa$ (we call x the *lower part* of p). $x \frown (A, H)$ and $x \frown (A^*, H^*)$ are compatible, so $\mathbb{P}$ is κ^+-c.c.

Proof of Prikry Lemma is just like for Prikry forcing. First show the analogue of the diagonal intersection theorem. Then

Step One: find (A, H) such that for any x, if there exists (B, G) such that $x \frown (B, G)$ decides ϕ then $x \frown (A, H)$ decides ϕ.

Step Two: consider $x \frown (\kappa, p) \frown (j_U(A), j_U(H))$. Use genericity to find $p = [H^*] \in F$ such that for all x if there is $q \leq p$ such that $x \frown (\kappa, q) \frown (j_U(A), j_U(H))$ decides $j_U(\phi)$ then $x \frown (\kappa, p) \frown (j_U(A), j_U(H))$ decides $j_U(\phi)$.

Step Three: Find $A^* \subseteq A$ such that for all x and all $\delta \in A^*$, if there is $q \leq H^*(\delta)$ such that $x \frown (\delta, q) \frown (A, H)$ decides ϕ then $x \frown (\delta, H^*(\delta)) \frown (A, H)$ decides ϕ

Step Four: for each x find $A^*_x \subseteq A^*$ such that all conditions $x \frown (\delta, H^*(\delta)) \frown (A, H)$ behave the same wrt ϕ. Diagonally intersect to get A^{**}.

Now consider an extension of $x \frown (A^{**}, H^*)$ which has minimal length and decides ϕ. If it is not direct then wlog it has form $y_0 \frown (\delta, q) \frown (A^{**}, H^*)$ where y_0 extends x, $\delta \in A^*_{y_0}$ and $q \leq H^*(\delta)$. By construction $y_0 \frown (\delta, H^*(\delta)) \frown (A^{**}, H^*)$ decides ϕ, but then $y_0 \frown (A^{**}, H^*)$ decides ϕ. Contradiction.

A more elaborate argument gives GCH up to $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$ by similar means. Idea: interleave $Coll(\kappa_i^{+5}, < \kappa_{i+1}) \times Coll(\kappa_{i+1}, \kappa_{i+1}^+)$ between successive points κ_i and κ_{i+1}. Need a more elaborate mechanism of constraint, functions of two variables representing a generic filter over the second iterated ultrapower for $Coll(\kappa^{+5}, < j(\kappa)) \times Coll(j(\kappa), j(\kappa^+))$. Pcf arguments show this idea will not generalise easily to get GCH up to $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+3}$.

Gitik-Magidor method.

Idea: want to add many Prikry sequences to large cardinal κ, without adding bounded subsets of κ. Problem: new reals may be coded by (say) the relationship between two Prikry sequences. Solution: arrange that if $\vec{x}$, $\vec{y}$ are two Prikry sequences then there is a sequence $\vec{z}$ and $\pi_x, \pi_y \in V$ such that up to finite perturbation $\pi_x \circ \vec{z}$ agrees with $\vec{x}$ and $\pi_y \circ \vec{z}$ agrees with $\vec{y}$

To be a little more precise we are given a κ^+-directed poset$(A, \prec)$ together with $\langle U_\alpha : \alpha \in A \rangle$ a system of measures on κ and $\langle \pi_{\alpha\beta} : \alpha \prec \beta \rangle$ a system of commuting projection maps. These have to satisfy some technical conditions. A will have a minimal element 0 with U_0 normal.

A condition p then prescribes

1. A set $g_p \in [A]^\kappa$, where there is $\gamma_p \in g_p$ such that $\forall \beta \in g_p\ \beta \prec \gamma_p$.

2. A set $p^\beta \in [\kappa]^{<\omega}$ for each $\beta \in g_p$.

3. A tree with stem p^{γ_p} and U_{γ_p}-large branching.

There are two special kinds of extension.

Direct: $q \leq^* p$. $g_q \supseteq g_p$ (maybe with a larger maximal element), and $q^\gamma = p^\gamma$ for all $\gamma \in g_p$.

"Projective": $q \leq^0 p$. $g^q = g^p$ and $q^\beta \setminus p^\beta$ is obtained (roughly) by projecting $q^\gamma \setminus p^\gamma$ along $\pi_{\beta\gamma}$.

Extension = projective followed by direct. This is transitive because the system $\vec{\pi}$ is "commutative enough".

The key facts:

1. $(\mathbb{P}, \leq)$ is κ^{++}-c.c. and $(\mathbb{P}, \leq^*)$ is κ-closed.

2. $(\mathbb{P}, \leq, \leq^*)$ is Prikry-like.

3. $\mathbb{P}$ adds $\langle x_\alpha : \alpha \in A \rangle$, distinct ω-sequences cofinal in κ.

4. $\mathbb{P}$ preserves κ^+.

How to get a suitable family of measures?

κ is λ-strong, λ successor ordinal or a cardinal with $\mathrm{cf}(\lambda) > \kappa$, and GCH holds. Can build a family of size $\kappa^{+\lambda}$.

$o(\kappa) = \kappa^{+\lambda}$ is enough if λ is successor or $\kappa < \mathrm{cf}(\lambda) < \lambda$.

GCH up to $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+\zeta+1}$, ζ countable.

Start with GCH and κ which is $(\kappa + \zeta + 1)$-strong.

To bring κ down to $\aleph_0$, interleave with collapses along the Prikry sequence for U_0. It is comparatively easy to get a suitably generic constraint filter because GCH holds.

If ζ is finite the construction is essentially as before.

If ζ is infinite then let $\zeta + 1 = \cup_n D_n$ with D_n finite. Build $\mathbb{P}$ so that if $\vec{\kappa}$ is the Prikry sequence for U_0 then (roughly) the cardinals $\{\kappa_n^{+i} : i \in D_n\}$ survive in $[\kappa_n, \kappa_{n+1})$.

$\mathbb{P}$ is now only $\kappa^{+\zeta+2}$-c.c. but for all $\delta \leq \zeta$ it embeds in a forcing that preserves $\kappa^{+\delta+1}$.

From a strong cardinal: no bound on 2^κ where κ is the least fixed point of order ω.

(Segal) Versions of the Gitik-Magidor construction which make κ singular of uncountable cofinality.

Example: $2^{\aleph_\alpha} = \aleph_{\alpha+3}$ for $\alpha \leq \omega_1$ limit, GCH holds except at limits and their successors up to $\aleph_{\omega_1}$.

(Gitik and Merimovich) Fine control.

Example: From κ which is $(\kappa + m)$-strong, $2^{\aleph_\omega} = \aleph_{\omega+m}$ and complete freedom below $\aleph_\omega$.

Lower bounds: core models, covering lemmas.

Core models: on the hypothesis “there is no X cardinal”, build a model K_X and

1. Prove that K_X has a nice internal structure. GCH, diamonds, squares, morasses etc.

2. Prove that there is some resemblance between K_X and V.

In general for $M \subseteq V$:

Strong covering: every uncountable set of ordinals is contained in a set of the same V-cardinality which lies in M.

Weak covering: M computes successors of singulars correctly.

If M is a model of GCH and strong covering holds then SCH is true in V.

$X = 0^\sharp$, $K_X = L$.

(Jensen) If $0^\sharp$ does not exist, then strong covering holds between L and V.

Once measurable cardinals are allowed into K then we can no longer ask for strong covering by K itself, because of Prikry forcing. Weak covering goes a long way.

$X = 0^\dagger$, K_X has maximal form $L[\mu]$.

(Dodd-Jensen) If $L[\mu]$ exists and $0^\dagger$ does not then EITHER strong covering holds between $L[\mu]$ and V OR there is $C \in V$ a maximal Prikry generic sequence over $L[\mu]$ such that strong covering holds between $L[\mu, C]$ and V.

Still a misleadingly simple example. When K contains many measures then there is no longer a uniform set of indiscernibles which works to cover all sets in V, and it is hard to see the set of indiscernibles as a generic object for some forcing.

Typical covering lemma for K a large core model:

$x \subseteq ON$ covered by a set $h``(\rho \cup I)$ where $h \in K$, $\rho \in ON$, $I \subseteq ON$.

I arises from iterating some small "$L[\vec{E}]$ model" up to a model in K. The analysis gets harder as K gets larger.

Lower bounds for failure of SCH.

(Mitchell) If there is κ singular strong limit with $2^\kappa > \kappa^+$ then in a suitable core model K, $\forall\alpha < \kappa\ \exists\nu < \kappa\ o(\nu) \geq \alpha$. If in addition $\mathrm{cf}(\kappa) > \omega$ $o(\kappa) = \kappa^{++}$ in K.

(Shelah) If λ is the least counterexample to SCH then $\lambda > 2^{\aleph_0}$, $\mathrm{cf}(\lambda) = \aleph_0$, $\mu^{\aleph_0} \leq \mu^+$ for $2^{\aleph_0} < \mu < \lambda$, and $pp(\lambda) \geq \lambda^{++}$.

(Gitik) If $\lambda > 2^{\aleph_0}$ is singular and $pp(\lambda) \geq \lambda^{++}$ then there is an inner model for $\exists\kappa\ o(\kappa) = \kappa^{++}$.

These results can be extended: need to understand "$o(\kappa)$" as a measure of how many extenders there are at κ.

Gitik and Mitchell: further results.

1) κ strong limit, $\mathrm{cf}(\kappa) = \delta < \kappa$, $2^\kappa \geq \lambda > \kappa^+$ where λ is not ρ^+ for $\mathrm{cf}(\rho) \leq \kappa$.

1. If $\delta > \omega_1$, there is an inner model where $o(\kappa) \geq \lambda + \delta$ (Optimal by work of Woodin, or by work of Segal).

2. If $\delta = \omega_1$, there is an inner model where $o(\kappa) \geq \lambda$ (and $\lambda + \omega_1$ suffices).

3. If $\delta = \omega$ then there is an inner model where either $o(\kappa) \geq \lambda$ or $\{\alpha < \kappa : o(\alpha) \geq \alpha^{+n}\}$ is unbounded for all $n < \omega$. The latter case can actually occur, by work of Gitik.

2) If $n > 0$, $\mathrm{cf}(\kappa) = \omega$, GCH holds below κ and $2^\kappa \geq \kappa^{+n+2}$ then there is an inner model where either $o(\kappa) \geq \kappa^{+n+2} + 1$ or $\{\alpha < \kappa : o(\alpha) \geq \alpha^{+m}\}$ is unbounded for all $m < \omega$.

3) If $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} > \aleph_{\omega_1}$ then there is an inner model for a Woodin cardinal (assuming that we can build the Steel core model).

Gitik: more on failure of GCH at a measurable.

1) κ measurable and $2^\kappa = \kappa^{+\alpha}$ for $\alpha \geq 2$. Then there is an inner model where $o(\kappa) \geq \kappa^{+\alpha}$ or some τ is strong up to the next measurable.

2) κ measurable and $2^\kappa = \kappa^{+\beta+1}$ for β singular, $\mathrm{cf}(\beta) \leq \kappa$. Then there is an inner model where $o(\kappa) \geq \kappa^{+\beta+1}+1$ or some τ is strong up to the next measurable.

Complementary consistency results: If $\alpha = \beta + 1$ for β a successor or $\mathrm{cf}(\beta) > \kappa$, or α is limit with $\mathrm{cf}(\alpha) > \kappa$ then $o(\kappa) = \kappa^{+\alpha}$ suffices to make κ measurable with $2^\kappa = \kappa^{+\alpha}$.

Work of Woodin shows $o(\kappa) = \kappa^\alpha + 1$ always suffices.

Combinatorial problems about regular cardinals tend to be hardest at successors of singulars. Examples:

1. Is there a κ-Aronszajn tree?

2. Is there a non-trivial non-reflecting stationary subset of κ?

3. $\kappa^{<\kappa} = \kappa$? (=SCH, essentially)

4. Is there a Jonsson algebra on κ?

Some ways to resolve problems about singular cardinals and their successors.

1. Reflection principles (typically obtained from large cardinals by forcing).

2. Squares (typically obtained by forcing or from inner models).

3. Pcf theory.

Q: How are they related?

Definition: Let κ be an uncountable cardinal. A $\square_\kappa$-*sequence* is a sequence $\langle C_\alpha : \alpha < \kappa^+, \lim(\alpha)\rangle$ such that for all $\alpha < \kappa^+$

1. C_α is closed and unbounded in α.

2. If $\mathrm{cf}(\alpha) < \kappa$, then $\mathrm{ot}(C_\alpha) < \kappa$.

3. For all $\beta \in \lim(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$.

We say that $\square_\kappa$ *holds* iff there exists a $\square_\kappa$-sequence.

CFM = Cummings, Foreman and Magidor.

CS = Cummings and Schimmerling.

FM = Foreman and Magidor.

Approachability (Shelah)

Definition: $\langle C_\alpha : \alpha < \kappa^+ \rangle$ is an AP$_\kappa$-*sequence* iff for a club of α

1. C_α is club in α, $\mathrm{ot}(C_\alpha) = \mathrm{cf}(\alpha)$.

2. $\forall \beta < \alpha \, \exists \gamma < \alpha \; C_\alpha \cap \beta = C_\gamma$.

Here we could just demand that C_α was unbounded and get an equivalent definition.

Very weak square (Foreman and Magidor)

Definition: $\langle C_\alpha : \alpha < \kappa^+\rangle$ is an VWS$_\kappa$-*sequence* iff for a club of $\alpha < \kappa^+$

1. $\sup(C_\alpha) = \alpha$.

2. For every bounded $x \in [C_\alpha]^{\aleph_0}$, there is $\beta < \alpha$ with $x = C_\beta$.

Here demanding that C_α be club in α gives a stronger property, the Not So Very Weak Square or NSVWS$_\kappa$.

By *scale* we will mean scale of length $\aleph_{\omega+1}$ in a product of the form $\prod_{n \in A} \aleph_n$ ordered by eventual domination.

(FM) $\mathsf{VWS}_{\aleph_\omega}$ implies that in any scale the set of non-good points of cofinality $\aleph_1$ is non-stationary.

$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ implies that in any scale the set of non-good points of cofinality $\aleph_1$ is stationary.

$\mathsf{AP}_{\aleph_\omega}$ implies that in any scale, the set of non-good points of uncountable cofinality is non-stationary.

Relations with large cardinals etc

(FM) If GCH holds and κ is supercompact then there is a class generic extension in which κ is supercompact, cardinals and cofinalities are preserved and VWS_λ holds for all singular λ. We can also preserve hugeness.

NSVWS_λ fails if λ is the supremum of ω supercompact cardinals.

Definition: Let κ be an uncountable regular cardinal. Let S be a stationary subset of κ.

1. S *reflects at* α iff $\alpha < \kappa$, $\mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α.

2. $\mathrm{Refl}(S)$ *holds* iff every stationary subset of S reflects at some α.

3. S is *non-reflecting* iff S does not reflect at any α.

Fact: Let $\Box_\kappa$ hold and let S be a stationary subset of κ^+. Then $\mathrm{Refl}(S)$ fails.

Definition: Let κ be a cardinal. A $\square_{\kappa,\lambda}$-*sequence* is a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+, \lim(\alpha) \rangle$ such that

1. $\mathcal{C}_\alpha \subseteq P(\alpha)$, $1 \leq |\mathcal{C}_\alpha| \leq \lambda$, and $\mathcal{C}_\alpha$ is a set of closed and unbounded subsets of α.

2. If $\mathrm{cf}(\alpha) < \kappa$ then $\forall C \in \mathcal{C}_\alpha \ \mathrm{ot}(C) < \kappa$.

3. $\forall C \in \mathcal{C}_\alpha \ \forall \beta \in \lim(C) \ C \cap \beta \in \mathcal{C}_\beta$.

A $\square_{\kappa,<\lambda}$-*sequence* is defined similarly, only we demand that $1 \leq |\mathcal{C}_\alpha| < \lambda$.

$\square_\kappa = \square_{\kappa,1}$, $\square^*_\kappa = \square_{\kappa,\kappa}$.

Fact: Let $\kappa^{<\lambda} = \kappa$ and let $\square_{\kappa,<\lambda}$ hold. If $S \subseteq \kappa^+$ is stationary then there exists a stationary set $T \subseteq S$ such that T does not reflect at any α with $\mathrm{cf}(\alpha) \geq \lambda$.

Theorem (CFM): Let κ be supercompact, and suppose $2^{\kappa^{+\omega}} = \kappa^{+\omega+1}$. Let μ, ν be two cardinals (one or both can be finite) such that $1 \leq \mu < \nu < \aleph_\omega$. Then there is a generic extension in which

1. All cardinals less than or equal to ν are preserved.

2. $\aleph_\omega = \kappa_V^{+\omega}$.

3. $\square_{\aleph_\omega,\nu}$ holds.

4. $\square_{\aleph_\omega,\mu}$ fails.

Failure of $\square_\kappa$ for κ singular requires substantial large cardinals. More information in Ernest Schimmerling's talk.

Fact (Solovay): If κ is supercompact and λ is a cardinal with $\kappa < \lambda$ then $\square_\lambda$ fails.

Fact (Shelah) If κ is supercompact and $\mathrm{cf}(\lambda) < \kappa < \lambda$ then $\square^*_\lambda$ fails.

Fact (Burke and Kanamori): If κ is supercompact and $\kappa, \mathrm{cf}(\lambda) < \lambda$ then $\square_{\lambda,<\mathrm{cf}(\lambda)}$ fails.

Fact (Magidor): PFA implies that $\square_{\kappa,\aleph_1}$ fails for $\kappa \geq \aleph_1$, while "PFA $+\forall\kappa \geq \aleph_2\ \square_{\kappa,\aleph_2}$" is consistent.

Definition: A $\square^{\mathrm{ind}}_{\lambda,\mathrm{cf}(\lambda)}$-*sequence* is a matrix of sets $\langle C_{\alpha,i} : \alpha < \lambda^+, i(\alpha) \leq i < \mathrm{cf}(\lambda)\rangle$ such that for some increasing sequence $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ of regular cardinals with limit λ

1. $i(\alpha) < \mathrm{cf}(\lambda)$ for all $\alpha < \lambda^+$.

2. $\mathrm{ot}(C_{\alpha,i}) < \lambda_i$ for all α.

3. $C_{\alpha,i}$ is club in α.

4. If $i(\alpha) \leq i < j < \mathrm{cf}(\lambda)$ then $C_{\alpha,i} \subseteq C_{\alpha,j}$.

5. If $i(\beta) \leq i < \mathrm{cf}(\lambda)$ and $\alpha \in \lim(C_{\beta,i})$ then $i(\alpha) \leq i$ and $C_{\alpha,i} = C_{\beta,i} \cap \alpha$.

6. If α and β are limit ordinals with $\alpha < \beta < \lambda^+$ then $\alpha \in \lim(C_{\beta,i})$ for all sufficiently large $i < \mu$.

$\square^{\mathsf{ind}}_{\lambda,\mathsf{cf}(\lambda)}$ implies $\square_{\lambda,\mathsf{cf}(\lambda)}$ and the transfer principle $(\aleph_1, \aleph_0) \longrightarrow (\lambda^+, \lambda)$.

Theorem (CFM): Let λ be a singular cardinal. Then there exists a forcing poset $\mathbb{P}$ such that

1. $\mathbb{P}$ is $\mathsf{cf}(\lambda)$-directed closed.

2. $\mathbb{P}$ is $< \lambda$-strategically closed.

3. $\Vdash_{\mathbb{P}}$ "$\square^{\mathsf{ind}}_{\lambda,\mathsf{cf}(\lambda)}$ holds"

Corollary: Let κ be a Laver indestructible supercompact cardinal and let $\kappa \leq \mathsf{cf}(\lambda) < \lambda$. Then there is a forcing extension in which κ is still supercompact, cardinals and cofinalities up to λ^+ are preserved, and $\square^{\mathsf{ind}}_{\lambda,\mathsf{cf}(\lambda)}$ holds.

Definition: $(\vec{\kappa}, \vec{f})$ is a *very good scale for* κ iff

1. $\vec{\kappa} = \langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ is an increasing sequence of regular cardinals cofinal in κ.

2. $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ is a scale in $\prod_i \kappa_i / J^{\mathrm{bd}}_{\mathrm{cf}(\kappa)}$.

3. For every point $\alpha < \kappa^+$ such that $\mathrm{cf}(\alpha) > \mathrm{cf}(\kappa)$ there exists a closed and unbounded set $C \subseteq \alpha$ and $i < \mathrm{cf}(\kappa)$ such that $\forall \beta, \gamma \in C\ \forall j > i\ (\beta < \gamma \implies f_\beta(j) < f_\gamma(j))$.

VGS_κ *holds* iff there exists a very good scale for κ.

Theorem (CFM): Let κ be singular, let $\lambda < \kappa$. Then $\square_{\kappa,\lambda}$ implies VGS_κ.

Theorem (CFM): Let κ be singular, and let VGS_κ hold. Then for every stationary $T \subseteq \kappa^+$ there are stationary $\langle T_i : i < \mathrm{cf}(\kappa) \rangle$ such that $T_i \subseteq T$ and the T_i do not reflect simultaneously at any point of cofinality greater than $\mathrm{cf}(\kappa)$.

Theorem (CFM): Let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of supercompact cardinals. Let $\kappa = \sup_n \kappa_n$. Then there is a generic extension in which

1. $\kappa = \aleph_\omega$, and κ^+ is preserved.

2. $\square^{\mathrm{ind}}_{\aleph_\omega,\omega}$ holds.

3. For every finite set f of stationary subsets of $\aleph_\omega$ there exists $N < \omega$ such that if $N \leq n < \omega$ then there exists α of cofinality $\aleph_n$ such that all sets in f reflect at α.

Theorem (CFM): Let $\mathbb{P}$ be Prikry forcing at κ. In V let $S_0 =_{\mathsf{def}} \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) < \kappa\}$ and let $S_1 = \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \kappa\}$. Then in $V^{\mathbb{P}}$

1. S_1 is a non-reflecting stationary set of cofinality ω ordinals.

2. If κ is κ^+-supercompact, then any finite set of stationary subsets of S_0 reflect simultaneously.

3. There are ω subsets of S_0 which do not reflect simultaneously.

Theorem (CS): $\square_{\kappa,\omega}$ holds in $V^{\mathbb{P}}$.

Definition: ADS_κ *holds* iff there exists $\langle A_\alpha : \alpha < \kappa^+ \rangle$ such that

1. A_α is unbounded in κ, $\mathrm{ot}(A_\alpha) = \mathrm{cf}(\kappa)$.

2. For all $\beta < \kappa^+$ there exists $g : \beta \longrightarrow \kappa$ such that the sequence $\langle A_\alpha \setminus g(\alpha) : \alpha < \beta \rangle$ consists of pairwise disjoint sets.

Theorem (CFM): If VGS_κ or $\square^*_\kappa$ holds then ADS_κ holds.

Theorem (CFM): Let κ be singular of cofinality ω. If ADS_κ holds then there is stationary $S \subseteq \mathrm{Refl}^*([\kappa^+]^{\aleph_0})$ which does not reflect to any X with $|X| = \mathrm{cf}(X) = \aleph_1$.

Theorem (CFM): Let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of supercompact cardinals. Let $\kappa = \sup_n \kappa_n$, and assume that GCH holds above κ.

Then there is a forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$

1. $\kappa = \aleph_\omega$ and GCH holds.

2. $\square^*_{\aleph_\omega}$ holds.

3. For all n, any $\aleph_n$ stationary subsets of the set $\aleph_{\omega+1} \cap \mathrm{cof}(< \aleph_n)$ reflect simultaneously at some point of cofinality $\aleph_n$.

Theorem (CFM): Let κ be supercompact. Then there is a generic extension W such that

1. $\kappa = \aleph_2^W$.

2. $\Box^*_{\aleph_\omega}$ fails in W.

3. If H is $Coll(\omega, \omega_1)$-generic over W then $\Box_{\aleph_\omega}$ holds in $W[H]$.

A similar argument shows that for λ regular we can create a situation in which $\Box_\lambda$ fails, and forcing with $Coll(\omega, \omega_1)$ makes $\Box_\lambda$ hold. However, in general it may not be possible to force $\Box_\lambda$ with mild forcing if we demand that $\Box^*_\lambda$ should fail in the ground model.

Theorem (CFM): Let $1 \leq n < \omega$ and let $\lambda = \aleph_n$. Let $\mathbb{P}$ be λ-c.c. and suppose that $\Vdash_{\mathbb{P}}$"$\Box_\lambda$ holds". Then $\Box^*_\lambda$ holds in V.

Theorem (CFM): Let MM^+ hold. Let κ be regular and uncountable. Then there is a forcing extension in which

- There are two stationary subsets of κ which do not reflect simultaneously.
- Stationary subsets of $[\lambda]^{\aleph_0}$ reflect for all $\lambda \geq \aleph_2$.