1. Iterated Forcing and Elementary Embeddings

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1. Introduction

In this chapter we present a survey of the area of set theory in which iterated forcing interacts with elementary embeddings. The original plan was to concentrate on forcing constructions which preserve large cardinal axioms, particularly *Reverse Easton* iterations.

However this plan proved rather restrictive, so we have also treated constructions such as Baumgartner's consistency proof for the Proper Forcing Axiom. The common theme of the constructions which we present is that they involve extending elementary embeddings.

We have not treated the preservation of large cardinal axioms by "Prikry-type" forcing, for example by Radin forcing or iterated Prikry forcing. For this we refer the reader to Gitik's chapter in this Handbook [22].

After some preliminaries, the bulk of this chapter consists of fairly short sections, in each of which we introduce one or two technical ideas and give one or more examples of the ideas in action. The constructions are generally of increasing complexity as we proceed and have more techniques at our disposal.

Especially at the beginning, we have adopted a fairly leisurely and discursive approach to the material. The impatient reader is encouraged to jump ahead and refer back as necessary. At the end of this introduction there is a brief description of the contents of each section.

Here is a brief review of our notation and conventions. We defer the discussion of forcing to Section 5.

- $P(X)$ is the power set of X. If X is a subset of a well-ordered set then $\mathrm{ot}(X)$ is the order-type of X. V_α is the set of sets with rank less than α. $\mathrm{tc}(X)$ is the transitive closure of X. H_θ is the set of x such that $\mathrm{tc}(\{x\})$ has cardinality less than θ.

- For τ a term and M a model, τ_M or τ^M denotes the result of interpreting the set-theoretic term τ in the model M, for example V_α^M or 2^ω_M. When $\tau_M = \tau \cap M$ we sometimes write "$\tau \cap M$" instead of "τ_M", especially when τ is a term of the form "$P(X)$" or "V_α".

- f is a *partial function f from X to Y* ($f : X \rightsquigarrow Y$) if and only if $f \subseteq X \times Y$ and for every $a \in X$ there is at most one $b \in Y$ with $(a, b) \in f$. f is a *total function from X to Y* ($f : X \to Y$) if and only if for every $a \in X$ there is exactly one $b \in Y$ with $(a, b) \in f$. As usual we write "$f(a) = b$" for "$(a, b) \in f$". id_X is the identity function on X.
- We use On for the class of ordinals, Card for the class of cardinals, Lim for the class of limit ordinals, Reg for the class of regular cardinals and Sing for the class of singular ordinals.
- If α is a limit ordinal then $\mathrm{cf}(\alpha)$ is the cofinality of α. If δ is a regular cardinal then $\mathrm{Cof}(\delta)$ is the class of limit ordinals α such that $\mathrm{cf}(\alpha) = \delta$. Expressions like "$\mathrm{Cof}({<}\kappa)$" have the obvious meaning.
- $|X|$ is the cardinality of X.
- ${}^X Y$ is the set of all functions from X to Y. If κ and λ are cardinals then $\kappa^\lambda = |{}^\lambda \kappa|$.
- We will make the following abuse of notation. When M, N are transitive models with $M \subseteq N$ we will write "$N \models {}^\beta M \subseteq M$" to mean that every β-sequence from M which lies in N actually lies in M, even in situations where possibly M is not definable in N. A similar convention applies when we write "$N \models {}^\beta \mathrm{On} \subseteq M$".
- $[X]^\lambda$ is the set of subsets of X of cardinality λ. Expressions like $[X]^{\leq\lambda}$ have the obvious meaning. If κ is regular and $\kappa \leq \lambda$ then $P_\kappa \lambda = \{a \in [\lambda]^{<\kappa} : a \cap \kappa \in \kappa\}$; this is a departure from the more standard notation in which the terms "$P_\kappa \lambda$" and "$[\lambda]^{<\kappa}$" are synonymous.
- A *tree* is a structure $(T, <_T)$ where $<_T$ is a well-founded strict ordering on T, and each element of T has a linearly ordered set of predecessors. T_α is the set of elements of height α, $T \restriction \alpha$ is the set of elements of height less than α.
- A tree is *normal* if and only if it is nonempty, has a unique minimal element, and has the properties that every element has two immediate successors and that every element of limit height is determined uniquely by the set of its predecessors in the tree. For κ regular a *κ-tree* is a normal tree of height κ, in which every level has size less than κ.
- ω_α is the α^{th} infinite cardinal.
- Throughout we use "inaccessible" to mean "strongly inaccessible" and "Mahlo" to mean "strongly Mahlo".

- An *ideal on* X is a non-empty family of subsets of X which is downwards closed and closed under finite unions; a *filter on* X is a non-empty family of subsets of X which is upwards closed and closed under finite intersections. An ideal I is *proper* if $X \notin I$, and a filter F is *proper* if $\emptyset \notin F$; most of the ideals and filters appearing in this chapter will be proper. If I is an ideal on X then $\{X \setminus A : A \in I\}$ is a filter on X, which is called the *dual filter* and will often be denoted by I^*; similarly if F is a filter then $F^* = \{X \setminus A : A \in F\}$ is an ideal.

  Ideals often arise in measure theory, where the class of measure zero sets for a (complete) measure on X is an ideal. If I is an ideal on X then we say that $A \subseteq X$ is *positive for* I or I*-positive* iff $A \notin I$, and we often write I^+ for the class of positive sets; we also sometimes say that A is *measure one for* I if $A \in I^*$. Similarly if F is a filter we say A is F*-positive* iff $A \notin F^*$, and is F*-measure one* iff $A \in F$.

- An *ultrafilter* on X is a maximal proper filter on X, or equivalently a filter U such that for all $A \subseteq X$ exactly one of the sets $A, X \setminus A$ is in U. An ultrafilter is *principal* if and only if it is of the form $\{A \subseteq X : a \in A\}$ for some $a \in X$.

- If I is an ideal and λ is a cardinal, then I is λ*-complete* if and only if I is closed under unions of length less than λ; similarly a filter F is λ-complete if and only if F is closed under intersections of length less than λ.

  If κ is a regular cardinal then a *measure on* κ is a κ-complete non-principal ultrafilter on κ. The measure U is *normal* if and only if it is closed under *diagonal intersections*, that is for every sequence $\langle X_i : i < \kappa \rangle$ with $X_i \in U$ for all $i < \kappa$, the *diagonal intersection* $\{\beta : \forall \alpha < \beta \; \beta \in X_\alpha\}$ of the sequence lies in U.

The prerequisites for reading this chapter are some familiarity with iterated forcing and the formulation of large cardinal axioms in terms of elementary embeddings. Knowledge of the material in Baumgartner's survey paper on iterated forcing [6, Sections 0, 1, 2 and 5] and Kanamori's book on large cardinals [43, Sections 5, 22, 23, 24 and 26] should be more than sufficient.

I learned much of what I know about elementary embeddings and forcing from Hugh Woodin, and would like to thank him for many patient explanations. I have also profited greatly from conversations with Uri Abraham, Arthur Apter, Jim Baumgartner, Matt Foreman, Sy Friedman, Moti Gitik, Aki Kanamori, Menachem Magidor, Adrian Mathias and Saharon Shelah.

Several people have been kind enough to read drafts of this chapter and give me their comments. Thanks in particular to Arthur Apter, Matt Foreman, Sy Friedman, Radek Honzik, Aki Kanamori, John Krueger, Peter Lumsdaine and Ernest Schimmerling; apologies to anyone whom I have omitted from this list.

We conclude this Introduction with the promised road map of the chapter.

- Section 2 discusses basic facts about elementary embeddings.
- Section 3 describes how we approximate elementary embeddings by ultrapowers and more generally by *extenders*, a special kind of limit ultrapower.
- Section 4 reviews some basic large cardinal axioms and their formulation in terms of elementary embeddings.
- Section 5 contains a discussion of the basics of forcing. Our convention (following Kunen [46]) is that a *notion of forcing* is a preordering with a designated largest element; we discuss the relationship with the other standard approaches to forcing. We review the basic closure, distributivity and chain condition properties and introduce some variants (the Knaster property and strategic closure) which are important later. We also introduce some basic forcing posets, Cohen forcing and the standard cardinal collapsing posets.
- Section 6 defines four forcing posets which enable us to distinguish different closure properties and will all play various roles later in the chapter. These are the posets to add a Kurepa tree, a non-reflecting stationary set, a square sequence and finally a club set disjoint from a prescribed co-stationary set in ω_1.
- Section 7 reviews iterated forcing, essentially following the approach of Baumgartner's survey [6]. We discuss the preservation of various closure and chain conditions and the idea of a factor iteration.
- Section 8 describes how to build generic objects over sufficiently closed inner models for sufficiently closed forcing posets. We apply this to construct a variant form of Prikry forcing first isolated by Foreman and Woodin in their work on the global failure of the GCH [21].
- Section 9 proves a key lemma of Silver's on lifting elementary embeddings to generic extensions, discusses the properties of the lifted embeddings and gives some easy applications.
- Section 10 discusses the key idea of a *generic elementary embedding*, constructs some examples and applies them to a discussion of stationary reflection at small cardinals.
- Section 11 describes Silver's idea of iterating forcing with Easton supports. As a first application we sketch a simpler proof of a theorem by Kunen and Paris [47], that under GCH a measurable cardinal κ may carry κ^{++} normal measures.
- Section 12 introduces another key idea of Silver's, that of a *master condition*. As a first example of a master condition argument we give something close to Silver's original consistency proof for the failure of

GCH at a measurable cardinal, starting from the hypothesis that there is a model of GCH in which some κ is κ^{++}-supercompact.

- Section 13 describes a technique, which is due to Magidor, for doing without a master condition under some circumstances. As an example we redo the failure of GCH at a measurable cardinal from the hypothesis that there is a model of GCH in which some κ is κ^{+}-supercompact.

- Section 14 describes how we may absorb κ-closed forcing posets into a large enough κ-closed collapsing poset, so that the quotient is also κ-closed. We then apply this to prove a theorem of Kunen [45] about saturated ideals, a theorem of the author from joint work with Džamonja and Shelah [11] about *strong non-reflection*, and Magidor's theorem [55] that consistently every stationary set in $\omega_{\omega+1}$ reflects.

- Section 15 discusses how to transfer generic filters between models of set theory, and sketches an application to constructing generalised versions of Prikry forcing.

- Section 16 shows that we may apply the ideas in this chapter in the context of weak large cardinal axioms such as weak compactness, and sketches a proof that GCH may first fail at a weakly compact cardinal.

- Section 17 proves two theorems of Jech, Magidor, Mitchell and Prikry [40]. The first result is that ω_1 may carry a precipitous ideal, the second is that in fact the non-stationary ideal on ω_1 may be precipitous. The argument for the second result uses the absorption idea from Section 14, and also involves iterating a natural forcing for *shooting club sets* through stationary sets.

- Section 18 sketches the proof of Gitik's result [23] that the precipitousness of NS_{ω_2} is equiconsistent with a cardinal of Mitchell order two.

- Section 19 gives two more applications of iterated club shooting, Jech and Woodin's result [41] that $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ can be κ^{+}-saturated for a Mahlo cardinal κ and Magidor's result [55] that consistently every stationary set of cofinality ω ordinals in ω_2 may reflect at almost all points of cofinality ω_1.

- Section 20 discusses some variant collapsing posets which are often useful, Kunen's *universal collapse* [45] and the *Silver collapse*. We sketch Kunen's proof [45] that ω_1 can carry an ω_2-saturated ideal, starting from the hypothesis of a huge cardinal.

- Section 21 sketches some results primarily due to Hamkins which put limits on what we can achieve by Reverse Easton forcing. As a sample application we sketch an easy case of Hamkins' *superdestructibility theorem* [32].

- Section 22 describes an idea of Laver's for introducing a kind of universal generic object by forcing with a poset of terms. As an application we sketch an unpublished proof by Magidor [51] of his celebrated theorem [52] that the least measurable cardinal can be strongly compact.
- Section 23 introduces the idea of analysing iterations by term forcing. As an example we introduce yet another collapsing poset and give a version of Mitchell's proof [57] that ω_2 may have the tree property.
- Section 24 discusses how to build universal iterations using prediction principles. We prove Laver's theorem that supercompact cardinals carry *Laver diamonds*, and use this to give Baumgartner's proof for the consistency of the Proper Forcing Axiom [15] and Laver's proof that a supercompact cardinal κ can be made indestructible under κ-directed closed forcing [49].
- Section 25 introduces an idea due to Woodin for altering generic objects, and then applies this to give Woodin's consistency proof for failure of GCH at a measurable from an optimal assumption.

2. Elementary Embeddings

We will be concerned with elementary embeddings $k : M \longrightarrow N$ where M, N are transitive models of ZFC and k, M, N are all classes of some universe of set theory. It will not in general be the case that k or N are classes of M or that $N \subseteq M$. In particular we will be interested in the situation of a "generic embedding" where $j : V \longrightarrow M \subseteq V[G]$ for $V[G]$ a generic extension of V, and j, M are defined in $V[G]$.

This notion is straightforward if M, N are sets but one needs to be a little careful when M, N are proper classes. We refer the reader to Kanamori's book [43, Sections 5 and 19] for a careful discussion of the metamathematical issues. From now on we will freely treat elementary embeddings between proper classes as if those classes were sets, a procedure which can be justified by the methods of [43]. We reserve the term "inner model" for a transitive class model of ZFC which contains all the ordinals.

We start by recalling a few basic facts about elementary embeddings.

2.1 Proposition. *Let M and N be transitive models of* ZFC *and let the map $k : M \longrightarrow N$ be elementary. Then*

1. *The pointwise image k“M is an elementary substructure of N, the Mostowski collapse of the structure $(k“M, \in)$ is M, and k is the inverse of the collapsing isomorphism from k“M to M.*
2. *$k(\alpha) \geq \alpha$ for all $\alpha \in M \cap \mathrm{On}$.*
3. *If $k \restriction (\beta + 1) = \mathrm{id}$ and $A \in M$ with $A \subseteq \beta$, then $k(A) = A$.*

Proof. Easy. ⊣

2.2 Proposition. *Let M be a transitive model of* ZFC*, let $x \in M$ and let $M \models$ "$x \in H_{\lambda^+}$" where λ is an infinite M-cardinal. Then there is a set $A \subseteq \lambda$ such that $A \in M$ and for any transitive model N of* ZF*, $A \in N$ implies that $x \in N$.*

Proof. Let $f \in M$ be an injection from $\mathrm{tc}(\{x\})$ to λ, let G be Gödel's pairing function and let

$$A = \{G(f(a), f(b)) : a, b \in \mathrm{tc}(\{x\}) \text{ and } a \in b\}.$$

If $A \in N$ then N can compute x by forming the Mostowski collapse of the well-founded extensional relation $\{(\alpha, \beta) : G(\alpha, \beta) \in A\}$, and then finding the element of maximal rank in this set. ⊣

We abbreviate the rather cumbersome assertion "A is a set of ordinals such that $\{(\gamma, \delta) : G(\gamma, \delta) \in A\}$ is a well founded relation whose transitive collapse is $\mathrm{tc}(\{x\})$" by "A codes x". The assertions "A codes x" and "A codes something" are both Δ_1^{ZFC} and are thus absolute between transitive models of ZFC.

2.3 Proposition. *Let M and N be transitive models of* ZFC *and let the map $k : M \longrightarrow N$ be elementary. If k"$(M \cap \mathrm{On})$ is cofinal in $N \cap \mathrm{On}$ then exactly one of the following is true:*

1. *$k = \mathrm{id}_M$ and $M = N$.*

2. *There exists an ordinal $\delta \in M \cap \mathrm{On}$ such that $k(\delta) > \delta$.*

Proof. Suppose the second alternative fails, so that $k \restriction (M \cap \mathrm{On})$ is the identity. Let $x \in M$ and find a set of ordinals $A \in M$ such that A codes x. Then $A = k(A)$ by Proposition 2.1, $k(A)$ codes $k(x)$ by elementarity, and so $k(x) = x$. Since x was arbitrary, $k = \mathrm{id}_M$.

Since $k = \mathrm{id}_M$, $M \cap \mathrm{On} = N \cap \mathrm{On}$ and $V_\beta^N = V_{k(\beta)}^N = k(V_\beta^M) = V_\beta^M$ for all $\beta \in M \cap \mathrm{On}$. So $M = N$. ⊣

From now on we will say that $k : M \longrightarrow N$ is *nontrivial* if $k \neq \mathrm{id}_M$.

2.4 Remark. It was crucial in Proposition 2.3 that k should map $M \cap \mathrm{On}$ cofinally into $N \cap \mathrm{On}$. For example the theory of sharps [43, Section 9] shows that if $0^\sharp$ exists then L_{ω_1}, L_{ω_2} are models of ZFC and $L_{\omega_1} \prec L_{\omega_2}$.

2.5 Remark. Let $k : M \longrightarrow N$ be elementary, where M is an inner model and N is transitive. Then N is an inner model, and the hypotheses of Proposition 2.3 are satisfied.

If $k : M \longrightarrow N$ is elementary then the least δ such that $k(\delta) > \delta$ (if it exists) is called the *critical point* of k and is denoted by $\mathrm{crit}(k)$. It is not hard to see that $\mathrm{crit}(k)$ is a regular uncountable cardinal in M.

It is natural to ask how much agreement there must be between the models M and N. The following proposition puts a lower bound on the level of agreement.

2.6 Proposition. *If $k : M \longrightarrow N$ is an elementary embedding between transitive models of* ZFC *and* $\mathrm{crit}(k) = \delta$*, then $H^M_{\delta^+} \subseteq N$.*

Proof. Let $x \in H^M_{\delta^+}$ and let $A \in M$ code x with $A \subseteq \delta$. Then for $\alpha < \delta$ we have

$$\alpha \in A \iff k(\alpha) \in k(A) \iff \alpha \in k(A),$$

so $A = k(A) \cap \delta \in N$. Therefore $x \in N$. ⊣

In general we cannot say much more, as illustrated by the following two examples. In Example 2.7 $M = N$, while in Example 2.8 M and N agree only to the extent indicated by Proposition 2.6.

2.7 Example. Suppose that $0^\sharp$ exists. Then there is a nontrivial elementary embedding $k : L \longrightarrow L$ [43, Section 9].

2.8 Example. It is consistent (from large cardinals) that there exist inner models M and N and an embedding $k : M \longrightarrow N$ such that $\mathrm{crit}(k) = \omega_1^M$ and $V_{\omega+1} \cap M \subsetneq V_{\omega+1} \cap N$. We will construct such an example in Theorem 10.2.

If the critical point is inaccessible in M we can say more:

2.9 Proposition. *If $k : M \longrightarrow N$ is an elementary embedding between transitive models of* ZFC*, and* $\mathrm{crit}(k) = \delta$ *where δ is inaccessible in M, then $V_\delta \cap M = V_\delta \cap N$.*

Proof. For $\alpha < \delta$, the set $V_\alpha \cap M$ is coded by a bounded subset of δ lying in M. In particular it is fixed by k, so as α is also fixed by elementarity $V_\alpha \cap M = V_\alpha \cap N$. ⊣

In the theory of large cardinals we are most interested in embeddings of the following type, where usually M will be an inner model.

2.10 Definition. An embedding $k : M \longrightarrow N$ is *definable* if and only if k and N are definable in M.

The analysis of these embeddings is due to Scott [61] and is summarised in the following proposition.

2.11 Proposition. *Let M, N be inner models and let $k : M \longrightarrow N$ be a nontrivial definable elementary embedding with* $\mathrm{crit}(k) = \delta$*. Let*

$$U = \{X \subseteq \delta : X \in M, \delta \in k(X)\}.$$

Then

1. *$U \in M$ and $M \models$ "U is a normal measure on δ".*

2. *$V^M_{\delta+1} = V^N_{\delta+1}$.*

3. *$k \restriction V^M_\delta = \mathrm{id}$.*

4. *For all $A \in V^M_{\delta+1}$, $A = k(A) \cap V^M_\delta$.*

Proof. See [43, Section 5]. ⊣

2.12 Remark. Neither of the embeddings from Examples 2.7 and 2.8 is definable.

3. Ultrapowers and Extenders

It will be important for us to be able to describe embeddings between models by ultrapowers and limit ultrapowers. We give a sketchy outline here and refer the reader to [43, Sections 19 and 26] for the details.

Let M be a transitive model of ZFC, let $X \in M$ and let U be an ultrafilter on $P(X) \cap M$. Then we may form $\mathrm{Ult}(M, U)$, the collection of U-equivalence classes of functions $f \in M$ with $\mathrm{dom}(f) = X$. As usual we let $[f]_U$ denote the class of f, and for $x \in M$ we let $j_U(x) = [f_x]_U$ where f_x is the function with domain X and constant value x. $\mathrm{Ult}(M, U)$ is made into a structure for the language of set theory by defining

$$[f]_U E [g]_U \iff \{x : f(x) \in g(x)\} \in U,$$

and we make a mild abuse of notation by writing "$\mathrm{Ult}(M, U)$" for the structure $(\mathrm{Ult}(M, U), E)$.

3.1 Remark. When M is an inner model $[f]_U$ is typically a proper class, which makes the definition of $\mathrm{Ult}(M, U)$ appear problematic. This can be fixed by *Scott's trick* in which $[f]_U$ is redefined as the set of functions with minimal rank which are equivalent to f modulo U. Similar remarks apply to ultrapowers throughout this chapter.

Since M is a model of ZFC Łoś's theorem holds, that is to say that for any formula $\phi(x_1, \ldots, x_n)$ and any functions $F_1, \ldots, F_n \in M$ with domain X,

$$\mathrm{Ult}(M, U) \models \phi([F_1]_U, \ldots, [F_n]_U)$$

if and only if

$$\{x : M \models \phi(F_1(x), \ldots, F_n(x))\} \in U.$$

In particular j_U is an elementary embedding from M to $\mathrm{Ult}(M, U)$. When $\mathrm{Ult}(M, U)$ is well-founded we will identify it with its transitive collapse. The following propositions are standard.

3.2 Proposition. *Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of* ZFC*, let $a \in N$ and let $B \in M$ with $a \in k(B)$. Let $E_a = \{A \subseteq B : A \in M, a \in k(A)\}$. Then*

1. *E_a is an ultrafilter on $P(B) \cap M$. For notational convenience we define $M_a = \mathrm{Ult}(M, E_a)$ and $j_a = j_{E_a}$.*
2. *If we define $k_a : M_a \longrightarrow N$ by $k_a([f]_{E_a}) = k(f)(a)$ then k_a is a well-defined elementary embedding and $k_a \circ j_a = k$. k_a and M_a do not depend on the choice of B.*
3. *M_a is isomorphic via k_a to X_a, where*
$$X_a = \{k(F)(a) : F \in M, \mathrm{dom}(F) = B\}.$$
4. *M_a is well-founded and when we identify it with its transitive collapse k_a is the inverse of the transitive collapsing map on X_a.*
5. *If k is definable then $E_a \in M$ and j_a is definable.*

3.3 Proposition. *Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of* ZFC*. Let $a_1 \in k(B_1)$, $a_2 \in k(B_2)$ and let E_1, E_2 be the associated ultrafilters. Suppose that $F : B_2 \longrightarrow B_1$ is such that $k(F)(a_2) = a_1$. Then F induces an elementary embedding*

$$F^* : \mathrm{Ult}(M, E_1) \longrightarrow \mathrm{Ult}(M, E_2),$$

where $F^*([g]_{E_1}) = [g \circ F]_{E_2}$. Moreover $j_{E_2} = F^* \circ j_{E_1}$.

3.4 Proposition. *Let $\lambda \in N \cap \mathrm{On}$ be such that $\lambda \leq \sup(k``(M \cap \mathrm{On}))$. For each $a \in [\lambda]^{<\omega}$ let μ_a be the least ordinal such that $a \subseteq j(\mu_a)$ and let*

$$E_a = \{A \subseteq [\mu_a]^{|a|} : A \in M, a \in k(A)\}.$$

Let M_a, j_a, k_a, and X_a be as in Proposition 3.2. If $a, b \in [\lambda]^{<\omega}$ and $a \subseteq b$ then define

$$F_{ab}(x) = \{\gamma \in x : \exists \gamma^* \in a \ \mathrm{ot}(x \cap \gamma) = \mathrm{ot}(b \cap \gamma^*)\}$$

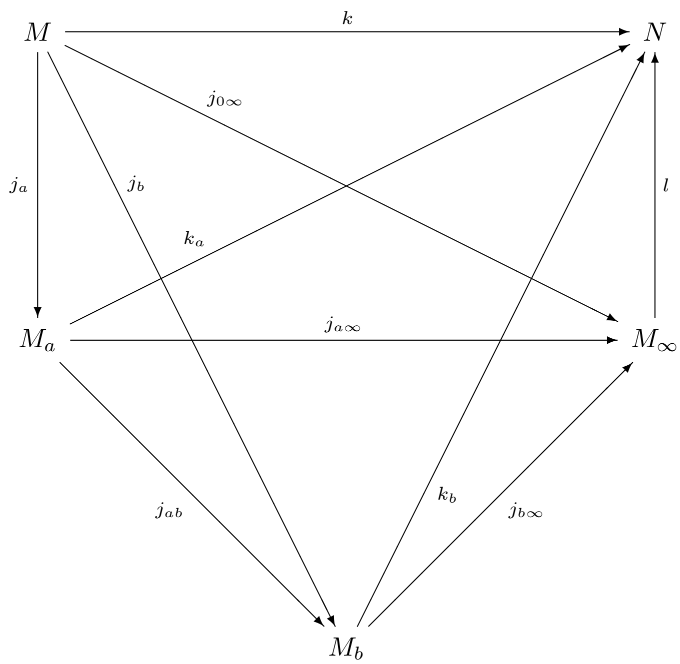
for $x \in [\mu_b]^{|b|}$. Then

1. *$F_{ab} : [\mu_b]^{|b|} \longrightarrow [\mu_a]^{|a|}$ and $k(F_{ab})(b) = a$. We let j_{ab} denote the embedding from M_a to M_b induced by F_{ab}.*
2. *$M_0 = M$, $k_0 = k$, $j_{0a} = j_a$.*
3. *The system of structures M_a and embeddings j_{ab} is a directed system, so has a direct limit M_∞. There are elementary embeddings $j_{a\infty} : M_a \longrightarrow M_\infty$ such that $M_\infty = \bigcup_a j_{a\infty}[M_a]$ and $j_{b\infty} \circ j_{ab} = j_{a\infty}$.*

4. *There is an elementary embedding* $l : M_\infty \longrightarrow M$ *such that* $l \circ j_{a\infty} = k_a$ *for all* a*.*

5. M_∞ *is isomorphic via* l *to* $X_\infty = \bigcup_a X_a$*, and* l *is the inverse of the Mostowski collapsing map on* X_∞*. In particular* M_∞ *is well-founded.*

6. *If* k *is definable and* M *is an inner model then* $j_{0\infty}$ *is definable.*



If $k : M \longrightarrow N$ is elementary where M, N are inner models then we may make X_∞ contain arbitrarily large initial segments of N by choosing λ sufficiently large. M_∞ is the transitive collapse of X_∞, l is the inverse of the collapsing map and $l \circ j_{0\infty} = k$. It follows that we may make $j_{0\infty}$ approximate k to any required degree of precision by a suitable choice of λ.

3.5 Definition. Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of ZFC with $\mathrm{crit}(k) = \delta$, and let $\lambda \leq \sup(k``(M \cap \mathrm{On}))$. If

$$E = \{E_a : a \in [\lambda]^{<\omega}\}$$

where E_a is defined as above, then we call E the M-(δ, λ)*-extender derived from* k.

It is possible [43, Section 26] to give an axiomatisation of the properties which are enjoyed by E as in Definition 3.5, thus arriving at the concept of an "M-(δ, λ)-extender". Given an M-(δ, λ)-extender E we can compute the limit ultrapower of M by E to get a well-founded structure $\mathrm{Ult}(M, E)$ and an embedding $j_E : M \longrightarrow \mathrm{Ult}(M, E)$.

If E is the extender derived from $k : M \longrightarrow N$ as in Proposition 3.4 then in the notation of that proposition $\mathrm{Ult}(M, E) = M_\infty$ and $j_E = j_{0\infty}$. If E is an M-(δ, λ)-extender and E' is the M-(δ, λ)-extender derived from the ultrapower map $j_E : M \longrightarrow \mathrm{Ult}(M, E)$ then $E = E'$.

When E is a V-(δ, λ)-extender lying in V we will just refer to E as a "(δ, λ)-extender".

3.6 Definition. An M-(δ, λ)-extender E is called *short* if all the measures E_a concentrate on $[\delta]^{<\omega}$, or equivalently if $\lambda \leq j_E(\delta)$.

We now make a couple of (non-standard) definitions which will give us a convenient way of phrasing some results later. See for example Propositions 3.9 and 15.1.

3.7 Definition. Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of ZFC, and let μ be an ordinal. The embedding k *has width* $\leq \mu$ if and only if every element of N is of the form $k(F)(a)$ for some $F \in M$, $a \in N$ where $M \models |\mathrm{dom}(F)| \leq \mu$.

3.8 Definition. Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of ZFC, and let $A \subseteq N$. The embedding k is *supported on* A if and only if every element of N is of the form $k(F)(a)$ for some $F \in M$ and $a \in A \cap \mathrm{dom}(k(F))$

The following easy Proposition will be useful later.

3.9 Proposition. *Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of* ZFC *with* $\mathrm{crit}(k) = \kappa$*, and let*

$$U = \{X \subseteq \kappa : X \in M, \kappa \in k(X)\}.$$

Then k is the ultrapower map computed from M and U if and only if k is supported on $\{\kappa\}$.

4. Large Cardinal Axioms

We briefly review some standard large cardinal axioms and their formulation in terms of elementary embeddings and ultrapowers. Once again we refer the reader to Kanamori's book [43] for the details.

We start with the characterisations in terms of elementary embeddings.

- κ is *measurable* if and only if there is a definable $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$.
- κ is λ*-strong* if and only if there is a definable $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \subseteq M$. κ is *strong* if and only if it is λ-strong for all λ.

- κ is λ-*supercompact* if and only if there is a definable $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$. κ is *supercompact* if and only if it is λ-supercompact for all λ.

- κ is λ-*strongly compact* if and only if there is a definable $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and there is a set $X \in M$ such that $M \models |X| < j(\kappa)$ and $j``\lambda \subseteq X$. κ is strongly compact if and only if it is λ-strongly compact for all λ.

- κ is *huge with target* λ if and only if there is a definable $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and ${}^{\lambda}M \subseteq M$. κ is *almost huge with target* λ if and only if there is a definable $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and ${}^{<\lambda}M \subseteq M$.

Each of these concepts can also be characterised using ultrafilters or extenders.

- κ is measurable if and only if there is a *measure on* κ (that is a normal κ-complete non-principal ultrafilter on κ).

- Assuming GCH, κ is $(\kappa + \beta)$-strong if and only if there is a short $(\kappa, \kappa^{+\beta})$-extender E such that $V_{\kappa+\beta} \subseteq \mathrm{Ult}(V, E)$.

- For $\lambda \geq \kappa$, κ is λ-supercompact if and only if there is a normal, fine and κ-complete ultrafilter on $P_\kappa\lambda$. We will generally refer to such an object as a *supercompactness measure on* $P_\kappa\lambda$.

- For $\lambda \geq \kappa$, κ is λ-strongly compact if and only if there is a fine and κ-complete ultrafilter on $P_\kappa\lambda$. We will generally refer to such an object as a *strong compactness measure on* $P_\kappa\lambda$.

- For $\lambda \geq \kappa$, κ is huge with target λ if and only if there is a normal, fine and κ-complete ultrafilter on $P^\kappa\lambda$, where $P^\kappa\lambda$ is the set of $X \subseteq \lambda$ with order type κ. Almost-hugeness has a rather technical characterisation in terms of a direct limit system of supercompactness measures on $P_\kappa\mu$ for $\mu < \lambda$.

4.1 Remark. If $j : V \longrightarrow M$ is a definable embedding such that $\mathrm{crit}(j) = \kappa$ and $j``\lambda \in M$, then $\{X \in P_\kappa\lambda : j``\lambda \in j(X)\}$ is a supercompactness measure.

4.2 Remark. Weak compactness may also be characterised in terms of elementary embeddings, we discuss this in Section 16.

For use later we record the definition of the *Mitchell ordering* $\lhd$ and a few basic facts about it.

4.3 Definition. Let κ be a measurable cardinal and let U_0, U_1 be measures on κ. Then $U_0 \lhd U_1$ if and only if $U_0 \in \mathrm{Ult}(V, U_1)$.

The theory of the Mitchell ordering is developed in Mitchell's chapter "Beginning inner model theory". The relation $\lhd$ is a strict well-founded partial ordering. If U is a measure then $o(U)$ is defined to be the height of U in $\lhd$, and the *Mitchell order* $o(\kappa)$ of κ is defined to be the height of $\lhd$. In the usual canonical inner models for large cardinals, $\lhd$ is a linear ordering.

The following propositions collect some easy but useful facts about the behaviour of elementary embeddings.

4.4 Proposition. *Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of* ZFC, *and let k have width $\leq \mu$. If $M \models \mathrm{cf}(\alpha) > \mu$ then $\sup(k``\alpha) = k(\alpha)$.*

If $\sup(k``\alpha) = k(\alpha)$ we will say that k is *continuous at α.*

4.5 Proposition. *If $U \in V$ is a countably complete ultrafilter on X, X has cardinality κ and $j : V \longrightarrow M$ is the associated ultrapower map then $|j(\mu)| < (|\mu|^\kappa)^+$ for all ordinals μ.*

4.6 Proposition. *If $E \in V$ is a short (κ, λ)-extender and $j : V \longrightarrow M$ is the associated ultrapower map then $|j(\mu)| < (\lambda \times |\mu|^\kappa)^+$ for all ordinals μ.*

4.7 Proposition. *Let M be an inner model of V. If ${}^\lambda M \subseteq M$ then the cardinals of V and M agree up to and including λ^+. If* GCH *holds, κ is inaccessible and $V_{\kappa+\beta} \subseteq M$ then the cardinals of V and M agree up to $\beth_\beta(\kappa)$.*

The following example illustrates how these ideas can be used. There are many similar calculations in later sections, where we will generally suppress the details.

4.8 Example. Let GCH hold and let U be a supercompactness measure on $P_\kappa \kappa^+$, with $j : V \longrightarrow M$ the associated ultrapower map. Then

1. j is continuous at κ^{++} and κ^{+++}.
2. $\kappa^{++} < j(\kappa)$.
3. $j(\kappa^{+++}) = \kappa^{+++}$.

Proof. $|P_\kappa \kappa^+| = \kappa^+$, so by Proposition 4.4 j is continuous at κ^{++} and κ^{+++}. By the definition of a supercompactness measure ${}^{\kappa^+}M \subseteq M$, and so by Proposition 4.7 $\kappa^{++} = \kappa^{++}_M$. By elementarity $j(\kappa)$ is an M-inaccessible cardinal greater than κ, and so $\kappa^{++} < j(\kappa)$.

For every $\eta < \kappa^{+++}$, Proposition 4.5 and GCH imply that $j(\eta) < \kappa^{+++}$. Since j is continuous at κ^{+++} we have $j(\kappa^{+++}) = \kappa^{+++}$ as required. ⊣

5. Forcing

We assume that the reader is familiar with forcing; in this section we establish our forcing conventions and review some of the basic definitions and facts.

We will essentially follow the treatment of forcing in Kunen's text [46]. Proofs of all the facts that we mention in this section can be found in at least one of the texts by Kunen [46] or Jech [39].

Our approach to forcing is based on posets with a largest element. We justify this by the sociological observation that when a set theorist writes down a new set of forcing conditions it is almost always of this form.

For technical reasons we sometimes work with preordered sets rather than partially ordered sets; recall that a *preordering* is a transitive and reflexive relation, and that if $\leq$ is a preordering of $\mathbb{P}$ we may form the quotient by the equivalence relation

$$pEq \iff p \leq q \leq p$$

to get a partially ordered set. We refer to this as the *quotient poset*.

A *largest element* in a preordered set $\mathbb{P}$ is an element b such that $a \leq b$ for all a. A preordering may have many largest elements, which will all be identified when we form the quotient poset.

A *notion of forcing* is officially a triple $(\mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}})$ where $\leq_{\mathbb{P}}$ is a preordering of $\mathbb{P}$ and $1_{\mathbb{P}}$ is a largest element. A *forcing poset* is a notion of forcing where $\leq_{\mathbb{P}}$ is a partial ordering; if $\mathbb{P}$ is a notion of forcing then the quotient poset is a forcing poset. If p, q are conditions in a notion of forcing $\mathbb{P}$ then $p \leq q$ means that p is stronger than q.

5.1 Remark. It might seem more natural just to use forcing posets in our discussion of forcing. However this would cause irritating problems when we come to discuss iterated forcing; for example in a two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ we may have $p \Vdash \dot{q}_1 = \dot{q}_2$, in which case the conditions $(p, \dot{q}_1)$ and $(p, \dot{q}_2)$ are equivalent but not identical.

For $p \in \mathbb{P}$ we denote by $\mathbb{P}/p$ the subset $\{q \in \mathbb{P} : q \leq p\}$ with the inherited ordering. It is a standard fact that there is a bijection between $\mathbb{P}$-generic filters G with $p \in G$, and $(\mathbb{P}/p)$-generic filters, in which G corresponds to $G \cap (\mathbb{P}/p)$. If $p \in G$ then $V[G] = V[G \cap (\mathbb{P}/p)]$.

We say $\mathbb{P} \subseteq \mathbb{Q}$ is *dense* if every condition in $\mathbb{Q}$ has an extension in $\mathbb{P}$. There is a bijection between $\mathbb{Q}$-generic filters G and $\mathbb{P}$-generic filters, in which G corresponds to $G \cap \mathbb{P}$ and $V[G] = V[G \cap \mathbb{P}]$.

If $\mathbb{P}$ is a notion of forcing then the class $V^{\mathbb{P}}$ of $\mathbb{P}$-names is defined recursively so that σ is a $\mathbb{P}$-name if and only if every element of σ has the form (τ, p) for some $\mathbb{P}$-name τ and condition $p \in \mathbb{P}$.

We denote by $i_G(\sigma)$ the result of interpreting the name σ with respect to the filter G, that is

$$i_G(\sigma) = \{i_G(\tau) : \exists p \in G \ (\tau, p) \in \sigma\}.$$

We let $\check{x}$ denote the standard forcing name for the ground model object x, that is $\check{x} = \{(\check{y}, 1_{\mathbb{P}}) : y \in x\}$. $\dot{G} = \{(\check{p}, p) : p \in \mathbb{P}\}$ is the standard name for the generic filter.

A notion of forcing is *non-trivial* if and only if it is forced by every condition that $V[G] \neq V$, or equivalently that $G \notin V$. *The trivial forcing* is the forcing poset with just one element; we usually denote the trivial forcing by "$\{1\}$".

It is easy to see that $p \Vdash \check{q} \in \dot{G}$ if and only if every extension of p is compatible with q; we will say that a notion of forcing is *separative* when $p \Vdash \check{q} \in \dot{G} \iff p \leq q$. It is routine to check that if $\mathbb{P}$ is a separative notion of forcing then the quotient forcing poset is also separative.

It is a standard fact that for any notion of forcing $\mathbb{P}$ there is a separative forcing poset $\mathbb{Q}$ and an order and incompatibility preserving surjection $h : \mathbb{P} \to \mathbb{Q}$. The map h and forcing poset $\mathbb{Q}$ are unique up to isomorphism, $\mathbb{Q}$ is called the *separative quotient* of $\mathbb{P}$ and forcing with $\mathbb{Q}$ is equivalent to forcing with $\mathbb{P}$.

If $\mathbb{P}$ is a separative forcing poset then the Boolean algebra $\mathrm{ro}(\mathbb{P})$ of regular open subsets of $\mathbb{P}$ is complete, and $\mathbb{P}$ is isomorphic to a dense set in $\mathrm{ro}(\mathbb{P}) \setminus \{0\}$. It follows that there is a bijection between $\mathbb{P}$-generic filters and $\mathrm{ro}(\mathbb{P})$-generic ultrafilters, so that forcing with the poset $\mathbb{P}$ is equivalent to forcing with the complete Boolean algebra $\mathrm{ro}(\mathbb{P})$. We sometimes abuse notation and write $\mathrm{ro}(\mathbb{P})$ for the regular open algebra of the separative quotient of a notion of forcing $\mathbb{P}$.

In general when $\mathbb{P}$ and $\mathbb{Q}$ are notions of forcing we will say that they are *equivalent* if and only if for every $\mathbb{P}$-generic filter G there is a $\mathbb{Q}$-generic filter H with $V[G] = V[H]$, and symmetrically for every $\mathbb{Q}$-generic filter H there is a $\mathbb{P}$-generic filter G with $V[H] = V[G]$. It is routine to see that this can be formulated in a first-order way which does not mention generic filters.

Complete Boolean algebras have the advantage that they allow a straightforward discussion of the relationship between different forcing extensions. If $\mathbb{P}$ and $\mathbb{Q}$ are notions of forcing then forcing with $\mathbb{P}$ is equivalent to forcing with $\mathbb{Q}$ if and only if $\mathrm{ro}(\mathbb{P})$ is isomorphic to $\mathrm{ro}(\mathbb{Q})$. For $\mathbb{C}$ a complete Boolean algebra and G a $\mathbb{C}$-generic ultrafilter over V, the models of ZFC intermediate between V and $V[G]$ are precisely the models of form $V[G \cap \mathbb{B}]$ for $\mathbb{B}$ a complete subalgebra of $\mathbb{C}$.

In particular when $\mathbb{B}$ is a complete subalgebra of $\mathbb{C}$ then $\dot{G}_{\mathbb{C}} \cap \mathbb{B}$ is a $\mathbb{C}$-name for a $\mathbb{B}$-generic ultrafilter. Conversely for any complete $\mathbb{B}$ and $\mathbb{C}$, a $\mathbb{C}$-name for a $\mathbb{B}$-generic ultrafilter gives a complete embedding of $\mathbb{B}$ into $\mathbb{C}$.

Since we are wedded to an approach to forcing via posets, it is helpful to have some sufficient conditions which guarantee that a $\mathbb{Q}$-generic extension contains a $\mathbb{P}$-generic one without mentioning the regular open algebras.

5.2 Definition. If $\mathbb{P}$ and $\mathbb{Q}$ are notions of forcing then a *projection* from $\mathbb{Q}$ to $\mathbb{P}$ is a map $\pi : \mathbb{Q} \to \mathbb{P}$ such that π is order-preserving, $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$, and for all $q \in \mathbb{Q}$ and all $p \leq \pi(q)$ there is $\bar{q} \leq q$ such that $\pi(\bar{q}) \leq p$.

The following facts are standard [2]:

1. If H is $\mathbb{Q}$-generic over V then π"H generates a $\mathbb{P}$-generic filter G.

2. Conversely if G is $\mathbb{P}$-generic over V and we set

$$\mathbb{Q}/G = \{q \in \mathbb{Q} : \pi(q) \in G\},$$

with the partial ordering inherited from $\mathbb{Q}$, then any $H \subseteq \mathbb{Q}/G$ which is $\mathbb{Q}/G$-generic over $V[G]$ is $\mathbb{Q}$-generic over V.

5.3 Remark. In general if $\mathbb{Q}$ and $\mathbb{P}$ are forcing posets such that forcing with $\mathbb{Q}$ adds a generic object for $\mathbb{P}$, then there is a projection from $\mathbb{Q}$ to the poset of nonzero elements of $ro(\mathbb{P})$.

5.4 Definition. If $\mathbb{P}$ and $\mathbb{Q}$ are notions of forcing then a *complete embedding* from $\mathbb{P}$ to $\mathbb{Q}$ is a function $i : \mathbb{P} \to \mathbb{Q}$ such that $i(1_\mathbb{P}) = 1_\mathbb{Q}$, and

$$p_1 \leq p_2 \iff i(p_1) \leq i(p_2)$$

for all p_1 and p_2 in $\mathbb{P}$, and for every $q \in \mathbb{Q}$ there is a condition $p \in \mathbb{P}$ such that $i(\bar{p})$ is compatible with q for all $\bar{p} \leq p$.

The following facts are standard [46]:

1. If H is $\mathbb{Q}$-generic over V then $G = i^{-1}``H$ is a $\mathbb{P}$-generic filter.
2. Conversely if G is $\mathbb{Q}$-generic over V and we set $\mathbb{Q}/G$ to be the set of $q \in \mathbb{Q}$ which are compatible with all elements of $i``G$, any H which is $\mathbb{Q}/G$-generic over $V[G]$ is $\mathbb{P}$-generic over V.

5.5 Remark. In the context of projections or complete embeddings as above $\mathbb{Q}/G$ may not be separative, even if $\mathbb{P}$ and $\mathbb{Q}$ both are.

5.6 Remark. We have overloaded the notation "$\mathbb{Q}/G$", defining it both in the setting of a projection from $\mathbb{Q}$ to $\mathbb{P}$ and of a complete embedding from $\mathbb{P}$ to $\mathbb{Q}$. This is (we assert) harmless in the sense that if we have both a projection $\pi : \mathbb{Q} \to \mathbb{P}$ and a complete embedding $i : \mathbb{P} \to \mathbb{Q}$, and $i \circ \pi = \mathrm{id}_\mathbb{P}$, then the two definitions of $\mathbb{Q}/G$ give equivalent notions of forcing.

We will make some use of the *Maximum principle*: if $\mathbb{P}$ is a notion of forcing and $p \in \mathbb{P}$ forces $\exists x\ \phi(x)$, then there is a term $\dot{\tau} \in V^\mathbb{P}$ such that $p \Vdash \phi(\dot{\tau})$. This needs the Axiom of Choice, but that presents no obstacle for us.

When we say that $\mathbb{P}$ *adds* some kind of object or *forces* some statement to hold, we mean that this is forced by every condition in $\mathbb{P}$, or equivalently it is forced by $1_\mathbb{P}$. This is important because some natural notions of forcing are highly inhomogeneous.

We will frequently use the standard forcing posets for adding subsets to a regular cardinal κ, and for collapsing cardinals to have cardinality κ. Each forcing poset consists of a family of partial functions ordered by reverse inclusion.

5.7 Definition. Let κ be a regular cardinal, and let λ be any ordinal.

1. (Cohen forcing) $\mathrm{Add}(\kappa, \lambda)$ is the set of all partial functions from $\kappa \times \lambda$ to 2 of cardinality less than κ.
2. $\mathrm{Col}(\kappa, \lambda)$ is the set of all partial functions from κ to λ of cardinality less than κ.
3. (The Lévy collapse) $\mathrm{Col}(\kappa, <\lambda)$ is the set of all partial functions p from $\kappa \times \lambda$ to λ such that
   (a) $|p| < \kappa$.
   (b) $p(\alpha, \beta) < \beta$ for all $(\alpha, \beta) \in \mathrm{dom}(p)$.

5.8 Definition. Let $\mathbb{P}$ be a notion of forcing and let κ be an uncountable cardinal. Then

1. $\mathbb{P}$ is *κ-chain condition (κ-c.c.)* if and only if $\mathbb{P}$ has no antichain of size κ.
2. $\mathbb{P}$ is *κ-closed* if and only if every decreasing sequence of conditions in $\mathbb{P}$ with length less than κ has a lower bound.
3. $\mathbb{P}$ is (κ, ∞)-*distributive* if and only if forcing with $\mathbb{P}$ adds no new $<\kappa$-sequence of ordinals.
4. $\mathbb{P}$ is *κ-directed closed* if and only if every directed set of size less than κ of conditions in $\mathbb{P}$ has a lower bound.

5.9 Remark. If $\mathbb{P}$ is separative, then $\mathbb{P}$ is (κ, ∞)-distributive if and only if every $<\kappa$-sequence of dense open subsets of $\mathbb{P}$ has a nonempty intersection.

The following fact is easy but crucial. See [39, Lemma 20.5] for a proof.

5.10 Fact (Easton's Lemma). Let κ be a regular uncountable cardinal. Let $\mathbb{P}$ be κ-c.c. and let $\mathbb{Q}$ be κ-closed. Then

1. $\Vdash_{\mathbb{P} \times \mathbb{Q}}$ "$\check{\kappa}$ is a regular uncountable cardinal".
2. $\Vdash_{\mathbb{Q}}$ "$\check{\mathbb{P}}$ is $\check{\kappa}$-c.c.".
3. $\Vdash_{\mathbb{P}}$ "$\check{\mathbb{Q}}$ is $(\check{\kappa}, \infty)$-distributive".

It is sometimes useful to consider a stronger form of the κ-c.c. See Kunen and Tall's paper [48] for more information about the following property.

5.11 Definition. Let κ be an uncountable regular cardinal. A poset $\mathbb{P}$ is *κ-Knaster* if and only if for every κ-sequence of conditions $\langle p_\alpha : \alpha < \kappa \rangle$ there is a set $X \subseteq \kappa$ unbounded such that $\langle p_\alpha : \alpha \in X \rangle$ consists of pairwise compatible conditions.

For example the standard Δ-system proof [46, Theorem 1.6] that the Cohen poset $\mathrm{Add}(\kappa, \lambda)$ is $(2^{<\kappa})^+$-c.c. actually shows that $\mathrm{Add}(\kappa, \lambda)$ is $(2^{<\kappa})^+$-Knaster. The following easy fact shows that the Knaster property is in some ways better behaved than the property of being κ-c.c. It is not in general the case that the product of two κ-c.c. posets is κ-c.c.

5.12 Fact. Let κ be regular and let $\mathbb{P}$, $\mathbb{Q}$ be two notions of forcing. Then

1. If $\mathbb{P}$ and $\mathbb{Q}$ are κ-Knaster then $\mathbb{P} \times \mathbb{Q}$ is κ-Knaster.

2. If $\mathbb{P}$ is κ-c.c. and $\mathbb{Q}$ is κ-Knaster then $\mathbb{P} \times \mathbb{Q}$ is κ-c.c.

5.13 Remark. In general the property of being κ-Knaster is stronger than that of being κ-c.c. For example if T is an ω_1-Suslin tree then $(T, \geq)$ is ω_1-c.c. but is not ω_1-Knaster, by Fact 5.12 and the easy remark that $T \times T$ is not ω_1-c.c.

We will also need some properties intermediate between κ-closure and (κ, ∞)-distributivity, involving the idea of a game on a poset. This concept was introduced by Jech [37] and studied by Foreman [18] and Gray [30] among others.

5.14 Definition. Let $\mathbb{P}$ be a notion of forcing and let α be an ordinal. We define $G_\alpha(\mathbb{P})$, a two-player game of perfect information. Two players Odd and Even take turns to play conditions from $\mathbb{P}$ for α many moves, with Odd playing at odd stages and Even at even stages (including all limit stages). Even must play $1_{\mathbb{P}}$ at move zero. Let p_β be the condition played at move β; the player who played p_β loses immediately unless $p_\beta \leq p_\gamma$ for all $\gamma < \beta$. If neither player loses at any stage $\beta < \alpha$, then player Even wins.

5.15 Definition. Let $\mathbb{P}$ be a notion of forcing and let κ be a regular cardinal.

1. $\mathbb{P}$ is *$<\kappa$-strategically closed* if and only if for all $\alpha < \kappa$, player Even has a winning strategy for $G_\alpha(\mathbb{P})$.

2. $\mathbb{P}$ is *κ-strategically closed* if and only if player Even has a winning strategy for $G_\kappa(\mathbb{P})$.

3. $\mathbb{P}$ is *$(\kappa+1)$-strategically closed* if and only if player Even has a winning strategy for $G_{\kappa+1}(\mathbb{P})$, where we note that it is player Even who must make the final move.

5.16 Remark. More general forms of strategic closure have been studied [18] and are sometimes useful, but this one is sufficient for us.

5.17 Remark. It is not difficult to see that the conclusions of Lemma 5.10 remain true when we weaken the hypothesis of κ-closure to κ-strategic closure. This *strategic Easton lemma* is part of the folklore.

6. Some Forcing Posets

It is easy to see that every κ-directed closed poset is κ-closed, every κ-closed poset is κ-strategically closed, every κ-strategically closed poset is $<\kappa$-strategically closed and every $<\kappa$-strategically closed poset is (κ, ∞)-distributive. The following examples illustrate that these concepts are distinct, and will all find some use later in this chapter.

The first example shows that κ-closure does not in general imply κ-directed closure.

6.1 Example (Adding a Kurepa tree at an inaccessible cardinal). Recall that if κ is inaccessible then a κ-*Kurepa tree* is a normal tree of height κ such that

- $|T_\alpha| \leq |\alpha| + \omega$ for $\alpha < \kappa$.
- T has at least κ^+ cofinal branches.

Devlin's book about constructibility [16] contains more information about Kurepa trees, including a discussion of when such trees exist in L. We note that if κ is *ineffable* (ineffability is a large cardinal axiom intermediate between weak compactness and measurability) then there is no κ-Kurepa tree, and that in L there is such a tree for every non-ineffable inaccessible κ.

6.2 Remark. It might have seemed more natural to generalise the definition of a Kurepa tree to inaccessible κ by simply asking for a κ-tree with more than κ cofinal branches. But this would be uninteresting because the complete binary tree of height κ is always such a tree.

6.3 Remark. It is very easy to see that there is no κ-Kurepa tree for κ measurable. For if T is such a tree and $j : V \longrightarrow M$ is elementary with critical point κ, then the map which takes each cofinal branch b to the unique point of $j(b)$ on level κ is one-to-one, so in M level κ of $j(T)$ has more than κ points.

Given κ inaccessible we define a forcing poset $\mathbb{P}$ to add a κ-Kurepa tree. Conditions are pairs (t, f) where

1. t is a normal tree of height $\beta + 1$ for some $\beta < \kappa$.
2. $|t_\alpha| \leq |\alpha| + \omega$ for all $\alpha \leq \beta$.
3. f is a function with $\mathrm{dom}(f) \subseteq \kappa^+$, $\mathrm{ran}(f) = t_\beta$ and $|\,\mathrm{dom}(f)| \leq |\beta| + \omega$.

Intuitively $f(\delta)$ is supposed to be the point in which branch δ meets t_β. Accordingly we say that $(u, g) \leq (t, f)$ if and only if

1. t is an initial segment of u.
2. $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$.

3. For all $\delta \in \mathrm{dom}(f)$, $f(\delta) \leq_u g(\delta)$.

It is easy to see that $\mathbb{P}$ is κ-closed and κ^+-c.c. and that $\mathbb{P}$ adds a κ-Kurepa tree. We claim that $\mathbb{P}$ is not κ-directed closed. To see this let $\{x_\alpha : \alpha < 2^\omega\}$ enumerate ${}^\omega 2$ and let S be the family of conditions (t, f) such that $t = {}^n\omega$ for some finite n, f has domain a countable subset of 2^ω and $f(\delta) = x_\delta \restriction n$ for all $\delta \in \mathrm{dom}(f)$. S is directed and $|S| = 2^\omega < \kappa$. However S cannot have a lower bound because, if (t, g) is a lower bound for S then t must have 2^ω points on level ω.

6.4 Remark. Similar arguments show that $\mathbb{P}$ has no dense κ-directed closed dense subset, and is not κ-directed closed below any condition. We will see in Theorem 24.12 that it is consistent for there to exist a measurable cardinal κ whose measurability is preserved by any κ-directed closed forcing, while by contrast forcing with the κ-closed poset $\mathbb{P}$ always destroys the measurability of κ.

Our next example shows that in general κ-strategic closure is a weaker property than κ-closure.

6.5 Example (Adding a non-reflecting stationary set). Let $\kappa = \mathrm{cf}(\kappa) \geq \omega_2$. We define a forcing poset $\mathbb{P}$ which aims to add a *non-reflecting stationary set of cofinality ω ordinals in κ*, that is to say a stationary $S \subseteq \kappa \cap \mathrm{Cof}(\omega)$ such that $S \cap \alpha$ is non-stationary for all $\alpha \in \kappa \cap \mathrm{Cof}(> \omega)$. $p \in \mathbb{P}$ if and only if p is a function such that

1. $\mathrm{dom}(p) < \kappa$, $\mathrm{ran}(p) \subseteq 2$.
2. If $p(\alpha) = 1$, $\mathrm{cf}(\alpha) = \omega$.
3. if $\beta \leq \mathrm{dom}(p)$ and $\mathrm{cf}(\beta) > \omega$ then there exists a set $c \subseteq \beta$ club in β such that $\forall \alpha \in c\ p(\alpha) = 0$.

It is easy to see that $\mathbb{P}$ is countably closed, and that it adds the characteristic function of a stationary subset of κ. It is also easy to see that if we let S be any stationary set of limit ordinals in ω_1, let $\chi_S : \omega_1 \to 2$ be the characteristic function of S, and define $p_\alpha = \chi_S \restriction \alpha$ for $\alpha < \omega_1$, then $\langle p_\alpha : \alpha < \omega_1 \rangle$ is a decreasing sequence of conditions in $\mathbb{P}$ with no lower bound and so $\mathbb{P}$ fails to be ω_2-closed.

We now claim that $\mathbb{P}$ is κ-strategically closed, which we will prove by exhibiting a winning strategy for Even. At stage α Even will compute $\gamma_\alpha = \mathrm{dom}(\bigcup_{\beta<\alpha} p_\beta)$, and will then define p_α by setting $\mathrm{dom}(p_\alpha) = \gamma_\alpha + 1$, $p_\alpha \restriction \gamma_\alpha = \bigcup_{\beta<\alpha} p_\beta$ and $p_\alpha(\gamma_\alpha) = 0$. This strategy succeeds because at every limit stage β of uncountable cofinality the set $\{\gamma_\alpha : \alpha < \beta\}$ is club in γ_β, and Even has ensured that p_β is 0 at every point of this club set.

The following example shows that in general the property of $<\kappa$-strategic closure is weaker than that of κ-strategic closure. The forcing is due to Jensen.

6.6 Example (Adding a square sequence). Let λ be an uncountable cardinal. Recall that a $\Box_\lambda$-*sequence* is a sequence $\langle C_\alpha : \alpha \in \lambda^+ \cap \mathrm{Lim}\rangle$ such that for all α

1. C_α is club in α.
2. $\mathrm{ot}(C_\alpha) \le \lambda$.
3. $\forall \beta \in \lim(C_\alpha)\ C_\alpha \cap \beta = C_\beta$.

We define a forcing poset $\mathbb{P}$ to add such a sequence. Conditions are initial segments of successor length of such a sequence and the ordering is extension. More formally $p \in \mathbb{P}$ iff

- $\mathrm{dom}(p) = (\beta + 1) \cap \mathrm{Lim}$ for some $\beta \in \lambda^+ \cap \mathrm{Lim}$.
- $p(\alpha)$ is club in α, $\mathrm{ot}(p(\alpha)) \le \lambda$ for all $\alpha \in \mathrm{dom}(p)$.
- If $\alpha \in \mathrm{dom}(p)$ then $\forall \beta \in \lim p(\alpha)\ p(\alpha) \cap \beta = p(\beta)$.

If $p, q \in \mathbb{P}$ then $q \le p$ if and only if $p = q \restriction \mathrm{dom}(p)$.

It can be checked that $\mathbb{P}$ is $<\lambda^+$-strategically closed, so that $\mathbb{P}$ preserves cardinals up to λ^+ and adds a $\Box_\lambda$-sequence. The author's joint paper with Foreman and Magidor [13] has a detailed discussion of the poset $\mathbb{P}$ and several variations.

We claim that $\mathbb{P}$ is not in general λ^+-strategically closed. To see this we observe that if player Even can win $G_{\lambda^+}(\mathbb{P})$, then the union of the sequence of the moves in a winning play is actually a $\Box_\lambda$-sequence. So if $\Box_\lambda$ fails then $\mathbb{P}$ is not λ^+-strategically closed. Ishiu and Yoshinobu [36] have observed that the principle $\Box_\lambda$ is in fact equivalent to the λ^+-strategic closure of $\mathbb{P}$.

6.7 Remark. The difference between the last two examples is essentially that "S is a stationary subset of κ" is a second-order statement in the structure (H_κ, S) while "$\vec{C}$ is a $\Box_\lambda$-sequence" is a first-order statement in the structure $(H_{\lambda^+}, \vec{C})$. This difference was exploited in [9].

Our final example shows that in general (κ, ∞)-distributivity is weaker than $<\kappa$-strategic closure. This forcing is due to Baumgartner, Harrington and Kleinberg [7].

6.8 Example (Killing a stationary subset of ω_1). Let $S \subseteq \omega_1$ be stationary and co-stationary. We define a forcing $\mathbb{P}$ to destroy the stationarity of S. The conditions in $\mathbb{P}$ are the closed bounded subsets c of ω_1 such that $c \cap S = \emptyset$.

We claim that $\mathbb{P}$ is (ω_1, ∞)-distributive. To see this let $\langle D_n : n < \omega\rangle$ be an ω-sequence of dense open sets and let $c \in \mathbb{P}$. Fix θ some large regular cardinal and $<_\theta$ a well-ordering of H_θ. Find an elementary substructure $N \prec (H_\theta, \in, <_\theta)$ such that

1. $p, \mathbb{P}, S, \vec{D} \in N$.

2. N is countable.

3. $N \cap \omega_1 \notin S$ (this is possible because S is co-stationary).

Let $\delta = S \cap \omega_1$ and fix $\langle \delta_n : n < \omega \rangle$ an increasing and cofinal sequence in δ. Now build a chain of conditions $\langle c_n : n < \omega \rangle$ as follows: $c_0 = c$ and c_{n+1} is the $<_\theta$-least condition d such that $d \leq c_n$, $d \in D_n$ and $\max(d) \geq \delta_n$. An easy induction shows that $c_n \in N$, so in particular $\max(c_n) \in N \cap \omega_1 = \delta$. It follows that if $c_\infty = \bigcup_n c_n \cup \{\delta\}$ then $c_\infty \in \mathbb{P}$, and by construction $c_\infty \in D_n$ for all n.

On the other hand $\mathbb{P}$ is not $< \omega_1$-strategically closed. To see this we show that for any $\mathbb{Q}$, if Even wins $G_{\omega+1}(\mathbb{Q})$ then $\mathbb{Q}$ preserves stationary subsets of ω_1. Let σ be a winning strategy for Even in $G_{\omega+1}(\mathbb{Q})$. Let $T \subseteq \omega_1$ be stationary. Let $q \Vdash_{\mathbb{Q}}$ "$\dot{C}$ is club in ω_1" and let $q, \mathbb{Q}, T, \dot{C}, \sigma \in N \prec (H_\theta, \in, <_\theta)$, where N is countable with $\delta = N \cap \omega_1 \in T$. Let $\langle E_n : n < \omega \rangle$ enumerate the dense subsets of $\mathbb{Q}$ which lie in N.

Now consider a run $\langle q_n : n \leq \omega \rangle$ of $G_{\omega+1}(\mathbb{Q})$ such that

1. $q_0 = q$.

2. Even plays according to σ.

3. For $n > 0$, q_{2n+1} is the $<_\theta$-least condition r such that $r \leq q_{2n}$ and $r \in E_{n-1}$.

It is easy to see that $q_n \in \mathbb{Q} \cap N$ for $n < \omega$. The condition q_ω forces that $\delta \in \lim(\dot{C})$, so $q_\omega \Vdash \delta \in \dot{C} \cap \check{T}$ and we have shown that the stationarity of T is preserved.

6.9 Remark. The question of preservation of stationarity by forcing is one to which we will return several times in this chapter. The argument of Example 6.8 shows that for any ordinal λ of uncountable cofinality, any stationary $S \subseteq \lambda \cap \mathrm{Cof}(\omega)$ and any $(\omega + 1)$-strategically closed $\mathbb{Q}$, forcing with $\mathbb{Q}$ preserves the stationarity of S. The situation is more complex for uncountable cofinalities, because if we build a structure N as in the last part of Example 6.8 and then try to build a suitable chain of conditions in N, we may in general wander out of N after ω steps. We will return to this topic in Lemma 10.6 and the proof of Theorem 14.10.

It will be convenient to fix some notation for the kind of forcing poset constructed in Example 6.8.

6.10 Definition. Let κ be a regular cardinal and let T be a stationary subset of κ. Then $\mathrm{CU}(\kappa, T)$ is the forcing poset whose conditions are closed bounded subsets of T, ordered by end-extension.

The poset of Example 6.8 is $\mathrm{CU}(\omega_1, \omega_1 \setminus S)$. For $\kappa > \omega_1$ the poset $\mathrm{CU}(\kappa, T)$ may not be well-behaved, in particular it may collapse cardinals; consider for example the situation where $\kappa = \omega_2$ and $T = \omega_2 \cap \mathrm{Cof}(\omega_1)$. See Section 18 for a detailed discussion of this issue.

7. Iterated Forcing

In this section we review the definition of iterated forcing and some basic facts about iterated forcing constructions. We will basically follow Baumgartner's survey paper [6] in our treatment of iterated forcing. Many readers may have learned iterated forcing from the excellent account in Kunen's book [46], and for their benefit we point out that there is one rather significant difference between the Baumgartner and Kunen treatments.

This involves the precise definition of a two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ where $\mathbb{P}$ is a notion of forcing and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a notion of forcing. In [46] the elements of $\mathbb{P} * \dot{\mathbb{Q}}$ are all pairs $(p, \dot{q})$ such that $p \in \mathbb{P}$ and $\dot{\mathbb{Q}}$ contains some pair of the form $(\dot{q}, r)$; Baumgartner [6] adopts a more liberal definition in which $\dot{q}$ is chosen from some set X of $\mathbb{P}$-names such that every $\mathbb{P}$-name for a member of $\dot{\mathbb{Q}}$ is forced to be equal to some name in X. This distinction makes for some (essentially trivial) differences in the theory, for example it is possible with the definition from [46] that $\mathbb{P}$ is countably closed and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is countably closed" but $\mathbb{P} * \dot{\mathbb{Q}}$ is not countably closed.

In the interests of precision we make the following definition, which really amounts to specifying the set of names X from the last paragraph.

7.1 Definition. Let $\mathbb{P}$ be a notion of forcing.

- A $\mathbb{P}$-name $\dot{x}$ is *canonical* iff there is no $\dot{y}$ such that $|\operatorname{tc}(\dot{y})| < |\operatorname{tc}(\dot{x})|$ and $\Vdash_{\mathbb{P}} \dot{x} = \dot{y}$.
- If $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a notion of forcing then $\mathbb{P} * \dot{\mathbb{Q}}$ is the set of all pairs such that $p \in \mathbb{P}$, $\Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$ and $\dot{q}$ is canonical.

The advantage of this convention will be that we get equality rather than just isomorphism in statements like Lemma 12.10 below.

We recall the standard facts about two step iterations:

1. $\mathbb{P} * \dot{\mathbb{Q}}$ is ordered as follows: $(p_0, \dot{q}_0) \leq (p_1, \dot{q}_1)$ if and only if $p_0 \leq p_1$ and $p_0 \Vdash \dot{q}_0 \leq \dot{q}_1$.
2. There is a bijection between V-generic filters for $\mathbb{P} * \dot{\mathbb{Q}}$ and pairs (G, H) where G is V-generic for $\mathbb{P}$, and H is $V[G]$-generic for $i_G(\dot{Q})$.

As we mentioned above we will follow the treatment of iterated forcing from Baumgartner's survey paper [6]. We give a brief review. We make the convention that whenever we have a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a notion of forcing, $\dot{1}_{\mathbb{Q}}$ names the specified largest element of $\dot{\mathbb{Q}}$.

An iteration of length α is officially an object of the form

$$(\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle)$$

where for every $\beta \leq \alpha$

- $\mathbb{P}_\beta$ is a notion of forcing whose elements are β-sequences.

- If $p \in \mathbb{P}_\beta$ and $\gamma < \beta$ then $p \restriction \gamma \in \mathbb{P}_\gamma$.
- If $\beta < \alpha$ then $\Vdash_{\mathbb{P}_\beta}$ "$\dot{\mathbb{Q}}_\beta$ is a notion of forcing".
- If $p \in \mathbb{P}_\beta$ and $\gamma < \beta$, then $p(\gamma)$ is a $\mathbb{P}_\gamma$-name for an element of $\dot{\mathbb{Q}}_\gamma$.
- If $\beta < \alpha$ then $\mathbb{P}_{\beta+1} \simeq \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$, via the map which takes $h \in \mathbb{P}_{\beta+1}$ to $(h \restriction \beta, h(\beta))$.
- If $p, q \in \mathbb{P}_\beta$ then $p \leq_{\mathbb{P}_\beta} q$ iff $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} p(\gamma) \leq_{\dot{\mathbb{Q}}_\gamma} q(\gamma)$ for all $\gamma < \beta$.
- $1_{\mathbb{P}_\beta}(\gamma) = \dot{1}_{\mathbb{Q}_\gamma}$ for all $\gamma < \beta$.
- If $p \in \mathbb{P}_\beta$, $\gamma < \beta$ and $q \leq_{\mathbb{P}_\gamma} p \restriction \gamma$ then $q ^\frown p \restriction [\gamma, \beta) \in \mathbb{P}_\beta$.

In a standard abuse of notation we will sometimes use "$\mathbb{P}_\alpha$" as a shorthand for the iteration $(\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle)$. We usually write "$\Vdash_\beta$" for "$\Vdash_{\mathbb{P}_\beta}$".

The key points in the definition of iteration are that if G_α is $\mathbb{P}_\alpha$-generic over V and $\beta < \alpha$ then

- $G_\beta =_{\text{def}} \{p \restriction \beta : p \in G_\alpha\}$ is $\mathbb{P}_\beta$-generic over V.
- $g_\beta =_{\text{def}} \{i_{G_\beta}(p(\beta)) : p \in G_\alpha\}$ is $i_{G_\beta}(\dot{\mathbb{Q}}_\beta)$-generic over $V[G_\beta]$.

7.2 Remark. It is sometimes useful to weaken the conditions in the definition of iteration, and to admit as a forcing iteration any pair $(\vec{\mathbb{P}}, \vec{\mathbb{Q}})$ where

1. $\mathbb{P}_\beta$ is a forcing poset whose conditions are β-sequences.
2. $\dot{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for a forcing poset.
3. $\mathbb{P}_{\beta+1} \simeq \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$, via the map which takes $h \in \mathbb{P}_{\beta+1}$ to $(h \restriction \beta, h(\beta))$.
4. The restriction map from $\mathbb{P}_\beta$ to $\mathbb{P}_\gamma$ for $\gamma < \beta$ is a projection.

Note that the "key properties" from the last paragraph will still be true in this setting. An important example is Prikry iteration with Easton support (see [22]) which are iterations in this more general sense,

7.3 Remark. Some arguments which we need to do subsequently work most smoothly with forcing posets which are separative partial orderings. The poset absorption argument of Section 14 is an example. In our definition of iterated forcing $\mathbb{P}_\alpha$ is just a notion of forcing. However it is routine to check that if we form an iteration such that each factor $\dot{\mathbb{Q}}_i$ is forced to be a separative partial ordering, then the quotient poset of $\mathbb{P}_\alpha$ is separative. We will sometimes blur the distinction between the preordering $\mathbb{P}_\alpha$ and its associated quotient partial ordering.

If $p \in \mathbb{P}_\alpha$ then the *support of* p (supp(p)) is $\{\beta < \alpha : p(\beta) \neq \dot{1}_{\mathbb{Q}_\beta}\}$.

Let λ be a limit ordinal and let an iteration of length λ be given. We define the *inverse limit* $\varprojlim \vec{\mathbb{P}} \restriction \lambda$ to be the set of sequences p of length λ such that $\forall \alpha < \lambda \; p \restriction \alpha \in \mathbb{P}_\alpha$. The *direct limit* $\varinjlim \vec{\mathbb{P}} \restriction \lambda$ is the subset of the inverse limit consisting of those p such that $p(\alpha) = \dot{1}_{\mathbb{Q}_\alpha}$ for all sufficiently large α. The definition of a forcing iteration implies that if we have an iteration of length greater than λ then

$$\varinjlim \vec{\mathbb{P}} \restriction \lambda \subseteq \mathbb{P}_\lambda \subseteq \varprojlim \vec{\mathbb{P}} \restriction \lambda.$$

To specify a forcing iteration it will suffice to describe the names $\dot{\mathbb{Q}}_\beta$ and to give a procedure for computing $\mathbb{P}_\lambda$ for λ limit. In many iterations the only kinds of limit which are used are direct and inverse ones.

7.4 Remark. Let κ be inaccessible, and suppose that we have an iteration of length κ where $\dot{\mathbb{Q}}_\beta \in V_\kappa$ for all $\beta < \kappa$ and a direct limit is taken at stage κ. Then

- $\mathbb{P}_\beta \subseteq V_\kappa$ for all $\beta < \kappa$.
- While it is not literally true that $\mathbb{P}_\kappa \subseteq V_\kappa$, for every $p \in \mathbb{P}_\kappa$ there exist $\beta < \kappa$ and $q \in \mathbb{P}_\beta$ such that $p(\alpha) = q(\alpha)$ for $\alpha < \beta$, $p(\alpha) = \dot{1}_{\mathbb{Q}_\alpha}$ for $\beta \leq \alpha < \kappa$. We will often blur the distinction between $\mathbb{P}_\kappa$ and $\bigcup_\alpha \mathbb{P}_\alpha$, which actually is a subset of V_κ.

7.5 Definition. If κ is regular then an *iteration with* $<\kappa$*-support* is an iteration in which direct limits are taken at limit stages of cofinality greater than or equal to κ, and inverse limits are taken at limit stages of cofinality less than κ. An *iteration with Easton support* is an iteration in which direct limits are taken at regular limit stages and inverse limits are taken elsewhere.

As this terminology would suggest, the support of a condition in an iteration with $<\kappa$-support has size less than κ. The support of a condition in an Easton iteration is an *Easton set*, that is to say a set of ordinals which is bounded in every regular cardinal.

The following are a few key facts about two-step iterations. Proofs are given in [6, Section 2] for 1 and 2, while the proofs for 3 are easy variations of the proof for 2.

7.6 Proposition. *Let* $\kappa = \mathrm{cf}(\kappa) > \omega$*, let* $\mathbb{P}$ *be a notion of forcing and let* $\Vdash_{\mathbb{P}}$ *"*$\dot{\mathbb{Q}}$ *is a notion of forcing".*

1. $\mathbb{P} * \dot{\mathbb{Q}}$ *is* κ*-c.c. iff* $\mathbb{P}$ *is* κ*-c.c. and* $\Vdash_{\mathbb{P}}$ *"*$\dot{\mathbb{Q}}$ *is* κ*-c.c."*
2. *If* $\mathbb{P}$ *is* κ*-closed and* $\Vdash_{\mathbb{P}}$ *"*$\dot{\mathbb{Q}}$ *is* κ*-closed" then* $\mathbb{P} * \dot{\mathbb{Q}}$ *is* κ*-closed.*
3. *Let* X *be any of the properties "*κ*-directed closed", "*$<\kappa$*-strategically closed" or "*κ*-strategically closed". If* $\mathbb{P}$ *is* X *and* $\Vdash_{\mathbb{P}}$ *"*$\dot{\mathbb{Q}}$ *is* X*" then* $\mathbb{P} * \dot{\mathbb{Q}}$ *is* X*.*

7.7 Proposition. *Let κ be inaccessible and $\mathbb{P} \in V_\kappa$. If $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{V}_\kappa$ then $\mathbb{P} * \dot{\mathbb{Q}} \in V_\kappa$, and if $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \subseteq V_\kappa$ then $\mathbb{P} * \dot{\mathbb{Q}} \subseteq V_\kappa$.*

In general the preservation of chain condition in iterations is a very delicate question. See [6, Section 4] and [62] to get an idea of the difficulties surrounding preservation of the ω_2-c.c. by countable support iterations. Fortunately we can generally get away with some comparatively crude arguments.

The following fact is proved in [6, Section 2].

7.8 Proposition. *Let α be limit and let $\mathbb{P}_\alpha = \varinjlim \vec{\mathbb{P}} \restriction \alpha$. Let $\kappa = \mathrm{cf}(\kappa) > \omega$. Suppose that*

- *For every $\beta < \alpha$, $\mathbb{P}_\beta$ is κ-c.c.*
- *If $\mathrm{cf}(\alpha) = \kappa$ then $\{\gamma < \alpha : \mathbb{P}_\gamma = \varinjlim \vec{\mathbb{P}} \restriction \gamma\}$ is stationary in α.*

Then $\mathbb{P}_\alpha$ is κ-c.c.

The following fact is proved in [6, Section 2] for the case when X equals "κ-closed". The proofs for the other closure properties are similar.

7.9 Proposition. *Let $\kappa = \mathrm{cf}(\kappa) > \omega$. Let X be any of the properties "κ-closed", "κ-directed closed", "$<\kappa$-strategically closed" or "κ-strategically closed". Suppose that*

- *$\Vdash_\beta$ "$\dot{\mathbb{Q}}_\beta$ is X" for $\beta < \alpha$.*
- *All limits are direct or inverse, and inverse limits are taken at every limit stage with cofinality less than κ.*

Then $\mathbb{P}_\alpha$ is X.

7.10 Remark. The moral of Proposition 7.8 is that one should take many direct limits to preserve chain condition properties, the moral of Proposition 7.9 is that one should take many inverse limits to preserve closure properties.

We will also need to analyse the quotient of an iteration by some initial segment. Once again we quote from [6, Section 5].

7.11 Proposition. *If $\beta < \alpha$ then there exists a term $\dot{\mathbb{R}}_{\beta,\alpha} \in V^{\mathbb{P}_\beta}$ such that*

1. *$\Vdash_\beta$ "$\mathbb{R}_{\beta,\alpha}$ is an iteration of length $\alpha - \beta$".*
2. *There is a dense subset of $\mathbb{P}_\beta * \dot{\mathbb{R}}_{\beta,\alpha}$ which is isomorphic to $\mathbb{P}_\alpha$.*

The definition of the iteration $\mathbb{R}_{\beta,\alpha}$ is simple at successor stages; we translate $\dot{\mathbb{Q}}_\gamma$ in the canonical way to a $\mathbb{P}_\beta$-name for an $\mathbb{R}_{\beta,\gamma}$-name for a notion of forcing and force with that poset at stage $\gamma - \alpha$. Limits are trickier because while a direct limit in V still looks like a direct limit in $V^{\mathbb{P}_\beta}$ the same may not be true in general of an inverse limit. We will usually write $\mathbb{P}_\alpha/G_\beta$ for $i_{G_\beta}(\dot{\mathbb{R}}_{\beta,\alpha})$.

The following proposition is proved in [6, Section 5] for the case when X equals "κ-closed", and once again can be proved in a very similar way for the other closure properties.

7.12 Proposition. *Let $\kappa = \mathrm{cf}(\kappa) > \omega$. Let X be any of the properties "κ-closed", "κ-directed closed", "$<\kappa$-strategically closed" or "κ-strategically closed". Let $\mathbb{P}_\alpha$ be an iteration of length α in which all limits are inverse or direct. Let $\beta < \alpha$ and suppose that*

1. *$\mathbb{P}_\beta$ is such that every set of ordinals of size less than κ in $V[G_\beta]$ is covered by a set of size less than κ in V.*
2. *For $\beta \le \gamma < \alpha$, $\Vdash_\gamma$ "$\dot{\mathbb{Q}}_\gamma$ is X".*
3. *Inverse limits are taken at all limit γ such that $\beta \le \gamma < \alpha$ and $\mathrm{cf}(\gamma) < \kappa$.*

Then $\Vdash_\beta$ "$\dot{\mathbb{R}}_{\beta,\alpha}$ is X".

The following result is easily proved by the methods of [6, Section 2].

7.13 Proposition. *Let κ be inaccessible and let $\mathbb{P}_\kappa$ be an iteration of length κ such that*

1. *$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \in V_\kappa$ for all $\alpha < \kappa$.*
2. *A direct limit is taken at κ and on a stationary set of limit stages below κ.*

Then

- *$\mathbb{P}_\kappa$ is κ-Knaster and has cardinality κ.*
- *If $\delta < \kappa$ then in $V[G_\delta]$ the quotient forcing $\mathbb{R}_{\delta,\kappa}$ is a κ-Knaster and has cardinality κ.*

7.14 Remark. The hypothesis 2 of Proposition 7.13 will be satisfied if the iteration is done with $<\lambda$-support for some $\lambda < \kappa$, and also if the iteration is done with Easton support and κ is Mahlo.

8. Building Generic Objects

A crucial fact about forcing is that if M is a countable transitive model of set theory and $\mathbb{P} \in M$ is a notion of forcing, then there exist filters which are $\mathbb{P}$-generic over M. We give an easy generalisation of this fact, which will be used very frequently in the constructions to follow.

8.1 Proposition. *Let M, N be two inner models with $M \subseteq N$ and let $\mathbb{P} \in M$ be a non-trivial notion of forcing. Let $\mathcal{A}$ be the set of $A \in M$ such that A is an antichain of $\mathbb{P}$, and note that $\mathcal{A} \in M$.*

Let $p \in \mathbb{P}$ and let λ be an N-cardinal. If

$$N \models \text{“}\mathbb{P} \text{ is } \lambda\text{-strategically closed and } |\mathcal{A}| \leq \lambda\text{”}$$

then there is a set in N of N-cardinality 2^λ of filters on $\mathbb{P}$, each one of which is generic over M.

Proof. We work in N. Let $\langle A_\alpha : \alpha < \lambda \rangle$ enumerate $\mathcal{A}$. Let σ be a winning strategy for player Even in the game $G_\lambda(\mathbb{P}/p)$.

We now build a binary tree $\langle p_s : s \in {}^{<\lambda}2 \rangle$ of conditions in $\mathbb{P}/p$ such that

1. $p_{\langle\rangle} = p$.
2. If $\mathrm{lh}(s)$ is even, say $\mathrm{lh}(s) = 2\alpha$, then $p_{s^\frown 0}$ and $p_{s^\frown 1}$ are incompatible and each of them refines some element of A_α.
3. If $\mathrm{lh}(s) = 2(1+\alpha)$, then p_s is the response dictated by σ at move 2α in the run of of the game $G_\lambda(\mathbb{P}/p)$ where $p_{s \restriction (2+i)}$ is played at move i for $i < 2\alpha$.

Then every branch generates a generic filter, and any two branches contain incompatible elements so generate distinct filters. ⊣

The following easy propositions will be useful in applications of Proposition 8.1.

8.2 Proposition. *Let M, N be inner models of* ZFC *such that $M \subseteq N$. Let $N \models$ “κ is a regular uncountable cardinal”. Then $N \models {}^{<\kappa}M \subseteq M$ if and only if $N \models {}^{<\kappa}\mathrm{On} \subseteq M$.*

8.3 Proposition. *Let M, N be inner models of* ZFC *such that $M \subseteq N$. Let $N \models$ “κ is a regular uncountable cardinal” and let $N \models {}^{<\kappa}M \subseteq M$. Let $\mathbb{P} \in M$ be a notion of forcing and let X be any of the properties “κ-directed closed”, “κ-closed”, “κ-strategically closed” and “$<\kappa$-strategically closed”.*

If $M \models$ “$\mathbb{P}$ is X” then $N \models$ “$\mathbb{P}$ is X”.

8.4 Proposition. *Let M and N be inner models of* ZFC *with $M \subseteq N$ and let $\mathbb{P} \in M$ be a notion of forcing.*

1. *If $N \models {}^{<\lambda}M \subseteq M$, $N \models$ “$\mathbb{P}$ is λ-c.c.” and G is $\mathbb{P}$-generic over N then $N[G] \models {}^{<\lambda}M[G] \subseteq M[G]$.*
2. *If $V_\lambda \cap M = V_\lambda \cap N$ and*

$$N \models \text{“Every canonical } \mathbb{P}\text{-name for a member of } V_\lambda^{N^{\mathbb{P}}} \text{ is in } V_\lambda\text{”}$$

then $V_\lambda \cap M[G] = V_\lambda \cap N[G]$.

We digress from our main theme to give a sample application of Proposition 8.1, namely building a generalised version of Prikry forcing.

8.5 Lemma. *Let κ be measurable with $2^\kappa = \kappa^+$, and let U be a normal measure on κ. Let $j : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map constructed from U, and let $\mathbb{Q} = \mathrm{Col}(\kappa^{++}, {<}j(\kappa))_M$. Then there is a filter $g \in V$ which is $\mathbb{Q}$-generic over M.*

Proof. In M, $\mathbb{Q}$ is a forcing of size $j(\kappa)$ which is $j(\kappa)$-c.c. Since $j(\kappa)$ is measurable in M it is surely inaccessible in M, and so

$$M \models \text{“}\mathbb{Q} \text{ has } j(\kappa) \text{ maximal antichains”}.$$

By Proposition 4.5 $V \models |j(\kappa)| = 2^\kappa = \kappa^+$, so

$$V \models \text{“}\mathbb{Q} \text{ has } \kappa^+ \text{ maximal antichains lying in } M\text{”}.$$

Clearly $M \models$ “$\mathbb{Q}$ is κ^{++}-closed”, and $V \models {}^\kappa M \subseteq M$. By Proposition 8.3 it follows that $V \models$ “$\mathbb{Q}$ is κ^+-closed”. Applying Proposition 8.1 we may therefore construct $g \in V$ which is $\mathbb{Q}$-generic over M. ⊣

The forcing we are about to describe is essentially a special case of the forcing $\mathbb{P}^\pi$ from Foreman and Woodin's paper on failure of GCH everywhere [21], and is also implicitly present in Magidor's work on failure of the SCH [53]. We learned this presentation from Woodin.

8.6 Example. Let κ be measurable with $2^\kappa = \kappa^+$. Then there is a κ^+-c.c. poset $\mathbb{P}$ such that $\Vdash_{\mathbb{P}} \check{\kappa} = \dot{\omega}_\omega$.

Sketch of proof. Let U, $\mathbb{Q}$ and g be as in Lemma 8.5. Conditions in $\mathbb{P}$ have the form $(p_0, \kappa_1, p_1, \ldots, \kappa_n, p_n, H)$ where

- The κ_i are inaccessible with $\kappa_1 < \ldots < \kappa_n < \kappa$.
-
  - $p_0 \in \mathrm{Col}(\omega_2, {<}\kappa_1)$.
  - $p_i \in \mathrm{Col}(\kappa_i^{++}, {<}\kappa_{i+1})$ for $0 < i < n$.
  - $p_n \in \mathrm{Col}(\kappa_n, {<}\kappa)$.
- H is a function such that $\mathrm{dom}(H) \in U$, $H(\alpha) \in \mathrm{Col}(\alpha^{++}, {<}\kappa)$ for $\alpha \in \mathrm{dom}(H)$ and $[H]_U \in g$.

We refer to n as the *length* of this condition.

Intuitively H constrains the possibilities for adding in new objects in the same way as the measure one set constrains new points in Prikry forcing. Formally $(q_0, \lambda_1, q_1, \ldots, \lambda_m, q_m, I)$ extends $(p_0, \kappa_1, p_1, \ldots, \kappa_n, p_n, H)$ iff

- $m \geq n$.
- For every $i \leq n$, $\lambda_i = \kappa_i$ and q_i extends p_i.

- For every i with $n < i \leq m$, $\lambda_i \in \mathrm{dom}(H)$ and $q_i \leq H(\lambda_i)$.
- $\mathrm{dom}(I) \subseteq \mathrm{dom}(H)$ and $I(\lambda)$ extends $H(\lambda)$ for every $\lambda \in \mathrm{dom}(I)$.

The second condition will be called a *direct* extension of the first if and only if $m = n$.

It is easy to see that $\mathbb{P}$ is κ^+-c.c. because any two elements in g are compatible. The poset $\mathbb{P}$ adds an increasing ω-sequence $\langle \kappa_i : i < \omega \rangle$ cofinal in κ (which is actually a Prikry-generic sequence for the measure U) and a sequence $\langle g_i : i < \omega \rangle$ where g_i is $\mathrm{Col}(\kappa_i^{++}, {<}\kappa_{i+1})$-generic over V.

The key lemma about $\mathbb{P}$ is that any statement in the forcing language can be decided by a direct extension. This is proved by an argument very similar to that for Prikry forcing. It can then be argued as in Magidor's paper [53] that below κ only the cardinals in the intervals $(\kappa_i^{++}, \kappa_{i+1})$ have collapsed. Thus $\mathbb{P}$ is a κ^+-c.c. forcing poset which makes κ into the ω_ω of the extension. ⊣

9. Lifting Elementary Embeddings

A key idea in this chapter is that it is sometimes possible to take an elementary embedding of a model of set theory and extend it to an embedding of some generic extension of that model. This idea goes back to Silver's consistency proof for the failure of GCH at a measurable, a proof which we will outline in Section 12.

9.1 Proposition. *Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of* ZFC. *Let $\mathbb{P} \in M$ be a notion of forcing, let G be $\mathbb{P}$-generic over M and let H be $k(\mathbb{P})$-generic over N. The following are equivalent:*

1. *$\forall p \in G\ k(p) \in H$.*
2. *There exists an elementary embedding $k^+ : M[G] \longrightarrow N[H]$, such that $k^+(G) = H$ and $k^+ \restriction M = k$.*

Proof. Clearly the second statement implies the first one. For the converse let $k``G \subseteq H$ and attempt to define k^+ by

$$k^+(i_G(\dot{\tau})) = i_H(k(\dot{\tau})).$$

To check that k^+ is well-defined, let $i_G(\dot{\sigma}) = i_G(\dot{\tau})$ and fix $p \in G$ such that $p \Vdash^M_{\mathbb{P}} \dot{\sigma} = \dot{\tau}$. Now by elementarity $k(p) \Vdash^N_{k(\mathbb{P})} k(\dot{\sigma}) = k(\dot{\tau})$, and since $k(p) \in H$ we have $i_H(k(\dot{\sigma})) = i_H(k(\dot{\tau}))$.

A similar proof shows that k^+ is elementary. If $x \in M$ and $\check{x}$ is the standard $\mathbb{P}$-name for x then $k(\check{x})$ is the standard $k(\mathbb{P})$-name for $k(x)$ and so $k^+(x) = k^+(i_G(\check{x})) = i_H(k(\check{x})) = k(x)$. Similarly if $\dot{G}$ is the standard $\mathbb{P}$-name for the $\mathbb{P}$-generic filter then $k(\dot{G})$ is the standard $k(\mathbb{P})$-name for the $k(\mathbb{P})$-generic filter, and so $k^+(G) = H$. ⊣

The following propositions give some useful structural information about the lifted embedding k^+. Recall that we defined the *width* and *support* of an embedding in Definitions 3.7 and 3.8.

9.2 Proposition. *Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of* ZFC *and let G, H, $k^+ : M[G] \longrightarrow N[H]$ be as in Proposition 9.1. Then $N \cap \operatorname{ran}(k^+) = \operatorname{ran}(k)$.*

Proof. Let $y \in N$ with $y = k^+(x)$ for some $x \in M[G]$. If $\alpha = 1 + \operatorname{rk}(x)$ then by elementarity $y \in V^N_{k(\alpha)}$. Since k^+ extends k and k is an elementary embedding, $k^+(V^M_\alpha) = k(V^M_\alpha) = V^N_{k(\alpha)}$. So $k^+(x) \in k^+(V^M_\alpha)$, and since k^+ is elementary $x \in V^M_\alpha$. So $x \in M$ and $y = k^+(x) = k(x)$, thus $y \in \operatorname{ran}(k)$. ⊣

9.3 Proposition. *Let $k : M \longrightarrow N$, G, H, $k^+ : M[G] \longrightarrow N[H]$ be as in Proposition 9.1. If k has width $\leq \mu$ then k^+ has width $\leq \mu$. If k is supported on A then k^+ is supported on A.*

Proof. Suppose first that k has width $\leq \mu$. Let $y \in N[H]$, so that $y = i_H(\dot\tau)$ for some $k(\mathbb{P})$-name $\dot\tau \in N$. By our assumptions about k, $\dot\tau = k(F)(a)$ where $F \in M$, $a \in N$ and $M \models |\operatorname{dom}(F)| \leq \mu$. Without loss of generality we may assume that $F(x)$ is a $\mathbb{P}$-name for all $x \in \operatorname{dom}(F)$.

Now we define a function $F^* \in M[G]$ by setting $\operatorname{dom}(F^*) = \operatorname{dom}(F)$ and $F^*(x) = i_G(F(x))$ for all $x \in \operatorname{dom}(F)$. By elementarity

$$k^+(F^*)(a) = i_{k^+(G)}(k^+(F)(a)) = i_H(\dot\tau) = y.$$

Therefore k^+ has width $\leq \mu$. The argument for the property "supported on A" is very similar. ⊣

In Section 4 we gave characterisations of various large cardinal axioms in terms of *definable* elementary embeddings. When we apply Proposition 9.1 to a definable embedding it is likely that definability will be lost; the next section gives an example of this phenomenon.

One of our major themes is forcing iterations which preserve large cardinal axiom, so we would like to preserve definability when applying Proposition 9.1. This motivates the following proposition, where the key hypothesis for getting a definable embedding is that we are choosing $H \in V[G]$.

9.4 Proposition. *Let $\kappa < \lambda$ and let $j : V \longrightarrow M$ be an elementary embedding with critical point κ. Let $\mathbb{P} \in V$ be a notion of forcing, and let G be $\mathbb{P}$-generic over V. Let H be $j(\mathbb{P})$-generic over M with $j``G \subseteq H$, and let $j^+ : V[G] \longrightarrow M[H]$ be the unique embedding with $j^+ \restriction V = j$ and $j^+(G) = H$. Let $H \in V[G]$.*

1. *If there is in V a short V-(κ, λ)-extender E such that j is the ultrapower of V by E, then there is in $V[G]$ a short $V[G]-(\kappa, \lambda)$-extender E^* such that j^+ is the ultrapower of $V[G]$ by E^*. Moreover $E_a = E^*_a \cap V$ for all $a \in [\lambda]^{<\omega}$.*

2. *If there is in V a supercompactness measure U on $P_\kappa\lambda$ such that j is the ultrapower of V by U, then there is in $V[G]$ a supercompactness measure U^* on $P_\kappa\lambda$ such that j^+ is the ultrapower of $V[G]$ by U^*. Moreover $U = U^* \cap V$.*

In both cases j^+ is definable.

Proof. Assume first that j is the ultrapower by some (κ, λ)-extender E. For each $a \in [\lambda]^{<\omega}$ let μ_a be minimal with $a \subseteq j(\mu_a)$. Arguing exactly as in the proof of Proposition 9.3,

$$M[H] = \{j^+(F)(a) : a \in [\lambda]^{<\omega}, F \in V[G], \mathrm{dom}(F) = [\mu_a]^{|a|}\}.$$

If we now let E^* be the (κ, λ)-extender derived from j^+ then it follows easily from the equation above and Proposition 3.4 that $\mathrm{Ult}(V[G], E^*) = M[H]$ and $j_{E^*} = j^+$. Since $H \in V[G]$ we see that $E^* \in V[G]$ and so j^+ is definable. Finally if $X \in V$ and $X \subseteq [\kappa]^{|a|}$ then $j(X) = j^+(X)$, so

$$X \in E_a \iff a \in j(X) \iff a \in j^+(X) \iff X \in E^*_a,$$

that is to say $E_a = E^*_a \cap V$.

The argument for j arising from a supercompactness measure is similar. $\dashv$

9.5 Remark. Either clause of Proposition 9.4 can be used to argue that κ is measurable in $V[G]$. By Remark 4.1 we may use the second clause to conclude without further work that κ is λ-supercompact in $V[G]$; preservation of some strength witnessed by E will need some argument about the resemblance between $V[G]$ and $M[H]$.

In what follows we will see a number of ways of arranging that $k``G \subseteq H$. We start by proving a classic result by Lévy and Solovay (which implies in particular that standard large cardinal hypotheses cannot resolve the Continuum Hypothesis).

9.6 Theorem (Lévy and Solovay [50]). *Let κ be measurable. Let $|\mathbb{P}| < \kappa$ and let G be $\mathbb{P}$-generic. Then κ is measurable in $V[G]$.*

Proof. Let U be a measure on κ and let $j : V \longrightarrow M = \mathrm{Ult}(M, U)$ be the ultrapower map. Without loss of generality $\mathbb{P} \in V_\kappa$, so that $j \restriction \mathbb{P} = \mathrm{id}_{\mathbb{P}}$ and $j(\mathbb{P}) = \mathbb{P}$. In particular $j``G = G$, and since $M \subseteq V$ and $\mathbb{P} \in M$ we have that G is $\mathbb{P}$-generic over M. Now by Proposition 9.1 we may lift j to get a new map $j^+ : V[G] \longrightarrow M[G]$. By Proposition 9.4 j^+ is definable in $V[G]$. j^+ extends j and so $\mathrm{crit}(j^+) = \mathrm{crit}(j) = \kappa$, and thus κ is measurable in $V[G]$. $\dashv$

9.7 Remark. Usually when we lift an embedding we will denote the lifted embedding by the same letter as the original one.

The Lévy-Solovay result actually applies to most other popular large cardinal axioms, for example "κ is λ-strong" or "κ is λ-supercompact".

9.8 Theorem. *Let $|\mathbb{P}| < \kappa$ and let $\lambda > \kappa$. Then forcing with $\mathbb{P}$ preserves the statements "κ is λ-strong" and "κ is λ-supercompact".*

Proof. Without loss of generality $\mathbb{P} \in V_\kappa$. Let G be $\mathbb{P}$-generic over V.

Suppose that κ is λ-strong. Let $j : V \longrightarrow M$ be such that $\mathrm{crit}(j) = \kappa$ and $V_\lambda \subseteq M$. Build $j^+ : V[G] \longrightarrow M[G]$ as in Theorem 9.6. By Proposition 8.4 we see that $V_\lambda^{V[G]} \subseteq M[G]$. By Proposition 9.4 j^+ is definable in $V[G]$. Since $\mathrm{crit}(j^+) = \kappa$, κ is λ-strong in $V[G]$.

The argument for λ-supercompactness is analogous. ⊣

9.9 Remark. Lévy and Solovay also showed that small forcing cannot create instances of measurability. We prove a generalisation of this in Theorem 21.1. In Theorem 14.6 a forcing poset of size κ makes a non-weakly compact cardinal κ measurable. See Section 21 for more discussion of these matters.

10. Generic Embeddings

It is a common situation that in some generic extension $V[G]$ we are able to define an elementary embedding $j : V \longrightarrow M \subseteq V[G]$. Such embeddings are usually known as *generic embeddings*. Foreman's chapter contains a wealth of information about generic embeddings.

10.1 Example. If I is an ω_2-saturated ideal on ω_1 and U is generic for the poset of I-positive sets, then in $V[U]$ the ultrapower $\mathrm{Ult}(V, U)$ is well-founded and we get a map $j : V \longrightarrow M \subseteq V[U]$ with $\mathrm{crit}(j) = \omega_1$ and $j(\omega_1) = \omega_2$. See Foreman's chapter in this Handbook for much more information.

We now honour a promise made in Section 2. The embedding that we describe is a generic embedding with critical point ω_1 and is added by a very simple poset. See Theorems 14.6, 23.2 and 24.11 for some applications of generic embeddings added by more elaborate posets.

10.2 Theorem. *Let κ be measurable, let U be a normal measure on κ and let $j : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map. Let $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa)$ and let G be $\mathbb{P}$-generic. There is a forcing poset $\mathbb{Q} \in M$ such that*

1. *For any H a $\mathbb{Q}$-generic filter over $V[G]$, j can be lifted to an elementary embedding $j_G : V[G] \longrightarrow M[G * H]$.*
2. *If*
$$U_G = \{X \in P(\kappa) \cap V[G] : \kappa \in j_G(X)\},$$
then U_G is a $V[G]$-κ-complete $V[G]$-normal $V[G]$-ultrafilter on κ. Also $M[G * H] = \mathrm{Ult}(V[G], U_G)$ and j_G is the ultrapower map.

Proof. By elementarity $j(\mathbb{P}) = \mathrm{Col}(\omega, {<}j(\kappa))$. Let $\mathbb{Q}$ be the set of finite partial functions q from $\omega \times (j(\kappa) \setminus \kappa)$ to $j(\kappa)$ such that $q(n, \alpha) < \alpha$ for all $(n, \alpha) \in \mathrm{dom}(p)$, ordered by reverse inclusion. Clearly the map which sends p to $(p{\restriction}(\omega\times\kappa), p{\restriction}(\omega\times(j(\kappa)\setminus\kappa)))$ sets up an isomorphism in M between $j(\mathbb{P})$ and $\mathbb{P}\times\mathbb{Q}$.

Now let H be $\mathbb{Q}$-generic over $V[G]$, so that by the Product Lemma $G\times H$ is $\mathbb{P}\times\mathbb{Q}$-generic over V. Let H^* be the $j(\mathbb{P})$-generic object which is isomorphic to $G \times H$ via the isomorphism from the last paragraph, that is

$$H^* = \{r \in j(\mathbb{P}) : r{\restriction}(\omega\times\kappa) \in G, r{\restriction}(\omega \times (j(\kappa)\setminus\kappa)) \in H\}.$$

If $p \in G$ then $\mathrm{dom}(p)$ is a finite subset of $\omega\times\kappa$ and $p(n, \alpha) < \alpha < \kappa$ for all $(n, \alpha) \in \mathrm{dom}(p)$. It follows that $\mathrm{dom}(j(p)) = j(\mathrm{dom}(p)) = \mathrm{dom}(p)$, and what is more if $p(n, \alpha) = \beta$ then $j(p)(n, \alpha) = j(p(n, \alpha)) = j(\beta) = \beta$. So $p = j(p) \in H^*$.

We now work in $V[G * H]$. By Proposition 9.1 we can lift j to get $j_G : V[G] \longrightarrow M[H^*] = M[G * H]$. It then follows from Proposition 3.9 $M[G*H] = \mathrm{Ult}(V[G], U_G)$ and j_G is precisely the ultrapower embedding. Of course U_G does not exist in $V[G]$; it is only definable in the generic extension $V[G * H]$. ⊣

10.3 Remark. This theorem provides an example of an elementary embedding $k : M_1 \longrightarrow M_2$ with critical point $\omega_1^{M_1}$ and $P(\omega)\cap M_1 \subsetneq P(\omega)\cap M_2$. This shows that Proposition 2.6 is sharp.

10.4 Remark. An important feature of the last proof was the product analysis of $j(\mathbb{P})$. In that proof we were careful to stress that $G \times H$ and H^* are isomorphic rather than identical.

In what follows we will follow the standard practice and be more cavalier about these issues. The cavalier way of writing the main point in the last proof is to say "$p \in G$ implies that $j(p) = p \in G \times H$".

Theorem 10.2 can be generalised in a way that is important for several later results.

10.5 Theorem. *Let κ be measurable, let U be a normal measure on κ and let $j : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map. Let δ be an uncountable regular cardinal less than κ. Let $\mathbb{P} = \mathrm{Col}(\delta, {<}\kappa)$ and let G be $\mathbb{P}$-generic. There is a δ-closed forcing poset $\mathbb{Q} \in M$ such that for any H a $\mathbb{Q}$-generic filter, j can be lifted to an elementary embedding $j_G : V[G] \longrightarrow M[G * H]$.*

The proof is just like that of Theorem 10.2. It will be useful later to know that some reflection properties of the original measurable cardinal κ survive in $V^{\mathbb{P}}$. We need a technical lemma on the preservation of stationary sets by forcing.

10.6 Lemma. *Let δ be regular with $\delta^{<\eta} = \delta$ for all regular $\eta < \delta$, and let $S \subseteq \delta^+ \cap \mathrm{Cof}({<}\delta)$ be stationary. Then the stationarity of S is preserved by δ-closed forcing.*

Proof. Let $\mathbb{P}$ be δ-closed and let $p \in \mathbb{P}$ force that $\dot{C}$ is a club set in δ^+. Fix $\eta < \delta$ such that $S \cap \mathrm{Cof}(\eta)$ is stationary in δ^+ and a large regular θ, then build $M \prec (H_\theta, \in)$ such that $p, \mathbb{P}, \dot{C} \in M$, $|M| = \delta$, M is closed under $<\eta$-sequences and $\gamma = M \cap \delta^+ \in S \cap \mathrm{Cof}(\eta)$. Now build a decreasing sequence p_i for $i < \eta$ of conditions in $\mathbb{P} \cap M$, and a sequence γ_i of ordinals increasing and cofinal in γ, such that $p_{i+1} \Vdash \gamma_i \in \dot{C}$. The construction is easy, using elementarity at successor stages and the closure of M at limit stages. Since $\mathbb{P}$ is δ-closed we may choose q a lower bound for the p_i and then $q \Vdash \gamma \in \dot{C}$. ⊣

10.7 Remark. In general it is *not* true for $\delta > \omega_1$ and $\gamma > \delta$ that every stationary subset of $\gamma^+ \cap \mathrm{Cof}(<\delta)$ is preserved by δ-closed forcing, even if we assume GCH. Shelah has given an incisive analysis of when we may expect stationarity to be preserved; the author's survey paper [10] contains an exposition of the resulting "$I[\lambda]$ theory".

10.8 Theorem (Baumgartner [5]). *In the model $V[G]$ of Theorem 10.5, where $\kappa = \delta^+$, every stationary $S \subseteq \kappa \cap \mathrm{Cof}(<\delta)$ reflects to a point of cofinality δ.*

Proof. Consider the generic embedding $j_G : V[G] \longrightarrow M[G * H]$ where H is generic for δ-closed forcing. We know that $j_G(S) \cap \kappa = S$, and since $M[G] \subseteq V[G]$ and $V[G] \models {}^\kappa M[G] \subseteq M[G]$ the set S is a stationary subset of $\kappa \cap \mathrm{Cof}(<\delta)$ in $M[G]$. The conditions of Lemma 10.6 apply (in fact $\delta^{<\delta} = \delta$) so S is stationary in $M[G * H]$, and so by the elementarity of j_G the set S has a stationary initial segment. Finally $\mathrm{cf}(\kappa) = \delta$ in $M[G * H]$ and $j_G(\delta) = \delta$, so stationarity reflects to an ordinal of cofinality δ. ⊣

10.9 Remark. Actually the conclusion of Theorem 10.8 holds if κ is only weakly compact, and this was the hypothesis used by Baumgartner. If κ is supercompact and we force with $\mathrm{Col}(\omega_1, <\kappa)$, then Shelah [5] observed that we get a model where for every regular $\lambda > \omega_1$ every stationary subset of $\lambda \cap \mathrm{Cof}(\omega)$ reflects.

Shelah [63] has also shown that it is consistent that (roughly speaking) "all stationary sets that can reflect do reflect". This is tricky because of the preservation problems alluded to in Remark 10.7.

10.10 Remark. The fact that we needed Lemma 10.6 to complete the proof of Theorem 10.8 is an example of a very typical phenomenon in the theory of generic embeddings, where we often need to know that the forcing which adds the embedding is in some sense "mild". See Theorem 23.2 for an example where the needed preservation lemmas involve not adding cofinal branches to trees.

11. Iteration with Easton Support

When defining an iterated forcing one of the key parameters is the type of support which is to be used. Silver realised that iteration with Easton support (see Definition 7.5) is a very useful technique in doing iterations which preserve large cardinal axioms. Easton [17] had already used Easton sets as the supports in products of forcings defined in V; the method of iteration with Easton supports has often been called "Reverse Easton", "Backwards Easton" or "Upwards Easton" to distinguish it from Easton's product construction.

We give an example of forcing with Easton support which is due in a slightly different form to Kunen and Paris [47]. The goal is to produce a measurable cardinal κ with the maximum possible number of normal measures; if we assume GCH for simplicity, then the maximal possible number of normal measures is $2^{2^\kappa} = \kappa^{++}$. Kunen's work on iterated ultrapowers [44] shows that if κ is measurable then in the canonical minimal model $L[U]$ in which κ is measurable, κ carries exactly one normal measure.

The arguments of Lévy and Solovay [50] show that if κ is measurable a forcing of size less than κ cannot increase the number of normal measures on κ. It follows that we need to force with a forcing poset of size at least κ. The simplest such poset which does not obviously destroy the measurability of κ is $\mathrm{Add}(\kappa, 1)$, however it is not hard to see that if we force over $L[U]$ this poset destroys the measurability of κ.

We will build an iterated forcing of size κ which adds subsets to many cardinals less than κ. As we will see shortly, we need the initial segments of the iteration to have a reasonable chain condition, and the final segments to have a reasonable degree of closure. Silver realised that the right balance between closure and chain condition could be achieved by doing an iteration with Easton support. We will assume that GCH holds in V. Assuming GCH is no burden because GCH is true in $L[U]$.

11.1 Theorem. *Let κ be measurable and let* GCH *hold. Then there exists $\mathbb{P}$ such that*

1. $|\mathbb{P}| = \kappa$.
2. $\mathbb{P}$ *is κ-c.c.*
3. GCH *holds in $V^{\mathbb{P}}$.*
4. *κ is measurable in $V^{\mathbb{P}}$.*
5. *κ carries κ^{++} normal measures in $V^{\mathbb{P}}$.*

The proof will occupy the rest of this subsection.

Let $A \subseteq \kappa$ be the set of those $\alpha < \kappa$ such that α is the successor of a singular cardinal. Let $j : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map. Since $M \subseteq V$ we see that κ is inaccessible in M, so that $\kappa \notin j(A)$ or equivalently

$A \notin U$. In the iteration we will add subsets to cardinals lying in A. The exact choice of A is irrelevant so long as it is a set of regular cardinals less than κ and is a set of measure zero for U.

We will now let $\mathbb{P} = \mathbb{P}_\kappa$ be an iteration of length κ with Easton support. For $\alpha < \kappa$ we let $\dot{\mathbb{Q}}_\alpha$ be a name for the trivial forcing unless $\alpha \in A$, in which case $\dot{\mathbb{Q}}_\alpha$ names $\mathrm{Add}(\alpha, 1)_{V^{\mathbb{P}_\alpha}}$. By Proposition 7.13, if $\delta \leq \kappa$ and δ is Mahlo then $\mathbb{P}_\delta$ is δ-c.c.

11.2 Lemma. *Let $\delta \leq \kappa$ be Mahlo and let $\lambda = \delta^{+\omega+1}$. As in Proposition 7.11 let $\dot{\mathbb{R}}_{\delta,\kappa}$ name the canonical iteration of length $\kappa - \delta$ such that $\mathbb{P}_\kappa \simeq \mathbb{P}_\delta * \dot{\mathbb{R}}_{\delta,\kappa}$. Then $V[G_\delta] \models$ "$\mathbb{R}_{\delta,\kappa}$ is λ-closed",*

Proof. If we fix $\delta \leq \kappa$ a Mahlo cardinal then it follows from the chain condition of $\mathbb{P}_\delta$ that every set of ordinals of size less than δ in $V[G_\delta]$ is covered by a set of size less than δ in V. $\lambda = \min(A \setminus \delta)$ and the iteration is only non-trivial at points of A, and so for all γ with $\delta \leq \gamma < \kappa$ we see that $\Vdash_\gamma$ "$\dot{\mathbb{Q}}_\gamma$ is λ-closed". By Proposition 7.12, $\mathbb{R}_{\delta,\kappa}$ is λ-closed in $V[G_\delta]$. ⊣

11.3 Remark. We already have enough information to see that all Mahlo cardinals $\delta \leq \kappa$ are preserved by $\mathbb{P}$. A more delicate analysis as in Hamkins' paper [31] shows that in fact this iteration preserves all cardinals and cofinalities.

11.4 Lemma. GCH *holds in $V[G_\kappa]$ above κ.*

Proof. If $\lambda \geq \kappa$ and $\Vdash_{\mathbb{P}} \dot{\tau} \subseteq \lambda$ then the interpretation of $\dot{\tau}$ is determined by $\{(p, \alpha) : p \Vdash \check{\alpha} \in \dot{\tau}\}$. There are only $2^{\kappa \times \lambda} = \lambda^+$ possibilities for this set. ⊣

11.5 Remark. With more care we can show that GCH holds everywhere.

We now need to compare $j(\mathbb{P})$ with $\mathbb{P}$. Elementarity implies that from the point of view of M, $j(\mathbb{P})$ is an Easton iteration of length $j(\kappa)$, with Easton support, in which we add a Cohen subset to each $\alpha \in j(A)$.

11.6 Lemma. *$j(\mathbb{P})_\kappa = \mathbb{P}_\kappa$ and $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_\kappa * \{1\}$.*

Proof. If $\alpha < \kappa$ then $\mathbb{P}_\alpha \in V_\kappa$ and so $j(\mathbb{P})_\alpha = j(\mathbb{P})_{j(\alpha)} = j(\mathbb{P}_\alpha) = \mathbb{P}_\alpha$. κ is inaccessible in M so a direct limit is taken at stage κ in the construction of $j(\mathbb{P})$. The direct limit construction is absolute so $j(\mathbb{P})_\kappa = \mathbb{P}_\kappa$. Finally $\kappa \notin j(A)$ and so $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_\kappa * \{1\}$. ⊣

Let G be $\mathbb{P}$-generic over V. Since $M \subseteq V$ and $\mathbb{P} \in M$, G is $\mathbb{P}$-generic over M and $M[G] \subseteq V[G]$.

11.7 Lemma. *Let $\mathbb{R} = i_G(\dot{\mathbb{R}}_{\kappa,j(\kappa)})$.*

$V[G] \models$ "$\mathbb{R}$ is κ^+-closed and has κ^+ maximal antichains lying in $M[G]$"

Proof. By Lemma 11.2 applied in M to $j(\mathbb{P})$, if $\lambda = \kappa^{+\omega+1}_{M[G]}$ then

$$M[G] \models \text{“}\mathbb{R} \text{ is } \lambda\text{-closed”}.$$

Since $\mathbb{P}$ is κ-c.c. it follows from Proposition 8.4 that $V[G] \models {}^{\kappa}M[G] \subseteq M[G]$. So $V[G] \models$ “$\mathbb{R}$ is κ^+-closed”.

$\mathbb{P}$ is κ-c.c. forcing poset with size κ, and in M we have $j(\mathbb{P}) \simeq \mathbb{P} * \dot{\mathbb{R}}$. It follows from Proposition 7.13 that in $M[G]$, $\mathbb{R}$ is $j(\kappa)$-c.c. forcing with size $j(\kappa)$, so if Z is the set of maximal antichains of $\mathbb{R}$ which lie in $M[G]$ then $M[G] \models |Z| = j(\kappa)$.

V is a model of GCH and so $V \models |j(\kappa)| = 2^{\kappa} = \kappa^+$. Therefore

$$V[G] \models \text{“}\mathbb{R} \text{ has } \kappa^+ \text{ maximal antichains lying in } M[G]\text{”}.$$

$\dashv$

From now on we work in $V[G]$. Applying Proposition 8.1 we construct a sequence $\langle H_\alpha : \alpha < \kappa^{++} \rangle$ of κ^{++} distinct $\mathbb{R}$-generic filters over $M[G]$. For each α the set $G * H_\alpha$ is $\mathbb{P} * \dot{\mathbb{R}}$-generic over M, and since $\mathbb{P} * \dot{\mathbb{R}}$ is canonically isomorphic to $j(\mathbb{P})$ in M we will regard $G * H_\alpha$ as a $j(\mathbb{P})$-generic filter over M.

11.8 Lemma. *For all $p \in G$ and all $\alpha < \kappa^{++}$, $j(p) \in G * H_\alpha$.*

Proof. $\mathbb{P}_\kappa = \varinjlim \vec{\mathbb{P}} \restriction \kappa$. Fix $\beta < \kappa$ such that $p(\gamma) = 1$ for $\beta \leq \gamma < \kappa$, and observe that $j(\beta) = \beta$ and so by elementarity $j(p)(\gamma) = 1$ for $\beta \leq \gamma < j(\kappa)$. What is more $p \restriction \beta \in V_\kappa$ and so $j(p) \restriction \beta = j(p \restriction \beta) = p \restriction \beta$. It follows that $j(p) \in G * H_\alpha$. $\dashv$

Accordingly we can find κ^{++} extensions $j_\alpha : V[G] \longrightarrow M[G * H_\alpha]$ with $j_\alpha \restriction V = j$ and $j_\alpha(G) = G * H_\alpha$. They are distinct because the filters H_α are distinct. $H_\alpha \in V[G]$ and so by Proposition 9.4 j_α is definable in $V[G]$. We will be done if we can show that each j_α is an ultrapower map computed from some normal measure on κ in $V[G]$.

11.9 Lemma. *For every α, j_α is the ultrapower of $V[G]$ by U_α where*

$$U_\alpha = \{X \subseteq \kappa : X \in V[G], \kappa \in j_\alpha(X)\}.$$

Proof. j is the ultrapower of V by the normal measure U, so that by Proposition 3.9 j is supported on $\{\kappa\}$. By Proposition 9.3 j_α is also supported on $\{\kappa\}$. By Proposition 3.9 again j_α is the ultrapower of $V[G]$ by U_α. $\dashv$

12. Master Conditions

We are now in a position to give Silver's proof that GCH can fail at a measurable cardinal. We will need Silver's idea of the *master condition*, which is a technique for arranging the compatibility between generic filters required to apply Proposition 9.1.

12.1 Definition. Let $k : M \longrightarrow N$ be elementary and let $\mathbb{P} \in M$. A *master condition for* k *and* $\mathbb{P}$ is a condition $q \in k(\mathbb{P})$ such that for every dense set $D \subseteq \mathbb{P}$ with $D \in M$, there is a condition $p \in D$ such that q is compatible with $k(p)$.

Suppose that q is a master condition for k, and H is any N-generic filter on $\mathbb{Q}$ with $q \in H$. It is routine to check that k^{-1}"H generates an M-generic filter G such that k"$G \subseteq H$, and so again Proposition 9.1 can be applied to lift k. In general different choices of H will give different filters G.

12.2 Definition. Let $k : M \longrightarrow N$ be elementary and let $\mathbb{P} \in M$. A *strong master condition for* k *and* $\mathbb{P}$ is a condition $q \in k(\mathbb{P})$ such that for every dense set $D \subseteq \mathbb{P}$ with $D \in M$, there is a condition $p \in D$ such that $q \leq k(p)$.

If q is a strong master condition then let $G = \{p \in \mathbb{P} : q \leq k(p)\}$. It is routine to check that G is an M-generic filter, and that k^{-1}"$H = G$ for any N-generic filter H on $\mathbb{Q}$ with $q \in H$. Under these circumstances we will often say that q is a *strong master condition for* k *and* G.

12.3 Remark. A similar distinction occurs in the theory of proper forcing. See Remarks 24.4 and 24.5 for more on this.

12.4 Remark. Most of the master conditions which we build will be of the strong persuasion.

For use later we record a remark on the connection between existence of strong master conditions and distributivity.

12.5 Theorem. *Let* $\pi : M \longrightarrow N$ *be elementary, let* $\mathbb{P} \in M$*, let* G *be* $\mathbb{P}$*-generic, and let* $q \in j(\mathbb{P})$ *be such that* $q \leq j$"G*. Then for every* $\delta < \mathrm{crit}(\pi)$*,* M *and* $M[G]$ *have the same* δ*-sequences of ordinals.*

Proof. Suppose not, and fix $p \in G$ and $\dot{\tau} \in M$ such that p forces $\dot{\tau}$ to be a new δ-sequence of ordinals. For each $i < \delta$ there is a condition $p_i \in G$ such that p_i determines $\dot{\tau}(i)$. By elementarity $\pi(p_i)$ determines $\pi(\dot{\tau})(i)$ for each $i < \pi(\delta) = \delta$, and so since $q \leq \pi(p_i)$ we have that q determines $\pi(\dot{\tau})(i)$ for all $i < \delta$, that is q forces that $\pi(\dot{\tau})$ is equal to some element of N; but $q \leq \pi(p)$ and by elementarity $\pi(p)$ forces that $\pi(\dot{\tau})$ is a new sequence of ordinals, contradiction. $\dashv$

It is easy to see that if U is a normal measure on κ and $2^\kappa \geq \kappa^{+n}$ then $\{\alpha < \kappa : 2^\alpha \geq \alpha^{+n}\} \in U$. In the light of this remark and the result of the last section, a natural strategy for producing a failure of GCH at a measurable is to start with a model of GCH with a measurable κ, and to do an iteration of length $\kappa + 1$ violating GCH on $A \cup \{\kappa\}$ for some suitably large A.

This strategy can be made to work but it is necessary to use a fairly strong large cardinal assumption. We will give here a version of Silver's original proof, using the hypothesis that GCH holds and there is a cardinal κ which is κ^{++}-supercompact. In Sections 13 and 25 we will see how to weaken this large cardinal assumption.

12.6 Theorem. *Let* GCH *hold and let* κ *be* κ^{++}*-supercompact. Then there is a forcing poset* $\mathbb{P}$ *such that*

1. $|\mathbb{P}| = \kappa^{++}$.
2. $\mathbb{P}$ *is* κ^+*-c.c.*
3. κ *is measurable in* $V^{\mathbb{P}}$.
4. $2^\kappa = \kappa^{++}$ *in* $V^{\mathbb{P}}$.

Proof. Let U be a κ-complete normal fine ultrafilter on $P_\kappa \kappa^{++}$, and define $j : V \longrightarrow M$ to be the associated ultrapower map. Arguing exactly as in Example 4.8, we have

1. $j(\kappa) > \kappa^{+++}$.
2. $j(\kappa^{+4}) = \kappa^{+4}$.

Let A be the set of inaccessible cardinals less than κ. As in the last section the exact choice of A is more or less irrelevant, so long as A is a set of inaccessible cardinals and $A \in U_0$, where $U_0 = \{X \subseteq \kappa : \kappa \in j(X)\}$.

We now let $\mathbb{P} = \mathbb{P}_{\kappa+1}$ be the iteration of length $\kappa+1$ with Easton supports in which $\dot{\mathbb{Q}}_\alpha$ names $\mathrm{Add}(\alpha, \alpha^{++})_{V^{\mathbb{P}_\alpha}}$ if $\alpha \in A \cup \{\kappa\}$, and names the trivial forcing otherwise. Let G_κ be $\mathbb{P}_\kappa$-generic over V, let g_κ be $\mathbb{Q}_\kappa$-generic over $V[G]$ and let $G_{\kappa+1} = G_\kappa * g_\kappa$.

The next lemma is similar to Lemma 11.2 from the last section, the crucial difference being that this time $\delta \in A$ and so the iteration $\mathbb{P}$ acts at stage δ.

12.7 Lemma. *Let* $\delta < \kappa$ *be Mahlo. Then*

1. $\mathbb{P}_\delta$ *is* δ*-c.c.*
2. $\mathbb{P}_{\delta+1}$ *is* δ^+*-c.c.*
3. *If* λ *is the least inaccessible greater than* δ *then*

$$V[G_{\delta+1}] \models \text{“}\mathbb{R}_{\delta+1,\kappa} \textit{ is } \lambda\textit{-closed”}.$$

Proof. By Proposition 7.13 $\mathbb{P}_\delta$ is δ-c.c. and has size δ. Then $V[G_\delta] \models \delta^{<\delta} = \delta$, and so $\Vdash_\delta$ “$\mathbb{Q}_\delta$ is δ^+-c.c.”. By Proposition 7.6 $\mathbb{P}_{\delta+1}$ is δ^+-c.c.

Since A is a set of inaccessible cardinals we are guaranteed that $\dot{\mathbb{Q}}_\alpha$ names the trivial forcing for $\delta < \alpha < \lambda$. Every set of ordinals of size less than λ in $V[G_{\delta+1}]$ is covered by a such a set in V, so by Proposition 7.12 $\mathbb{R}_{\delta+1,\kappa}$ is λ-closed in $V[G_{\delta+1}]$. ⊣

The next lemma follows by exactly the same argument as that for Lemma 11.4 in the last section.

12.8 Lemma. $\mathbb{P}_\kappa$ *is* κ*-c.c. with size* κ*, and* GCH *holds above* κ *in* $V^{\mathbb{P}_\kappa}$.

The standard arguments counting names also give us

12.9 Lemma. $\mathbb{P}$ *is* κ^+*-c.c. with size* κ^{++}*, and* GCH *holds above* κ^+ *in* $V^{\mathbb{P}}$.

We now need to analyse the iteration $j(\mathbb{P})$.

12.10 Lemma. $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_{\kappa+1}$.

Proof. We can argue exactly as in Lemma 11.6 that $j(\mathbb{P})_\kappa = \mathbb{P}_\kappa$. By Proposition 8.4 we see that $V[G] \models {}^{\kappa^{++}}M[G] \subseteq M[G]$, so that

$$\mathrm{Add}(\kappa, \kappa^{++})_{V[G]} = \mathrm{Add}(\kappa, \kappa^{++})_{M[G]}.$$

Every condition in $\mathrm{Add}(\kappa, \kappa^{++})_{V^{\mathbb{P}_\kappa}}$ has a name which lies in $H_{\kappa^{+++}}$, and $H_{\kappa^{+++}} \subseteq M$ so that $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_{\kappa+1}$. ⊣

12.11 Lemma. *Let* $\mathbb{R} = i_{G_{\kappa+1}}(\dot{\mathbb{R}}_{\kappa+1,j(\kappa)})$*. Then*

$$V[G_{\kappa+1}] \models \text{“}\mathbb{R} \textit{ is } \kappa^{+++}\textit{-closed”}$$

and

$$V[G_{\kappa+1}] \models \text{“}\mathbb{R} \textit{ has } \kappa^{+++} \textit{ maximal antichains lying in } M[G_{\kappa+1}]\text{”}.$$

Proof. By Lemma 12.7 applied in M to $j(\mathbb{P}_\kappa)$, if λ is the least M-inaccessible greater than κ then $M[G_{\kappa+1}] \models$ “$\mathbb{R}$ is λ-closed”. Since $\mathbb{P}$ is κ^+-c.c. it follows from Proposition 8.4 that $V[G_{\kappa+1}] \models {}^{\kappa^{++}}M[G_{\kappa+1}] \subseteq M[G_{\kappa+1}]$. So $V[G_{\kappa+1}] \models$ “$\mathbb{R}$ is κ^{+++}-closed”.

$\mathbb{P}_\kappa$ is κ-c.c. with size κ, and in M we have $j(\mathbb{P}_\kappa) \simeq \mathbb{P}_{\kappa+1} * \dot{\mathbb{R}}$. It follows from Proposition 7.13 that in $M[G_{\kappa+1}]$, $\mathbb{R}$ is $j(\kappa)$-c.c. with size $j(\kappa)$, so if Z is the set of maximal antichains of $\mathbb{R}$ which lie in $M[G_{\kappa+1}]$ then $M[G_{\kappa+1}] \models |Z| = j(\kappa)$.

By Proposition 4.5, $V \models |j(\kappa)| = \kappa^{+++}$. So

$$V[G_{\kappa+1}] \models \text{“}\mathbb{R} \text{ has } \kappa^{+++} \text{ maximal antichains in } M[G_{\kappa+1}]\text{”}.$$

⊣

Applying Proposition 8.1 we may find a filter $H \in V[G_{\kappa+1}]$ such that H is $\mathbb{R}$-generic over $M[G_{\kappa+1}]$. Let $G_{j(\kappa)} = G_{\kappa+1} * H$, so that $G_{j(\kappa)}$ is $j(\mathbb{P}_\kappa)$-generic over M. The argument of Lemma 11.8 shows that $j``G_\kappa \subseteq G_{j(\kappa)}$, so that by Proposition 9.1 we may lift $j : V \longrightarrow M$ and obtain an elementary embedding $j : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}]$.

To finish the proof we need to construct a filter $h \in V[G_{\kappa+1}]$ such that

1. h is $\mathrm{Add}(j(\kappa), j(\kappa^{++}))_{M[G_{j(\kappa)}]}$-generic over $M[G_{j(\kappa)}]$.

2. $j``g_\kappa \subseteq h$.

The first of these conditions can be met using methods we have seen already, once we have done some counting arguments.

12.12 Lemma. $V[G_{\kappa+1}] \models {}^{\kappa^{++}}M[G_{j(\kappa)}] \subseteq M[G_{j(\kappa)}]$.

Proof. $\mathbb{P}_{\kappa+1}$ is κ^+-c.c. and so $V[G_{\kappa+1}] \models {}^{\kappa^{++}}\mathrm{On} \subseteq M[G_{\kappa+1}]$ by Proposition 8.4. $M[G_{\kappa+1}] \subseteq M[G_{j(\kappa)}]$ so $V[G_{\kappa+1}] \models {}^{\kappa^{++}}\mathrm{On} \subseteq M[G_{j(\kappa)}]$. The result follows by Proposition 8.2. ⊣

12.13 Lemma. *Let* $\mathbb{Q} = \mathrm{Add}(j(\kappa), j(\kappa^{++}))_{M[G_{j(\kappa)}]}$. *Then*

$$V[G_{\kappa+1}] \models \text{“}\mathbb{Q} \textit{ is } \kappa^{+++}\textit{-closed”}$$

and

$$V[G_{\kappa+1}] \models \text{“}\mathbb{Q} \textit{ has } \kappa^{+++} \textit{ maximal antichains in } M[G_{j(\kappa)}]\text{”}.$$

Proof. $M[G_{j(\kappa)}] \models$ “$\mathbb{Q}$ is $j(\kappa)$-closed”, so by Proposition 8.3 it follows that $V[G_{\kappa+1}] \models$ “$\mathbb{Q}$ is κ^{+++}-closed”.

Lemma 12.8 and an easy counting argument give that

$$V[G_\kappa] \models \text{“Add}(\kappa, \kappa^{++}) \text{ has } \kappa^{+++} \text{ maximal antichains”}.$$

$j : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}]$ is elementary and so

$$M[G_{j(\kappa)}] \models \text{“}\mathbb{Q} \text{ has } j(\kappa^{+++}) \text{ maximal antichains”}.$$

Since $V \models |j(\kappa^{+++})| = \kappa^{+++}$,

$$V[G_{\kappa+1}] \models \text{“}\mathbb{Q} \text{ has } \kappa^{+++} \text{ maximal antichains in } M[G_{j(\kappa)}]\text{”}.$$

and we are done. ⊣

We can now build $h \in V[G_{\kappa+1}]$ which is suitably generic. To ensure that $j``g_\kappa \subseteq h$ we use the “strong master condition” idea from Definition 12.2.

12.14 Lemma. *There is a strong master condition for the elementary embedding* $j : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}]$ *and the generic object* g_κ.

Proof. If $p \in g_\kappa$ then p is a partial function from $\kappa \times \kappa^{++}$ to 2 with size less than κ, so in particular $j(p) = j``p$. $g_\kappa \in M[G_{j(\kappa)}]$ and $j \restriction (\kappa \times \kappa^{++}) \in M$, so that $j``g_\kappa \in M[G_{j(\kappa)}]$. Working in $M[G_{j(\kappa)}]$ the cardinality of $j``g_\kappa$ is $\kappa^{++}$, $j``g_\kappa$ is a directed subset of $\mathbb{Q}$ and $\mathbb{Q}$ is $j(\kappa)$-directed closed; it follows that we may find a condition $r \in \mathbb{Q}$ such that $r \leq j(p)$ for all $p \in g_\kappa$. ⊣

12.15 Remark. We can give an explicit description of an r with this property; let $\mathrm{dom}(r) = \kappa \times j``\kappa^{++}$ and $r(\alpha, j(\beta)) = j(F(\alpha, \beta)) = F(\alpha, \beta)$ where $F : \kappa \times \kappa^{++} \longrightarrow 2$ is given by $F = \bigcup g_\kappa$.

We now use Proposition 8.1 to build h which is $\mathbb{Q}$-generic over $M[G_{j(\kappa)}]$ with $r \in h$. Let $G_{j(\kappa)+1} = G_{j(\kappa)} * h$. Then by construction we have $j``g \subseteq h$, so that we may lift $j : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}]$ to $j : V[G_{\kappa+1}] \longrightarrow M[G_{j(\kappa)+1}]$. $G_{j(\kappa)+1} \in V[G_{\kappa+1}]$ and so by Proposition 9.4 the elementary embedding $j : V[G_{\kappa+1}] \longrightarrow M[G_{j(\kappa)+1}]$ is definable in $V[G_{\kappa+1}]$, that is to say it is a definable embedding in the sense of Definition 2.10. It follows that κ is measurable in $V[G_{\kappa+1}]$. ⊣

12.16 Remark. By Proposition 9.4 and Remark 4.1, κ is actually κ^{++}-supercompact in $V[G_{\kappa+1}]$.

12.17 Remark. If we had forced with $\mathrm{Add}(\alpha, 1)$ instead of $\mathrm{Add}(\alpha, \alpha^{++})$ at each stage in $A \cup \{\kappa\}$, then we could have proved that the measurability of κ was preserved assuming only that GCH holds and κ is measurable in the ground model. Of course we would not have violated GCH this way, and indeed it is known [59, 24] that to violate GCH at a measurable cardinal requires the strength of a cardinal κ with Mitchell order κ^{++}.

13. A Technique of Magidor

In this section we describe a technique due to Magidor [54] for lifting elementary embeddings in situations where we do not have enough closure to build a strong master condition. The trick will be to build an "increasingly masterful" sequence of conditions into our final generic filter. As an example we will redo the result from the last section from a weaker large cardinal hypothesis.

We assume that GCH holds and that κ is κ^+-supercompact, and we let $j : V \longrightarrow M$ be the ultrapower map arising from some κ^+-supercompactness measure on $P_\kappa \kappa^+$. As in Example 4.8 we see that

- $\kappa^{++} = \kappa^{++}_M < j(\kappa) < j(\kappa^+) < j(\kappa^{++}) < j(\kappa^{+++}) = \kappa^{+++}$.
- j is continuous at κ^{++} and κ^{+++}.
- j is discontinuous at every limit ordinal of cofinality κ^+.

We now perform exactly the same forcing construction as in the last section, namely we perform an Easton support iteration of length $\kappa + 1$ in which we add α^{++} Cohen subsets to every inaccessible $\alpha \leq \kappa$. We let G_κ be $\mathbb{P}_\kappa$-generic over V and g_κ be $\mathbb{Q}_\kappa$-generic over $V[G_\kappa]$.

As in Lemma 12.10 from the last section we see that $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_{\kappa+1}$. As in the last section we let $\mathbb{R} = i_{G_{\kappa+1}}(\dot{\mathbb{R}}_{\kappa+1,j(\kappa)})$, that is to say $\mathbb{R}$ is the forcing that one would do over $M[G_{\kappa+1}]$ to produce a $j(\mathbb{P}_\kappa)$-generic object extending $G_{\kappa+1}$.

Modifying the proof of Lemma 12.11 we see that

$$V[G_{\kappa+1}] \models \text{“}\mathbb{R} \text{ is } \kappa^{++}\text{-closed”}$$

and

$$V[G_{\kappa+1}] \models \text{“}\mathbb{R} \text{ has } \kappa^{++} \text{ maximal antichains lying in } M[G_{\kappa+1}]\text{”}.$$

By Proposition 8.1 we build a filter $H \in V[G_{\kappa+1}]$ which is $\mathbb{R}$-generic over $M[G_{\kappa+1}]$, and lift $j : V \longrightarrow M$ to get $j : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}]$ where $G_{j(\kappa)} = G_\kappa * g_\kappa * H$. We observe that $V[G_{\kappa+1}] \models {}^{\kappa^+}M[G_{j(\kappa)}] \subseteq M[G_{j(\kappa)}]$.

As in the last section we may apply Proposition 8.1 to build $h \in V[G_{\kappa+1}]$ which is $j(\mathbb{Q}_\kappa)$-generic over $M[G_{j(\kappa)}]$, and the remaining problem is to build h is such a way that $j``g_\kappa \subseteq h$. At this point we can no longer imitate the proof of the last section because we no longer have enough closure.

We do some analysis of an antichain A of $j(\mathbb{Q}_\kappa)$ with $A \in M[G_{j(\kappa)}]$. Let $A = j(F)(j``\kappa^+)$ where $F \in V[G_\kappa]$, $\mathrm{dom}(F) = P_\kappa\kappa^+$, and without loss of generality $F(x)$ is a maximal antichain in $\mathbb{Q}_\kappa$ for all x. Working in $V[G_\kappa]$, for each $\zeta < \kappa^{++}$ we let $X_\zeta = \mathrm{Add}(\kappa, \zeta)$, so that $X_\zeta \subseteq \mathbb{Q}_\kappa$, $|X_\zeta| = \kappa^+$, and $\mathbb{Q}_\kappa = \bigcup_\zeta X_\zeta$. A routine argument in the style of the Löwenheim-Skolem theorem shows that for each x there is a club subset C_x of κ^{++} such that for all $\alpha \in C_x \cap \mathrm{Cof}(\kappa^+)$ the antichain $F(x) \cap X_\alpha$ is maximal in X_α. Let C be the intersection of the C_x for $x \in P_\kappa\kappa^+$, then C is club and for every $\alpha \in C \cap \mathrm{Cof}(\kappa^+)$ the antichain A is maximal in $j(X_\alpha) = \mathrm{Add}(j(\kappa), j(\alpha))_{M[G_{j(\kappa)}]}$.

Now we work in $V[G_{\kappa+1}]$ to build a suitable filter h. Define Q a partial function from $j(\kappa) \times j``\kappa^{++}$ by setting $Q$ to be the union of $j(p)$ for $p \in g_\kappa$. It is routine to check that $\mathrm{dom}(Q) = \kappa \times j``\kappa^{++}$, and while Q is not even in $M[G_{j(\kappa)}]$, for all $\zeta < \kappa^{++}$ the partial function $q_\zeta = Q \restriction (j(\kappa) \times j(\zeta))$ is in $j(X_\zeta)$ and is a strong master condition for j and $g_\kappa \cap X_\zeta$.

Working in $V[G_{\kappa+1}]$, we may enumerate all the maximal antichains of $j(\mathbb{Q}_\kappa)$ as $\langle A_i : i < \kappa^{++} \rangle$. Using the analysis of such antichains given above we choose an increasing sequence $\alpha_i \in \kappa^{++} \cap \mathrm{Cof}(\kappa^+)$ such that $A_i \cap j(X_{\alpha_i})$ is maximal in $j(X_{\alpha_i})$ for all $i < \kappa^{++}$. Now we build a decreasing sequence of conditions $r_i \in j(\mathbb{Q}_\kappa)$ such that for each $i < \kappa^{++}$

1. $r_i \in j(X_{\alpha_i})$.
2. $r_i \leq q_{\alpha_i}$.
3. r_i extends some member of A_i.

The construction of r_i goes as follows. We start by forming $r = \bigcup_{j<i} r_j$, where we note that the support of r is contained in $j(\kappa) \times \sup_{j<i} j(\alpha_j)$. We claim that r is compatible with q_{α_i}. To see this let $(\delta, j(\gamma))$ be an arbitrary point in the domain of q_{α_i}, that is $\gamma < \alpha_i$ and $\delta < \kappa$. If $\gamma < \alpha_j$ for some j then since $r \leq r_j \leq q_j$ we have

$$q_{\alpha_i}(\delta, j(\gamma)) = Q(\delta, j(\gamma)) = r(\delta, j(\gamma)),$$

while if $\gamma \geq \alpha_j$ for all $j < i$ then $(\delta, j(\gamma)) \notin \mathrm{dom}(r)$.

So we may take the union $r \cup q_{\alpha_i}$ to get a condition in $j(X_{\alpha_i})$ and since $A_i \cap j(X_{\alpha_i})$ is maximal in $j(X_{\alpha_i})$ we may choose $r_i \leq r \cup q_{\alpha_i}$ so that $r_i \in j(X_{\alpha_i})$ and r_i extends some condition in A_i.

It is now easy to see that the sequence of r_i generates a generic filter h with $h \supseteq j``g$. We may then proceed as in the previous section to lift the embedding to $V[G_{\kappa+1}]$.

13.1 Remark. In fact κ is still κ^+-supercompact in $V[G_{\kappa+1}]$.

13.2 Remark. The forcing technique described here has many applications in the theory of precipitous and saturated ideals. See sections 17 and 18, and also Foreman's chapter.

14. Absorption

In this section we discuss an idea which is used in many forcing constructions (for example in building Solovay's model in which every set is Lebesgue measurable [65]) and is particularly useful for our purposes, namely the idea of embedding a complex poset into a simple one. This is one area of the subject where the presentation of forcing in terms of complete Boolean algebras is very helpful.

The "simple posets" into which we typically absorb more complex ones are Cohen forcing $\mathrm{Add}(\kappa, \lambda)$ and the collapsing poset $\mathrm{Col}(\kappa, \lambda)$. We note that for any regular κ the forcing $\mathrm{Col}(\kappa, \kappa)$ is equivalent to $\mathrm{Add}(\kappa, 1)$ so we phrase our whole discussion in terms of the collapsing poset.

The following universal property of the collapsing poset is key:

14.1 Theorem. *Let κ be regular. Let $\lambda \geq \kappa$ and let $\mathbb{P}$ be a separative forcing poset such that $\mathbb{P}$ is κ-closed, $|\mathbb{P}| = \lambda$, every condition in $\mathbb{P}$ has λ incompatible extensions and $\mathbb{P}$ adds a surjection from κ to λ.*

Then $\mathbb{P}$ is equivalent to the collapsing poset $\mathrm{Col}(\kappa, \lambda)$.

Notice that if $\lambda > \kappa$ and λ is regular, then the demand that $\mathbb{P}$ adds a surjection from κ to λ implies that for no $p \in \mathbb{P}$ can $\mathbb{P}/p$ be λ-c.c., and so the demand that every condition should have λ incompatible extensions follows from the other conditions.

Proof. Let $\dot{f}$ name a surjective map from κ to $\dot{G}$, where $\dot{G}$ names the generic filter on $\mathbb{P}$. We will build a dense subset of $\mathrm{ro}(\mathbb{P}) \setminus \{0\}$ which is isomorphic to $\mathrm{Col}(\kappa, \lambda)$. Let $\mathbb{P}^*$ be the canonical isomorphic copy of $\mathbb{P}$ in $\mathrm{ro}(\mathbb{P})$, so that $\mathbb{P}^*$ is a dense κ-closed subset of $\mathrm{ro}(\mathbb{P}) \setminus \{0\}$.

We will build a family b_s indexed by $s \in \mathrm{Col}(\kappa, \lambda)$ with the following properties:

1. $b_0 = 1$, and $b_s \in \mathrm{ro}(\mathbb{P}) \setminus \{0\}$ for all s.
2. For all s and t, $t \leq s$ implies that $b_t \leq b_s$.
3. For all $\alpha < \kappa$, $\{b_s : \mathrm{dom}(s) = \alpha\}$ is a maximal antichain.
4. For all s with $\mathrm{dom}(s)$ a successor ordinal, $b_s \in \mathbb{P}^*$.
5. For all $\alpha < \kappa$ and s with domain α, b_s determines $\dot{f} \restriction \alpha$.
6. For all s with $\mathrm{dom}(s)$ a limit ordinal μ, $b_s = \bigwedge_{i<\mu} b_{s \restriction i}$.

We will construct $\{b_s : \mathrm{dom}(s) = \alpha\}$ by recursion on α. At successor stages we construct $\{b_{s^\frown j} : j < \lambda\}$ to be a maximal antichain below b_s, consisting of conditions that lie in $\mathbb{P}^*$ and determine $\dot{f}\restriction(\mathrm{dom}(s)+1)$.

For limit μ we define (as we are compelled to) b_s as the infimum of $\{b_{s\restriction i} : i < \mu\}$ for all s with $\mathrm{dom}(s) = \mu$. By κ-closure of $\mathbb{P}$, $b_s \neq 0$. Clearly $\{b_s : \mathrm{dom}(s) = \mu\}$ is an antichain. To show it is maximal, it will suffice to show that it meets every generic G. Let G be generic, and let $s : \mu \to \lambda$ be the unique function with $b_{s\restriction i} \in G$ for all $i < \mu$; by closure $s \in V$, so s is a condition in $\mathrm{Col}(\kappa, \lambda)$ and by genericity $b_s \in G$.

This completes the construction, and it remains to see that the set of all b_s is dense. Let $p \in \mathbb{P}$, so that p forces $p \in G$, and find a condition $q \leq p$ and an ordinal $i < \kappa$ such that $q \Vdash \dot{f}(i) = p$. The condition q is compatible with b_s for s such that $\mathrm{dom}(s) = i + 1$. Now b_s determines $\dot{f}(i)$, so b_s forces that $\dot{f}(i) = p$, in particular $b_s \Vdash p \in G$ and so by separativity $b_s \leq p$. $\dashv$

In particular a separative κ-closed forcing poset of cardinality κ is equivalent to $\mathrm{Add}(\kappa, 1)$ and a separative forcing poset of size λ which makes λ countable is equivalent to $\mathrm{Col}(\omega, \lambda)$. Moreover if $\mathbb{P}$ is κ-closed then for a sufficiently large μ we see that $\mathbb{P} \times \mathrm{Col}(\kappa, \mu)$ is equivalent to $\mathrm{Col}(\kappa, \mu)$; this is the key point in Theorems 14.2 and 14.3.

Theorem 14.1 has the following corollaries. We separate the cases of $\mathrm{Col}(\omega, {<}\kappa)$ and $\mathrm{Col}(\delta, {<}\kappa)$ for uncountable δ because (as detailed below) we may say significantly more in the former case.

14.2 Theorem. *Let κ be an inaccessible cardinal and let $\mathbb{C} = \mathrm{Col}(\omega, {<}\kappa)$. Let $\mathbb{P}$ be a separative forcing poset with $|\mathbb{P}| < \kappa$ and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a separative forcing poset of size less than κ. Then*

1. *There is a complete embedding $i : \mathrm{ro}(\mathbb{P}) \to \mathrm{ro}(\mathbb{C})$.*
2. *For any such i and any $\mathbb{P}$-generic g, $\mathbb{C}/i(g)$ is equivalent in $V[g]$ to $\mathrm{Col}(\omega, {<}\kappa)$.*
3. *Any such i may be extended to a complete $j : \mathrm{ro}(\mathbb{P} * \dot{\mathbb{Q}}) \to \mathrm{ro}(\mathbb{C})$.*

In the general case we have:

14.3 Theorem. *Let κ be an inaccessible cardinal , let $\delta < \kappa$ be regular and let $\mathbb{D} = \mathrm{Col}(\delta, {<}\kappa)$. Let $|\mathbb{P}| < \kappa$ where $\mathbb{P}$ is a δ-closed separative forcing poset, and and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a δ-closed separative forcing poset of size less than κ. Then*

1. *There is a complete embedding $i : \mathrm{ro}(\mathbb{P}) \to \mathrm{ro}(\mathbb{D})$ such that $\mathbb{D}/i(g)$ is equivalent in $V[g]$ to $\mathrm{Col}(\delta, {<}\kappa)$.*
2. *Any such i may be extended to a complete $j : \mathrm{ro}(\mathbb{P} * \dot{\mathbb{Q}}) \to \mathrm{ro}(\mathbb{D})$ such that $\mathbb{D}/i(g * h)$ is equivalent in $V[g * h]$ to $\mathrm{Col}(\delta, {<}\kappa)$.*

Notice that Theorem 14.2 asserts that *however* we embed the small poset $\mathbb{P}$ in the collapse $\mathrm{Col}(\omega, <\kappa)$, the quotient forcing is $\mathrm{Col}(\omega, <\kappa)$. Theorem 14.3 just asserts that there is *some* way of embedding the small δ-closed poset $\mathbb{P}$ in the collapse $\mathrm{Col}(\delta, <\kappa)$, so that the quotient forcing is $\mathrm{Col}(\delta, <\kappa)$.

To underline the difference between these theorems we consider the following example, which implies that we can find an embedding of $\mathrm{Add}(\omega_1, 1)$ into $\mathrm{Add}(\omega_1, 1)$ where the quotient is not countably closed or even proper.

14.4 Example. Let CH hold, let $\mathbb{P} = \mathrm{Add}(\omega_1, 1)$. If $F : \omega_1 \to 2$ is the function added by $\mathbb{P}$ and $S = \{\alpha : F(\alpha) = 1\}$ then we claim that S is stationary in $V[F]$. To see this let p force that $\dot{C}$ is club and build a chain of extensions $p_0 = p \geq p_1 \geq p_2 \geq \ldots$ so that p_{n+1} forces some ordinal $\alpha_n > \mathrm{dom}(p_n)$ into $\dot{C}$. Then if $p^* = \bigcup p_n$ and $\alpha^* = \sup_n \alpha_n$ we have that $\mathrm{dom}(p^*) = \alpha^*$ and $p^* \Vdash \alpha^* \in \dot{C}$, and may then extend to force that $\alpha^* \in S$. A similar argument shows that S^c is also stationary.

Working in $V[F]$, we now let $\mathbb{Q}$ be the forcing from Example 6.8 to add a club set $D \subseteq S^c$. We claim that $\mathbb{P} * \dot{\mathbb{Q}}$ has a countably closed dense subset of size ω_1, namely the set of pairs $(p, \check{c})$ where $\mathrm{dom}(p) = \max(c) + 1$ and $p(\alpha) = 0$ for all $\alpha \in c$: the proof is just like the proof that S is stationary from the last paragraph. So now $\mathbb{P} * \mathbb{Q}$ is equivalent to $\mathrm{Add}(\omega_1, 1)$, while $\mathbb{Q}$ destroys the stationarity of S so is not countably closed (or even proper).

Now we give some examples of the absorption idea in action. The first one is due to Kunen [45]. Since there is a detailed account in Foreman's chapter in this Handbook we shall only sketch the result.

As motivation we recall some facts about saturated ideals, weakly compact cardinals, and stationary reflection.

1. If κ is strongly inaccessible and carries a λ-saturated ideal for some $\lambda < \kappa$ then κ is measurable [45].
2. If κ is weakly compact and carries a κ-saturated ideal then κ is measurable [45].
3. If κ is weakly compact then every stationary subset of κ reflects.
4. If $V = L$ then for every regular κ, κ is weakly compact if and only if every stationary subset of κ reflects [42].
5. If $\mathbb{P} \times \mathbb{P}$ is κ-c.c. and κ is measurable in $V^{\mathbb{P}}$ then κ is measurable in V (see Theorem 21.1)

In the model which we present there is an inaccessible cardinal κ which carries a κ-saturated ideal and reflects stationary sets, and there is also a κ-Suslin tree T (so in particular κ is not weakly compact). It follows that 1 and 2 above are close to optimal, and that in general the conclusion of 4 fails. The key property of the model will be that adding a branch through the κ-Suslin tree T resurrects the measurability of κ so that 5 is also close to optimal.

We will use a standard device due to Kunen [45] for manufacturing saturated ideals.

14.5 Lemma. *Let $\mathbb{P}$ be λ-c.c. and let $\dot{U}$ be a $\mathbb{P}$ name for a V-ultrafilter on κ. Let I be the set of those $X \in P(\kappa) \cap V$ such that $\Vdash_{\mathbb{P}} \kappa \setminus X \in \dot{U}$. Then*

1. *I is a λ-saturated ideal.*
2. *If $\dot{U}$ is forced to be V-κ-complete then I is κ-complete, and if $\dot{U}$ is forced to be V-normal then I is normal.*

Often we have a $\mathbb{P}$-name for a generic embedding $j : V \longrightarrow M \subseteq V[G_{\mathbb{P}}]$, and U will be $\{X \in P(\kappa) \cap V : \kappa \in j(X)\}$. This kind of *induced ideal* is discussed at length in Foreman's chapter in this Handbook.

We use a result by Kunen [45].

14.6 Fact. Let α be inaccessible. Then there is a forcing poset $\mathbb{P}_0(\alpha)$ such that

1. $\mathbb{P}_0(\alpha)$ has cardinality α and adds an α-Suslin tree T_α.
2. If $\mathbb{P}_1(\alpha) \in V^{\mathbb{P}_0(\alpha)}$ is the forcing poset $(T_\alpha, \geq_{T_\alpha})$, then $\mathbb{P}_0(\alpha) * \dot{\mathbb{P}}_1(\alpha)$ has a dense subset isomorphic to the Cohen poset $\mathrm{Add}(\alpha, 1)$.

We assume that κ is measurable and GCH holds. We do an iteration with Easton support of length $\kappa + 1$. For $\alpha < \kappa$ we let $\dot{\mathbb{Q}}_\alpha$ name the trivial forcing unless α is inaccessible, in which case $\dot{\mathbb{Q}}_\alpha$ names $\mathrm{Add}(\alpha, 1)_{V^{\mathbb{P}_\alpha}}$. We let $\dot{\mathbb{Q}}_\kappa$ name $\mathbb{P}_0(\kappa)_{V^{\mathbb{P}_\kappa}}$ where $\mathbb{P}_0(\kappa)$ is the forcing from Fact 14.6.

As usual let G_κ be $\mathbb{P}_\kappa$-generic over V and let g_κ be $\mathbb{Q}_\kappa$-generic over $V[G_\kappa]$.

14.7 Claim. *In $V[G_{\kappa+1}]$*

- *κ is not weakly compact.*
- *κ carries a normal κ-saturated ideal.*
- *κ reflects stationary sets.*

Proof. Let T be the tree added by the $\mathbb{P}_0(\kappa)$-generic filter g_κ. Since T is a κ-Suslin tree in $V[G_{\kappa+1}]$, κ is not weakly compact in $V[G_{\kappa+1}]$.

Let H be generic over $V[G_{\kappa+1}]$ for $\mathbb{P}_1(\kappa)$, that is to say $(T, \geq_T)$. Since T is a κ-Suslin tree, H is generic for κ-c.c. (κ, ∞)-distributive forcing over $V[G_{\kappa+1}]$.

Since $\mathbb{P}_0(\kappa) * \mathbb{P}_1(\kappa)$ is isomorphic to $\mathrm{Add}(\kappa, 1)$, $V[G_{\kappa+1} * H]$ is a model obtained by forcing with $\mathrm{Add}(\alpha, 1)_{V[G_\alpha]}$ at every inaccessible $\alpha \leq \kappa$. By Remark 12.17 κ is measurable in $V[G_{\kappa+1} * H]$, and we may fix $U \in V[G_{\kappa+1} * H]$ which is a normal measure on κ. Let $\dot{U}$ be a $\mathbb{P}_1(\kappa)$-name for U.

Working in $V[G_{\kappa+1}]$ we now define an ideal I on κ by

$$X \in I \iff \Vdash_{\mathbb{P}_1(\kappa)} \kappa \setminus X \in U.$$

By Lemma 14.5 this is a normal κ-saturated ideal.

Finally let $V[G_{\kappa+1}] \models$ "S is a stationary subset of κ". κ-c.c. forcing preserves the stationarity of stationary subsets of κ and so S is stationary in $V[G_{\kappa+1} * H]$. Measurable cardinals reflect stationary sets and so there is an ordinal $\alpha < \kappa$ such that $V[G_{\kappa+1} * H] \models$ "$S \cap \alpha$ is a stationary subset of α". It follows easily that

$$V[G_{\kappa+1}] \models \text{"}S \cap \alpha \text{ is a stationary subset of } \alpha\text{"}.$$

$\dashv$

As our second example we sketch a result from the author's program of joint work [11, 14] with Džamonja and Shelah on *strong non-reflection*. The argument has the interesting feature that we are creating a strong master condition by forcing.

14.8 Definition. Let $\kappa < \lambda < \mu$ be regular cardinals. Then the *Strong Non Reflection* principle $\mathrm{SNR}(\kappa, \lambda, \mu)$ is the assertion that there is a function F from $\mu \cap \mathrm{Cof}(\kappa)$ to λ, such that for every $\delta \in \mu \cap \mathrm{Cof}(\lambda)$ there is a set C club in δ with $F \restriction (C \cap \mathrm{Cof}(\kappa))$ strictly increasing.

It is easy to see that if F witnesses $\mathrm{SNR}(\kappa, \lambda, \mu)$, $S \subseteq \mu \cap \mathrm{Cof}(\kappa)$ is stationary and we use Fodor's lemma to find stationary $T \subseteq S$ with F constant on T, then T reflects at no point of cofinality λ. The next theorem shows that this idea can be used to make fine distinctions between stationary reflection principles. The hypothesis can be improved to the existence of a weakly compact cardinal with a little more work.

14.9 Theorem. *Suppose that it is consistent that there exists a measurable cardinal. Then it is consistent that every stationary subset of* $\omega_3 \cap \mathrm{Cof}(\omega)$ *reflects to a point of cofinality* ω_2*, while at the same time every stationary subset of* $\omega_3 \cap \mathrm{Cof}(\omega_1)$ *contains a stationary set which reflects at no point of cofinality* ω_2*.*

Proof. We start with κ a measurable cardinal. Fix U a normal measure on κ and let $j : V \longrightarrow M$ be the ultrapower map. We let $\mathbb{P} = \mathrm{Col}(\omega_2, {<}\kappa)$. As we saw in Theorem 10.8 in $V^{\mathbb{P}}$ every stationary subset of $\omega_3 \cap (\mathrm{Cof}(\omega) \cup \mathrm{Cof}(\omega_1))$ reflects to a point of cofinality ω_2.

Let $\mathbb{Q}$ be the natural poset to add a witness to $\mathrm{SNR}(\omega_1, \omega_2, \omega_3)$ by initial segments. More precisely the elements of $\mathbb{Q}$ are partial functions f with domain an initial segment of $\omega_3 \cap \mathrm{Cof}(\omega_1)$ and the property that if $\alpha \leq \mathrm{dom}(f)$ and $\alpha \in \mathrm{Cof}(\omega_2)$ then there is a set C club in α with $f \restriction (C \cap \mathrm{Cof}(\omega_1))$ strictly increasing. The ordering is end-extension.

It is easy to see that $\mathbb{Q}$ is ω_2-closed and that player II wins the strategic closure game of length $\omega_2 + 1$; to see the second claim consider a strategy where player II moves as follows: at every stage $\alpha \in \omega_2 \cap \mathrm{Cof}(\omega_1)$, player II extends the existing function f_α to $f_{\alpha+1} = f_\alpha \cup \{\mathrm{dom}(f_\alpha), \alpha)\}$. In particular $\mathbb{Q}$ adds no ω_2-sequences and so preserves cardinals up to and including ω_3.

We now make a suggestive false start. As usual we factor $j(\mathbb{P}) = \mathbb{P} \times \mathbb{R}$ where $\mathbb{R}$ is an ω_2-closed forcing poset collapsing cardinals in the interval $[\kappa, j(\kappa))$. If G is $\mathbb{P}$-generic and g is $\mathbb{Q}$-generic then by Theorem 14.3 we may absorb $\mathbb{P} * \dot{\mathbb{Q}}$ into $j(\mathbb{P})$ so that the quotient is ω_2-closed, and so build an embedding $j : V[G] \longrightarrow M[G * g * h]$ where h is generic for ω_2-closed forcing.

If F is the function added by g then $F = \bigcup g = \bigcup j``g$. It is natural to try and use F as a strong master condition. Since $\mathrm{cf}(\kappa) = \omega_2$ in $M[G * g * h]$ we need to know that F is increasing on a club set to see that $F \in j(\mathbb{Q})$, but this is not immediately clear.

To resolve this problem we work in $V[G * g]$ and define a poset $\mathbb{S}$ as follows: conditions in $\mathbb{S}$ are closed bounded subsets c of κ such that $|c| \leq \omega_1$ and $F \restriction (c \cap \mathrm{Cof}(\omega_1))$ is strictly increasing. It is easy to see that $\mathbb{S}$ is countably closed in $V[G * g]$. We claim that in $V[G]$ there is a dense ω_2-closed set of conditions in $\mathbb{Q} * \dot{\mathbb{S}}$, consisting of those conditions $(f, \check{c})$ such that $\mathrm{dom}(f) = (\max(c) + 1)) \cap \mathrm{Cof}(\omega_1)$ and f is strictly increasing on $c \cap \mathrm{Cof}(\omega_1)$. The proof is routine.

We now force over $V[G * g]$ with $\mathbb{S}$ to obtain a club set $C \subseteq \kappa$ such that C has order type ω_2 and $F \restriction (C \cap \mathrm{Cof}(\omega_1))$ is increasing. Since $\mathbb{Q} * \dot{\mathbb{S}}$ has an ω_2-closed dense set we may absorb $G * g * C$ into $j(\mathbb{P})$ with an ω_2-closed quotient and then lift to obtain $j : V[G] \longrightarrow M[G * g * C * h]$ where h is generic for ω_2-closed forcing. C serves as witness that $F \in j(\mathbb{Q})$ so we may force with $j(\mathbb{Q})/F$ to obtain a generic g^+ and then lift to get $j : V[G * g] \longrightarrow M[G * g * C * h * g^+]$.

This elementary embedding exists in a generic extension of $V[G * g]$ by countably closed forcing, so exactly as in Theorem 10.8 in $V[G * g]$ every stationary set in $\kappa \cap \mathrm{Cof}(\omega)$ reflects to a point of cofinality ω_2. By construction we also have $\mathrm{SNR}(\omega_1, \omega_2, \omega_3)$ in $V[G * g]$ so we are done. $\dashv$

As a third example we sketch Magidor's proof that consistently every stationary subset of $\omega_{\omega+1}$ reflects.

14.10 Theorem. *If it is consistent that there exist ω supercompact cardinals then it is consistent that every stationary subset of $\omega_{\omega+1}$ reflects.*

Proof. We start by fixing an increasing sequence $\langle \kappa_n : 0 < n < \omega \rangle$ of supercompact cardinals. We also fix $j_n : V \longrightarrow M_n$ witnessing that κ_n is λ^+-supercompact where $\lambda = \sup_n \kappa_n$. We then define a full support iteration of length ω by setting $\mathbb{P}_1 = \mathbb{Q}_0 = \mathrm{Col}(\omega, {<}\kappa_1)$, $\mathbb{Q}_n = \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})_{V^{\mathbb{P}_n}}$ for all $n > 0$, $\mathbb{P}_{n+1} = \mathbb{P}_n * \mathbb{Q}_n$, $\mathbb{P}_\omega = \varprojlim \mathbb{P}_n$.

Let G_ω be $\mathbb{P}_\omega$ generic, let G_n be the $\mathbb{P}_n$-generic filter induced by G_ω and let g_n be the corresponding $\mathbb{Q}_n$-generic filter over $V[G_n]$. The following claims are easy:

- $\kappa_n = \omega_n$, $\lambda = \omega_\omega$, and $\lambda^+ = \omega_{\omega+1}$ in $V[G_\omega]$.
- For every n, $\mathbb{P}_\omega / G_n$ is κ_n-directed-closed in $V[G_n]$.

- For every $n > 0$ let us factor G_ω as $G_{n-1} * g_{n-1} * H_n$. Then j_n can be lifted to an elementary embedding $j_n : V[G_{n-1}] \longrightarrow M_n[G_{n-1}]$, and in $V[G_{n-1}]$ we may embed $\mathbb{P}_\omega/G_{n-1}$ into $j_n(\mathbb{Q}_n)$ so that the quotient forcing is κ_{n-1}-closed.

It follows from this discussion that we may lift j_n to an embedding with domain $V[G_\omega]$ in three steps:

1. Lift to $j_n : V[G_{n-1}] \longrightarrow M_n[G_{n-1}]$.
2. Lift to $j_n : V[G_{n-1} * g_{n-1}] \longrightarrow M_n[G_{n-1} * g_{n-1} * H_n * I_n] = M_n[j_n(G_n)]$ where I_n is generic over $V[G_\omega]$ for κ_{n-1}-closed forcing.
3. Use the closure of M_n to show that $j_n``H_n \in M_n[j_n(G_n)]$, and then use the fact that $j_n(\kappa_n) > |H_n|$ and directedness to find a suitable strong master condition r. Then force with $j_n(\mathbb{P}_\omega/G_n)/r$ and lift once more to $j_n : V[G_n * H_n] \longrightarrow M_n[j_n(G_n) * j_n(H_n)]$.

The key points are that

1. By forcing over $V[G_\omega]$ with κ_{n-1}-closed forcing we have added a generic embedding $j_n : V[G_\omega] \longrightarrow M_n[j_n(G_\omega)]$ with critical point κ_n.
2. $j_n``\lambda^+ \in M_n$.

It remains to argue that in $V[G_\omega]$ every stationary subset of λ^+ reflects. By the completeness of the club filter, every stationary set in λ^+ has a stationary subset of ordinals with a constant cofinality, so it will suffice to show that for all n any stationary subset S of $\lambda^+ \cap \mathrm{Cof}(\omega_n)$ reflects.

We consider the generic embedding $j_{n+2} : V[G_\omega] \longrightarrow M_n[j_n(G_\omega)]$ constructed above. It is easy to see that if $\gamma = \sup j_n``\lambda^+$ then $\gamma < j_n(\lambda^+)$ and $j``S \cap \gamma$ is stationary in $M_n[G_\omega]$, because the map j_{n+2} is continuous at points of cofinality κ_n. The only problem is to see that S (and hence $j``S \cap \gamma$) is still stationary in $M_n[j_n(G_\omega)]$, so it will certainly suffice to see that the stationarity of S is preserved by any ω_{n+1}-closed forcing.

Unfortunately it is not true in general [10] that κ^+-closed forcing preserves stationary subsets of $\mu \cap \mathrm{Cof}(\kappa)$ when μ is the successor of a singular cardinal. We address this problem using an idea of Shelah to show that in our model $V[G_\omega]$ every stationary subset of $\omega_{\omega+1} \cap \mathrm{Cof}(\omega_n)$ is preserved by ω_{n+1}-closed forcing.

We start by fixing in V for every $\beta < \lambda^+$ a decomposition $\beta = \bigcup_{i<\omega} b^\beta_i$ where the b^β_i are disjoint and $|b^\beta_i| \leq \kappa_i$. We define $F(\alpha, \beta)$ to be the unique $i < \omega$ with $\alpha \in b^\beta_i$. The key technical claim is that in $V[G_\omega]$ any ordinal $\rho < \lambda^+$ with uncountable cofinality contains an unbounded homogeneous set for F.

We fix such a ρ and let n be the unique integer such that in V we have $\kappa_n \leq \mathrm{cf}(\rho) < \kappa_{n+1}$. We note that if $\sigma =_{\mathrm{def}} \sup(j_n``\rho)$ then $\sigma < j(\rho)$, and so we may define in V an ultrafilter $U =_{\rm def} \{A \subseteq \rho : \sigma \in j(A)\}$. Clearly $\rho \setminus \alpha \in U$ for all $\alpha < \rho$, and U is κ_n-complete in V.

We now fix for each $\alpha \in \rho$ a U-large set A_α on which $F(\alpha, -)$ is constant. In $V[G_\omega]$ we have that $\mathrm{cf}(\rho) = \kappa_n$, and we will build by recursion an increasing and cofinal sequence $\langle \alpha_i : i < \kappa_n \rangle$ in ρ such that $\alpha_j \in A_{\alpha_i}$ for $i < j$. This is possible because $\langle A_{\alpha_i} : i < j \rangle$ is in $V[G_n]$ so is covered by a subset Y_j of U which lies in V and has size less than κ_n; the intersection of Y_j is in U, and any element of this intersection will do as α_j. It is then easy to thin out the "tail homogeneous" sequence of α_i to a cofinal homogeneous set.

To finish we show the needed stationary preservation fact. We work in $V[G_\omega]$. Let $T \subseteq \lambda^+ \cap \mathrm{Cof}(\kappa_n)$ be stationary, let $\mathbb{Q}$ be κ_{n+1}-closed, let $\dot{C}$ be $\mathbb{Q}$-name for a club subset of λ^+. We build $N \prec H_\theta$ for some large θ such that N contains all the relevant parameters, $|N| = \lambda$, all bounded subsets of λ are in N and $\delta = N \cap \lambda^+ \in T$. Fix $A \subseteq \delta$ a cofinal set of order type κ_n and $i \in \omega$ so that A is i-homogeneous for F. We claim that all proper initial segments of A lie in N: for if $\beta \in A$ then $A \cap \beta \subseteq b^\beta_i$, and since $b^\beta_i \in N$ with $|b^\beta_i| \leq \kappa_i$ and also $P(\kappa_i) \subseteq N$ we see easily that $A \cap \beta \in N$.

The endgame of the argument is now very similar to the proof of Lemma 10.6. We enumerate the elements of A in increasing order as α_i for $i < \kappa_n$. We then build a decreasing sequence $\langle q_j : j < \kappa_n \rangle$ of conditions in $\mathbb{Q} \cap N$, where q_j is the least condition which both determines $\min(\dot{C} \setminus \alpha_j)$ and is below q_i for all $i < j$. We need to see that $q_j \in N$ for all $j < \kappa_n$; the key point is that $\langle q_i : i < j \rangle$ is definable from $A \cap \alpha_j$, and so can be computed in N. To finish we choose q a lower bound for $\langle q_i : i < \kappa_n \rangle$, and observe that $q \Vdash \alpha \in \dot{C} \cap T$. $\dashv$

15. Transfer and Pullback

It is sometimes possible to transfer a generic filter over one model to another model along an elementary embedding, and then to lift that elementary embedding. The following proposition makes this precise

15.1 Proposition. *Let $k : M \longrightarrow N$ have width $\leq \mu$, and let $\mathbb{P} \in M$ be a separative notion of forcing such that*

$$M \models \text{"}\mathbb{P} \text{ is } (\mu^+, \infty)\text{-distributive"}.$$

Let G be $\mathbb{P}$-generic over M and let H be the filter on $k(\mathbb{P})$ which is generated by $k``G$. Then H is $k(\mathbb{P})$-generic over N.

Proof. Let $D \in N$ be a dense open subset of $k(\mathbb{P})$. Let $D = k(F)(a)$ for some $a \in N$ and some $F \in M$ such that $M \models |\operatorname{dom}(F)| \leq \mu$; we may as well assume that for every $x \in \operatorname{dom}(F)$, $F(x)$ is a dense open subset of $\mathbb{P}$.

Now let $E = \bigcap_{x \in \operatorname{dom}(F)} F(x)$. By the distributivity assumption E is a dense subset of $\mathbb{P}$, and clearly $E \in M$, so that $E \cap G \neq \emptyset$. If $p \in G \cap E$ then $k(p) \in k(F)(a) = D$, so that $H \cap D \neq \emptyset$ and so H is generic as claimed. $\dashv$

15.2 Remark. Given the conclusion of the last Proposition, it follows from Proposition 9.1 that k can be lifted to get $k^+ : M[G] \longrightarrow N[H]$.

As an example of this proposition in action, we prove a result reminiscent of Lemma 8.5.

15.3 Lemma. *Let* GCH *hold. Let E be a (κ, κ^{++}) extender and let the map $j : V \longrightarrow M = \mathrm{Ult}(V, E)$ be the ultrapower. Let $\mathbb{Q} = \mathrm{Col}(\kappa^{+++}, {<}j(\kappa))_M$. Then there is a filter $g \in V$ which is $\mathbb{Q}$-generic over M.*

Proof. Let $U = \{X \subseteq \kappa : \kappa \in j(X)\}$ and let $i : V \longrightarrow N = \mathrm{Ult}(V, U)$. As in Proposition 3.2 we may define an elementary embedding $k : N \longrightarrow M$ by $k([F]_U) = j(F)(\kappa)$, and $j = k \circ i$.

Let $\lambda = \kappa_N^{++}$. It is easy to see that

$$\begin{aligned} M &= \{j(F)(a) : a \in [\kappa^{++}]^{<\omega}, \mathrm{dom}(F) = [\kappa]^{|a|}\} \\ &= \{k(H)(a) : a \in [\kappa^{++}]^{<\omega}, \mathrm{dom}(H) = [\lambda]^{|a|}\} \end{aligned}$$

It follows that k is an embedding of width at most λ.

Now let $\mathbb{Q}_0 = \mathrm{Col}(\lambda^+, {<}i(\kappa))_N$, and notice that $k(\mathbb{Q}_0) = \mathbb{Q}$. By exactly the same argument as in Lemma 8.5 there is $g_0 \in V$ which is $\mathbb{Q}_0$-generic over N. By Proposition 15.1 k“g_0 generates a filter g which is $\mathbb{Q}$-generic over M. ⊣

15.4 Remark. This lemma can be used to construct posets along the lines of the generalised Prikry forcing from Example 8.6, collapsing κ to become for example ω_{ω_1}. See [8] and [27] for details.

15.5 Remark. See Sections 22 and 25 for applications of Proposition 15.1 in Reverse Easton constructions.

Proposition 15.1 admits a kind of dual in which the traffic goes the other way:

15.6 Proposition. *Let $k : M \longrightarrow N$ have critical point δ, let $\mathbb{P} \in M$ be a notion of forcing such that*

$$M \models \text{“}\mathbb{P} \textit{ is } \delta\text{-c.c.”}.$$

Let H be $k(\mathbb{P})$-generic over N and let $G = k^{-1}$“H. Then G is $\mathbb{P}$-generic over M.

Proof. Let $A \in M$ be a maximal antichain of $\mathbb{P}$. Then $k(A) = k$“A and it is maximal in $k(\mathbb{P})$, so k“A meets H and hence A meets G. It is routine to check that G is a filter. ⊣

16. Small Large Cardinals

One of the main themes of this chapter has been preservation of large cardinal axioms in forcing extensions, using the characterisation of those large cardinal axioms in terms of elementary embeddings. It might seem that this method can only work for large cardinal hypotheses at least as strong as the existence of a measurable cardinal, because after all the critical point of any definable $j : V \longrightarrow M$ is always measurable (and even the existence of a generic embedding $j : V \longrightarrow M \subseteq V[G]$ implies the existence of an inner model with a measurable cardinal).

However it turns out that we can get down to the level of weakly compact cardinals by working with elementary embeddings whose domains are sets which do not contain the full powerset of the critical point. We record a number of equivalent characterisations of weak compactness. The last one (which is due to Hauser [34]) has the surprising feature that the target model of the embedding contains the embedding itself, a fact which can be used to good effect in master condition arguments [35, 34].

16.1 Theorem. *The following are equivalent for an inaccessible cardinal κ:*

1. *κ is weakly compact.*
2. *κ is Π^1_1-indescribable.*
3. *κ has the tree property.*
4. *For every transitive set M with $|M| = \kappa$, $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$ there is an elementary embedding $j : M \longrightarrow N$ where N is transitive, $|N| = \kappa$, ${}^{<\kappa}N \subseteq N$ and $\mathrm{crit}(j) = \kappa$.*
5. *For every transitive set M with $|M| = \kappa$, $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$ there is an elementary embedding $j : M \longrightarrow N$ where N is transitive, $|N| = \kappa$, ${}^{<\kappa}N \subseteq N$, $\mathrm{crit}(j) = \kappa$ and in addition j and M are both elements of N.*

Proof. The equivalence of the first four statements is standard [43]. So we only show that the last one follows from weak compactness. Given M a transitive set with $|M| = \kappa \in M$ and ${}^{<\kappa}M \subseteq M$ we find a transitive $\bar{M}$ with the same properties so that $M \in \bar{M}$. We fix in $\bar{M}$ a well founded relation E on κ so that (κ, E) collapses to $(M, \in)$.

By weak compactness we may find an embedding $j : \bar{M} \longrightarrow \bar{N}$ with critical point κ such that $|\bar{N}| = \kappa$ and ${}^{<\kappa}\bar{N} \subseteq \bar{N}$. Let $N = j(M)$ and $i = j \restriction M$ so that $i : M \longrightarrow N$ is elementary. Since $j(E) \in \bar{N}$ it is easy to see that M and i are both in $\bar{N}$; but by elementarity N is closed under κ-sequences in $\bar{N}$ so that M and i are in N. $\dashv$

16.2 Example. We show that it is consistent for the first failure of the GCH to occur at a weakly compact cardinal. This needs a little work. For example if $V = L$ and we add κ^{++} Cohen subsets to a weakly compact cardinal κ then this destroys the weak compactness of κ. The point is that for $X \subseteq \kappa$ the statement "$X \notin L$" is Π^1_1 in $(V_\kappa, \in, X)$, so that if κ is weakly compact and $X \notin L$ then by Π^1_1-indescribability some initial segment of X is not in L.

We will assume that GCH holds in V and that κ is weakly compact. We will force with Easton support to add α many Cohen subsets to each inaccessible $\alpha < \kappa$, and will then add κ^{++} many Cohen subsets to κ. Let $\mathbb{P}_\kappa$ be the iteration up to κ and let $\mathbb{Q} = \mathrm{Add}(\kappa, \kappa^{++})_{V^{\mathbb{P}_\kappa}}$. Let G be $\mathbb{P}_\kappa$-generic over V and let g be $\mathbb{Q}$-generic over $V[G]$.

For the sake of variety we show that κ has the tree property in $V[G * g]$. Let $T \in V[G*g]$ be a κ-tree. T is essentially a subset of κ, and so by the κ^+-c.c. there is in V a set $X \subseteq \kappa^{++}$ with $|X| = \kappa$ such that $T \in V[G * g_0]$ where $g_0 = g \restriction (\kappa \times X)$. Without loss of generality we may as well assume that $X = \kappa$. So now $T \in V[G*g_0]$, where $g_0 = g \restriction (\kappa \times \kappa)$ and g_0 is $\mathbb{Q}_0 = \mathrm{Add}(\kappa, \kappa)$-generic,

Working in V we fix a suitable transitive model M such that $\dot{T} \in M$, and then choose $j : M \longrightarrow N$ as in article 5 of Theorem 16.1. We now proceed to lift j. We need to be slightly careful about issues of closure. Our models are less closed than in the context of measurable cardinals, but since they are themselves small sets this is not a problem.

Since $\mathbb{P}_\kappa$ is κ-c.c. and $V \models {}^{<\kappa}N \subseteq N$, we have by Proposition 8.4 that $V[G] \models {}^{<\kappa}N[G] \subseteq N[G]$. $\mathbb{Q}_0$ adds no $<\kappa$-sequences so by Proposition 8.2 $V[G * g_0] \models {}^{<\kappa}N[G * g_0] \subseteq N[G * g_0]$. Since $|N[G * g_0]| = \kappa$ and the factor iteration $j(\mathbb{P}_\kappa)/G * g_0$ is $<\kappa$-closed in $V[G * g_0]$, we may as usual build $H \in V[G * g_0]$ suitably generic and lift to get $j : M[G] \longrightarrow N[G * g_0 * H]$. As usual $V[G*g_0] \models {}^{<\kappa}N[G*g_0*H] \subseteq N[G*g_0*H]$. Finally since $j \restriction g_0 = \mathrm{id}$ we may use $r = \bigcup g_0$ as a strong master condition, construct a suitable generic filter for $j(\mathbb{Q}_0)/R$ and lift the embedding onto $M[G * g_0]$. Since $j(T) \restriction \kappa = T$, we may use any point on level κ of $j(T)$ to generate a cofinal branch of T lying in $V[G * g_0]$.

17. Precipitous Ideals I

In this section we prove some theorems about precipitous ideals due to Jech, Magidor, Mitchell and Prikry [40]. As a warmup we show it is consistent that there exists a precipitous ideal (precipitousness is defined below) on ω_1, then we show that the non-stationary ideal on ω_1 can be precipitous. The hypothesis used is the existence of a measurable cardinal, which is known [40] to be optimal.

The proof has several very interesting technical features including:

- The use of the universal properties of the Lévy collapsing poset, an idea which goes back to Solovay's proof that every set of reals can be measurable [65].

- The use of forcing to add simultaneously filters G and H such that an embedding $M \longrightarrow N$ lifts to an embedding $M[G] \longrightarrow N[H]$.
- The use of an iterated club-shooting forcing to make the club filter exhibit properties that are characteristic of filters derived from elementary embeddings.

17.1. A Precipitous Ideal on ω_1

We refer readers to Foreman's chapter in this Handbook for the basic theory of precipitous ideals. We recall that if I is an ideal on κ then we may force with the forcing poset $P(\kappa)/I \setminus \{0\}$ (equivalence classes of I-positive sets modulo I) to add a V-ultrafilter U such that $U \cap I = \emptyset$. Working in $V[U]$ we may then form $\mathrm{Ult}(V, U)$ using functions in V ordered modulo U. The ideal I is said to be *precipitous* if and only if $\mathrm{Ult}(V, U)$ is forced to be well-founded. We will follow a common practice and abuse notation by saying that the ultrafilter U is "$P(\kappa)/I$-generic".

The following fact is key for us: to show that an ideal I on a cardinal κ is precipitous, it suffices to produce (typically by forcing) for every $A \notin I$ a V-ultrafilter U on κ such that $A \in U$, U is $P(\kappa)/I$-generic, and $\mathrm{Ult}(V, U)$ is well-founded. The point is that if I fails to be precipitous there is $A \notin I$ which forces this, and for such an A no U as above can exist.

We will reuse an example from earlier in this chapter. Assume that κ is measurable, and let $j : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map from a normal measure U on κ. Let $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa)$. Then as we saw in Theorem 10.2:

1. $j(\mathbb{P})$ is isomorphic to $\mathbb{P} \times \mathbb{Q}$ where $\mathbb{Q}$ is the poset which adds a surjection from ω onto each ordinal in $[\kappa, j(\kappa))$ with finite conditions. We will usually be careless and identify the posets $j(\mathbb{P})$ and $\mathbb{P} \times \mathbb{Q}$.
2. If G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over $V[G]$ then $G * H$ is $j(\mathbb{P})$-generic over V, and $j``G \subseteq G * H$, so in $V[G * H]$ we can lift our original j to $j_G : V[G] \longrightarrow M[G * H]$ with $j_G(G) = G * H$. So from the point of view of $V[G]$ the embedding j_G is a generic embedding added by forcing with $\mathbb{Q}$.
3. Since $M = \{j(f)(\kappa) : f \in V\}$ we have that

$$M[G * H] = \{j_G(f)(\kappa) : f \in V[G]\},$$

so that $M[G * H]$ is the ultrapower $\mathrm{Ult}(V[G], U_G)$ where

$$U_G = \{X \in P(\kappa) \cap V[G] : \kappa \in j_G(X)\}$$

Here U_G is a $V[G]$-normal $V[G]$-κ-complete $V[G]$-ultrafilter and j_G is the associated ultrapower map.

Foreman's chapter in this Handbook gives a rather general framework for defining precipitous ideals by way of generic elementary embeddings. In the interests of being self-contained, we describe how this plays out in the setting of the embedding from Theorem 10.2.

We caution the reader that the following arguments involve viewing the universe $V[G * H]$ both as an extension of V by $j(\mathbb{P})$ and an extension of $V[G]$ by $\mathbb{Q}$. We are quietly identifying $j(\mathbb{P})$-names in V with $\mathbb{Q}$-names in $V[G]$, resolving any possible confusion by making explicit which model we are forcing over and with which poset.

Working in $V[G]$ we define an ideal I on $\omega_1^{V[G]} (= \kappa)$ by

$$I = \{X \subseteq \kappa : \Vdash_{\mathbb{Q}}^{V[G]} \kappa \notin j_G(X)\}.$$

Equivalently I consists of those sets which are forced by $\mathbb{Q}$ not to be in the ultrafilter U_G.

Working in $V[G]$ we define a Boolean algebra homomorphism from $P(\kappa)$ to $\mathrm{ro}(\mathbb{Q})$ which maps X to the truth value $[X \in \dot{U}_G]_{\mathrm{ro}(\mathbb{Q})}$. The kernel of this map is exactly I so we may induce a map ι from $P(\kappa)/I$ to $\mathrm{ro}(\mathbb{Q})$.

The key point is that, as we see in a moment, the range of ι is dense. From this it follows that for any H which is $\mathbb{Q}$-generic over $V[G]$, the ultrafilter U_G is $P(\kappa)/I$-generic over $V[G]$; in fact it follows from the truth lemma that $X \in U_G \iff \iota(X) \in H$, so that in a very explicit way forcing with $\mathbb{Q}$ is equivalent to forcing with $P(\kappa)/I$.

To establish that the range of ι is dense recall that $\mathbb{Q}$ is densely embedded in $\mathrm{ro}(\mathbb{Q})$. Let $q \in \mathbb{Q}$, so that $q = j(F)(\kappa)$ where $F \in V$ is a function such that $F(\alpha) \in \mathrm{Col}(\omega, [\alpha, \kappa))$ for all $\alpha < \kappa$. Working in $V[G]$, define $X = \{\alpha : F(\alpha) \in G\}$. Since j_G extends j, $q = j_G(F)(\kappa)$ and so for any H we have that

$$\iota(X) \in H \iff \kappa \in j_G(X) \iff q \in j_G(G) \iff q \in H.$$

It is an immediate conclusion that I must be precipitous. For if $A \notin I$ then we may choose H inducing j_G such that $\kappa \in j_G(A)$. Arguing as above we get that U_G is $P(\kappa)/I$-generic with $A \in U_G$, so we are done.

It is interesting to note that the ideal I is precisely the ideal generated in $V[G]$ by the ideal dual to the ultrafilter U. It is immediate that I contains this ideal, so we only need to prove that I is contained in this ideal.

Let $p \Vdash_{\mathbb{P}}^{V} \dot{X} \in \dot{I}$. We claim that if we define $A = \{\alpha : p \Vdash \alpha \notin \dot{X}\}$ then $A \in U$. For if not then we may define a function F on A^c such that $F(\alpha) \leq p$ and $F(\alpha) \Vdash \alpha \in \dot{X}$. But then if we let $q = j(F)(\kappa)$ and force to get $G * H$ containing q, we obtain a situation in which $p \in G$ and yet $\kappa \in j_G(X)$, so that $X \in U_G$ and we have a contradiction.

17.1 Remark. Really we have just worked through a very special case of Foreman's Duality Theorem. See Foreman's chapter in this Handbook for more on this subject.

17.2. Iterated Club Shooting

In subsection 17.1 we produced a precipitous ideal I on ω_1. It is not hard to see that this ideal is not the non-stationary ideal. For example if in V we define $S = \kappa \cap cof(\omega)$, then S is stationary in $V[G]$ by the κ-c.c. of the collapsing poset $\mathbb{P}$. Since $\kappa \notin j(S)$ we see that $\Vdash_{\mathbb{Q}} \check{S} \notin \dot{U}_G$, so that $S \in I$.

To make the non-stationary ideal precipitous, we will iteratively shoot clubs so as to destroy the stationarity of inconvenient sets such as the set S from the last paragraph. The argument is somewhat technical so we give an overview before launching into the details.

Overview

Working in $V[G]$ we build a countable support iteration $\mathbb{R}$ of length κ^+ (that is the ω_2 of $V[G]$). At each stage we shoot a club set through some stationary subset of ω_1. A key point will be that this iteration adds no ω-sequences of ordinals; from this it will follow that

1. ω_1 is preserved.
2. At each stage of the iteration $\mathbb{R}$, the conditions in the club shooting forcing used at that stage (which are closed and bounded subsets of ω_1) actually lie in the model $V[G]$.

Recall from subsection 17.1 that $\mathbb{P} = Coll(\omega, < \kappa)$, $j(\mathbb{P}) = \mathbb{P} \times \mathbb{Q}$, and for any H which is $\mathbb{Q}$-generic over $V[G]$ we may lift $j : V \to M$ and get $j_G : V[G] \to M[G * H]$. It is the existence of these generic embeddings which is responsible for the precipitousness of I in $V[G]$.

For each α we will find an embedding of $\mathbb{P} * \mathbb{R}_\alpha$ into $j(\mathbb{P})$, and we will use this to produce generic embeddings $j_\alpha : V[G * g_\alpha] \to M[G * H * h_\alpha]$ where g_α is $\mathbb{R}_\alpha$-generic and $G * g_\alpha$ is embedded into $G * H$. From these embeddings j_α we will define normal ideals $I_\alpha \in V[G * g_\alpha]$ which are analogous to the ideal I; the construction will be organised so that

1. I_α increases with α.
2. In the final model $\bigcup_\alpha I_\alpha$ is the non-stationary ideal.

To finish we will use the embeddings j_α to show that the non-stationary ideal is precipitous. This argument, which is similar to but more complicated than that of subsection 17.1, appears as Lemma 17.4 below.

Details

Working in $V[G]$ we construct by induction on $\alpha < \kappa^+$

1. A countable support iteration $\mathbb{R}_\alpha$.

2. An embedding $i_\alpha : \mathbb{P} * \mathbb{R}_\alpha \to j(\mathbb{P})$, extending the identity embedding of $\mathbb{P}$ into $j(\mathbb{P})$. Our convention in what follows is that H is always some $\mathbb{Q}$-generic filter over $V[G]$ and g_α is always the $\mathbb{R}_\alpha$-generic filter induced by H via the embedding i_α.

3. A $\mathbb{Q}$-name $\dot{D}_\alpha$ for a strong master condition appropriate for the embedding $j_G : V[G] \to M[G * H]$ and generic object g_α; that is to say $\dot{D}_\alpha$ names a condition in $j(\mathbb{R}_\alpha)$ which is a lower bound for $j``g_\alpha$.

4. A $\mathbb{Q} * j(\mathbb{R}_\alpha)/D_\alpha$-name for $j_\alpha : V[G * g_\alpha] \to M[G * H * h_\alpha]$ extending $j_G : V[G] \to M[G * H]$.

5. An $\mathbb{R}_\alpha$-name $\dot{I}_\alpha$ for a normal ideal on κ; this is defined as a *master condition ideal* of the sort discussed in Foreman's chapter of this handbook, to be more precise it is defined to be the set of those $X \subseteq \kappa$ in $V[G*g_\alpha]$ such that it is forced over $V[G * g_\alpha]$ by $j(\mathbb{P})/(G * g_\alpha) * j(\mathbb{R}_\alpha)/D_\alpha$ that $\kappa \notin j_\alpha(X)$.

6. An $\mathbb{R}_\alpha$-name $\dot{S}_\alpha$ for a set in the filter dual to the ideal I_α.

We will maintain the hypotheses that

1. $\mathbb{R}_\alpha$ adds no ω-sequences of ordinals and has the κ^+-c.c.

2. For $\beta < \gamma \leq \alpha$
   (a) i_γ extends i_β (from which it follows that g_γ extends g_β).
   (b) It is forced over $V[G]$ (by the appropriate forcing posets) that $D_\gamma \restriction j(\beta) = D_\beta$, $j_\gamma \restriction V[G * g_\beta] = j_\beta$, and $I_\gamma \cap V[G_\beta] = I_\beta$.

3. The set of *flat* conditions is dense in $\mathbb{R}_\alpha$, where a condition r in $\mathbb{R}_\alpha$ is *flat* if
   (a) For every η in the support of r, $r(\eta)$ is a canonical $\mathbb{R}_\eta$-name $\check{d}_\eta$ for some $d_\eta \in V[G]$, where d_η is a closed and bounded subset of κ.
   (b) There is an ordinal $\gamma < \kappa$ such that $\gamma = \max(d_\eta)$ for every η in the support of p.

We will explain why some of the hypotheses are maintained and then give the details of the construction. Since D_α is a strong master condition for $\mathbb{R}_\alpha$ it follows from Theorem 12.5 that forcing with $\mathbb{R}_\alpha$ adds no ω-sequences of ordinals. As we see shortly D_α is flat, and since $D_\alpha \leq j``g_\alpha$ it follows from elementarity that the set of flat conditions is dense. Standard Δ-system arguments show that the set of flat conditions has the κ^+-c.c. and so since this set is dense $\mathbb{R}_\alpha$ has the κ^+-c.c. The remaining "coherence" hypotheses will be satisfied by construction.

- At successor stages we take $\mathbb{R}_{\alpha+1} \simeq \mathbb{R}_\alpha * CU(\kappa, S_\alpha)$. For α limit, $\mathbb{R}_\alpha$ is constructed as the direct limit of $\langle \mathbb{R}_\beta : \beta < \alpha \rangle$ if α has uncountable cofinality and the inverse limit if α has countable cofinality.

- i_α is defined using the universal properties of the Lévy collapse as in Theorem 14.2.

- Recall that j_α exists in the extension of $V[G * g_\alpha]$ by the forcing poset $j(\mathbb{P})/(G * g_\alpha) * j(\mathbb{R}_\alpha)/D_\alpha$. Working in $V[G * g_\alpha]$ we define I_α to be the ideal of those $X \subseteq \kappa$ such that it is forced over $V[G * g_\alpha]$ by $j(\mathbb{P})/(G * g_\alpha) * j(\mathbb{R}_\alpha)/D_\alpha$ that $\kappa \notin j_\alpha(X)$.

- Let $\langle C_\eta : \eta < \alpha \rangle$ be the sequence of club sets added by g_α. We construct D_α as follows: the support of D_α is $j``\alpha$, and $D_\alpha(j(\eta))$ is the canonical $j(\mathbb{R}_\eta)$-name for $C_\eta \cup \{\kappa\}$.

  Of course we need to check that D_α is a strong master condition. The salient points are that

  - Since $\alpha < \kappa^+$, $j``\alpha \in M$.
  - Since $\kappa^+ < j(\kappa)$, $j``\alpha$ is countable in $M[G * H]$. In particular D_α has countable support.
  - If $f \in g_\alpha$ then $f \in V[G]$ and the support of f has size less than $\mathrm{crit}(j_G)$. Hence the support of $j_G(f)$ is $j``\mathrm{dom}(f)$.
  - For every $\beta < \alpha$, $D_\beta = D_\alpha \restriction j(\beta)$ is a lower bound for $j``g_\beta$. In particular it is immediate for α limit that D_α is a strong master condition, so we may concentrate on the case when $\alpha = \beta + 1$.

    Recall that by induction $\mathbb{P}_\beta$ adds no ω-sequences of ordinals. Let $r \in g_\alpha$, then it follows from the distributivity of $\mathbb{P}_\beta$ that we may write $r = r_0 \frown r_1$ where $r_0 \in g_\beta$ and (without loss of generality) r_1 is the canonical name for $C_\beta \cap (\eta + 1)$ where $\eta \in C_\beta$. By induction $D_\beta \leq j(r_0)$. By the distributivity of $\mathbb{P}_\beta$ again, every initial segment of C_β is in $V[G]$, and is fixed by j_G. So $D_\beta (= D_\alpha \restriction j(\beta))$ forces that $D_\alpha(j(\beta))$ is an end-extension of $j(r_1)$.

    It only remains to check that D_β forces that $C_\beta \cup \{\kappa\}$ is a legitimate condition in $j_\beta(CU(\kappa, S_\beta))$. But this is immediate because S_β was chosen to lie in the filter dual to I_β.

- D_α is flat. This is straightforward: we have already checked that it is a condition, and by construction each entry is a canonical name for a closed set of ordinals with maximum element κ.

- j_α is defined in the standard way as a lifting of $j_G : V[G] \to M[G * H]$. The fact that D_α is a strong master condition ensures the necessary compatibility of generic filters.

- The sets S_α are chosen according to a suitable book-keeping scheme so that after κ^+ steps it is forced that every element of $\bigcup_\alpha I_\alpha$ has become non-stationary.

17.2 Remark. The trick of using a dense subset of conditions which are "flat" (in a suitable sense) is very often useful in situations when we are iteratively shooting clubs through stationary sets. It would have been tempting to define $\mathbb{P}_\alpha$ as the set of flat conditions with a suitable ordering, but this raises problems of its own; in particular we would have needed to verify that the flat poset is an iteration of club shooting forcing, which amounts to showing that flat conditions are dense.

In several subsequent arguments that involve iterated club shooting we have cheated (in a harmless way) by just defining the set of flat conditions, and leaving it to the reader to check that this set is dense in the corresponding iteration. See in particular Lemma 17.4 and the results of Sections 18 and 19.

17.3 Remark. The ideals I_α are *master condition ideals* of the sort discussed at some length in Foreman's chapter.

Precipitousness

It remains to see that the non-stationary ideal is precipitous in $V[G * g_{\kappa^+}]$. The argument runs parallel to that for the precipitous ideal in the preceding section, but is harder because we now need a generic elementary embedding with domain $V[G * g_{\kappa^+}]$. The main technical difficulties are that

1. $j``\kappa^+ \notin M$, indeed it is cofinal in $j(\kappa^+)$, so that we cannot hope to cover it by any countable set in $M[G * H]$. So there is no chance of building a strong master condition.
2. The method for doing without a strong master condition which we described in Section 13 uses a reasonably large amount of closure but $\mathbb{P}_{\kappa^+}$ is not even countably closed.
3. $\mathbb{P}_{\kappa^+}$ is not sufficiently distributive to transfer a generic object as in Section 15, nor does it obey a strong enough chain condition to pull back a generic object as in Proposition 15.6.

Since we will use the same set of ideas again in Section 18 when we build a model in which NS_{ω_2} is precipitous, we state a rather general lemma about constructing precipitous ideals by iterated club-shooting. This is really just an abstraction of an argument from [40]. In the applications which we are making of this lemma the preparation forcing $\mathbb{P}$ will make κ into the successor of some regular $\delta < \kappa$.

17.4 Lemma. *Suppose that κ is measurable and $2^\kappa = \kappa^+$. Let U be a normal measure on κ and let $j : V \longrightarrow M$ be the associated ultrapower map. Let $\mathbb{P}$ be a κ-c.c. poset with $\mathbb{P} \subseteq V_\kappa$. As usual $\mathbb{P}$ is completely embedded in $j(\mathbb{P})$, so that if G is $\mathbb{P}$-generic and H is $j(\mathbb{P})/G$-generic then $j : V \longrightarrow M$ can be lifted to an elementary embedding $j_G : V[G] \longrightarrow M[G * H]$.*

Let $\langle \dot{\mathbb{Q}}_\alpha : \alpha \le \kappa^+ \rangle$ *be an* $\mathbb{P}$*-name for a sequence of forcing posets such that in* $V[G]$

1. $\mathbb{Q}_\alpha$ *is a complete subposet of* $\mathbb{Q}_\beta$ *for* $\alpha \le \beta \le \kappa^+$.
2. *Forcing with* $\mathbb{Q}_\alpha$ *adds no* $< \kappa$*-sequences of ordinals.*
3. $\mathbb{Q}_{\kappa^+}$ *is* κ^+*-c.c.*
4. *Every condition in* $\mathbb{Q}_\alpha$ *is a partial function* q *such that* $\mathrm{dom}(q) \subseteq \alpha$, $|\mathrm{dom}(q)| < \kappa$ *and* $q(\eta)$ *is a closed and bounded subset of* κ *for all* $\eta \in \mathrm{dom}(q)$.
5. *If* g *is a* $\mathbb{Q}_\alpha$*-generic filter then* $\bigcup g$ *is a sequence* $\langle C_\beta : \beta < \alpha \rangle$ *of club subsets of* κ.

Suppose further that there are sequences $\langle i_\alpha : \alpha < \kappa^+ \rangle$ *and* $\langle \dot{D}_\alpha : \alpha < \kappa^+ \rangle$ *such that*

1. i_α *is a complete embedding of* $\mathbb{P} * \dot{\mathbb{Q}}_\alpha$ *into* $j(\mathbb{P})$*, with* $i_0 = \mathrm{id}$.
2. i_β *extends* i_α *for* $\alpha \le \beta \le \kappa^+$.
3. $\dot{D}_\alpha$ *is a* $j(\mathbb{P})/\mathbb{P} * \dot{\mathbb{Q}}_\alpha$*-name for a condition* $Q \in j_G(\mathbb{Q}_\alpha)$ *such that* $\mathrm{dom}(Q) = j``\alpha$*, and for every* $\eta \in j``\alpha$ $Q(\eta) = C_\eta \cup \{\kappa\}$*, where* $\langle C_\beta : \beta < \alpha \rangle$ *is the sequence of club subsets of* κ *added by* $\mathbb{Q}_\alpha$.
4. *It is forced that* $\dot{D}_\beta$ *extends* $\dot{D}_\alpha$ *for* $\alpha \le \beta \le \kappa^+$.

Let G *be* $\mathbb{P}$*-generic over* V*, and let* H *be* $j(\mathbb{P})/G$*-generic over* $V[G]$*. For each* $\alpha < \kappa^+$*, let* g_α *be the filter on* $\mathbb{Q}_\alpha$ *induced by* i_α *and* H *(so that* g_α *is* $V[G]$*-generic).*

For each $\nu < \kappa^+$ *the hypotheses above imply that it is forced over* $V[G*g_\nu]$ *by* $j(\mathbb{P})/G * g_\nu * j_G(\mathbb{Q}_\nu)/D_\nu$ *that* j_G *can be extended to a generic embedding* j_ν *with domain* $V[G * g_\nu]$*. Let* $J_\nu \in V[G * g_\nu]$ *be the ideal of those* $X \subseteq \kappa$ *such that it is forced that* $\kappa \notin j_\nu(X)$*. Let* $g = \bigcup_\nu g_\nu$ *and* $J = \bigcup_\nu J_\nu$.

Then

1. g *is* $\mathbb{Q}_{\kappa^+}$*-generic over* $V[G]$.
2. J *is precipitous in* $V[G * g]$.

17.5 Remark. In the intended applications, J will end up being the non-stationary ideal. However it may be (as will be the case in Section 18) that this is accomplished by a more sophisticated strategy than arranging that every $A \in J$ is disjoint from some C_i.

Proof. The induced filters g_ν are compatible in the sense that if $\rho \le \sigma$ then $g_\rho = g_\sigma \cap \mathbb{Q}_\rho$. It follows easily from the κ^+-c.c that if we set $g = \bigcup_\rho g_\rho$ then g is $\mathbb{Q}_{\kappa^+}$-generic. Let D_ρ be the strong master condition computed from g_ρ.

We will construct K which is $j_G(\mathbb{Q}_{\kappa^+})$-generic over $M[G*H]$, and is compatible with g in the sense that $j_G``g \subseteq K$. We do this as follows: define $\mathbb{Q}^*$ to be the subset of $j_G(\mathbb{Q}_{\kappa^+})$ consisting of those F such that for some $\mu < \kappa^+$ we have $F \in j_G(\mathbb{Q}_\mu)/D_\mu$. Of course $\mathbb{Q}^* \subseteq M[G*H]$, but $\mathbb{Q}^* \notin M[G*H]$ since its definition requires a knowledge of $j \restriction \kappa^+$.

We force to get K_0 which is $\mathbb{Q}^*$-generic over $V[G*H]$. Let K be the upwards closure of K_0 in $j_G(\mathbb{Q}_{\kappa^+})$, then we claim that K is $j_G(\mathbb{Q}_{\kappa^+})$-generic over $M[G*H]$. Let $A \in M[G*H]$ be a maximal antichain in $j_G(\mathbb{Q}_{\kappa^+})$. We will show that conditions extending some element of A are dense in $\mathbb{Q}^*$. Let $F \in \mathbb{Q}^*$, and fix μ such that $F \in j_G(\mathbb{Q}_\mu)/D_\mu$. By a familiar chain condition argument we may fix $\rho > \mu$ such that A is a maximal antichain in $j_G(\mathbb{Q}_\rho)$. Working in $M[G*H]$ we first extend F to $F' = F \cup D_\rho \in j_G(\mathbb{Q}_\rho)/D_\rho$, and then extend F' to a condition $F'' \in j(\mathbb{Q}_\rho)$ which also extends some member of A.

In the usual way we may now extend $j_G : V[G] \longrightarrow M[G*H]$ to an embedding $j^* : V[G*g] \longrightarrow M[G*H*K]$. From the point of view of the model $V[G*g]$ this is a generic embedding added by forcing with $j(\mathbb{P})/(G*g)*\mathbb{Q}^*$. It is routine to check that J is the ideal on κ induced by this generic embedding; more explicitly for every $X \in V[G*g]$

$$X \in J \iff \Vdash^{V[G*g]}_{j(\mathbb{P})/(G*g)*\mathbb{Q}^*} \kappa \notin j^*(X).$$

For any $X \notin J$ we may therefore force to obtain some embedding j^* such that $\kappa \in j(X)$. To finish it will suffice to show that for any generic embedding j^* as above, if we define a $V[G*g]$ -ultrafilter U^* by

$$U^* = \{Y : \kappa \in j^*(Y)\}$$

then U^* is $P(\kappa)/J$-generic over $V[G*g]$ and gives a well-founded ultrapower. Since as in the last section we have $\mathrm{Ult}(V[G*g], U^*) = M[G*H*K]$, the well-foundedness is immediate.

The argument that U^* is $P(\kappa)/J$-generic is similar to that from the last section but there are some extra subtleties. We note in particular that the embedding j^* is defined in a generic extension of V by $j(\mathbb{P}) * \mathbb{Q}^*$, but is a lifting of j to a map from the extension of V by $\mathbb{P} * \mathbb{Q}_{\kappa^+}$ to the extension of M by $j(\mathbb{P}) * j(\mathbb{Q}_{\kappa^+})$.

Let $\dot{\mathcal{A}}$ be a $\mathbb{P} * \mathbb{Q}_{\kappa^+}$-name for a maximal antichain of J-positive sets and suppose towards a contradiction that some condition $(P, \dot{F}) \in j(\mathbb{P})*\mathbb{Q}^*$ forces that for every $B \in \mathcal{A}$, $\kappa \notin j^*(B)$. We will find a J-positive set T which has J-small intersection with every $B \in \mathcal{A}$, contradicting the maximality of $\mathcal{A}$.

Since $\mathbb{Q}^* \subseteq j(\mathbb{P}_{\kappa^+})$ we see that $(P, \dot{F}) \in j(\mathbb{P} * \mathbb{Q}_{\kappa^+})$, and so we may choose in V a function $R : \kappa \to \mathbb{P} * \mathbb{Q}_{\kappa^+}$ such that $j(R)(\kappa) = (P, \dot{F})$. Let $\dot{T}$ name the set $\{\alpha : R(\alpha) \in G*g\}$, then it is easy to see that is forced by $j(\mathbb{P}) * \mathbb{Q}^*$ that for every $B \in \mathcal{A}$, $\kappa \notin j^*(B \cap T)$; the key point is that by construction $\kappa \in j^*(T) \iff (P, F) \in G*H*K$.

We now force to get $G * H$ which is $j(\mathbb{P})$-generic over V with $P \in G * H$, and from this we obtain as usual g which is $\mathbb{Q}_{\kappa^+}$-generic over $V[G]$. Moving to $V[G * g]$ and using the fact that J is the ideal induced by j^*, we see that $B \cap T \in J$ for all $B \in \mathcal{A}$.

To finish the argument we show that $T \notin J$. By forcing over $V[G*H]$ with $\mathbb{Q}^*/F$ we obtain an elementary embedding $j^* : V[G * g] \longrightarrow M[G * H * K]$ where $(P, F) \in G * H * K$, so that $\kappa \in j^*(T)$ by the construction of R and T. Since J is the ideal induced by j^*, $T \notin J$ and we are done. ⊣

17.6 Remark. The technique used in this lemma is discussed in a more general and abstract setting in Foreman's chapter of this Handbook.

18. Precipitous Ideals II

In this section we discuss some work of Moti Gitik in which he obtains various results of the form "$\mathrm{NS}_\kappa \restriction \mathrm{Cof}(\mu)$ can be precipitous" from hypotheses which are optimal or close to optimal. We will describe in some detail the proof of

18.1 Theorem (Gitik [23]). *The precipitousness of* NS_{ω_2} *is equiconsistent with the existence of a cardinal of Mitchell order two.*

We will then give a much less detailed discussion of some of Gitik's consistency and equiconsistency results for cardinals greater than ω_2, which use many of the same ideas. Throughout this section, we will be using the general machinery of Lemma 17.4 to construct precipitous ideals. We will focus on the technical problems that need to be overcome to invoke this machinery, and on their solutions.

18.2 Remark. As discussed in Foreman's chapter, the simplest known model [20] for the precipitousness of NS_κ is obtained by taking a Woodin cardinal $\delta > \kappa$ and forcing with $\mathrm{Col}(\kappa, {<}\delta)$. The point here is to use the optimal hypotheses, which turn out to be much weaker.

18.3 Remark. We note that NS_{ω_2} is precipitous if and only if both of the restrictions $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega)$ and $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$ are precipitous.

18.1. A Lower Bound

We start by sketching a proof of a lower bound for the strength of "NS_{ω_2} is precipitous". Suppose for a contradiction that NS_{ω_2} is precipitous and there is no inner model with a cardinal κ such that $o(\kappa) = 2$, and let K be the core model for sequences of measures constructed by Mitchell [59]. Let $\lambda = \omega_2^V$ and let $\mathcal{F}$ be the measure sequence of K. We recall the key facts that K is definable and invariant under set forcing, and that any elementary $i : K \longrightarrow N \subseteq V$ is an iterated ultrapower of K by $\mathcal{F}$.

By precipitousness of $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$, we may force to get a V-ultrafilter U which concentrates on ordinals of cofinality ω_1 and has $M = \mathrm{Ult}(V, U)$

wellfounded. Let $j : V \longrightarrow M \subseteq V[U]$ be the ultrapower map. By the usual arguments $\mathrm{crit}(j) = \lambda = [\mathrm{id}]_U$, $P(\lambda)^V \subseteq M$, and $\mathrm{cf}^M(\lambda) = \omega_1^M = \omega_1^V$. Note also that if A is an ω-club subset of λ in V, then the same is true of A in M.

Let $i = j \restriction K$, then by the properties of K mentioned above we know that $i : K \longrightarrow K' = K^M$ and i is an iterated ultrapower of K with critical point λ. In particular λ is measurable in K, and so $\mathcal{F}(\lambda, 0)$ exists. By our initial hypotheses λ is not measurable in K', and so in K the only measure on λ is $\mathcal{F}(\lambda, 0)$. Note also that $P(\lambda)^K = P(\lambda)^{K'}$

Let C be the ω-club filter on λ as computed in V, let $A \in \mathcal{F}(\lambda, 0)$ and let W be an arbitrary V-generic ultrafilter added by forcing with C-positive sets. Then by precipitousness of $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega)$, we get an elementary embedding $j_W : V \longrightarrow \mathrm{Ult}(V, W)$, and if $i_W = j_W \restriction K$ then i_W is an iterated ultrapower of K with critical point λ; since in K the only measure on λ is $\mathcal{F}(\lambda, 0)$, we see that $\kappa \in j_W(A)$, that is $A \in W$. Since it is forced that $A \in W$, we have that $A \in C$. So $\mathcal{F}(\lambda, 0) \subseteq C \cap K$, and since the left hand side is a K-ultrafilter in fact $\mathcal{F}(\lambda, 0) = C \cap K$.

Now let D be the ω-club filter on λ as computed in M. This makes sense because $\mathrm{cf}^M(\lambda) = \omega_1$. We know that $C \subseteq D$ and $P(\lambda)^K = P(\lambda)^{K'}$, so easily $\mathcal{F}(\lambda, 0) = D \cap K'$. Since D is a countably complete filter in M we see that $M' = \mathrm{Ult}(K', \mathcal{F}(\lambda, 0))$ is wellfounded and we get in M an elementary embedding $j' : K' \longrightarrow M'$ with critical point λ; since $K' = K^M$ this is an iteration of K', but that is impossible because λ is not measurable in K'.

18.2. Precipitousness for $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$

We have established that if NS_{ω_2} is precipitous then there is an inner model with a cardinal κ such that $o(\kappa) = 2$. We will prove that this is an equiconsistency, but before we do that we warm up with a sketch of the easier argument that starting from a measurable cardinal $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$ can be precipitous [40].

We have already introduced in Section 17 most of the ideas needed to show that $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$ can be precipitous. What is still missing is a discussion of how we should shoot club sets through stationary subsets of ω_2. The arguments of Lemmas 18.5 and 18.6 are due to Stavi (see [3]).

As we saw in Section 6 if S is a stationary subset of ω_1 then it is possible to add a club set C with $C \subseteq S$, using a forcing poset which does not add any ω-sequences of ordinals. Suppose now that instead S is a stationary subset of ω_2. In general we may not be able to shoot a club set through S without collapsing cardinals, for example if $S = \omega_2 \cap \mathrm{Cof}(\omega_1)$.

In a way the rather trivial example from the last paragraph is misleading. If we aim to make $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$ precipitous then we need to take a stationary $S \subseteq \omega_2 \cap \mathrm{Cof}(\omega_1)$ and add, without collapsing ω_1 or ω_2, a club subset C of ω_2 such that $C \cap \mathrm{Cof}(\omega_1) \subseteq S$. This is fairly easy.

We recall that $\mathrm{CU}(\delta, A)$ is the forcing poset whose conditions are closed bounded subsets of δ which are contained in A, ordered by end-extension. We will need a technical lemma on the existence of countably closed structures.

18.4 Lemma. *Let* CH *hold and let* $S \subseteq \omega_2 \cap \mathrm{Cof}(\omega_1)$ *be stationary. Let* θ *be a large regular cardinal and let* $x \in H_\theta$*. Then there exists* $N \prec H_\theta$ *such that* $\omega_1 \cup \{x\} \subseteq N$*,* $|N| = \omega_1$*,* ${}^\omega N \subseteq N$ *and* $N \cap \omega_2 \in S$*.*

Proof. We build an increasing and continuous chain $\langle N_j : j < \omega_2 \rangle$ such that $N_j \prec H_\theta$, $\omega_1 \cup \{x\} \subseteq N_0$, $|N_j| = \omega_1$ and ${}^\omega N_j \subseteq N_{j+1}$. Since $\omega_1 \subseteq N_j$ we see that $N_j \cap \omega_2 \in \omega_2$, and so by continuity and the stationarity of S we may choose j such that $cf(j) = \omega_1$ and $N_j \cap \omega_2 \in S$; it is easy to see that ${}^\omega N_j \subseteq N_j$. ⊣

18.5 Lemma. *Let* CH *hold, and let* $S \subseteq \omega_2 \cap \mathrm{Cof}(\omega_1)$ *be stationary. Let* $\mathbb{P} = \mathrm{CU}(\omega_2, (\omega_2 \cap \mathrm{Cof}(\omega)) \cup S)$*. Then* $\mathbb{P}$ *is countably closed and adds no* ω_1*-sequences of ordinals.*

Proof. Countable closure is immediate, so suppose that c forces that $\dot\tau$ is a function from ω_1 to On. By Lemma 18.4 we may find $N \prec H_\theta$ for some large θ so that N contains everything relevant, $|N| = \omega_1$, ${}^\omega N \subseteq N$, and $\delta =_{\mathrm{def}} N \cap \omega_2$ lies in S. Now we build a decreasing chain of conditions $\langle c_i : i < \omega_1 \rangle$ so that $c_i \in N$, c_{i+1} decides $\tau(i)$ and the sequence $\langle \delta_i : i < \omega_1 \rangle$ where $\delta_i =_{\mathrm{def}} \max(c_i)$ is cofinal in δ. If $\lambda < \omega_1$ is a limit stage there is no problem because ${}^\omega N \subseteq N$, and we may safely choose $\delta_\lambda = \sup_{i<\lambda} \delta_i$, $c_\lambda = \bigcup_{i<\lambda} c_i \cup \{\delta_\lambda\}$. To finish we choose $d = \bigcup_{i<\omega_1} c_i \cup \{\delta\}$, which is legal since $\delta \in S$, and then d is a condition which refines c and determines $\dot\tau$. ⊣

An equivalent formulation would be that we are shooting an ω_1-club set through S by forcing with bounded ω_1-closed subsets of S. Using the forcing of Lemma 18.5 and the ideas of Section 17, it is now fairly straightforward to show that starting with a measurable cardinal κ we may produce a model where $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$ is precipitous. We force first with $\mathrm{Col}(\omega_1, {<}\kappa)$ and then iterate club shooting, absorb forcing posets and construct strong master conditions more or less exactly as in Section 17.

If we are interested in the full ideal NS_{ω_2} then we need to shoot club sets rather than ω_1-club sets. This is more subtle; a little thought shows that if $S \subseteq \omega_2$ and we wish to shoot a club set through S without adding ω_1-sequences, then there must be stationarily many $\alpha \in S \cap \mathrm{Cof}(\omega_1)$ such that $S \cap \alpha$ contains a closed cofinal set of order type ω_1. The next result shows that (at least under CH) this is the only obstacle.

18.6 Lemma. *Let* CH *hold, and let* $S \subseteq \omega_2$ *be such that for stationarily many* $\alpha \in S \cap \mathrm{Cof}(\omega_1)$ *there exists a set* $C \subseteq S \cap \alpha$ *with* C *club in* α*. Let* $\mathbb{P} = \mathrm{CU}(\omega_2, S)$*. Then* $\mathbb{P}$ *adds no* ω_1*-sequences of ordinals.*

Proof. The proof is similar to that of Lemma 18.5. Let T be the stationary set of $\alpha \in S \cap \mathrm{Cof}(\omega_1)$ such that there exists a set $C \subseteq S \cap \alpha$ with C club in α. Suppose that c forces that $\dot\tau$ is a function from ω_1 to On. Build an elementary $N \prec H_\theta$ for some large θ so that N contains everything relevant, $|N| = \omega_1$,

${}^\omega N \subseteq N$, $\alpha =_{\rm def} N \cap \omega_2 \in T$. By hypothesis there is a set $C \subseteq S \cap \alpha$ with C club in α. We build a strictly decreasing chain of conditions $\langle c_i : i < \omega_1 \rangle$ so that $c_i \in N$, c_{i+1} decides $\tau(i)$ and $\delta_i \in C$ where $\delta_i =_{\rm def} \max(c_i)$. To finish we choose $d = \bigcup_{i<\omega_1} c_i \cup \{\alpha\}$, so that d is a condition which refines c and determines $\dot{\tau}$. ⊣

18.3. Outline of the Proof and Main Technical Issues

We will use measures $U_0 \lhd U_1$ of Mitchell orders zero and one respectively. We let B be the set of $\alpha < \kappa$ with $o(\alpha) = 1$, so that $B \in U_1$ and $B \notin U_0$. We fix measures W_i for $i \in B$ so that W_i is a measure of order zero on i (so in particular $B \cap i \notin W_i$) and $\langle W_i : i \in B \rangle$ represents U_0 in $\mathrm{Ult}(V, U_1)$, or more concretely for $X \subseteq \kappa$

$$X \in U_0 \iff \{i : X \cap i \in W_i\} \in U_1.$$

The rough idea is this: we start with some preparation forcing which adds no reals, makes κ into ω_2, makes all inaccessible α lying in B into ordinals of cofinality ω_1, and makes all inaccessible α not lying in B into ordinals of cofinality ω. We then iterate shooting club sets so that U_0 extends to the ω-club filter and U_1 extends to the ω_1-club filter. Roughly speaking U_0 will be responsible for the precipitousness of $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega)$ and U_1 will be responsible for the precipitousness of $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$.

There are several technical obstacles to be overcome.

- In the proof sketched above that $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$ can be precipitous, the forcing which is being iterated to shoot ω_1-club subsets of ω_2 is countably closed. In particular it can be absorbed into any sufficiently large countably closed collapsing poset. This means that the "preparation stage" of the preceding construction can be the simple forcing $\mathrm{Col}(\omega_1, {<}\kappa)$. In the construction to follow we will be shooting club subsets of ω_2 in a way which destroys stationary subsets of $\omega_2 \cap \mathrm{Cof}(\omega)$, so that the forcing can not be embedded into any countably closed forcing (or even any proper forcing). This is one reason why the preparation stage for the construction to follow has to be more complicated.

- The measure U_0 will be extended to become the ω-club filter. So we need to shoot ω-club sets through (at least) all $A \in U_0$, and we will therefore need to shoot closed sets of order type ω_1 through many initial segments of A, in order to appeal to a suitable version of Lemma 18.6. We need some way of organising the construction so that all $A \in U_0$ are anticipated.

  Recall that if $A \in U_0$ then there are many $i \in B$ such that $A \cap i \in W_i$. At many $i \in B$ we will add a club subset of i which has order type ω_1, and is eventually contained in every member of W_i.

- To build the preparation forcing, we need some way of iterating forcings which change cofinality without adding reals. This will require an appeal to Shelah's machinery of revised countable support iteration.

- In the arguments for the precipitousness of NS_{ω_1} and $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$, we iterated to shoot club sets through stationary sets which were measure one for certain "master condition ideals" (in the sense of Foreman's chapter) arising along the way. In the current setting it is not clear that we can do this in a distributive way, so we finesse the question and shoot club sets through some more tractable sets, then argue that this is enough.

  To be a bit more precise, suppose that $V[G_\kappa]$ is the result of the preparation stage. We will build a κ^+-c.c. iteration $\mathbb{Q}_{\kappa^+}$, shooting club sets through subsets of κ. At successor stages we will shoot club sets through certain sets of inaccessibles from the ground model of the form $X \cup Y$ where $X \in U_0$, $Y \in U_1$, $X \subseteq B^c$ and $Y \subseteq B$.

  As the construction proceeds we will show, by induction on ν, that the embeddings j_i can be lifted onto the extension of $V[G_\kappa]$ by $\mathbb{Q}_\nu$. For limit ν we will write $\mathbb{Q}_\nu$ as the union $\bigcup_{\alpha<\kappa} \mathbb{Q}_\nu^\alpha$ of a continuous sequence of subsets each of size less than κ. The existence of the lifted embeddings implies that if H_ν is $\mathbb{Q}_\nu$-generic over $V[G_\kappa]$, then there are many $\alpha < \kappa$ such that $H_\nu \cap \mathbb{Q}_\nu^\alpha$ is $\mathbb{Q}_\nu^\alpha$-generic over $V[G_\alpha]$; we will ensure that a club set is shot through each such set of "generic points".

  We then argue that using the club sets which are added in this process, for each set in one of the relevant master condition ideals we may define a club set which is disjoint from it. Below, at the end of subsection 18.6, we will work through a toy example which illustrates this central idea.

- In order to realise the idea of the last item, we need that the closed sets of order type ω_1 added to points of B during the preparation stage have an additional property. Namely if $i \in B$ and c is the club set in i which is added at stage i during the preparation, then we require that for every $\beta \in \lim(c)$ the set $c \cap \beta$ must intersect every club subset of β which lies in $V[G_\beta]$.

- Let $j_i : V \longrightarrow M_i$ be the ultrapower by U_i for $i = 0, 1$. We will need to embed $\mathbb{P}_\kappa * \mathbb{Q}_\nu$ into both $j_0(\mathbb{P}_\kappa)$ and $j_1(\mathbb{P}_\kappa)$. Naturally the iterations $j_0(\mathbb{P}_\kappa)$ and $j_1(\mathbb{P}_\kappa)$ differ at stage κ; the forcing at stage κ will change the cofinality of κ to the values ω and ω_1 respectively.

18.4. Namba Forcing, RCS Iteration and the S and $\mathbb{I}$ Conditions

Several important ingredients in the proof come from Shelah's work [64] on iterated forcing. The technical issue is that in the preparation stage we need to iterate forcing posets which change cofinalities to ω and add no reals, in such a way that the whole iteration adds no reals. A detailed discussion of Shelah's techniques would take us too far afield, so we content ourselves with a very brief overview.

The preparation iteration will be done using Shelah's *Revised Countable Support* (RCS) technology. This is a version of countable support iteration in which (very roughly speaking) we allow the supports of conditions used in the iteration to be countable sets which are not in the ground model, but arise in the course of the iteration: the point of doing this is to cope gracefully with iteration stages δ such that $\mathrm{cf}(\delta) > \omega$ in V but the cofinality of δ is changed to ω in the course of the iteration.

For motivation, consider the case of Namba forcing. The conditions are trees $T \subseteq {}^{<\omega}\omega_2$ with a unique *stem element* $\mathrm{stem}(T)$, such that every element of T is comparable with $\mathrm{stem}(T)$ and every element extending $\mathrm{stem}(T)$ has ω_2 immediate successors in T. Forcing with these conditions adds an ω-sequence cofinal in ω_2.

Namba [60] showed that under CH this forcing poset adds no reals; we sketch an argument for this which is due to Shelah. Let $\dot{r}$ be a name for a real and let S be a condition. We first find a refinement $T \leq S$ such that $\mathrm{stem}(T) = \mathrm{stem}(S)$, and T forces that $\dot{r}(n)$ is determined by the first n points of the generic branch. We then appeal to a partition theorem for trees (proved from CH, by applying Borel determinacy to each of a family of ω_1 "cut and choose" games played on T) to find a refinement $U \subseteq S$ such that $\mathrm{stem}(U) = \mathrm{stem}(S)$ and every branch through U determines the same real r, so that $U \Vdash \dot{r} = \check{r}$.

The decisive points for the arguments of the last paragraph were that Namba forcing satisfies a version of the fusion lemma and that (under CH) the ideal of bounded subsets of ω_2 is $(2^\omega)^+$-complete. Motivated by these ideas Shelah formulated a technical condition on forcing posets known as the *S-condition*, where S is some set of regular cardinals; this is an abstract form of fusion, saying *very* roughly that a tree of conditions which has cofinally many λ-branching points for each $\lambda \in S$ can be fused. Shelah also showed that under the right circumstances an RCS iteration of S-condition forcing does not add reals. The variant of Namba forcing in which conditions are subtrees of ${}^{<\omega}\omega_2$ such that cofinally many points have ω_2 successors satisfies the S-condition for $S = \{\omega_2\}$.

An important ingredient in the proof we are describing that NS_{ω_2} can be precipitous is a variant Nm' of Namba forcing. Conditions in Nm' are subtrees T of ${}^{<\omega}\omega_3$, such that for $i \in \{2, 3\}$ there are cofinally many points $t \in T$ with $\{\alpha : t^\frown\alpha \in T\}$ an unbounded subset of ω_i.

The salient facts about Nm' are encapsulated in the following result. The first fact in this list is quite hard, but the remaining ones follow easily.

18.7 Lemma. *Let* CH *hold. Then*

1. Nm' *satisfies Shelah's S-condition for $S = \{\omega_2, \omega_3\}$, in particular it*

adds no reals (and so preserves ω_1).

2. Nm$'$ *adds cofinal ω-sequences in ω_2^V and ω_3^V.*

3. *In the generic extension ω_3^V can be written as the union of ω many sets which each lie in V and have V-cardinality ω_1.*

4. *Assuming that $2^{\omega_2} = \omega_3$ in V, in the generic extension by* Nm$'$ *there is an ω-sequence $\langle E_n \rangle$ such that*

   (a) *$E_n \in V$ and $V \models$ "E_n is a club subset of ω_2" for each $n < \omega$.*

   (b) *For every $E \in V$ such that $V \models$ "E is a club subset of ω_2" there is an integer n such that $E_n \subseteq E$.*

5. *Assuming that $2^{\omega_2} = \omega_3$ in V, if $\mathbb{R}$ is any forcing poset of size ω_2 which adds no ω_1-sequences then forcing with* Nm$'$ *adds a generic filter for the poset $\mathbb{R}$.*

For use later we note that Gitik and Shelah defined a generalised version of the S-condition known as the $\mathbb{I}$-*condition*, where $\mathbb{I}$ is a family of ideals on some set S of regular cardinals. The $\mathbb{I}$-condition is just like the S-condition except that the branching in the fusion trees now has to be positive cofinally often with respect to every ideal in $\mathbb{I}$. Gitik and Shelah extended the iteration theorems for RCS iteration which we mentioned above to cover the $\mathbb{I}$-condition, subject to additional technical conditions.

18.5. The Preparation Iteration

We will start by forcing with an RCS iteration $\mathbb{P}_\kappa$. Among the important features of this forcing poset will be that

1. $\mathbb{P}_\kappa$ adds no reals.

2. For every inaccessible $\alpha \leq \kappa$

   (a) $\mathbb{P}_\alpha$ is isomorphic to the direct limit of $\langle \mathbb{P}_\beta : \beta < \alpha \}$.

   (b) $\mathbb{P}_\alpha \subseteq V_\alpha$.

   (c) $\mathbb{P}_\alpha$ is α-c.c.

   (d) $\mathbb{P}_\alpha$ collapses α to become $\omega_2^{V[G_\alpha]}$.

   (e) After forcing with $\mathbb{P}_{\alpha+1}$, α is an ordinal of cardinality ω_1, which has cofinality ω for $\alpha \notin B$ and cofinality ω_1 for $\alpha \in B$.

   (f) For $\alpha \in B$, $V[G_\alpha]$ and $V[G_{\alpha+1}]$ have the same ω-sequences of ordinals.

18.8 Remark. It follows from the properties of $\mathbb{P}_\kappa$ we just listed that

1. All bounded subsets of κ in $V[G_\kappa]$ appear in $V[G_\beta]$ for some $\beta < \kappa$.

2. All elements of ${}^{\omega}\alpha$ which are in $V[G_\kappa]$ already appear in $V[G_{\alpha+1}]$, and if $\alpha \in B$ then such ω-sequences are actually in $V[G_\alpha]$.

As usual it suffices to define the poset which is used at each stage i of the iteration.

- Case 1: If i is not inaccessible we force with $\mathrm{Col}(\omega_1, 2^{\omega_1})^{V[G_i]}$.
- Case 2: If i is inaccessible and $i \notin B$ then we force with $(\mathrm{Nm}')^{V[G_i]}$ (where we note that in $V[G_i]$ we will have $i = \omega_2$ and $i^+ = \omega_3$).
- Case 3: If i is inaccessible and $i \in B$ then we force with $\mathbb{P}^*[W_i]$ defined as follows: conditions are pairs (c, A) such that c is a countable closed subset of $B^c \cap i$ consisting of V-inaccessibles, $A \subseteq B^c \cap i$ with $A \in W_i$, and for every $\beta \in \lim(c)$ the set $c \cap \beta$ meets every club subset of β lying in the model $V[G_\beta]$. The condition (c', A') extends (c, A) if and only if c' end-extends c, $A' \subseteq A$ and $c' - c \subseteq A$.

18.9 Remark. Let i fall under case 3, let g be a $V[G_i]$-generic subset of $\mathbb{P}^*[W_i]$ and let $e = \bigcup\{c : \exists A\ (c, A) \in g\}$. Then e is a club subset of i with order type ω_1, e is eventually contained in every element of W_i, and every element of e falls under case 2.

18.10 Remark. The definition of $\mathbb{P}^*[W_i]$ can be simplified by the observation that (by the β-c.c.) every club subset of β in $V[G_\beta]$ contains a club subset of β in V.

A key technical point (which we are glossing over here) is that the poset $\mathbb{P}^*[W_i]$ satisfies a suitable version of Gitik and Shelah's $\mathbb{I}$-condition [28]. In fact the argument we are describing was one of the main motivations for the development of the $\mathbb{I}$-condition. Once it is has been checked that Nm' has the S-condition and $\mathbb{P}^*[W_i]$ satisfies the $\mathbb{I}$-condition for suitable S and $\mathbb{I}$, an appeal to standard facts about RCS iterations lets us conclude that $\mathbb{P}_\kappa$ has the properties listed above.

18.11 Lemma. *If $i \in B$ then forcing with $\mathbb{P}^*[W_i]$ adds no ω-sequences of ordinals to $V[G_i]$.*

Sketch of proof. Take a $\mathbb{P}_i$- name for a sequence $\langle D_n : n < \omega\rangle \in V[G_i]$ of dense open subsets of $\mathbb{P}^*[W_i]$. Working in V we fix an elementary chain of models M_β for $\beta \in i$ such that $M_\beta \prec (H_\theta, \ldots)$, M_0 contains everything relevant and $M_\beta \cap i \in i$. Now we choose an inaccessible $\beta \notin B$ such that $M_\beta \cap i = \beta$ and $\beta \in A$ for every $A \in M_\beta \cap W_i$. Since $\mathbb{P}_\beta$ is β-c.c. and $\mathbb{P}_\beta \subseteq M_\beta$, routine arguments as in the theory of proper forcing show that $M_\beta[G_\beta] \prec H_\theta[G_\kappa]$ and $M_\beta[G_\beta] \cap V = M_\beta$.

As we observed already β must fall under case 2 in the definition of the preparation iteration $\mathbb{P}_\kappa$, so that by Lemma 18.7 there is in $V[G_i]$ an ω-sequence $\langle E_m : m < \omega\rangle$ which "diagonalises" the club subsets of β lying in $V[G_\beta]$. We may now construct a sequence $\langle (c_n, A_n) : n < \omega \rangle$ of conditions in $\mathbb{P}^*[W_i] \cap M_\beta[G_\beta]$ such that $c_{2n+1} \in D_n$ and $\max(c_{2n+2}) \in E_n$. Let $d =_{\text{def}} \bigcup_n c_n \cup \{\beta\}$ and $A^* = \bigcap (M_\beta \cap W_i)$, then (d, A^*) is a condition in $\mathbb{P}^*[W_i]$ which lies in the intersection of the D_n. ⊣

18.6. A Warmup for the Main Iteration

Throughout the discussion that follows we are working in $V[G_\kappa]$, in particular $\kappa = \omega_2$ and $\kappa^+ = \omega_3$. We will eventually describe an iteration of length κ^+ in which we shoot club sets through subsets of κ without adding bounded subsets of κ. Before we do that, for purposes of motivation we will describe a much simpler three step iteration $\mathbb{R}_0 * \dot{\mathbb{R}}_1 * \dot{\mathbb{R}}_2$ of club-shooting forcing, and sketch proofs of its salient properties which contain most of the ideas needed for the full iteration.

To describe $\mathbb{R}_0$ we fix sets of inaccessibles $X \in U_0$ and $Y \in U_1$ such that $X \subseteq B^c$ and $Y \subseteq B$. Let $A = X \cup Y$ and define $\mathbb{R}_0 = \mathrm{CU}(\kappa, A)$, the poset of closed and bounded subsets of A ordered by end-extension.

18.12 Lemma. *Forcing with $\mathbb{R}_0$ over $V[G_\kappa]$ adds no ω_1-sequences of ordinals.*

Proof. Working in V, let $T = \{\beta \in Y : X \cap \beta \in W_\beta\}$. Then $T \in U_1$, because $X \in U_0$ and U_0 is represented by $\langle W_i : i \in B \rangle$ in $\mathrm{Ult}(V, U_1)$. In particular T is stationary in κ. The poset $\mathbb{P}_\kappa$ is κ-c.c. and so T is stationary in $V[G_\kappa]$. For each $\beta \in T$, the preparation forcing added a closed set of order type ω_1 which is contained in $X \cap \beta$, and we are done by Lemma 18.6. ⊣

One of the key ideas in Gitik's arguments is that of a "local master condition". We give a more precise formulation in a moment, but the rough idea is to look at conditions which induce generic filters over a submodel of the universe for subposets of a forcing poset. The idea is similar to that of a strongly generic condition in proper forcing (see Remark 24.5) but the relevant submodels here are the classes $V[G_\beta]$ for $\beta < \kappa$. We will construct our iterations so that there are many local master conditions; as we see at the end of this section, this is vital when it comes to lifting the elementary embeddings j_0 and j_1 in the required way.

The set T defined in the proof of Lemma 18.12 is stationary, so by the usual reflection arguments the set of points where T reflects is a measure one set for any normal measure. We let $A' = X' \cup Y'$, where

$$X' = \{\beta \in X : T \cap \beta \text{ is stationary in } \beta\},$$

$$Y' = \{\beta \in Y : T \cap \beta \text{ is stationary in } \beta\}.$$

For $\beta < \kappa$ we define $\mathbb{R}_{0,\beta}$ to be the set of $d \in \mathbb{R}_0$ such that $\max(d) < \beta$ and $d \in V[G_\beta]$. It is easy to see that $\mathbb{R}_0 = \bigcup_{\beta < \kappa} \mathbb{R}_{0,\beta}$, and that

$$\mathbb{R}_{0,\gamma} = \bigcup_{\beta < \gamma} \mathbb{R}_{0,\beta} = \mathbb{R}_0 \cap V_\gamma[G_\gamma]$$

when γ is V-inaccessible.

18.13 Remark. By the usual conventions, for λ an uncountable regular cardinal and X a set with $|X| = \lambda$, a *filtration* of X is an increasing and continuous sequence $\langle X_i : i < \lambda \rangle$ such that $X_i \subseteq X$, $X = \bigcup_i X_i$, and $|X_i| < \lambda$. The key property is that given filtrations X_i, X'_i we have $X_j = X'_j$ for a club set of j.

Technically the sequence of posets $\mathbb{R}_{0,\beta}$ is not a filtration of $\mathbb{R}_0$ because it is only continuous at V-inaccessible points. Until the end of this section we will abuse notation and refer to such sequences as filtrations.

A local version of the argument of Lemma 18.12 shows immediately that

18.14 Lemma. *For every $\beta \in A'$, forcing with $\mathbb{R}_{0,\beta}$ over $V[G_\beta]$ adds no ω_1-sequences of ordinals.*

The next lemma can be seen as a more refined version of this result. Let $\beta \in A'$. We will say that $c \in \mathbb{R}_0$ is a *β-master condition for* $\mathbb{R}_0$ if $\max(c) = \beta$, and $\{c \cap (\alpha + 1) : \alpha \in \beta \cap \lim(c)\}$ is a $V[G_\beta]$-generic subset of $\mathbb{R}_{0,\beta}$.

18.15 Lemma. *For every $\beta \in A'$ and every $d \in \mathbb{R}_{0,\beta}$ there is a β-master condition $c \leq d$ with $c \in V[G_{\beta+2}]$.*

Proof. In $V[G_\beta]$ we have $\beta = \omega_2$ and $(\beta^+)^V = \omega_3$. By the previous lemma, $\mathbb{R}_{0,\beta}$ is (ω_1, ∞)-distributive in $V[G_\beta]$. We distinguish the cases $\beta \in X'$ and $\beta \in Y'$.

$\beta \in X'$: At stage β in the preparation forcing we forced with Nm'. So we are done by an appeal to clause 5 of Lemma 18.7, and in fact we can build a suitable c in $V[G_{\beta+1}]$.

$\beta \in Y'$: Again $\mathbb{R}_{0,\beta}$ is (ω_1, ∞)-distributive in $V[G_\beta]$. In $V[G_{\beta+2}]$ we have $\mathrm{cf}(\beta) = \mathrm{cf}(\beta^+) = \omega_1$; so if $\mathcal{D}$ is the set of dense open subsets of $\mathbb{R}_{0,\beta}$ which lie in $V[G_\beta]$, working in $V[G_{\beta+2}]$ we may write $\mathcal{D} = \bigcup_{i<\omega_1} \mathcal{D}_i$ where $\mathcal{D}_i \in V[G_\beta]$ and $V[G_\beta] \models |\mathcal{D}_i| = \omega_1$.

We fix $D \in V[G_{\beta+2}]$ such that D is a club subset of β of order type ω_1 and $D \subseteq X \cap \beta$. Now we build a chain of conditions $c_i \in \mathbb{R}_{0,\beta}$ such that $\max(c_i) \in D$ and $c_{i+1} \in \bigcap \mathcal{D}_i$ for all i. Since $V[G_{\beta+2}]$ and $V[G_\beta]$ have the same ${}^\omega\beta$, there is no problem at limit stages. As usual we may now set $c = \bigcup_i c_i \cup \{\beta\}$ to finish. $\dashv$

We now define $\mathbb{R}_1$. Let E be the generic club subset of κ added by $\mathbb{R}_0$. Then $\mathbb{R}_1$ is the set of those closed bounded sets d such that $d \subseteq E \cap A'$, and $E \cap (\beta + 1)$ is a β-master condition for every $\beta \in d$.

18.16 Remark. We remind the reader of the discussion of the "flat condition trick" in Remark 17.2. We will be using that trick heavily in what follows. In particular when we get to the main construction in Section 18.7 we will just define the set of flat conditions and leave all the details to the reader.

We define a suitable concept of flatness for conditions in the two-step iteration $\mathbb{R} =_{\text{def}} \mathbb{R}_0 * \dot{\mathbb{R}}_1$. The flat conditions are pairs $(c, \check{d})$ where $c \in \mathbb{R}_0$, $c \Vdash \check{d} \in \mathbb{R}_1$ and $\max(c) = \max(d)$. We define $\mathbb{R}_\gamma = \mathbb{R} \cap V_\gamma[G_\gamma]$ for inaccessible $\gamma < \kappa$.

18.17 Lemma. *Forcing with $\mathbb{R}$ over $V[G_\kappa]$ adds no ω_1-sequences of ordinals, and the set of flat conditions is dense in $\mathbb{R}$.*

Proof. Working in V we fix a $\mathbb{P}_\kappa$-name $\dot{D}$ for an ω_1-sequence of dense subsets of $\mathbb{R}$, where we may as well assume that $\dot{D} \subseteq V_\kappa$. By routine arguments there is a club set $F \subseteq \kappa$ in V such that for every inaccessible $\gamma \in F$, $\dot{D} \cap V_\gamma$ is a $\mathbb{P}_\gamma$-name for a sequence of dense sets in $\mathbb{R}_\gamma$.

Now we choose $\gamma \in Y'$ such that $F \cap X' \cap \gamma \in W_\gamma$. By the definition of the preparation forcing there is a club set $e \subseteq \gamma$ in $V[G_{\gamma+1}]$ such that $\text{ot}(e) = \omega_1$, $e \subseteq F \cap X'$, and for every $\beta \in \lim(e)$ the set $e \cap \beta$ meets every club subset of β lying in $V[G_\beta]$.

We will now work in $V[G_\kappa]$. Let r be an arbitrary condition in $\mathbb{R}$; we will show that r can be extended to a flat condition which lies in the intersection of the dense sets D_i for $i < \omega_1$, establishing both of our claims about $\mathbb{R}$. We will build a decreasing sequence of conditions $(c_i, \dot{d}_i)$ for $i \leq \omega_1$, such that

1. $r = (c_0, \dot{d}_0)$.
2. For every $i < \lambda$,
   (a) $(c_i, \dot{d}_i) \in \mathbb{R}_\gamma$
   (b) The condition c_{i+1} determines $\dot{d}_i$, that is $c_{i+1} \Vdash \dot{d}_i = \check{d}_i$ for some $d_i \in V[G_\kappa]$.
   (c) $(c_{i+1}, \dot{d}_{i+1}) \in D_i$.
   (d) The condition c_{i+1} forces that $\max(\dot{d}_{i+1}) > \max(c_i)$.
   (e) The ordinal $\beta_i =_{\text{def}} \max(c_i)$ lies in the set e, and c_i is a β_i-master condition.
3. The sequence $\langle \beta_i : i \leq \omega_1 \rangle$ is increasing and continuous.
4. For every limit $\lambda \leq \omega_1$, $(c_\lambda, \dot{d}_\lambda)$ is a flat condition.

The successor steps in this construction are easy by an appeal to Lemmas 18.14 and 18.15, and the fact we reflected the density of the dense sets down to each β_j.

The subtle point is that for a limit ordinal $\lambda \leq \omega_1$ we are safe to set $\beta_\lambda = \sup_{i<\lambda} \beta_i$, $c_\lambda = \bigcup_{i<\lambda} c_i \cup \{\beta_\lambda\}$ and $\dot{d}_\lambda$ equal to the canonical name for $d_\lambda = \bigcup_{i<\lambda} d_i \cup \{\beta_\lambda\}$. The issue is to check that c_λ is a β_λ-master condition, so we set $\beta = \beta_\lambda$ and fix a $\mathbb{P}_\beta$ name $Z \subseteq V_\beta$ for a dense subset of $\mathbb{R}_{0,\beta}$. We then find a club set $C_Z \subseteq \beta$ such that if $\alpha \in C_Z$ is inaccessible then $Z \cap V_\alpha$ names a dense subset of $\mathbb{R}_{0,\alpha}$. Now the key point is that $\beta \in \lim(e)$ so $e \cap \beta$

meets C_Z, and we have $i < \lambda$ such that $\beta_i \in C_Z$. Let $\alpha = \beta_i$, then we are done since c_i is an α-master condition and it generates a filter which meets the dense set named by $Z \cap \mathbb{R}_{0,\alpha}$ (which is an initial segment of the dense set named by Z itself). ⊣

We may now define the notion of a β-master condition for $\mathbb{R}$ and prove analogues of Lemmas 18.15 and 18.17. To be a bit more explicit, we say that (c, d) is a *β-master condition* for $\mathbb{R}$ if and only if it is flat, $\max(c) = \max(d) = \beta$, and $\{(c \cap (\alpha+1), d \cap (\alpha+1)) : \alpha \in d\}$ is $\mathbb{R}_\beta$-generic over $V[G_\beta]$. We define T', X'', Y'', A'' from X' and Y' in just the same way that T, X', Y', A' were defined from X and Y. Then the analogue of Lemma 18.15 says that if $\beta \in A''$ any condition in $\mathbb{R}_\beta$ extends to a β-master condition, and there is a similar generalisation of Lemma 18.17.

We now sketch the main ideas in the argument that we can make the restriction of NS_{ω_2} to $\mathrm{Cof}(\omega)$ precipitous. Similar arguments apply to the restriction to $\mathrm{Cof}(\omega_1)$.

Applying the elementary embedding j_0 to the result of Lemma 18.15, we obtain the result that every condition in $\mathbb{R}_0$ can be extended in M_0 to a κ-master condition in $j_0(\mathbb{R}_0)$. Implicitly this defines an embedding of $\mathbb{P}_\kappa * \mathbb{R}_0$ into $j_0(\mathbb{P}_\kappa)$, and a strong master condition suitable for lifting the elementary embedding j_0 to the extension by $\mathbb{P}_\kappa * \mathbb{R}_0$. A similar argument applies to the iteration $\mathbb{R}_0 * \mathbb{R}_1$.

We now return to a point which we already mentioned in subsection 18.3, namely that can achieve the same kind of effect as in the construction of subsection 17.2 by performing an iteration where every step is either like $\mathbb{R}_0$ or like $\mathbb{R}_1$. To fix ideas let H_0 be $\mathbb{R}_0$-generic over $V[G_\kappa]$, and let $a \in V[G_\kappa * H_0]$ be in the master condition ideal for j_0. Explicitly this means that it is forced that $\kappa \notin j_0^+(a)$ where j_0^+ is the lifting of j_0 onto $V[G_\kappa * H_0]$ described in the preceding paragraph. We will show how to add an ω-club set disjoint from a.

Let $\dot{a}$ be a $\mathbb{P}_\kappa * \dot{\mathbb{R}}_0$-name for a and let $(p, q) \in G_\kappa * H_0$ force that $\dot{a}$ is in the master condition ideal. That is to say, (p, q) forces that "it is forced that $\kappa \notin j_0^+(a)$". Analysing the lifting construction and viewing p now as a condition in $j_0(\mathbb{P}_\kappa)$, p forces over M_0 that for every κ-master condition $Q \leq j_0(q)$, Q forces that $\kappa \notin j_0(\dot{a})$.

Now let C be the set of $\alpha < \kappa$ such that

1. $q \in \mathbb{R}_{0,\alpha}$.

2. p forces (over V for the forcing poset $\mathbb{P}_\kappa$) that for every $Q \leq q$ which is an α-master condition for $\mathbb{R}_0$, Q forces (over $V[G_\kappa]$ for the forcing poset $\mathbb{R}_0$) that $\alpha \notin \dot{a}$.

By Łoś's theorem we see that $C \in U_0$.

Define $\mathbb{R}_2$ to be similar to $\mathbb{R}_0$, adding a club contained in $C \cup D$ for some $D \in U_1$. One can do an analysis of $\mathbb{R}_0 * \mathbb{R}_1 * \mathbb{R}_2$ which is similar to the analyses of $\mathbb{R}_0$ and $\mathbb{R}_0 * \mathbb{R}_1$ given above. Let E_i be the club set added by $\mathbb{R}_i$. Then

- By the construction of $\mathbb{R}_1$, $E_0 \cap (\alpha + 1)$ is an α-master condition for every $\alpha \in E_1$.
- By the construction of $\mathbb{R}_2$, for every $\alpha \in E_2 \cap \mathrm{Cof}(\omega)$ we have $\alpha \in C$.
- So for every $\alpha \in E_1 \cap E_2 \cap \mathrm{Cof}(\omega)$, it follows from the definition of C that $\alpha \notin a$.

We have argued that in the extension by $\mathbb{R}_0 * \mathbb{R}_1 * \mathbb{R}_2$ there is an ω-club set disjoint from a. In the next subsection we will show how to iterate and achieve the same effect for every set which appears in some master condition ideal during the course of the iteration.

18.7. The Main Iteration

Recall from the last section that we defined $\mathbb{R}_0$ from a set $A = X \cup Y$ and then $\mathbb{R}_1$ from a set $A' \subseteq A$, where $\beta \in A'$ if $\beta \in A$ and there are stationarily many $\gamma < \beta$ such that $X \cap \gamma \in W_\gamma$. The poset $\mathbb{R}_0$ shot a club set E through A, and the poset $\mathbb{R}_1$ shot a club set through the set of points $\beta \in E \cap A'$ such that $E \cap \beta$ was $\mathbb{R}_{0,\beta}$-generic over $V[G_\beta]$. The main iteration, which we will only describe in outline, can be viewed as iterating this kind of construction many times for every possible A simultaneously. The main difficulty in defining the iteration is that when we have iterated ν times and have obtained an iteration $\mathbb{Q}_\nu$, we need to define a suitable notion of *β-master condition for* $\mathbb{Q}_\nu$; this requires choosing a filtration of $\mathbb{Q}_\nu$, and the filtrations for different values of ν must fit together nicely.

The main iteration is defined from some parameters $\langle A_\nu, i_\nu, C_\nu : \nu < \kappa^+ \rangle$, which are chosen in V. They must satisfy a long list of technical conditions, most of which we are omitting. In particular

1. A_ν is the union of sets of inaccessibles $X_\nu \subseteq B^c$ and $Y_\nu \subseteq B$, with $X_\nu \in U_0$ and $Y_\nu \in U_1$.
2. Every set of inaccessibles $X \subseteq B^c$ with $X \in U_0$ is enumerated as X_ν for some successor ν, and similarly every set of inaccessibles $Y \subseteq B$ with $Y \in U_1$ is enumerated as Y_ν for some successor ν.
3. i_ν is a surjection from κ to ν, which is also injective for $\nu \geq \kappa$. Note that for any normal measure on κ, the map which takes $\beta < \kappa$ to the order-type of i_ν“β represents ν in the ultrapower.
4. C_ν is club in κ.
5. If $\kappa \leq \nu_1 < \nu_2$, $\beta \in C_{\nu_2}$ and $\nu_1 \in i_{\nu_2}$“β, then $\beta \in C_{\nu_1}$.

We define $X_{\nu,\beta} = i_\nu``\beta$ for $\beta < \kappa$, so that the $X_{\nu,\beta}$'s form a filtration of ν.

We define by recursion posets $\mathbb{Q}_\nu$ for $\nu < \kappa^+$, and for each $\mathbb{Q}_\nu$ also a filtration in which $\mathbb{Q}_\nu$ is written as the union of subsets $\mathbb{Q}_{\nu,\beta}$ for $\beta < \kappa$.

18.18 Remark. Once again the remarks about the "flat condition trick" in Remark 17.2 are somewhat applicable. We are defining a sequence of posets, whose conditions are comprised of closed bounded sets from the ground model, and claiming that they can be considered as an iteration. However in this instance it would be hard to write down a genuine iteration and then identify our conditions as a dense subset. To give a complete account of the proof we would have to check that the sequence of posets $\mathbb{Q}_\nu$ can be considered as an iteration, but this is only one of many details that we are omitting.

Conditions in $\mathbb{Q}_\nu$ are sequences of the form $q = \langle q_\alpha : \alpha \in X_{\nu,\beta} \rangle$ where (omitting one condition for the moment)

I. $\beta \in C_\nu$ (we will denote this ordinal β by β_q in what follows).

II. For successor α in the support of q, $q_\alpha \in \mathrm{CU}(\kappa, A_\alpha)$.

III. For limit α in the support of q, $q_\alpha \in \mathrm{CU}(\kappa, A_\alpha \cap C_\alpha)$.

IV. For limit α in the support of q, for every $\eta \in q_\alpha$

(a) $\eta \leq \beta_q$.

(b) $X_{\alpha,\eta} \subseteq X_{\nu,\beta_q}$.

(c) $\eta \in q_\tau$ for every $\tau \in X_{\alpha,\eta}$.

To qualify as a member of $\mathbb{Q}_\nu$ a sequence q as above must satisfy a fifth property (property V.), whose description we defer until we have made a few definitions.

Once we have defined $\mathbb{Q}_\nu$, we define $\mathbb{Q}_{\nu,\beta}$ for $\beta < \kappa$ to be the set of those $p \in \mathbb{Q}_\nu$ such that

1. $p \in V[G_\beta]$.

2. $\beta_p < \beta$.

3. For every τ in the support X_{ν,β_p} of p, p_τ is bounded in β.

If $q \in \mathbb{Q}_\nu$, α is a limit ordinal in the support X_{ν,β_q} of q and $\beta \in q_\alpha$ then we define $q \restriction (\alpha, \beta) = \langle q_\tau \cap (\beta + 1) : \tau \in X_{\alpha,\beta} \rangle$. Notice that by the conditions we imposed on q we have that the support $X_{\alpha,\beta}$ of $q \restriction (\alpha, \beta)$ is contained in the support X_{ν,β_q} of q; also $\beta \in q_\tau$ for all $\tau \in X_{\alpha,\beta}$.

The intuition here is that $q \restriction (\alpha, \beta)$ is of the right general shape to be a β-*master condition* for $\mathbb{Q}_\alpha$. To be a bit more formal we say that r is a β-master condition for $\mathbb{Q}_\alpha$ if

1. The support of r is $X_{\alpha,\beta}$.
2. For every $\tau \in X_{\alpha,\beta}$, $\beta = \max(r_\tau)$.
3. The set of conditions $p \in \mathbb{Q}_\alpha \restriction \beta$ such that p_τ is an initial segment of $r_\tau \cap \beta$ for all τ is a $V[G_\beta]$-generic filter on $\mathbb{Q}_\alpha \restriction \beta$.

Now we can complete the description of $\mathbb{Q}_\nu$. Intuitively the following condition says that at limit stages we are shooting clubs through certain sets of "generic points".

V. For every limit α in the support X_{ν,β_q} of q and every $\beta \in q_\alpha$, $q \restriction (\alpha, \beta)$ is a β-master condition for $\mathbb{Q}_\alpha$.

If $p, q \in \mathbb{Q}_\nu$ then $p \leq_\nu q$ iff $\beta_p \geq \beta_q$ and p_α end-extends q_α for all $\alpha \in X_{\nu,\beta_q}$.

The key lemmas are proved by similar means to those used in the last section.

18.19 Lemma. *For $\nu < \mu < \kappa^+$, $\mathbb{Q}_\nu$ is a complete subordering of $\mathbb{Q}_\mu$. Defining $\mathbb{Q}_{\kappa^+} = \bigcup_\nu \mathbb{Q}_\nu$, $\mathbb{Q}_{\kappa^+}$ has the κ^+-c.c.*

The following lemma is the technical heart of the whole construction. The proof (which we omit) is by a very intricate double induction on the pairs (μ, β) with $\beta \in A_\mu \cap C_\mu$, ordered lexicographically.

18.20 Lemma. *If ν is limit, $\alpha \in A_\nu \cap C_\nu$, $p \in \mathbb{Q}_{\nu,\alpha}$ then there exists a condition $q = \langle q_\tau : \tau \in X_{\nu,\alpha}\rangle \leq_\nu p$ in $V[G_{\alpha+2}]$ such that q is an α-master condition for $\mathbb{Q}_\nu$.*

The following is an easy corollary:

18.21 Lemma. *Let $\nu < \kappa^+$ be limit and let $\alpha \in A_\nu \cap C_\nu$. Forcing over $V[G_\alpha]$ with $\mathbb{Q}_{\nu,\alpha}$ adds no ω_1-sequence of ordinals.*

Let $j_i : V \longrightarrow M$ be the ultrapower by the normal measure U_i, and observe that since $V \models {}^\kappa M \subseteq M$ and $\mathbb{P}_\kappa$, $V[G_\kappa] \models {}^\kappa M_i[G_\kappa] \subseteq M_i[G_\kappa]$. Observe also that by normality $\kappa \in j(A_\nu \cap C_\nu)$ for all limit $\nu < \kappa^+$. Accordingly we see that

18.22 Lemma. *For every limit $\nu < \kappa^+$, in $M_i[G_{\kappa+1}]$ there is a condition $q \in \mathbb{Q}_{j(\nu)}$ such that $q = \langle q_\tau : \tau \in j``\nu\rangle$, $q$ induces a $j(\mathbb{Q})_{\nu,\kappa}$-generic filter over $V[G_\kappa]$, and $\max(q_\tau) = \kappa$ for every $\tau \in j``\nu$.*

Each condition in $\mathbb{Q}_\nu$ is an object of size less than κ. It follows easily that

18.23 Lemma. *For every limit $\nu < \kappa^+$ and every $\alpha \in A_\nu \cap C_\nu$, there is an isomorphism between $\mathbb{Q}_{\nu,\alpha}$ and $j(\mathbb{Q})_{\nu,\alpha}$ in $M_i[G_\kappa]$.*

18.24 Lemma. *There exists an isomorphism between $\mathbb{Q}_\nu$ and $j(\mathbb{Q})_{\nu,\kappa}$ in $M_i[G_\kappa]$.*

Putting these various pieces of information together, we get

18.25 Lemma. *For every limit $\nu < \kappa^+$, there is a $\mathbb{Q}_\nu$-generic filter over $V[G_\kappa]$in $M_i[G_{\kappa+1}]$, which is induced by a condition as in Lemma 18.22.*

18.8. Precipitousness of the Non-Stationary Ideal

We are now in precisely the situation of Lemma 17.4, so we have produced two precipitous ideals I_0 and I_1, where I_a concentrates on points of cofinality ω_a. It remains to be seen that these are in fact restrictions of the non-stationary ideal. We will show that the ideal I_0 induced by the construction with j_0 is the ω-nonstationary ideal, the argument for I_1 is exactly the same. We worked through a simple case of the argument at the end of Section 18.6, the idea here is very similar.

Let H be generic over $V[G_\kappa]$ for $\mathbb{Q}_{\kappa^+}$. We work in $V[G_\kappa * H]$. We denote by $H \restriction \nu$ the $\mathbb{Q}_\nu$-generic object induced by H. Let t_j be the club set added by H at stage j.

Since I_0 is a normal ideal concentrating on points of cofinality ω, I_0 must contain the ω-nonstationary ideal. The other direction is trickier, since we did not explicitly shoot ω-club sets through every I_0-large set.

18.26 Claim. *I_0 is contained in the ω-nonstationary ideal.*

Proof. Suppose that a is in I_0. Unwrapping the definition, this means that at some stage ν we have that $a \in V[G_\kappa * H_\nu]$ and it is forced that $\kappa \notin j_{0,\nu}(a)$ where $j_{0,\nu}$ is the lifting of j_0 to $V[G_\kappa * H_\nu]$.

We now fix $\dot{a}$ a $\mathbb{P}_\kappa * \mathbb{Q}_\nu$-name name for a and a condition $(p, q) \in G_\kappa * H_\nu$ forcing (over M_0 for $\mathbb{P}_\kappa * \mathbb{Q}_\nu$) that "it is forced (over $M_0[\dot{G}_\kappa][\dot{H}_\nu]$ by the forcing poset $(j_{0,\nu}(\mathbb{P}_\kappa)/\dot{G}_\kappa * \dot{H}_\nu) * j_{0,\nu}(\mathbb{Q}_\nu)/\dot{m}_\nu)$, where m_ν is the master condition) that $\kappa \notin j_{0,\nu}(\dot{a})$". Regarding p as a condition in $j_0(\mathbb{P}_\kappa)$, p forces (over M_0 for $j_0(\mathbb{P}_\kappa)$) that for every κ-master condition $Q \le j_0(q)$ for $j_0(\mathbb{Q})_\nu$, Q forces (over $M_0[\dot{G}_{j_0(\kappa)}]$ for $j_0(\mathbb{Q})_\nu$) that $\kappa \notin j_0(\dot{a})$.

Now we apply Łoś's theorem to see that $R \in U_0$, where R is the set of α such that $q \in \mathbb{Q}_{\nu,\alpha}$ and p forces (over V for $\mathbb{P}_\kappa$) that for every $Q \le q$ with Q an α-master condition for $\mathbb{Q}_\nu$, Q forces (over $V[\dot{G}_\kappa]$ for $\mathbb{Q}_\nu$) that $\alpha \notin \dot{a}$.

Let $\eta > \nu$ be some limit stage. The construction of the forcing poset implies that for all sufficiently large $\alpha \in t_\eta$, there is a condition $Q \le q$ in H which is an α-master condition for $\mathbb{Q}_\nu$. So for all sufficiently large $\alpha \in t_\eta \cap R$, $\alpha \notin a$.

In the construction we enumerated R as $X_{\bar{\eta}}$ for some $\bar{\eta}$. By definition $A_{\bar{\eta}} = X_{\bar{\eta}} \cup Y_{\bar{\eta}}$, and in $V[G_\kappa]$ the preparation forcing arranged that all points of $X_{\bar{\eta}}$ have cofinality ω while all points of $Y_{\bar{\eta}}$ have cofinality ω_1. At stage $\bar{\eta}$ in the main iteration we added a club set $t_{\bar{\eta}} \subseteq A_{\bar{\eta}}$, so $t_{\bar{\eta}} \cap \mathrm{Cof}(\omega) \subseteq R$.

Combining these results, all sufficiently large $\alpha \in t_\eta \cap t_{\bar{\eta}} \cap Cof(\omega)$ fail to be in a. We conclude that a is ω-nonstationary in $V[G_\kappa * H]$, as required. $\dashv$

18.9. Successors of Larger Cardinals

Gitik [25, 26] has also obtained rather similar equiconsistency results for regular cardinals $\kappa > \omega_2$. The idea is broadly the same, but the preparation forcing is an iteration of Prikry-style forcing with Easton supports followed by an iteration of Cohen forcing (for κ inaccessible) or a Lévy collapse (for κ a successor cardinal). The main iteration is essentially the same.

We content ourselves with quoting some of the main results. When stating the lower bounds we assume throughout that there is no inner model with a cardinal λ such that $o(\lambda) = \lambda^{++}$, and we let K be the Mitchell core model for sequences of measures and $\mathcal{F}$ its measure sequence. Let $\vec{U}$ be a coherent sequence of measures. An ordinal α is an (ω, δ) *repeat point over* κ if and only if $\mathrm{cf}(\alpha) = \omega$ and for every $A \in \bigcap\{U(\kappa, \zeta) : \alpha \leq \zeta < \alpha + \delta\}$ there are unboundedly many $\gamma < \alpha$ such that $A \in \bigcap\{U(\kappa, \zeta') : \gamma \leq \zeta' < \gamma + \delta\}$.

The result for successors of regular cardinals greater than ω_1 is exact.

18.27 Theorem (Gitik [26]). *Let $\lambda = \mathrm{cf}(\lambda) < \kappa$ and suppose that* GCH *holds and there is a measure sequence with an $(\omega, \lambda + 1)$-repeat point over κ. Then there is a generic extension in which* GCH *holds, cardinals up to and including λ are preserved, $\kappa = \lambda^+$ and NS_κ is precipitous.*

18.28 Theorem (Gitik [25]). *Suppose that $\mu = \mathrm{cf}(\mu) > \omega_1$,* GCH *holds and NS_κ is precipitous where $\kappa = \mu^+$. Then in K there is an $(\omega, \mu + 1)$-repeat point over κ.*

Interestingly enough, the proof uses only the precipitousness of the restrictions of NS_κ to cofinality ω and cofinality μ. When κ is inaccessible the strength of "NS_κ is inaccessible" is bounded from above by an $(\omega, \kappa + 1)$-repeat and from below by an $(\omega, {<}\kappa)$-repeat.

19. More on Iterated Club Shooting

In this section we give sketches of two more theorems obtained by iterated club shooting. The first theorem is due to Jech and Woodin [41] and shows that it is consistent for $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ to be a κ^+-saturated ideal. The second is due to Magidor [55] and shows that it is consistent for every stationary subset of $\omega_2 \cap \mathrm{Cof}(\omega)$ to reflect at almost every point of $\omega_2 \cap \mathrm{Cof}(\omega_1)$. Apart from their intrinsic interest we have included them because they illustrate some new ideas: the theorem by Jech and Woodin involves embedding one iteration in another "universal" iteration, while the theorem by Magidor gives another example of shooting clubs to make a natural filter (defined in this case via stationary reflection) become the club filter.

As some motivation for Theorem 19.1 we sketch a proof that if κ is weakly compact then $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ is not κ^+-saturated. We start by recalling a classical result of Solovay: if κ is a regular uncountable cardinal and $S \subseteq \kappa$ is stationary then $T = \{\alpha \in S : S \cap \alpha \text{ is non-stationary in } \alpha\}$ is stationary (given a club C look at the first place where $\lim(C)$ meets S). In particular $T \cap \alpha$ is non-stationary in α for every $\alpha \in T$, in what follows we refer to stationary sets which reflect at no point of themselves as *thin.*

We now consider an ordering on stationary subsets of inaccessible cardinals investigated by Jech [38]. Given an inaccessible cardinal κ and stationary subsets $S, T \subseteq \kappa$ we write $S < T$ when $S \cap \alpha$ is stationary for almost every $\alpha \in T$ (modulo the club filter). It is easy to check that $<$ is well-founded, and by the result of Solovay from the last paragraph $<$ is irreflexive. If $S < T$ with S and T both thin, then clearly $S \cap T$ is non-stationary.

Assume now that κ is weakly compact. We will produce a $<$-increasing sequence $\langle S_\alpha : \alpha < \kappa^+ \rangle$ of thin stationary sets of regular cardinals. Let $S_0 = \kappa \cap \mathrm{Reg}$. At stage α fix a surjection f from κ to α, and use Π^1_1-indescribability to show that

$$S = \{\delta : \forall \gamma < \delta \ S_{f(\gamma)} \cap \delta \text{ is stationary in } \delta\}$$

is stationary. Then choose S_α to be a thin stationary subset of this set S. If $\beta < \alpha$ then $S_\beta \cap \delta$ is stationary for all large $\delta \in S_\alpha$, so $S_\beta < S_\alpha$. Since the S_α for $\alpha < \kappa^+$ have pairwise non-stationary intersections, $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ is not κ^+-saturated.

The proof we just gave shows essentially that if κ is κ^+-Mahlo then $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ is not κ^+-saturated. Jech and Woodin showed [41] that for any $\alpha < \kappa^+$ we may have κ which is α-Mahlo with $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ κ^+-saturated, starting from a measurable cardinal of Mitchell order α. This is known [38] to be optimal.

19.1 Theorem. *Let κ be measurable and let* GCH *hold. Then in a suitable generic extension* $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ *is κ^+-saturated.*

Proof. Let δ be inaccessible and let $S \subseteq \mathrm{Reg} \cap \delta$. We define a forcing poset $\mathrm{CU}_{\mathrm{Reg}}(\delta, S) = \mathrm{CU}(\delta, (\mathrm{Sing} \cap \delta) \cup S)$; to be more explicit conditions are closed bounded subsets c of δ such that $c \cap \mathrm{Reg} \subseteq S$, ordered by end-extension.

It is easy to see that for every $\gamma < \delta$ the set of conditions c with $\max(c) > \gamma$ is dense and γ-closed, so that $\mathrm{CU}_{\mathrm{Reg}}(\delta, S)$ forces that almost every regular cardinal is in S while adding no $<\delta$-sequences.

We now describe a kind of "universal" iteration of this forcing. To be more precise we define by recursion $\mathbb{Q}_\alpha$ for $\alpha \le \delta^+$ and $\mathbb{Q}_\alpha$-names $\dot{S}_\alpha$ for $\alpha < \delta^+$ so that

1. $f \in \mathbb{Q}_\alpha$ if and only if
   (a) f is a partial function on α.
   (b) $\mathrm{dom}(f)$ has size less than δ, and $f(\beta)$ is a closed bounded subset of δ for all $\beta \in \mathrm{dom}(f)$.
   (c) For all $\alpha \in \mathrm{dom}(f)$, $f \restriction \alpha \Vdash_{\mathbb{Q}_\alpha} f(\alpha) \cap \mathrm{Reg} \subseteq \dot{S}_\alpha$.
2. For conditions $f, g \in \mathbb{Q}_\alpha$, $f \le g$ if and only if $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and $f(\beta)$ end-extends $g(\beta)$ for all $\beta \in \mathrm{dom}(g)$.
3. (Universality) Every $\mathbb{Q}_{\delta^+}$-name for a subset of δ is equivalent to $\dot{S}_\alpha$ for unboundedly many $\alpha < \delta^+$.

For every α, $\mathbb{Q}_\alpha$ is δ^+-c.c by an easy Δ-system argument. Also for all γ and α the set of $f \in \mathbb{Q}_\alpha$ such that $\max(f(\beta)) > \gamma$ for all $\beta \in \mathrm{dom}(f)$ is dense and γ-closed. The GCH assumption and the δ^+-c.c make it possible to satisfy universality.

19.2 Remark. We are cheating slightly, in the sense that we should really verify that $\mathbb{Q}_\alpha$ is equivalent to an iteration of club-shooting forcing. See the remarks on the "flat condition trick" in Section 17.

19.3 Lemma. *Let $\mathbb{Q}^*_{\delta^+}$ be built in a similar way from a sequence of names $\dot{S}^*_\alpha$ satisfying clauses 1 and 2 above. Then there is a complete embedding of $\mathbb{Q}^*_{\delta^+}$ into $\mathbb{Q}_{\delta^+}$.*

Sketch of proof. This is almost immediate if we use the flat conditions trick to regard $\mathbb{Q}_{\delta^+}$ and $\mathbb{Q}^*_{\delta^+}$ as dense sets in iterations of club shooting forcing. We may also proceed quite explicitly by constructing for each α a complete embedding i_α of $\mathbb{Q}_\alpha$ into $\mathbb{Q}^*_{\beta_\alpha}$ for a suitable $\alpha < \delta^+$. At successor stages we use i_α to identify the $\mathbb{Q}^*_\alpha$-name $\dot{S}^*_\alpha$ with a $\mathbb{Q}_{\beta_\alpha}$-name, use universality to find $\gamma > \beta_\alpha$ such that this name is $\dot{S}_\gamma$, and then set $\beta_{\alpha+1} = \gamma + 1$ and extend to $i_{\alpha+1} : \mathbb{Q}^*_{\alpha+1} \to \mathbb{Q}_{\gamma+1}$ in the obvious way; at limits we just take a suitable limit of the embeddings i_α and check that everything works. $\dashv$

We are now ready to build the model. We will do a reverse Easton iteration of length $\kappa + 1$. For $\alpha < \kappa$ we let $\dot{\mathbb{Q}}_\alpha = \{0\}$ unless α is inaccessible, in which case we let $\dot{\mathbb{Q}}_\alpha$ name some universal iteration as above for α.

We fix some normal measure U and let $j : V \longrightarrow M$ be the associated ultrapower map. Let $\dot{\mathbb{Q}}$ be the member of M represented by $\langle \mathbb{Q}_\alpha : \alpha < \kappa \rangle$. Since ${}^\kappa M[G_\kappa] \subseteq M[G_\kappa]$, it is routine to check that $\mathbb{Q}$ is a universal iteration in $V[G_\kappa]$; we let $\dot{S}_j$ be the set which is used at stage j.

The last step $\mathbb{Q}_\kappa$ in our iteration will be a certain sub-iteration of $\mathbb{Q}$. The idea is to build a submodel $V[G * g_0]$ of $V[G * g]$ (where g is $\mathbb{Q}$-generic) and an embedding j which is defined in $V[G * g]$ and has domain $V[G * g_0]$, in such a way that if $S \in V[G * g_0]$ then $\Vdash \kappa \in j(S)$ if and only if S contains a club. A slightly subtle point is that as the construction proceeds we can anticipate in $V[G_\kappa]$ which of the names $\dot{S}_i$ are naming sets S of this type, and pick out the sub-iteration $\mathbb{Q}_\kappa$ so that we shoot a club through each one.

By the usual arguments $\mathbb{P}_\kappa * \mathbb{Q}$ is an initial segment of $j(\mathbb{P}_\kappa)$. If $G_\kappa * g$ is $\mathbb{P}_\kappa * \mathbb{Q}$-generic then as usual we may build in $V[G_\kappa * g]$ a $M[G_\kappa * g]$-generic filter H for the factor iteration $j(\mathbb{P}_\kappa)/G * g$, and then extend to get $j : V[G_\kappa] \longrightarrow M[G_\kappa * g * H]$.

Working in $V[G_\kappa]$, we construct an increasing sequence $\langle \alpha_i : i < \kappa^+ \rangle$ of ordinals, subiterations $\mathbb{Q}^*_i$ of $\mathbb{Q}$ and names for conditions $r_i \in j(\mathbb{Q}^*_i)$ as follows:

1. $\mathbb{Q}^*_i$ is the subiteration of $\mathbb{Q}$ which adds a club subset $C_j \subseteq \kappa$ with $C_j \cap \mathrm{Reg} \subseteq S_{\alpha_j}$ for each $j < i$.
2. r_i is a $j(\mathbb{P}_\kappa)$-name for a condition in $j(\mathbb{Q}^*_i)$ which is a strong master condition for j and the $\mathbb{Q}^*_i$-generic object g_i.

3. α_i is chosen least so that S_{α_i} is a $\mathbb{Q}^*_i$-name and

$$\Vdash_{(j(\mathbb{P}_\kappa)/(\mathbb{P}_\kappa * \mathbb{Q}^*_i)) * (j(\mathbb{Q}^*_i)/r_i)} \kappa \in j(S_{\alpha_i}).$$

4. The domain of r_i is $j``i$, and if $\langle C_k : k < i \rangle$ is the sequence of club sets added by $\mathbb{Q}^*_i$ then $r(j(k)) = C_k \cup \{\kappa\}$.

The construction is very similar to that of Section 17.2 and we omit all details.

Let $\mathbb{Q}_\kappa = \mathbb{Q}^*_{\kappa^+}$ and let g_0 be $\mathbb{Q}_\kappa$-generic over $V[G_\kappa]$. By forcing over $V[G_\kappa * g_0]$ with $\mathbb{Q}/g_0$ we may obtain g which is $\mathbb{Q}$-generic over $V[G]$, and working in $V[G * g]$ we may lift to get $j : V[G] \longrightarrow M[G * g * H]$ as above. Using Magidor's method from Section 13 and the sequence of partial strong master conditions r_i, we may build in $V[G*g]$ an $M[G*g*H]$-generic filter I on $j(\mathbb{Q}_\kappa)$ with $j``g_0 \subseteq I$ and then lift to get $j : V[G*g_0] \longrightarrow M[G*g*H*I]$.

The construction guarantees that for any $T \in V[G*g_0]$ with $T \subseteq \mathrm{Reg} \cap \kappa$, T is non-stationary if and only if $\Vdash_{\mathbb{Q}/g_0} \kappa \notin j(T)$. Since $\mathbb{Q}/g_0$ has the κ^+-c.c. it follows by Lemma 14.5 that $\mathrm{NS}{\restriction}\mathrm{Reg}$ is κ^+-saturated. ⊣

We now sketch Magidor's result that consistently every stationary subset of $\omega_2 \cap \mathrm{Cof}(\omega)$ reflects almost everywhere in $\omega_2 \cap \mathrm{Cof}(\omega_1)$. The construction is quite similar to that for the precipitousness of NS_{ω_1}; we use this as the pretext for omitting many details.

19.4 Remark. Magidor used the optimal hypothesis of weak compactness; to simplify the exposition we use a measurable cardinal.

19.5 Theorem. *If κ is measurable, then in some generic extension $\kappa = \omega_2$ and for every $S \subseteq \omega_2 \cap \mathrm{Cof}(\omega)$ there is a club set C such that $S \cap \alpha$ is stationary for all $\alpha \in C \cap \mathrm{Cof}(\omega_1)$.*

Proof. Let $\mathbb{P} = \mathrm{Col}(\omega_1, {<}\kappa)$ and let $j : V \longrightarrow M$ be the ultrapower map arising from some normal measure U on κ. The idea of the proof is that after forcing with $\mathbb{P}$ every stationary set reflects stationarily often, and we may then shoot club sets to arrange the desired result. Of course new stationary sets will arise as we iterate so some care is required.

Much as in Section 17.2 we will work in $V[G]$ where G is $\mathbb{P}$-generic over V, and define $\mathbb{Q}$ which has the effect of iterating club-shooting with supports of size ω_1. We will be constructing certain strong master conditions as we go, whose existence will imply by Theorem 12.5 that no ω_1-sequences of ordinals are added to $V[G]$ by $\mathbb{Q}$. This is why we can set things up so that the conditions in $\mathbb{Q}$ are just functions in $V[G]$.

Explicitly in $V[G]$ we define by recursion $\mathbb{Q}_\alpha$ and $\mathbb{Q}_\alpha$-names $\dot{S}_\alpha$ such that

1. $\dot{S}_\alpha$ is a $\mathbb{Q}_\alpha$-name for a stationary subset of $\omega_2 \cap \mathrm{Cof}(\omega)$.
2. $f \in \mathbb{Q}_\alpha$ if and only if
   (a) f is a partial function on α with $|\mathrm{dom}(f)| \leq \omega_1$.

(b) For all $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is a closed bounded subset of ω_2 and $f \restriction \beta$ forces that

$$f(\beta) \subseteq \mathrm{Cof}(\omega) \cup \{\gamma \in \mathrm{Cof}(\omega_1) : S_\beta \cap \gamma \text{ is stationary in } \gamma\}$$

Clearly $\mathbb{Q}_\alpha$ is countably closed and an easy Δ-system argument shows that it is κ^+-c.c.

19.6 Remark. Once again we are cheating slightly in the definition of the forcing by using only "flat" conditions. See the remarks on the "flat condition trick" in Section 17.

Exactly as in Section 17.2 we will build embeddings i_α of $\mathbb{P} * \mathbb{Q}_\alpha$ into $j(\mathbb{P})$, with the added wrinkle that we use Theorem 14.3 to ensure that the quotient forcing for prolonging a $\mathbb{P} * \mathbb{Q}_\alpha$-generic to a $j(\mathbb{P})$-generic is countably closed. As we see soon this is crucial for the success of the master condition argument.

At a stage $\alpha < \kappa^+$, if $\langle C_\beta : \beta < \alpha \rangle$ is the sequence of club sets added by $\mathbb{Q}_\alpha$, then we define r_α as follows: $\mathrm{dom}(r_\alpha) = j``\alpha$, and $r_\alpha(j(\beta)) = C_\beta \cup \{\kappa\}$ for every $\beta < \alpha$. We verify that r_α is a strong master condition just as in Section 17.2, the only sticky point is that since $\mathrm{cf}(\kappa) = \omega_1$ after forcing with $j(\mathbb{P})$ we need to know that r_β forces that $j(S_\beta) \cap \kappa$ is stationary. This is easy because (by virtue of being a master condition) r_β forces that $j(S_\beta) \cap \kappa = S_\beta$, and since we are in a countably closed extension of $V[G * g_\beta]$ we see that the stationarity of S_β is preserved.

It is now easy to see that forcing with $\mathbb{Q}_{\kappa^+}$ adds no ω_1-sequences of ordinals to $V[G]$, so that κ is preserved. By the usual book-keeping we may arrange that every $\mathbb{Q}_{\kappa^+}$-name for a stationary subset of $\kappa \cap \mathrm{Cof}(\omega)$ appears as S_α for some $\alpha < \kappa^+$. If H is $\mathbb{Q}_{\kappa^+}$-generic over $V[G]$ then $V[G * H]$ is as required. $\dashv$

20. More on Collapses

We have seen many applications of the Lévy collapse. In this section we discuss two situations where the Lévy collapse cannot be used, one involving master conditions and the other involving absorbing "large" forcing posets into a collapsing poset. We shall describe some more exotic collapsing posets which can sometimes be used in these situations, namely the Silver collapse and Kunen's universal collapse. We then show how these can be applied by sketching Kunen's consistency proof [45] for an ω_2-saturated ideal on ω_1.

We have seen many situations where we are given $\mathbb{P} = \mathrm{Col}(\delta, {<}\kappa)$ and an elementary embedding j with critical point κ, and wish to lift j to the extension by $\mathbb{P}$. Here there is no master condition issue because $j \restriction \mathbb{P} = \mathrm{id}$ and $\mathbb{P}$ is just an initial segment of $j(\mathbb{P})$.

But now consider the following situation: $k : M \longrightarrow N$ has critical point κ, $\mathbb{P} = \mathrm{Col}(\kappa, {<}\lambda)_M$, G is $\mathbb{P}$-generic over M and both G and $k \restriction \lambda$ are in N. Certainly we may form in N a partial function $Q = \bigcup k``G$, where

$\mathrm{dom}(Q) = \kappa \times k``\lambda$; but if $\lambda \geq k(\kappa)$ then Q has the wrong shape to be a condition in $k(\mathbb{P})$.

To fix this we consider a cardinal collapsing poset due to Silver, which was first used by him in the consistency proof for Chang's Conjecture.

20.1 Definition. Let κ be inaccessible and let $\delta = \mathrm{cf}(\delta) < \kappa$. The *Silver collapse* $\mathbb{S}(\delta, {<}\kappa)$ is the set of those partial functions f on $\delta \times \kappa$ such that $\mathrm{dom}(f) = \alpha \times X$ for some $\alpha < \delta$ and some $X \in [\kappa]^\delta$, and $f(\beta, \gamma) < \gamma$ for all $\beta < \alpha$ and $\gamma \in X$. The ordering is extension.

It is easy to see that $\mathbb{S}(\delta, {<}\kappa)$ is δ-closed and κ-c.c.

Returning for a moment to the discussion preceding Definition 20.1, if we let $\mathbb{P} = \mathbb{S}(\kappa, {<}\lambda)$ where $\lambda = k(\kappa)$ then it *is* possible to build a strong master condition. We will use this shortly, but first we discuss another problem with the Lévy collapse.

Suppose that $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa)$ and that $\mathbb{B}$ is a complete subalgebra of $\mathrm{ro}(\mathbb{P})$. Then as we saw in Theorem 14.2 we can embed $\mathbb{B} * \dot{\mathbb{C}}$ into $\mathbb{P}$ when $\dot{\mathbb{C}}$ names an algebra of size less than κ. However there is no guarantee that this is possible when $\dot{\mathbb{C}}$ has size κ, even if $\mathbb{C}$ is forced to have the κ-c.c.

Kunen [45] showed that it is possible to construct a poset with stronger universal properties. We sketch a version of his construction. Let κ be an inaccessible cardinal, and let U be a function which returns for each complete Boolean algebra $\mathbb{B}$ of size less than κ a $\mathbb{B}$-name $U(\mathbb{B})$ for a κ-Knaster poset of size κ. We aim to build a κ-c.c. poset $\mathbb{P}$ of size κ such that for every complete subalgebra $\mathbb{B}$ of $\mathrm{ro}(\mathbb{P})$ with size less than κ, the inclusion embedding of $\mathbb{B}$ into $\mathrm{ro}(\mathbb{P})$ extends to a complete embedding of $\mathbb{B} * \mathrm{ro}(U(\mathbb{B}))$ into $\mathrm{ro}(\mathbb{P})$.

To construct the universal collapse we build a finite support κ-c.c. iterated forcing poset $\mathbb{P}_\kappa$ of length and cardinality κ, where each step $\mathbb{P}_\alpha$ is κ-c.c. with cardinality κ. At stage α we choose by some book-keeping scheme some $\mathbb{B}_\alpha$ which is a complete subalgebra of $\mathrm{ro}(\mathbb{P}_\alpha)$ with $|\mathbb{B}_\alpha| < \kappa$. Given an $\mathbb{B}_\alpha$-generic filter g we may form in $V[g]$ the product $\mathbb{P}_\alpha / g \times U(\mathbb{B}_\alpha)$, which is κ-c.c. by Theorem 5.12. Back in V we see that $\mathbb{B}_\alpha * (\mathbb{P}_\alpha / \dot{g} \times U(\mathbb{B}_\alpha))$ is κ-c.c. and embeds both $\mathbb{P}_\alpha$ and $\mathbb{B}_\alpha * U(\mathbb{B}_\alpha)$, and choose $\mathbb{P}_{\alpha+1}$ accordingly. With appropriate book-keeping we may arrange that every small subalgebra of $\mathrm{ro}(\mathbb{P}_\kappa)$ has appeared as $\mathbb{B}_\alpha$ for some α, giving the desired universal property for $\mathbb{P}_\kappa$. Preservation of κ-c.c. is easy since we are iterating with finite support.

20.2 Remark. The construction of the universal collapse is an example of "iteration with amalgamation", a technique which is frequently used in forcing constructions to build saturated ideals. Note that in the construction we amalgamated $\mathbb{P}_\alpha$ and $U(\mathbb{B}_\alpha)$ over $\mathbb{B}_\alpha$. The point in applications will typically be that we can absorb an iteration $\mathbb{P} * \dot{\mathbb{Q}}$ into $j(\mathbb{P})$ in a context where $\mathbb{P} * \dot{\mathbb{Q}}$ is "large".

20.3 Remark. Laver showed that is sometimes possible to build λ-closed collapsing posets with similar universal properties. Naturally one needs to be a little more careful about the chain condition.

We are now ready to sketch Kunen's consistency proof for an ω_2-saturated ideal on ω_1. More details will be found in Foreman's chapter in this Handbook.

20.4 Theorem. *Let κ be a huge cardinal with target λ. Then in some generic extension κ is ω_1, $\lambda = \omega_2$ and there is an ω_2-saturated ideal on ω_1.*

Proof. We fix an elementary embedding $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$, and ${}^{\lambda}M \subseteq M$. We start by constructing as above a κ-c.c. poset $\mathbb{P}$ of size κ such that for every subalgebra $\mathbb{B}$ of $\mathrm{ro}(\mathbb{P})$ with $|\mathbb{B}| < \kappa$, the inclusion embedding extends to an embedding of $\mathbb{B} * \mathrm{ro}(\mathbb{S}(\omega_1^{V^{\mathbb{B}}}, {<}\kappa))$. For convenience we assume (as we clearly may) that $\mathbb{P} \subseteq V_\kappa$.

It is easy to see that after forcing with $\mathbb{P}$, κ is the new ω_1. Let G be generic for $\mathbb{P}$, let $\mathbb{Q} = \mathbb{S}(\kappa, {<}\lambda)_{V[G]}$ and let H be $\mathbb{Q}$-generic over $V[G]$.

We will show that there is a λ-saturated ideal on κ in $V[G * H]$. Working in M we fix an embedding of $\mathrm{ro}(\mathbb{P} * \mathbb{Q})$ into $\mathrm{ro}(j(\mathbb{P}))$ extending the identity embedding of $\mathrm{ro}(\mathbb{P})$. Since $V[G * H] \models {}^{\lambda}M[G * H] \subseteq M[G * H]$, we see that $j(\mathbb{P})/G * H$ is λ-c.c. in $V[G * H]$. Forcing with this poset over $V[G * H]$, we obtain an embedding $j^{+} : V[G] \longrightarrow M[G*H*I]$ in $V[G*H*I]$. Since $\mathbb{Q}$ is a Silver collapse, and H and $j \restriction \lambda$ are both in M, we may construct a strong master condition $r = \bigcup j``H \in j(\mathbb{Q})$ and force with $j(\mathbb{Q})/r$ to obtain a compatible generic object J and an embedding $j^{++} : V[G * H] \longrightarrow M[G * H * I * J]$.

Unfortunately this is not quite enough because the $V[G * H]$-ultrafilter $U = \{X \in P(\kappa) \cap V[G * H] : \kappa \in j^{++}(X)\}$ lives in the extension by $j(\mathbb{P})/G * H * j(\mathbb{Q})/r$, which is not λ-c.c. in $V[G * H]$. To fix this we note that in $V[G * H * I]$ the $V[G * H]$-powerset of κ has size λ, and $j^{+}(\mathbb{Q})$ is λ-closed; so we may build a decreasing sequence $\langle r_i : i < \lambda\rangle$ with $r_0 = r$ deciding whether $\kappa \in j^{++}(X)$ for all $X \in P(\kappa) \cap V[G * H]$, and then let $U_0 = \{X : \exists i \ r_i \Vdash \kappa \in j^{++}(X)\}$. Then U_0 is a $V[G * H]$-ultrafilter which lives in $V[G * H * I]$, so that we may derive a λ-saturated ideal by Lemma 14.5. ⊣

20.5 Remark. A Woodin cardinal is all that is required to get the consistency of an ω_2-saturated ideal on ω_1. The argument given here was generalised by Laver to get saturated ideals on larger cardinals. Magidor [54] showed that the kind of argument given here can be done from an almost huge cardinal.

21. Limiting Results

In this section we sketch some results which put limits on the effects which we can achieve in Reverse Easton constructions. We are not sure to whom the following result should be attributed; it has a family resemblance to some results by Kunen [45] on the question of whether an inaccessible cardinal λ can carry a λ-saturated ideal.

21.1 Theorem. *If $\mathbb{P} \times \mathbb{P}$ is κ-c.c. and $\mathbb{P}$ forces that κ is measurable then κ is measurable.*

Proof. Clearly κ is inaccessible in V. Let $\dot{U}$ name a normal measure and suppose that κ is not measurable in V. If A is a potential member of $\dot{U}$ then it can be split into two disjoint potential members of $\dot{U}$, otherwise we could read off a measure on κ in V. Using this we build a binary tree of height κ with root node κ such that the levels form increasingly fine partitions of κ into fewer than κ many pieces. At successor steps every node is partitioned by its two immediate successors, and if a node is a potential member of $\dot{U}$ then so are both of its immediate successors; at limit steps λ, every branch through the binary tree of height λ which has been constructed so far is continued by putting at level λ the intersection of the nodes on that branch.

Now let G be $\mathbb{P}$-generic and realise $\dot{U}$ as U_G; then there is a unique branch through the tree consisting of members of U_G. Choosing for each A on the branch a condition which forces the successor of A which is not in U_G into $\dot{U}$, we build an antichain of size κ in $\mathbb{P}$, contradicting our assumption that $\mathbb{P} \times \mathbb{P}$ has the κ-c.c. $\dashv$

We now sketch some results of Hamkins [33]. The key technical result is Theorem 21.3 which involves two notions of resemblance between inner models of ZFC.

21.2 Definition. Let $M \subseteq N$ with M and N inner models of ZFC. Let δ be a regular uncountable cardinal in N. Then

1. *δ-covering holds between M and N* if and only if for every set $A \subseteq \mathrm{On}$ such that $A \in N$ and $N \models |A| < \delta$, there exists a set $B \subseteq \mathrm{On}$ such that $B \in M$, $A \subseteq B$ and $M \models |B| < \delta$.

2. *δ-approximation holds between M and N* if and only if for every $A \subseteq \mathrm{On}$ with $A \in N$, if $A \cap a \in M$ for all $a \in M$ with $M \models |a| < \delta$, then $A \in M$.

21.3 Theorem. *Let V and $\bar{V}$ be inner models with $V \subseteq \bar{V}$. Let $j : \bar{V} \longrightarrow \bar{M}$ be a definable elementary embedding with $\mathrm{crit}(j) = \kappa$, and let $M = \bigcup j``V$ so that $j \restriction V$ is an elementary embedding from V to M.*

If there is a cardinal $\delta < \kappa$ regular in $\bar{V}$ such that $\bar{V} \models {}^{\delta}\bar{M} \subseteq \bar{M}$, and the δ-covering and δ-approximation properties hold between V and $\bar{V}$, then

1. *$M = \bar{M} \cap V$, in particular $V \models {}^{\delta}M \subseteq M$.*

2. *$j \restriction A \in V$ for all $A \in V$.*

Proof. Throughout the proof we work in $\bar{V}$. In particular all cardinalities are computed in $\bar{V}$ unless otherwise specified. By elementarity and the fact that $\delta < \kappa$, the δ-covering and δ-approximation properties hold between M and $\bar{M}$.

We claim that every set of ordinals A with $|A| < \delta$ is contained in a set of ordinals $B \in V \cap M$ such that $|B| \leq \delta$. To see this we build (starting with A) an increasing and continuous chain of length δ consisting of sets of size less than δ, with even successor elements in V and odd successor elements in M. If B is the union then by the approximation property $B \in V \cap M$.

Next we claim that for every set of ordinals A with $|A| < \delta$, $A \in V$ if and only if $A \in M$. To see this find a set $B \in V \cap M$ with $A \subseteq B$ and $\gamma = \mathrm{ot}(B) < \delta^+$. Since $\gamma < \kappa$ and κ is inaccessible in V, it follows from Proposition 2.9 that $P(\gamma) \cap M = P(\gamma) \cap V$.

Now we claim that $M = \bar{M} \cap V$. Let $A \in M$ where by Proposition 2.2 we may assume that A is a set of ordinals. Clearly $A \in \bar{M}$. Let $a \in V$ with $|a| < \delta$. Applying the preceding claim $a \in M$, hence $A \cap a \in M$, hence by another application of the preceding claim $A \cap a \in V$. By the approximation property $A \in V$. Conversely let $A \in \bar{M} \cap V$ be a set of ordinals; arguing just as before $A \cap a \in M$ for all $a \in M$ with $|a| < \delta$, so that $A \cap a \in M$.

To finish we show that $j \restriction A \in V$ for all sets of ordinals $A \in V$. By approximation it will suffice to show that $j \restriction a \in V$ for all $a \in V$ with $a \subseteq A$ and $|a| < \delta$. Since $\mathrm{ot}(a) < \kappa$ we see that $j``a = j(a) \in M \subseteq V$, and since $j \restriction a$ is the order-isomorphism between $a$ and $j``a$ we have $j \restriction a \in V$. $\dashv$

21.4 Corollary. *Under the hypotheses of Theorem 21.3, if $\bar{V}$ is a set-generic extension of V then $j \restriction V$ is definable in V. It is also easy to see that if j witnesses the λ-supercompactness or λ-strongness of κ in $\bar{V}$ then $j \restriction V$ will do the same in V.*

Of course the interest of Theorem 21.3 hinges on there being some examples of extensions with the covering and approximation properties. The following result [33] shows that many extensions by Reverse Easton iterations have these properties.

21.5 Theorem. *Let δ be a cardinal. Let $\mathbb{P} * \dot{\mathbb{Q}}$ be a forcing iteration where $|\mathbb{P}| \leq \delta$, $\mathbb{P}$ is non-trivial and $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is $(\delta + 1)$-strategically closed. Then the δ^+-covering and δ^+-approximation properties hold between V and the extension by $\mathbb{P} * \dot{\mathbb{Q}}$.*

Proof. The covering is easy so we concentrate on the approximation. Let $G * H$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-generic filter and let $S : \theta \to 2$ be such that $S \in V[G * H]$ and $S \restriction a \in V$ for all $a \in V$ with $|a| \leq \delta$. By induction we may assume that $S \restriction \lambda \in V$ for all $\lambda < \theta$. Let $\dot{S}$ name S.

If $\mathrm{cf}(\theta) \leq \delta$ then $S \in V[G]$, so without loss of generality $\dot{S}$ is a $\mathbb{P}$-name. Consider the tree T of potential proper initial segments of $\dot{S}$; it is easy to see that there are at most δ many sequences t such that both $t^\frown 0$ and $t^\frown 1$ are in T. So by specifying δ many bits in S we determine S, hence $S \in V$.

If $\mathrm{cf}(\theta) > \delta$ in V, we note that this remains true in $V[G * H]$. So since $|G| \leq \delta$, there is a condition $p \in G$ such that for all $i < \theta$ there is a condition $q \in H$ so that (p, q) determines $\dot{S} \restriction i$. We may thus find a condition $(p, \dot{q}_0) \in G * H$ forcing that $\dot{S} \notin \check{V}$ and that p has this property; so easily for all i and all $(p, \dot{q}_1) \leq (p, \dot{q}_0)$ there is a condition $(p, \dot{q}_2) \leq (p, \dot{q}_1)$ determining $\dot{S} \restriction i$.

Using the non-triviality of $\mathbb{P}$ we can find a function $h \in V[G] \setminus V$ such that $h : \beta \to 2$ for some $\beta \leq \delta$, where (by choosing β to be minimal) we may also assume that $h \restriction j \in V$ for all $j < \beta$. Using the strategic closure of $\dot{Q}$, the choice of p and the fact that $\dot{S}$ is forced to be new we build $\langle \dot{q}_t : t \in {}^{<\beta}2 \rangle$ and $\langle \dot{\beta}_t : t \in {}^{<\beta}2 \rangle$ such that

1. For each t, $\dot{q}_t$ is a $\mathbb{P}$-name for a condition in $\mathbb{Q}$, and $\dot{\beta}_t$ is a $\mathbb{P}$-name for an element of ${}^{<\theta}2 \cap V$.
2. The sequences $\langle \dot{q}_t : t \in {}^{<\beta}2 \rangle$ and $\langle \dot{\beta}_t : t \in {}^{<\beta}2 \rangle$ lie in V.
3. It is forced by p that for any branch x of the tree ${}^{<\beta}2 \cap V$, the sequence $\langle q_{x \restriction j} : j < \beta \rangle$ has a lower bound.
4. $(p, q_{t^\frown i})$ forces that $\beta_t^\frown i$ is an initial segment of S.

Now working in $V[G]$ we choose a lower bound q for $\langle q_{h \restriction j} : j < \beta \rangle$. If we force so that H contains q we obtain a situation in which h can be computed from a proper initial segment of S, contradiction! ⊣

As an example of these ideas in action we sketch an easy case of the *superdestructibility* theorem of Hamkins [32]. A supercompact cardinal κ is said to be *Laver indestructible* if it is supercompact in every extension by κ-directed closed forcing; we show in Section 24 that any supercompact cardinal can be made indestructible.

21.6 Corollary. *Let κ be supercompact and let $\mathbb{P} = \mathrm{Add}(\omega, 1)$. Then κ is not Laver indestructible after forcing with $\mathbb{P}$.*

Proof. Let g be $\mathbb{P}$-generic, and let $\mathbb{Q} = \mathrm{Add}(\kappa, 1)_{V[g]}$. Let G be $\mathbb{Q}$-generic over $V[g * G]$. we show that κ is not measurable in $V[g * G]$.

Let $\bar{V} = V[g*G]$ and suppose that $j : \bar{V} :\longrightarrow \bar{M}$ is the ultrapower by some normal measure in $\bar{V}$. By Theorems 21.5 and 21.3 we have $j \restriction V : V \longrightarrow M$ where $M \subseteq V$. Now easily $\bar{M} = M[g * j(G)]$, by the closure of ultrapowers $G \in \bar{M}$, and by the closure of $j(\mathbb{Q})$ we have $G \in M[g]$. This is impossible as $M \subseteq V$ and $G \notin V[g]$. ⊣

22. Termspace Forcing

In this section we introduce a very useful idea due to Laver, that of the *term forcing* or *termspace forcing*. The idea is roughly that given a two step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ we can add by forcing over V a sort of "universal generic object", from which given any G which is $\mathbb{P}$-generic over V we may compute in a uniform way an H which is $i_G(\dot{\mathbb{Q}})$-generic over $V[G]$.

Magidor [52] showed by iterated Prikry forcing that the least measurable cardinal can be strongly compact. In unpublished work Magidor [51] gave an alternative proof, using term forcing and an Easton iteration of the forcing from Example 6.5. We outline the proof here, a more detailed account is given in a joint paper by Apter and the author [4] which further exploits these ideas.

22.1 Definition. Let $\mathbb{P}$ be a notion of forcing and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a notion of forcing. Then $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$ is the notion of forcing whose underlying set is the set of canonical $\mathbb{P}$-terms for members of $\dot{\mathbb{Q}}$, with the ordering being given by

$$\dot{\sigma} \leq_{\mathrm{A}(\mathbb{P},\dot{\mathbb{Q}})} \dot{\tau} \iff \Vdash_{\mathbb{P}} \dot{\sigma} \leq_{\dot{\mathbb{Q}}} \dot{\tau}.$$

22.2 Remark. Several notations for the termspace forcing are in use, for example $\mathbb{Q}^*$ and $\mathbb{Q}^{\mathbb{P}}$. We follow Foreman's paper [19] in using $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$, emphasising the importance of $\mathbb{P}$.

The following proposition is the key to the applications of term forcing.

22.3 Proposition. *Let G be $\mathbb{P}$-generic over V and let H be $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$-generic over V. Define $I = \{i_G(\dot{\tau}) : \dot{\tau} \in H\}$. Then I is an $i_G(\dot{\mathbb{Q}})$-generic filter over $V[G]$.*

Proof. We begin by checking that I is a filter. If $\dot{\sigma}$ and $\dot{\tau}$ are in H then there is a term $\dot{\rho} \in H$ such that $\Vdash_{\mathbb{P}} \dot{\rho} \leq \dot{\sigma}, \dot{\tau}$. It follows that $i_G(\dot{\rho}) \leq i_G(\dot{\sigma}), i_G(\dot{\tau})$ so that I is a directed set.

If $i_G(\dot{\sigma}) \leq i_G(\dot{\tau})$ with $\dot{\sigma} \in H$ then we fix $p \in G$ such that $p \Vdash \dot{\sigma} \leq \dot{\tau}$. Let $\dot{\rho}$ be a name which is interpreted as $\dot{\tau}$ if p is in the generic filter and as the trivial condition otherwise, so that $p \Vdash \dot{\rho} = \dot{\tau}$ and $\Vdash_{\mathbb{P}} \dot{\sigma} \leq \dot{\rho}$. Then $\dot{\tau} \in H$ and so $i_G(\dot{\rho}) = i_G(\dot{\tau}) \in I$. It follows that I is upwards closed, and so is a filter.

Finally let $D = i_G(\dot{D})$ where $\dot{D}$ is forced to be a dense subset of $\dot{\mathbb{Q}}$. If $E = \{\dot{\sigma} : \Vdash_{\mathbb{P}} \dot{\sigma} \in \dot{D}\}$ then by the maximum principle E is a dense subset of $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$. We find a term $\dot{\sigma} \in E \cap H$, and observe that $i_G(\dot{\sigma}) \in D \cap I$. It follows that I is $i_G(\dot{\mathbb{Q}})$-generic over $V[G]$ as required. ⊣

The next result is an easy application of the maximum principle.

22.4 Proposition. *If it is forced by $\mathbb{P}$ that $\dot{\mathbb{Q}}$ is κ-strategically closed then $\mathrm{A}(\mathbb{P}, \mathbb{Q})$ is κ-strategically closed.*

Foreman's paper "More saturated ideals" [19] contains a wealth of other structural results about $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$. We quote some here.

22.5 Proposition. *Let $\mathbb{P}$ be a poset and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a poset.*

1. *If $\mathbb{P}$ is non-trivial and it is not forced that $\dot{\mathbb{Q}}$ is κ-c.c. then $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$ is not 2^κ-c.c.*

2. *If κ is inaccessible, $\mathbb{P}$ is κ-c.c. and it is forced that $\mathbb{Q}$ is κ-c.c. then $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$ is κ-c.c.*

3. *If $\langle P_i, \dot{\mathbb{Q}}_i \rangle$ is a forcing iteration with supports in an ideal I, then the limit of the iteration can be completely embedded in the product of the termspace posets $\mathrm{A}(\mathbb{P}_i, \dot{\mathbb{Q}}_i)$ taken with supports in I.*

We will now use term forcing to give a proof (due to Magidor) that the least measurable cardinal can be strongly compact. The idea of the proof is to shoot a non-reflecting stationary set through each measurable cardinal below a supercompact cardinal κ, and then argue that the strong compactness of κ is preserved and no new measurable cardinals are created.

To get an embedding witnessing strong compactness we use the following easy result.

22.6 Proposition. *Let $j : V \longrightarrow M$ be an embedding with critical point κ, and let $\lambda \geq \kappa$ be such that $j``\lambda \in M$ and $\lambda < j(\kappa)$. Let $k : M \longrightarrow N$ be any embedding with $\mathrm{crit}(k) \geq \kappa$ and let $X = k(j``\lambda)$. Then $\mathrm{crit}(k \circ j) = \kappa$, $X \in N$, $(k \circ j)``\lambda \subseteq X$ and $N \models \mathrm{ot}(X) < k \circ j(\kappa)$.*

In particular if $k \circ j$ is definable then $k \circ j$ witnesses that κ is λ-strongly compact. If $V[G]$ is a generic extension of V, and $i : V[G] \longrightarrow M[H]$ is an embedding definable in $V[G]$ extending $k \circ j$, then i witnesses that κ is λ-strongly compact in $V[G]$.

We now fix a ground model in which GCH holds, κ is supercompact, and there is no measurable cardinal greater than κ. This last hypothesis is a technical one which simplifies some later arguments; it entails no loss of generality because we can truncate the universe at the least measurable greater than κ if such a cardinal exists. Notice that since κ is supercompact there are unboundedly many measurable cardinals less than κ.

Let A be the set of $\alpha < \kappa$ which are measurable in V. We will define an iteration $\mathbb{P}_\kappa$ of length κ with Easton support, in which $\dot{\mathbb{Q}}_\alpha$ names the trivial forcing unless $\alpha \in A$. If $\alpha \in A$ then $\dot{\mathbb{Q}}_\alpha$ names the poset from Example 6.5 to add a non-reflecting stationary set to α, as defined in $V[G_\alpha]$. It is clear that this iteration will destroy the measurability of every α in A. We will show that no new measurable cardinals are created.

Let G_κ be $\mathbb{P}_\kappa$-generic over V and suppose for a contradiction that $\alpha < \kappa$ and α is measurable in $V[G_\kappa]$. By construction $\alpha \notin A$, and if γ is the least measurable greater than α then arguments as in Lemma 11.2 show that $V[G_\kappa]$ is an extension of $V[G_\alpha]$ by γ-strategically closed forcing. In particular α is measurable in $V[G_\alpha]$, from which it easily follows that α must be a Mahlo cardinal in V. Since α is Mahlo, by Proposition 7.13 $\mathbb{P}_\alpha$ is α-Knaster. It follows that $\mathbb{P}_\alpha \times \mathbb{P}_\alpha$ is α-c.c. By Theorem 21.1 α must be measurable in V, which is a contradiction as $\alpha \notin A$.

To finish, we show that κ is still strongly compact in $V[G_\kappa]$. We fix a regular cardinal $\lambda > \kappa$ and let $j : V \longrightarrow M$ be the ultrapower of V by a supercompactness measure on $P_\kappa\lambda$. The argument of Example 4.8 shows that $V \models |j(\kappa)| = \lambda^+$.

By GCH $\lambda \geq 2^\kappa$ and so κ is measurable in M, and we may find a measure U on κ such that $U \in M$ and U is minimal in the Mitchell ordering [58];

we let $k : M \longrightarrow N$ be the ultrapower of M by U, so that in particular $N \models$ "κ is not measurable". It is easy to see that $\kappa \notin k \circ j(A)$.

Consider the iteration $j(\mathbb{P}_\kappa)$, which is an iteration defined in M in which a non-reflecting stationary set is added to each $\alpha \in j(A)$. The cardinal κ is measurable in M, so $\kappa \in j(A)$ and $j(\mathbb{P})_\kappa$ adds a set at κ. There are no measurable cardinals above κ in V and $P(\lambda) \subseteq M$, so if γ is the least M-measurable cardinal greater than κ then $\gamma > \lambda$.

Notice that since we are aiming to show that κ is strongly compact (and so a fortiori measurable) in $V[G_\kappa]$ we cannot hope to find a $\mathbb{Q}_\kappa$-generic filter over $M[G_\kappa]$ in $V[G_\kappa]$. It is at this point that we use term forcing.

Working in M we may factor $j(\mathbb{P}_\kappa)$ as $\mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, where $\dot{\mathbb{Q}}$ adds a non-reflecting stationary subset of κ. Working in $M[G_\kappa]$ we get a factorisation of the rest of the iteration as $\mathbb{Q} * \dot{\mathbb{R}}$.

22.7 Lemma. *$\dot{\mathbb{R}}$ is $j(\kappa)$-c.c. and γ-strategically closed in $M[G_\kappa]^{\mathbb{Q}}$.*

Proof. It follows from Proposition 7.13 that $\mathbb{R}$ is $j(\kappa)$-c.c. The closure follows from Proposition 7.12. ⊣

22.8 Lemma. *In $M[G_\kappa]$, $\mathrm{A}(\mathbb{Q}, \dot{\mathbb{R}})$ is $j(\kappa)$-c.c. and λ^+-strategically closed.*

Proof. We work in the model $M[G_\kappa]$. The strategic closure follows by Proposition 22.4.

For the chain condition, assume for a contradiction that $\langle \dot{r}_\alpha : \alpha < j(\kappa) \rangle$ is an antichain in $\mathrm{A}(\mathbb{Q}, \dot{\mathbb{R}})$. If $\alpha < \beta$ then $\dot{r}_\alpha$ and $\dot{r}_\beta$ are incompatible, which means that there is no term for a condition forced to refine both of them; by the maximum principle this is equivalent to saying that $\dot{r}_\alpha$ and $\dot{r}_\beta$ are not forced to be compatible in $\dot{\mathbb{R}}$.

For $\alpha < \beta$ we choose $q_{\alpha\beta} \in \mathbb{Q}$ such that $q_{\alpha\beta} \Vdash^{M[G_\kappa]}_{\mathbb{Q}} \dot{r}_\alpha \perp \dot{r}_\beta$. $j(\kappa)$ is measurable in M and so by the Lévy-Solovay Theorem [50] $j(\kappa)$ is measurable in $M[G_\kappa]$. By Rowbottom's theorem we may therefore find a fixed $q \in \mathbb{Q}$ and $X \subseteq j(\kappa)$ unbounded such that $q_{\alpha\beta} = q$ for all $\alpha, \beta \in X$. q forces that $\{\dot{r}_\alpha : \alpha \in X\}$ is an antichain of size $j(\kappa)$ in $\mathbb{R}$, contradicting Lemma 22.7. So $\mathrm{A}(\mathbb{Q}, \dot{\mathbb{R}})$ is $j(\kappa)$-c.c. in $M[G_\kappa]$. ⊣

Appealing to Proposition 8.1 we may now build $H \in V[G_\kappa]$ which is $\mathrm{A}(\mathbb{Q}, \dot{\mathbb{R}})$-generic over $M[G_\kappa]$.

We now consider the embedding k and the iteration $k(\mathbb{P}_\kappa)$. Since κ is not a point at which this iteration adds a set, we may argue exactly as in Section 11 to build $g \in M[G_\kappa]$ such that $G_\kappa * g$ is $k(\mathbb{P}_\kappa)$-generic over $N[G_\kappa]$, and may lift to get $k : M[G_\kappa] \longrightarrow N[G_\kappa * g]$. By similar arguments we may also build $h \in M[G_\kappa]$ which is $k(\mathbb{Q})$-generic over $N[G_\kappa * g]$.

By Proposition 9.3 this lifted embedding has width $\leq \kappa$, so by Proposition 15.1 we may transfer H along k to get H^+ which is $k(\mathrm{A}(\mathbb{Q}, \mathbb{R}))$-generic over $N[G_\kappa * g]$. If we let $I = \{i_h(\dot{\sigma}) : \dot{\sigma} \in H^+\}$ then I is $k(\mathbb{R})$-generic over $N[G_\kappa * g * h]$.

Putting everything together we get $G_\kappa * g * h * I$ which is $k \circ j(\mathbb{P}_\kappa)$-generic over N, and then as in Section 11 we may lift $k \circ j$ to get a map from $V[G_\kappa]$ to $N[G_\kappa * g * H * I]$. This map is definable by Proposition 9.4, and so by Proposition 22.6 we see that κ is λ-strongly compact in $V[G_\kappa]$.

23. More on Term Forcing and Collapsing

In this section we show that the term forcing ideas of Section 22 may be used to analyse iterations. We also introduce yet another cardinal collapsing poset, this time one due to Mitchell [57]

We give an outline of Mitchell's model [57] in which there are no ω_2-Aronszajn trees. Our treatment of this material owes much to Abraham [2]. For simplicity we build the model using a measurable cardinal. Mitchell actually used a weakly compact cardinal and this is known to be optimal [57].

Throughout this section we assume that κ is measurable. We recall the easy proof that κ has the tree property; let T be a κ-tree, let $j : V \longrightarrow M$ have critical point κ, then $j(T)\restriction\kappa$ is isomorphic to T and any point on level κ of $j(T)$ gives us a branch through T.

We start by making an instructive false start. Let $\mathbb{P} = \mathrm{Col}(\omega_1, {<}\kappa)$ and as in Theorem 10.5 factor $j(\mathbb{P})$ as $\mathbb{P} \times \mathbb{Q}$. If $G * H$ is $j(\mathbb{P})$-generic then we may build as usual an embedding $j : V[G] \longrightarrow M[G * H]$. If $T \in V[G]$ is a κ-tree then as above $j(T)\restriction\kappa$ is isomorphic to κ, so by choosing any point on level κ we may determine a branch b of T.

It is well-known that CH implies there is a special ω_2-Aronszajn tree, and since $V[G]$ is a model of CH and $\kappa = \omega_2$ there is a κ-Aronszajn tree in $V[G]$. This is not a contradiction to the argument of the previous paragraph; the point is that $j(T)$ only exists in $M[G*H]$, so the branch b that we constructed is a member of $V[G * H]$ but not in general a member of $V[G]$.

To put the problem more abstractly, we need to create a situation in which a generic embedding with critical point ω_2 is added by a poset which does not add any branches through any ω_2-Aronszajn tree. By the remarks made above we also need the continuum to be at least ω_2.

Before the main argument we need a technical fact about trees.

23.1 Lemma. *Let $2^\omega = \omega_2$ and let T be an ω_2-Aronszajn tree. Let $\mathbb{S}$ and $\mathbb{T}$ be forcing posets such that*

1. *$\mathbb{T}$ is countably closed forcing and collapses ω_2.*
2. *$\mathbb{S}$ is ω_1-Knaster in $V^{\mathbb{T}}$.*

Then forcing with $\mathbb{S} \times \mathbb{T}$ does not add a cofinal branch of T.

Proof. Let $G_{\mathbb{S}} \times G_{\mathbb{T}}$ be $\mathbb{S} \times \mathbb{T}$-generic. We claim first that T has no cofinal branch in $V[G_{\mathbb{T}}]$. To see this suppose $p \in \mathbb{T}$ forces that $\dot{b}$ is a cofinal branch, and use the fact that $b \notin V$ to build a binary tree $\langle p_s : s \in {}^{<\omega}2\rangle$ and increasing $\langle \alpha_n : n < \omega\rangle$ such that $p_0 = p$ and for each n the conditions $\{p_s : s \in {}^n 2\}$ decide where $\dot{b}$ meets level α_n in 2^n different ways. Then let $\alpha = \sup_n \alpha_n$ and observe that level α must have at least 2^ω elements, contradicting our assumptions that T is an ω_2-tree and $2^\omega = \omega_2$.

Choose in $V[G_{\mathbb{T}}]$ a sequence β_j for $j < \omega_1$ which is cofinal in ω_2^V. Suppose for a contradiction that some $q \in \mathbb{S}$ forces over $V[G_{\mathbb{T}}]$ that $\dot{c}$ is cofinal in T, and then choose for each j a condition $q_j \leq q$ deciding where the branch $\dot{c}$ meets level β_j. In $V[G_{\mathbb{T}}]$ a subfamily of size ω_1 of $\{q_j\}$ must be pairwise compatible, but this implies that there is a cofinal branch in $V[G_{\mathbb{T}}]$. ⊣

23.2 Theorem. *Let κ be measurable. Then in some ω_1-preserving generic extension, $2^\omega = \omega_2 = \kappa$ and κ has the tree property.*

Proof. Let

$$\mathbb{P} = \mathrm{Add}(\omega, \kappa),\ \mathbb{P}_\alpha = \mathrm{Add}(\omega, \alpha),\ \mathbb{R}_\alpha = \mathrm{Add}(\omega_1, 1)_{V^{\mathbb{P}_\alpha}}.$$

We define $\mathbb{Q}$ as follows; a condition is a pair (p, f) where $p \in \mathbb{P}$, f is a partial function on κ with countable support, and $f(\alpha)$ is a $\mathbb{P}_\alpha$-name for a condition in $\mathbb{R}_\alpha$. $(p_2, f_2) \leq (p_1, f_1)$ iff $p_2 \leq p_1$ in $\mathbb{P}$, $\mathrm{supp}(f_1) \subseteq \mathrm{supp}(f_2)$, and $p_2 \restriction (\omega \times \alpha) \Vdash f_2(\alpha) \leq f_1(\alpha)$ for all $\alpha \in \mathrm{supp}(f_1)$.

It is easy to see that $\mathbb{Q}$ is κ-c.c. Since adding a Cohen subset of ω_1 collapses the continuum to ω_1, it is also easy to see that $\mathbb{Q}$ collapses every α between ω_1 and κ. It may not be immediately clear that $\mathbb{Q}$ preserves ω_1. This will fall out from the product analysis of $\mathbb{Q}$ which we give below.

For any inaccessible $\delta < \kappa$ we may truncate the forcing at δ in the obvious way, to get $\mathbb{Q}\restriction\delta$ which forces $2^\omega = \omega_2 = \delta$. We note that if G_δ is $\mathbb{Q}\restriction\delta$-generic then $\mathbb{Q}/G_\delta$ is very similar to $\mathbb{Q}$.

To analyse $\mathbb{Q}$ we define a variation of the sort of term forcing we studied in Section 22. Let $\mathbb{R}$ be the set of g such that g is a function on κ with countable support, and $g(\alpha)$ is $\mathbb{P}_\alpha$-name for an element of $\mathbb{R}_\alpha$. Order $\mathbb{R}$ by setting $r_2 \leq r_1$ if and only if $\mathrm{supp}(r_1) \subseteq \mathrm{supp}(r_2)$, and $\Vdash r_2(\alpha) \leq r_1(\alpha)$ for all $\alpha \in \mathrm{supp}(r_1)$. It is routine to check that the identity is a projection map from $\mathbb{P} \times \mathbb{R}$ to $\mathbb{Q}$. It follows that if G is $\mathbb{Q}$-generic with projection g on the first coordinate then we may view $V[G]$ as a submodel of $V[g \times h]$ where $g \times h$ is $\mathbb{P} \times \mathbb{R}$-generic. By Easton's lemma all countable sequences from $V[G]$ are in $V[g]$, so in particular ω_1 is preserved.

We now finish the argument by showing there are no ω_2-Aronszajn trees in $V[G]$. To do this we start by noting that (morally speaking) $\mathbb{Q} \subseteq V_\kappa$, so that we may regard $\mathbb{Q}$ as an initial segment of $j(\mathbb{Q})$ where $j : V \longrightarrow M$ is the ultrapower by some normal measure on κ. As usual we may then build a generic embedding $j : V[G] \longrightarrow M[G * H]$ where H is $j(\mathbb{Q})/G$-generic.

Suppose for contradiction that $T \in V[G]$ is a κ-tree. By the usual chain condition arguments $T \in M[G]$, and since $j(T) \in M[G * H]$ we see that forcing over $M[G]$ with $j(\mathbb{Q})/G$ has added a branch to the κ-Aronszajn tree

T. We observe that in $M[G]$ we have that $2^\omega = \kappa = \omega_2$. It is not hard to see that $j(\mathbb{Q})/G$ is susceptible exactly to the same kind of product analysis as $\mathbb{Q}$ or $j(\mathbb{Q})$, so that by Lemma 23.1 it is not possible for $j(\mathbb{Q})/G$ to add a branch through T. This concludes the proof. ⊣

23.3 Remark. Abraham [2] showed that it is consistent for both ω_2 and ω_3 to simultaneously have the tree property. Foreman and the author [12] built a model where ω_n has the tree property for $1 < n < \omega$. Magidor and Shelah [56] showed that $\omega_{\omega+1}$ may have the tree property. Foreman and the author [12] constructed a model where ω_ω is strong limit and $\omega_{\omega+2}$ has the tree property.

Mitchell also showed that if κ is Mahlo and we force with the poset $\mathbb{Q}$ of Theorem 23.2, then in the extension there is no special ω_2-Aronszajn tree. By work of Jensen there is a special ω_2-Aronszajn tree if and only if the weak square principle $\square^*_{\omega_1}$ holds, so in Mitchell's model $\square^*_{\omega_1}$ fails. We sketch a proof (due to Mitchell) that an even weaker version of square fails in the model; for more on the ideal $I[\lambda]$ see [10].

Recall that $I[\omega_2]$ is the (possibly improper) ideal of $A \subseteq \omega_2$ such that there exist $\langle x_\alpha : \alpha < \omega_2 \rangle$ and a club set $C \subseteq \omega_2$, such that for every $\alpha \in C \cap A \cap Cof(\omega_1)$ there is a set $d \subseteq \alpha$ with d club in α, $\mathrm{ot}(d) = \omega_1$, and every proper initial segment of d appearing as x_β for some $\beta < \alpha$. It is easy to see that if $\square^*_{\omega_1}$ then $\omega_2 \in I[\omega_2]$.

23.4 Theorem. *If κ is Mahlo and we force with $\mathbb{Q}$ as in Theorem 23.2 then in the extension $\omega_2 \notin I[\omega_2]$.*

Proof. Let G be $\mathbb{Q}$-generic. An argument similar to that of Theorem 21.5 shows that if $\alpha < \kappa$ is inaccessible and $X \in P(\alpha) \cap V[G]$ with $X \cap \beta \in V[G_\alpha]$ for all $\beta < \alpha$, then $X \in V[G_\alpha]$. Suppose for contradiction that $\langle x_\alpha : \alpha < \kappa \rangle$ and C witness in $V[G]$ that $\omega_2 \notin I[\omega_2]$.

Then since β is Mahlo and $\mathbb{Q}$ is β-c.c. there is a V-inaccessible cardinal $\beta \in C$ such that $\langle x_\alpha : \alpha < \beta \rangle \in V[G_\beta]$, and so there is in $V[G]$ a club subset $d \subseteq \beta$ such that $\mathrm{ot}(d) = \omega_1$ and every initial segment of d is in $V[G_\beta]$. By the remarks of the last paragraph we have $d \in V[G_\beta]$, which is impossible because $\beta = \omega_2^{V[G_\beta]}$. ⊣

24. Iterations with Prediction

In this section we look at some theorems proved using the powerful reflection properties of supercompact cardinals. Both of the results we prove depend on the following theorem of Laver [49] which may be viewed as a kind of diamond principle.

24.1 Theorem. *Let κ be a supercompact cardinal, then there exists a function $f : \kappa \to V_\kappa$ such that for all $\lambda \geq \kappa$ and all $x \in H_{\lambda^+}$ there is a supercompactness measure U on $P_\kappa\lambda$ such that $j_U(f)(\kappa) = x$.*

Proof. Fix a well-ordering $\prec$ of V_κ. We define $f(\alpha)$ by recursion on α. We set $f(\alpha) = 0$ unless there exists a cardinal λ with $\alpha \leq \lambda < \kappa$ and $x \in H_{\lambda^+}$, such that for no supercompactness measure U on $P_\alpha \lambda$ does $j_U(f \restriction \alpha)(\alpha) = x$. In this case we choose the minimal such λ and then the $\prec$-minimal such $x \in H_{\lambda^+}$, and set $f(\alpha) = x$.

Suppose for a contradiction that there exist $\lambda \geq \kappa$ and $x \in H_{\lambda^+}$ such that for no supercompactness measure U on $P_\kappa \lambda$ does $j_U(f)(\kappa) = x$. Let $\rho = 2^{2^\lambda}$, let W be a supercompactness measure on $P_\kappa \rho$, and let the ultrapower by W be $j : V \longrightarrow N = \mathrm{Ult}(V, W)$. Observe that $H_{\lambda^+} \subseteq (V_{j(\kappa)})_N$.

All supercompactness measures on $P_\kappa \lambda$ and all functions from $P_\kappa \lambda$ to V_κ lie in N. It follows easily that

$$N \models \text{“ for no supercompactness measure } U \text{ on } P_\kappa \lambda \text{ does } j_U(f)(\kappa) = x\text{”}.$$

Let μ be minimal such that for some $y \in H_{\mu^+}$ there is no supercompactness measure U on $P_\kappa \mu$ with $j_U(f)(\kappa) = y$; clearly $\mu \leq \lambda$, so in particular $y \in (V_{j(\kappa)})_N$. Let y be $j(\prec)$-minimal such $y \in H_{\mu^+}$. By elementarity, the definition of f, and the agreement between V and N we may conclude that $j(f)(\kappa) = y$.

Now we define $U = \{X \subseteq P_\kappa \mu : j“\mu \in j(X)\}$ so that U is a supercompactness measure on $P_\kappa \mu$. Let $i : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map, and observe that by Proposition 3.2 there is an elementary embedding $k : M \longrightarrow N$ given by $k : [F]_U = j(F)(j“\mu)$. We also have that $k \circ i = j$.

We now analyse the embedding k. The definition of k gives easily that $j“V \subseteq \mathrm{ran}(k)$ and $j“\mu \in \mathrm{ran}(k)$. If $X \subseteq \mu$ then

$$X = \{\mathrm{ot}(\gamma \cap j“\mu) : \gamma \in j“\mu \cap j(X)\},$$

so that $X \in \mathrm{ran}(k)$. It follows that $H_{\mu^+} \subseteq \mathrm{ran}(k)$ and so in particular $k \restriction H_{\mu^+} = \mathrm{id}$.

Since $y \in H_{\mu^+}$, $k(y) = y$. We also know that $k(\kappa) = \kappa$ and $k \circ i = j$, so $k(i(f)(\kappa)) = j(f)(\kappa) = y$. Contradiction!

It follows that for all $\lambda \geq \kappa$ and all $x \in H_{\lambda^+}$ there is a supercompactness measure U on $P_\kappa \lambda$ with $j_U(f)(\kappa) = x$. ⊣

24.2 Remark. Using extenders in the place of supercompactness measures it is possible to prove a similar result for strong cardinals. See Gitik and Shelah's paper [29] for this result and some applications.

In this section we prove the consistency of the Proper Forcing Axiom (defined below) and of the statement “the supercompactness of κ is indestructible under κ-directed closed forcing”. These statements have in common that they involve a universal quantification over a proper class; they will both be proved by doing a set forcing and using some reflection arguments.

In each of the two consistency proofs we will begin with a supercompact cardinal κ. We fix a function f as in Theorem 24.1 (a *Laver function*) and use this function as a guide in building an iteration of length κ which anticipates a proper class of possibilities for what may happen at stage κ.

The details of the constructions are of course somewhat different, but they each involve taking a generic object for some forcing we may do stage κ, and copying it via some supercompactness embedding j to a filter on the image of that forcing under j. In the argument for PFA the existence of this filter is reflected back to give a witness for the truth of PFA, while in the indestructibility theorem the filter is used to construct a strong master condition and lift the embedding j.

We now give Baumgartner's consistency proof [15] for the Proper Forcing Axiom. We begin with a brief review of proper forcing.

24.3 Definition. Let θ be regular with $\mathbb{P} \in H_\theta$. Let $<_\theta$ be a well-ordering of H_θ and let $\mathbb{P} \in N \prec (H_\theta, \in, <_\theta)$ where N is countable. $p \in \mathbb{P}$ is $(N, \mathbb{P})$-*generic* if and only if for every maximal antichain A of $\mathbb{P}$ with $A \in N$, $A \cap N$ is predense below p.

24.4 Remark. This notion is closely related to the ideas about lifting embeddings from Proposition 9.1. Let $\bar{N}$ be the Mostowski collapse of N, let $\pi : \bar{N} \longrightarrow N$ be the inverse of the Mostowski collapse and let $\bar{x}$ be the collapse of x for $x \in N$.

Then it is easy to see that p is $(N, \mathbb{P})$-generic if and only if p forces that $\bar{G} =_{\rm def} \{\bar{p} : p \in G \cap N\}$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$; that is, p is a master condition for π in the sense of Definition 12.1. The definition of $\bar{G}$ implies that $\pi``\bar{G} \subseteq G$. Therefore if p is $(N, \mathbb{P})$-generic and $p \in G$ for some G which is $\mathbb{P}$-generic over V, then π can be lifted to a map $\pi^+ : \bar{N}[\bar{G}] \longrightarrow N[G]$ which is the inverse of the Mostowski collapse map for $N[G]$.

We note that for example in the Martin's Maximum paper [20] $(N, \mathbb{P})$-generic conditions are referred to as "$(N, \mathbb{P})$-master conditions". We have chosen to follow the conventions of Shelah's book on proper forcing [64].

24.5 Remark. In the study of proper forcing it is often interesting to look at conditions which are *strongly* $(N, \mathbb{P})$*-generic*, where (adopting the notation of the last remark) a condition $p \in \mathbb{P}$ is strongly $(N, \mathbb{P})$-generic if and only if $g_p =_{\rm def} \{\bar{q} : q \in N \cap \mathbb{P}, p \leq q\}$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$. Such a condition is precisely a strong master condition for g_p and π in the sense of Definition 12.2.

24.6 Definition. $\mathbb{P}$ is *proper* if and only if for all large θ, all countable N with $\mathbb{P} \in N \prec (H_\theta, \in, <_\theta)$, and all $p \in \mathbb{P} \cap N$ there exists a condition $q \leq p$ which is $(N, \mathbb{P})$-generic.

See Abraham's chapter in this Handbook for an exposition of proper forcing. The only fact about properness we will need is that a countable support iteration of proper forcing is proper.

24.7 Definition. The Proper Forcing Axiom (PFA) is the statement: for every proper $\mathbb{P}$ and every sequence $\langle D_\alpha : \alpha < \omega_1 \rangle$ of dense subsets of $\mathbb{P}$ there exists a filter F on $\mathbb{P}$ such that $F \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

Before we prove the consistency of PFA we make a few remarks.

24.8 Remark. It would be hopeless to ask to meet ω_2 sets in the statement of PFA, because we could then apply the axiom to the proper forcing poset $\mathrm{Col}(\omega_1, \omega_2)$ and produce a surjection from ω_1 onto ω_2.

24.9 Remark. The axiom PFA is known [66] to have a very high consistency strength. One way of seeing this is that by work of Todorčević [67] PFA implies the failure of $\Box_\kappa$ for κ singular, which implies in turn that the weak covering lemma fails over any reasonable core model.

24.10 Remark. In the consistency proof for MA_{ω_1} [6] the first step is to observe that we only need to deal with forcing posets of size ω_1. The point is that the property of being c.c.c. is inherited by any completely embedded subposet, and so to deal with ω_1 dense subsets of an arbitrary $\mathbb{P}$ we may as well work in some subposet Q of size ω_1 in which all those dense sets remain dense.

This is not so for proper forcing. Notice that by Example 6.6 we may force $\Box_\kappa$ without changing H_{κ^+}, so that PFA cannot in general be "localised" to a statement in H_{κ^+} for any κ.

24.11 Theorem. *Let κ be supercompact. Then there is a forcing iteration of length κ such that in $V^{\mathbb{P}_\kappa}$*

1. PFA *holds.*
2. $2^\omega = \kappa = \omega_2$.

Proof. Let $f : \kappa \longrightarrow V_\kappa$ be a function as in Theorem 24.1. The poset $\mathbb{P}_\kappa$ will be an inductively defined iteration of length κ with countable support, with each $\mathbb{Q}_\alpha$ forced to be proper in $V^{\mathbb{P}_\alpha}$. The name $\dot{\mathbb{Q}}_\alpha$ will name the trivial forcing unless $f(\alpha)$ is a $\mathbb{P}_\alpha$-name for a proper forcing poset, in which case $\mathbb{Q}_\alpha = f(\alpha)$.

By the Properness Iteration Theorem [1] the poset $\mathbb{P}_\kappa$ is proper, so preserves ω_1. By Proposition 7.13 $\mathbb{P}_\kappa$ is κ-c.c. with cardinality κ, so in particular κ is preserved.

Now let $\dot{Q}$ be the standard $\mathbb{P}_\kappa$-name for $\mathrm{Add}(\omega, 1)$. Let $\lambda = 2^{2^\kappa}$ and find a supercompactness measure U on $P_\kappa \lambda$ such that $j_U(f)(\kappa) = \dot{Q}$. Arguing as in Lemma 11.6, $j_U(\mathbb{P}_\kappa)$ is an iteration of length $j(\kappa)$ in $\mathrm{Ult}(V, U)$ whose first κ stages are exactly those of $\mathbb{P}_\kappa$.

$\mathrm{Ult}(V, U)$ agrees that $\dot{Q}$ is a $\mathbb{P}_\kappa$-name for $\mathrm{Add}(\omega, 1)$, so by the usual reflection argument there are unboundedly many $\alpha < \kappa$ such that $f(\alpha)$ is a $\mathbb{P}_\alpha$-name for $\mathrm{Add}(\omega, 1)$. Since $\mathrm{Add}(\omega, 1)$ is proper, there are unboundedly many α where $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for $\mathrm{Add}(\omega, 1)$, so that in the course of the iteration $\mathbb{P}_\kappa$ we add κ many subsets of ω. A very similar argument shows that $\mathbb{Q}_\alpha = \mathrm{Col}(\omega_1, \alpha)_{V^{\mathbb{P}_\alpha}}$ for many $\alpha < \kappa$, so that $2^\omega = \kappa = \omega_2$ in $V^{\mathbb{P}_\kappa}$.

To finish the proof we need to show that PFA holds in $V^{\mathbb{P}_\kappa}$. Let G be $\mathbb{P}_\kappa$-generic over V, and let $\mathbb{Q} = i_G(\dot{\mathbb{Q}})$ where $\dot{\mathbb{Q}}$ is a $\mathbb{P}_\kappa$-name for a proper forcing poset. Find a cardinal λ such that $\dot{\mathbb{Q}} \in H_\lambda$, let $\mu = 2^{2^\lambda}$, and let U be a supercompactness measure on $P_\kappa \mu$ such that $j_U(f)(\kappa) = \dot{\mathbb{Q}}$. Let $j : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map.

By Proposition 8.4 we see that $V[G] \models {}^\mu M[G] \subseteq M[G]$, so in particular $\Vdash^M_{\mathbb{P}_\kappa}$ "$\dot{\mathbb{Q}}$ is proper". It follows that in the iteration $j(\mathbb{P}_\kappa)$, the forcing which is used at stage κ is precisely $\dot{\mathbb{Q}}$.

Now let g be $\mathbb{Q}$-generic over $V[G]$. Working in $M^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}}$ let $\dot{\mathbb{R}}$ name the canonical forcing such that $j(\mathbb{P}_\kappa) \simeq \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, and let $\mathbb{R} = i_{G*g}(\dot{\mathbb{R}})$. Let H be $\mathbb{R}$-generic over $V[G * g]$.

Since $\mathbb{P}_\kappa$ is an iteration with countable support, the support of every condition in G is bounded in κ. This implies that $j``G \subseteq G * g * H$, and so we may lift j to get a map $j_G : V[G] \longrightarrow M[G * g * H]$.

Now let $\vec{D} = \langle D_\alpha : \alpha < \omega_1 \rangle$ be an ω_1-sequence of dense subsets of $\mathbb{Q}$, with $\vec{D} \in V[G]$. Since g is generic over $V[G]$, $g \cap D_\alpha \neq \emptyset$ for each α. By the choice of μ and U we know that $j \restriction \dot{\mathbb{Q}} \in M$, from which it follows by the definition of j_G in Proposition 9.1 that $j_G \restriction \mathbb{Q} \in M[G * g * H]$.

Now let F be the filter on $j_G(\mathbb{Q})$ generated by $j_G``g$. By the arguments of the last paragraph we see that $F \in M[G*g*H]$. Since $\mathrm{crit}(j_G) = \mathrm{crit}(j) = \kappa$, we see that $j_G(\vec{D}) = \langle j_G(D_\alpha) : \alpha < \omega_1 \rangle$. By genericity $g \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$, and so by elementarity $F \cap j_G(D_\alpha) \neq \emptyset$ for all $\alpha < \omega_1$.

It follows that

$$M[G * g * H] \models \text{“}F \text{ meets every set in } j_G(\vec{D})\text{”}.$$

By the elementarity of j_G,

$$V[G] \models \text{“}\exists f\ f \text{ meets every set in } \vec{D}\text{”}.$$

It follows that $V[G]$ is a model of PFA. ⊣

We will prove the following theorem of Laver.

24.12 Theorem (Laver [49]). *Let κ be supercompact and let $\delta < \kappa$. There is a forcing iteration $\mathbb{P}_\kappa$ of length κ such that*

1. *κ is supercompact in $V^{\mathbb{P}_\kappa}$, and in any extension of $V^{\mathbb{P}_\kappa}$ by κ-directed closed forcing.*
2. *$\mathbb{P}_\kappa$ has cardinality κ, is κ-c.c. and is δ-directed closed.*

24.13 Remark. The proof of Theorem 24.12 is similar in its outline to the consistency proof for PFA. One significant difference is that we will leave long gaps in the iteration in which nothing happens. This is natural when we consider that

- κ is supposed to be supercompact in $V^{\mathbb{P}_\kappa}$.
- $\mathbb{P}_\kappa$ will have no effect above κ on cardinals, cofinalities and the continuum function.

An easy reflection argument shows that there should be arbitrarily long intervals (α, β) below κ in which $V^{\mathbb{P}_\kappa}$ should resemble V.

Proof. Let $f : \kappa \longrightarrow V_\kappa$ be a function as in Theorem 24.1. The poset $\mathbb{P}_\kappa$ will be an iteration of length κ with Easton support, such that for each α

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ is } \alpha\text{-directed closed”}.$$

$\dot{\mathbb{Q}}_\alpha$ will name the trivial forcing unless

1. $\alpha \geq \delta$.
2. $f(\alpha)$ is a pair $(\lambda, \mathbb{Q})$ where $\mathbb{Q}$ is a $\mathbb{P}_\alpha$-name for an α-directed closed forcing poset and λ is an ordinal.
3. For all $\beta < \alpha$, if $f(\beta)$ is an ordered pair whose first entry $f(\beta)_0$ is an ordinal, then $f(\beta)_0 < \alpha$.

In this case we let $\mathbb{Q}_\alpha = \mathbb{Q}$.

By Proposition 7.13 $\mathbb{P}_\kappa$ is κ-c.c. with cardinality κ, and by Proposition 7.9 $\mathbb{P}_\kappa$ is δ-directed closed.

Let G be $\mathbb{P}_\kappa$-generic over V. We need to show that the supercompactness of κ is indestructible; accordingly we fix $\mathbb{Q} \in V[G]$ such that

$$V[G] \models \text{“}\mathbb{Q} \text{ is a } \kappa\text{-directed closed forcing poset”},$$

and a cardinal λ with $\lambda \geq \kappa$, and we prove that κ is λ-supercompact in $V[G]^{\mathbb{Q}}$. Notice that the trivial forcing is (trivially) κ-directed closed, so that our proof will show in particular that κ is supercompact in $V[G]$.

Let $\mathbb{Q} = i_G(\dot{\mathbb{Q}})$, where (increasing λ if necessary) we may as well assume that $\dot{\mathbb{Q}} \in H_\lambda$. Let $\mu = 2^{2^\lambda}$. Let W be a supercompactness measure on $P_\kappa \mu$ such that $j_W(f)(\kappa) = (\mu, \mathbb{Q})$. Let $j = j_W$ and $N = \mathrm{Ult}(V, W)$.

Let g be $\mathbb{Q}$-generic over V. Working in N let $\dot{\mathbb{R}}$ be the standard name for the iteration such that $\mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{R}} \simeq j(\mathbb{P}_\kappa)$, let $\mathbb{R} = i_{G*g}(\dot{\mathbb{R}})$ and let H be $\mathbb{R}$-generic over $V[G * g]$.

Arguing exactly as in the consistency proof for PFA, we may lift j to get $j_G : V[G] \longrightarrow N[G * g * H]$. We may also argue exactly as before that $M[G] \models |Q| < \lambda$, $V[G*g] \models {}^{\mu}M[G*g] \subseteq M[G*g]$ and $j_G{\restriction}\mathbb{Q} \in M[G*g*H]$.

We now need to lift j_G to an embedding of $V[G*g]$, which we will do using Silver's master condition argument as in the proof of Theorem 12.6. By the last paragraph $j_G``g \in M[G * g * H]$, and clearly $j_G``g$ is a directed set of conditions in $j_G(\mathbb{Q})$. We recall that $j(\kappa) > \lambda$, $M[G] \models |\mathbb{Q}| < \lambda$ and by the elementarity of j_G

$$M[G * g * H] \models \text{“}j(\mathbb{Q}) \text{ is } j(\kappa)\text{-directed closed”}.$$

It follows that there is a condition $q \in j_G(\mathbb{Q})$ such that $\forall p \in g\ q \leq j(p)$, that is to say q is a strong master condition for g and j_G. Let h be $j_G(\mathbb{Q})$-generic over $V[G * g * G]$ with $q \in h$. We may lift j_G as usual to get an elementary embedding $j^{++} : V[G * g] \longrightarrow M[G * g * H * h]$.

The argument is not finished at this point because j^{++} can only be defined in $V[G * g * H * h]$. The final stage of the proof is to find an approximation to j^{++} which can be defined in $V[G * g]$. For notational simplicity let $V^* = V[G * g]$ and $M^* = M[G * g * H * h]$. Let

$$U = \{X \subseteq P_\kappa\lambda : X \in V^*, j``\lambda \in j^{++}(X)\}.$$

As in Proposition 3.2 we may factor j^{++} as $k \circ j_U$ where j_U is the ultrapower of V^* by U.

Recall that $|\mathbb{P}_\kappa * \dot{\mathbb{Q}}| < \lambda$. The definition of the iteration $j(\mathbb{P}_\kappa)$ and the fact that $j(f)(\kappa) = (\mu, \dot{\mathbb{Q}})$ together imply that in the iteration $j(\mathbb{P}_\kappa)$ we do trivial forcing at every stage between κ and μ. It follows that

$$M[G * g] \models \text{``}\mathbb{R} * j(\dot{\mathbb{Q}}) \text{ is } \mu\text{-closed''}$$

Since $V^* \models {}^{\mu}M[G * g] \subseteq M[G * g]$, V^* agrees that $\mathbb{R} * j(\dot{\mathbb{Q}})$ is μ-closed.

Now since $V^* \models \lambda^{<\kappa} < \mu$, the arguments of the last paragraph imply that $U \in V^*$. It is easy to check that

$$V^* \models \text{``}U \text{ is a supercompactness measure on } P_\kappa\lambda\text{''}.$$

This concludes the proof that κ is λ-supercompact in $V[G * g]$. $\dashv$

24.14 Remark. It is easy to see that if κ is measurable then there is no κ-Kurepa tree. Since Example 6.1 shows that a κ-Kurepa tree can be added by a κ-closed forcing poset for any inaccessible κ, it is not possible to improve Laver's result to cover all κ-closed forcing posets.

25. Altering Generic Objects

In this final section we introduce an idea due to Woodin, namely that it is sometimes possible to alter generic objects so as to enforce the compatibility requirements of Proposition 9.1. Returning to the theme of failure of GCH at a measurable cardinal, we will prove a result of Woodin which gets GCH to fail at a measurable cardinal from the optimal large cardinal hypothesis.

25.1 Theorem. *Let* GCH *hold and let* $j : V \longrightarrow M$ *be a definable embedding such that* $\text{crit}(j) = \kappa$, ${}^{\kappa}M \subseteq M$ *and* $\kappa^{++} = \kappa_M^{++}$. *Then there is a generic extension in which* κ *is measurable and* GCH *fails.*

25.2 Remark. The hypotheses of Theorem 25.1 can easily be had from a cardinal κ which is $(\kappa + 2)$-strong. Work of Gitik [24] shows that they can be forced starting with a model of $o(\kappa) = \kappa^{++}$, and by work of Mitchell [59] this is optimal.

Proof of Theorem 25.1. By the arguments of Section 3, we may assume that $j = j_E^V$ for some (κ, κ^{++})-extender E. We define $U = \{X : \kappa \in j(X)\}$ and form the ultrapower map $i : V \longrightarrow N \simeq \mathrm{Ult}(V, U)$. We write $j = k \circ i$ where $k : N \longrightarrow M$ is given by $k([F]) = j(F)(\kappa)$.

Let $\lambda = \kappa_N^{++}$. Then $\lambda < i(\kappa)$, since $i(\kappa)$ is inaccessible in N. Since the GCH holds $i(\kappa) < \kappa^{++}$. On the other hand $k(\lambda) = \kappa_M^{++} = \kappa^{++}$, so that $\mathrm{crit}(k) = \lambda$. It is also easy to see that k is an embedding of width $\leq \lambda$.

As before we will let $\mathbb{P} = \mathbb{P}_{\kappa+1}$ be an iteration with Easton support, where for $\alpha \leq \kappa$ we let $\mathbb{Q}_\alpha = \mathrm{Add}(\alpha, \alpha^{++})_{V[G_\alpha]}$ for α inaccessible and let it be the trivial forcing otherwise.

Let G be $\mathbb{P}_\kappa$-generic over V and let g be $\mathbb{Q}_\kappa$-generic over $V[G]$. The iterations $\mathbb{P}$, $i(\mathbb{P})$ and $j(\mathbb{P})$ agree up to stage κ.

The following lemmas are easy.

25.3 Lemma. $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_{\kappa+1}$.

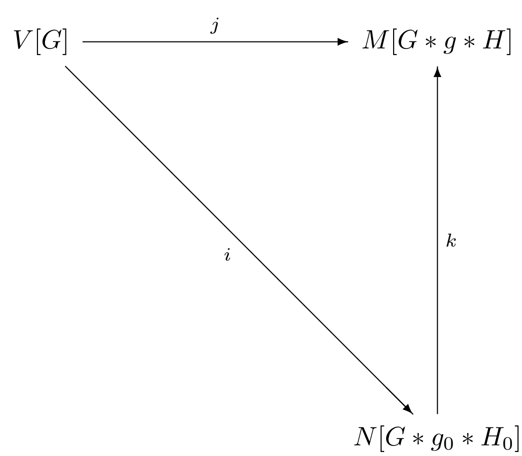
25.4 Lemma. $i(\mathbb{P})_{\kappa+1} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa^*$, *where* $\mathbb{Q}_\kappa^* = \mathrm{Add}(\kappa, \lambda)_{V[G_\kappa]}$.

Let $g_0 = g \cap \mathbb{Q}_\kappa^*$, then g_0 is $\mathbb{Q}_\kappa^*$-generic over $V[G]$ and also over $N[G]$. It follows from Proposition 8.4 that $V[G * g_0] \models {}^\kappa N[G * g_0] \subseteq N[G * g_0]$.

Let $\mathbb{R}_0 = \mathbb{R}^N_{\kappa+1, i(\kappa)}$ be the factor forcing to prolong $G * g_0$ to a generic object for $i(\mathbb{P}_\kappa)$. By Proposition 8.1 we may build $H_0 \in V[G * g_0]$ which is $\mathbb{R}_0$-generic over $N[G * g_0]$.

Since $\mathrm{crit}(k) = \lambda$ it is easy to see that k"$G = G$, and we may lift $k : N \longrightarrow M$ to get $k : N[G] \longrightarrow M[G]$. Since $\lambda = \kappa_N^{++}$, if $q \in \mathbb{Q}_\kappa^*$ then the support of q is contained in $\kappa \times \mu$ for some $\mu < \lambda$, and so $k(q) = q$. It follows that k"$g_0 = g_0 \subseteq g$, and so we may lift again to get $k : N[G*g_0] \longrightarrow M[G*g]$.

Since $N[G * g_0] \models$ "$\mathbb{R}_0$ is λ^+-closed" and we bounded the width of k, we may now appeal to Proposition 15.1 and transfer H_0 along k. We obtain H which is $\mathbb{R}$-generic over $M[G * g]$, where $\mathbb{R} = \mathbb{R}^M_{\kappa+1 j(\kappa)}$. We may then build a commutative triangle



Let $\mathbb{S}_0 = i(\mathbb{Q}_\kappa)$, that is $\mathbb{S}_0 = \mathrm{Add}(i(\kappa), i(\kappa^{++}))_{N[G*g_0*H_0]}$.

25.5 Lemma. $\mathbb{S}_0$ *is* κ^+*-closed and* κ^{++}*-Knaster in the model* $V[G * g_0]$.

Proof. The closure is easy because $V[G * g_0] \models {}^\kappa N[G * g_0] \subseteq N[G * g_0]$. Let $\langle p_\alpha : \alpha < \kappa^{++} \rangle$ be a sequence of conditions, and let $p_\alpha = i(f_\alpha)(\kappa)$ where $f_\alpha : \kappa \longrightarrow \mathbb{Q}_\kappa$, $f_\alpha \in V[G]$. An easy Δ-system argument shows that κ^{++} of the functions f_α are pointwise compatible, from which it follows that κ^{++} of the conditions p_α are compatible. ⊣

25.6 Remark. A more delicate analysis shows that $\mathbb{S}_0$ is isomorphic to $\mathrm{Add}(\kappa^+, \kappa^{++})_{V[G*g_0]}$.

25.7 Lemma. $\mathbb{S}_0$ *is* (κ^+, ∞)*-distributive and* κ^{++}*-c.c. in* $V[G * g]$.

Proof. $V[G * g]$ is a generic extension of $V[G * g_0]$ by a forcing isomorphic to $\mathbb{Q}_\kappa$. The poset $\mathbb{Q}_\kappa$ is κ^+-c.c. in $V[G * g_0]$ and so by Easton's lemma $\mathbb{S}_0$ is (κ^+, ∞)-distributive in $V[G * g]$. By Proposition 5.12 $\mathbb{S}_0 \times \mathbb{Q}_\kappa$ is κ^{++}-c.c. in $V[G * g_0]$ and so by Easton's lemma again $\mathbb{S}_0$ is κ^{++}-c.c. in $V[G * g]$. ⊣

Now we force over $V[G * g]$ with $\mathbb{S}_0$, and denote the generic object by f_0. By the last lemma forcing with $\mathbb{S}_0$ preserves cardinals. Since k has width $\leq \lambda$, we may use Proposition 15.1 and transfer f_0 to get f which is $\mathbb{S}$-generic over $M[G * g * H]$, where $\mathbb{S} = \mathrm{Add}(j(\kappa), j(\kappa^{++}))$. The problem is now that we wish to lift j, but it may not be the case that $j``g \subseteq f$.

There is no hope of using any of our previous methods for doing without a master condition, since f_0 is built by forcing (and even if we could build f_0 in a suitably compatible way, this would not guarantee compatibility for f). We adopt a different approach based on the observation that we only need to adjust any given condition in f on a small set to make it agree with $j``g$. We will work in $V[G * g * f_0]$ and construct a suitable f^*, by altering each element of f to conform with j and g.

To be precise let $Q = \bigcup j``g$, so that $Q$ is a partial function from $\kappa \times j``\kappa^{++}$ to 2. Let $p \in f$, so that $p = j(P)(a)$ for some $a \in [\kappa^{++}]^{<\omega}$ and some function $P : [\kappa]^{|a|} \to \mathbb{Q}_\kappa$ with $P \in V[G]$. If $(\gamma, j(\delta)) \in \mathrm{dom}(p)$ then by elementarity $(\gamma, \delta) \in \mathrm{dom}\, P(x)$ for at least one $x \in [\kappa]^{|a|}$, so that there are at most κ many points in the intersection of $\mathrm{dom}(Q)$ and $\mathrm{dom}(p)$. If we then define p^* to be the result of altering p on $\mathrm{dom}(p) \cap \mathrm{dom}(Q)$ to agree with Q, then since $V[G * g] \models {}^\kappa M[G * g * H] \subseteq M[G * g * H]$ we see that $p^* \in M[G * g]$ and hence $p^* \in j(\mathbb{Q}_\kappa)$.

It remains to see that $f^* = \{p^* : p \in f\}$ is $j(\mathbb{Q}_\kappa)$-generic over $M[G * g * H]$. To see this we work temporarily in $V[G]$. Let $\delta < \kappa$ and let D be dense in $\mathbb{Q}_\kappa$. Let E be the set of those $p \in D$ such that every q obtained by altering p on a set of size δ is in D. An easy argument shows that E is also dense. Returning to $M[G * g * H]$ and applying this remark with κ in place of δ, we see that f^* meets every dense set in $M[G * g * H]$.

We may now lift to get $j : V[G * g] \longrightarrow M[G * g * H * f^*]$. We are not quite done because f^* only exists in the extension $V[G * g * f_0]$. However since f_0 is generic for (κ, ∞)-distributive forcing and j has width $\leq \kappa$, we may transfer f_0 to get a suitable generic object h and finish by lifting to get $j : V[G * g * f_0] \longrightarrow M[G * g * H * f^* * h]$. $\dashv$

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