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SOME RESULTS IN POLYCHROMATIC RAMSEY THEORY

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§1. Introduction. *Classical Ramsey theory* (at least in its simplest form) is concerned with problems of the following kind: given a set X and a colouring of the set $[X]^n$ of unordered n-tuples from X, find a subset $Y \subseteq X$ such that all elements of $[Y]^n$ get the same colour. Subsets with this property are called *monochromatic* or *homogeneous,* and a typical positive result in Ramsey theory has the form that when X is large enough and the number of colours is small enough we can expect to find reasonably large monochromatic sets.

Polychromatic Ramsey theory is concerned with a "dual" problem, in which we are given a colouring of $[X]^n$ and are looking for subsets $Y \subseteq X$ such that any two distinct elements of $[Y]^n$ get *different* colours. Subsets with this property are called *polychromatic* or *rainbow*. Naturally if we are looking for rainbow subsets then our task becomes easier when there are many colours. In particular given an integer k we say that a colouring is *k-bounded* when each colour is used for at most k many n-tuples.

At this point it will be convenient to introduce a compact notation for stating results in polychromatic Ramsey theory. We recall that in classical Ramsey theory we write $\kappa \rightarrow (\alpha)^n_k$ to mean "every colouring of $[\kappa]^n$ in k colours has a monochromatic set of order type α". We will write $\kappa \rightarrow^{\text{poly}} (\alpha)^n_{k-\text{bd}}$ to mean "every k-bounded colouring of $[\kappa]^n$ has a polychromatic set of order type α". We note that when κ is infinite and k is finite a k-bounded colouring will use exactly κ colours, so we may as well assume that κ is the set of colours used.

Polychromatic Ramsey theory in the finite case has been extensively studied by finite combinatorists [3, 7, 11, 12], sometimes under the name "Rainbow Ramsey theory" or "Sub-Ramsey theory". In particular the quantity $\text{sr}(K_n, k)$, which in our notation is the least m such that $m \rightarrow^{\text{poly}} (n)^2_{k-\text{bd}}$, has been investigated; it grows *much* more slowly than the corresponding classical Ramsey number, for example it is known [3] that $\text{sr}(\text{K}_\text{n}, \text{k}) \leq \frac{1}{4}\text{n}(\text{n}-1)(\text{n}-2)(\text{k}-1)+3$.

In this paper we investigate polychromatic versions of some classic results in infinite Ramsey theory. Interestingly we will also often find that, as in the finite case, polychromatic Ramsey numbers grow more slowly than their classical counterparts.

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• The polychromatic Ramsey theory of the infinite was first investigated by Galvin [10]. In Section 2 we sketch a couple of results which motivated our work in the area.

We describe some machinery (*dual colourings*) for translating positive results in monochromatic Ramsey theory into positive results in polychromatic Ramsey theory. We use dual colourings to show that if $\kappa \to (\lambda)^n_k$ then $\kappa \to^{\rm poly} (\lambda)^n_{k-\rm bd}$. In particular the classical infinite Ramsey theorem implies that $\omega \to^{\rm poly} (\omega)^n_{\rm k-bd}$ for all n and k. We then show that assuming the Continuum Hypothesis (CH) we have the negative relation $\omega_1 \not\to^{\rm poly} (\omega_1)^2_{2-\rm bd}$.

• It is natural to ask whether CH was necessary to show $\omega_1 \not\to^{\rm poly} (\omega_1)^2_{2-\rm bd}$. We recall that Todorčević [19] showed in ZFC that $\omega_1 \not\to [\omega_1]^2_{\omega_1}$, which gives a very strong counterexample to $\omega_1 \to (\omega_1)^2_2$.

The polychromatic situation is quite different. Todorčević [17] proved a result which implies that, assuming the Proper Forcing Axiom (PFA), we have the positive relation $\omega_1 \to^{\rm poly} (\omega_1)^2_{2-\rm bd}$. In Section 3 we will give an alternative proof of this implication from PFA; Todorčević's methods give the consistency of $\omega_1 \to^{\rm poly} (\omega_1)^2_{2-\rm bd}$ relative to the consistency of ZFC, it is not clear that the methods of Section 3 will do this. See the discussion at the end of Section 3.

• Since PFA implies that $\omega_1 \to^{\rm poly} (\omega_1)^2_{2-\rm bd}$, it is natural to ask whether Martin's Axiom (MA) suffices for this partition relation. In Section 4 we show that this relation is in fact independent of MA. The proof builds on ideas of Abraham and Todorčević [2], and shows that we may force the existence of a 2-bounded colouring of $[\omega_1]^2$ which has no uncountable polychromatic set in any ccc forcing extension.

• After hearing the results of Section 4, Sy Friedman asked for a concrete example of a 2-bounded colouring of $[\omega_1]^2$ which has no uncountable polychromatic set, but which acquires one in some ccc forcing extension. In Section 5 we construct such a colouring from the assumption that CH holds and there is a Souslin tree.

• The argument of Section 3 actually shows that under PFA every 2-bounded colouring of pairs from ω_1 has a *stationary* polychromatic set. So a 2-bounded colouring of pairs from ω_2 has many polychromatic sets of order type ω_1 which are stationary in their supremum. In Section 6 we show that under MM a 2-bounded colouring of pairs from ω_2 has a polychromatic set of order type ω_1 which is *club* in its supremum.

• In Section 7 we show that PFA does not suffice for the result of Section 6. We do this by isolating a consequence of square which is strong enough to build a 2-bounded colouring of pairs from ω_2 with no polychromatic closed copy of ω_1, yet weak enough to be consistent with PFA.

• Finally in Section 8 we prove some positive partition results from GCH assumptions, which are stronger than the easy results obtained from the Erdős-Rado theorem by the dual colouring argument. The results give a fairly clear picture of which polychromatic partition relations hold assuming GCH, at least at successors of regular cardinals.

Notation: our notation is standard. One point is worthy of comment, namely that when f is a colouring of pairs of ordinals from some set X and $\alpha, \beta \in X$ with $\alpha < \beta$ we will typically write "$f(\alpha, \beta)$" instead of the more correct "$f(\{\alpha, \beta\})$".

Whenever we use this convention it is safe to assume that the first argument of f is less than the second argument.
Directions for further research:

1. It follows from the results in Section 8 that under CH we have the relation $\omega_2 \to^{\text{poly}} (\alpha)^2_{\omega-\text{bd}}$ for every $\alpha < \omega_2$. In a projected sequel to this paper we show that a wide range of possibilities is consistent with not-CH; in particular each of the relations $\omega_2 \to^{\text{poly}} (\omega_1 + 1)^2_{\omega-\text{bd}}$ and $\omega_2 \not\to^{\text{poly}} (\omega_1)^2_{2-\text{bd}}$ is consistent with $2^\omega = \omega_2$.
2. Jindrich Zapletal and Otmar Spinas pointed out that polychromatic Ramsey theory and classical Ramsey theory both fall under the umbrella of "canonical Ramsey theory", in which we are given an arbitrary colouring of $[X]^n$ and look for a large $Y \subseteq X$ so that the colouring is "simple" on $[Y]^n$. One way to look at the results of this paper is to see them as fragments of a canonical Ramsey theory for $[\omega_1]^2$ and $[\omega_2]^2$; such a theory will clearly be fraught with independence results. On a related note it may be interesting to explore connections with "structural Ramsey theory".
3. It is natural to ask for a polychromatic analogue of the Galvin-Prikry theorem. Otmar Spinas pointed out that such a result follows easily from a canonical colouring result of Prömel and Voigt [13]. To be precise if we are given a 2-bounded Borel colouring of $[\omega]^\omega$ then there is an infinite $P \subseteq \omega$ such that the colouring is 1-1 on $[P]^\omega$.
4. Many natural questions remain open, we give a sampling:

(a) Construct in ZFC a counterexample to $\omega_2 \to^{\text{poly}} (\omega_2)^2_{2-\text{bd}}$.
(b) What can we say about colourings of $[\omega_1]^3$? See the remarks after Theorem 1.
(c) Suppose that every 2-bounded colouring of $[\omega_2]^2$ has an uncountable polychromatic set. Must every 2-bounded colouring of $[\omega_1]^2$ have an uncountable polychromatic set?
(d) Under what circumstances can "ccc indestructibly bad" colourings of the sort constructed in Section 4 exist? Are they compatible with CH? Do they exist in L?
(e) For which inaccessible κ does $\kappa \to^{\text{poly}} (\kappa)^2_{2-\text{bd}}$ hold? In particular does this partition relation characterise the weakly compact inaccessible cardinals?
(f) The GCH results of Section 8 only work at successors of regular cardinals. What happens at successors of singular cardinals?

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§2. Results of Galvin.

In this section we sketch the proofs of some basic results due to Galvin [10].

2.1. Dual colourings.

Definition 1. Let X be a set and let f be a k-bounded colouring of $[X]^n$. We fix a linear ordering $\prec$ of $[X]^n$ and define the *dual colouring* $f^* : [X]^n \to k$ as follows: if $f(a) = c$ then enumerate the n-tuples with colour c in $\prec$-increasing order as $a_0 \prec a_1 \prec \dots a_j$, and define $f^*(a) = i$ where $a = a_i$.

It is easy to see that if X is monochromatic for f^* then it is polychromatic for f. It follows immediately that

$$\kappa \to (\lambda)^n_k \Longrightarrow \kappa \to^{\text{poly}} (\lambda)^n_{k-\text{bd}}.$$

So every theorem in classical Ramsey theorem implies a corresponding theorem in polychromatic Ramsey theory. In particular we recall that Baumgartner and Hajnal [4] showed that $\omega_1 \to (\alpha)^2_2$ for every countable ordinal α. It follows immediately that $\omega_1 \to^{\text{poly}} (\alpha)^2_{2-\text{bd}}$ for every countable α.

The following result shows that in general we can not improve on this; the proof follows a standard paradigm for building "wild" colourings using cardinal arithmetic assumptions.

2.2. A CH example.

Theorem 1. *Let CH hold. Then there is a 2-bounded colouring of $[\omega_1]^2$ such that for all uncountable $X \subseteq \omega_1$ there are uncountably many colours which appear twice on pairs in $[X]^2$. In particular $\omega_1 \not\to^{\text{poly}} (\omega_1)^2_{2-\text{bd}}$ under CH.*

Proof. Using CH enumerate $[\omega_1]^\omega$ as $\langle A_\alpha : \alpha < \omega_1 \rangle$. Now we build the colouring inductively, at step γ assigning colours to pairs $\{\beta, \gamma\}$ for $\beta < \gamma$. We assume that the colouring of $[\gamma]^2$ which we have built when we reach stage γ is 2-bounded.

We enumerate the sets A_α where $A_\alpha \subseteq \gamma$ and $\alpha < \gamma$ as $\langle B_j : j < \omega \rangle$. We also choose colours $\langle c_k : k < \omega \rangle$ which were not yet used. Now we choose the first two points in B_0 and colour the pairs they form with γ using colour c_0, take the first two points not yet coloured in B_1 and colour the pairs they form with γ using c_1 and so on. If at the end there are points $\alpha < \gamma$ such that $\{\alpha, \gamma\}$ was not yet coloured then enumerate these pairs and colour them inductively giving each one a colour not yet used.

To see that this works let X be unbounded in ω_1, and let $Y = A_\alpha$ be the initial segment of X consisting of the first ω points. Let $\gamma \in X$ with $\gamma > \alpha, \sup(Y)$. Then by construction there exist two points ρ and σ in Y such that the pairs $\{\rho, \gamma\}$ and $\{\sigma, \gamma\}$ are given the same colour. $\dashv$

Remark 1. Matt Foreman pointed out that the theorem of Baumgartner and Hajnal [4] used above is specific to colourings of pairs: if we fix injections $f_\gamma : \gamma \hookrightarrow \omega$ for each countable γ and then colour $\{\alpha, \beta, \gamma\}_<$ red or blue according to whether $f_\gamma(\alpha) < f_\gamma(\beta)$ or $f_\gamma(\alpha) > f_\gamma(\beta)$, then there is no red-homogeneous set of type $\omega + 2$ and no blue-homogeneous set of type $\omega + 1$. It is unclear what is the situation regarding the relation $\omega_1 \to^{\text{poly}} (\alpha)^3_{2-\text{bd}}$ for $\alpha \leq \omega_1$.

Remark 2. Mirna Džamonja pointed out that the use of CH in the argument above is an overkill, and that much weaker guessing principles consistent with large continuum will do. For example the *stick principle* asserts that there is $\mathscr{X} \subseteq [\omega_1]^\omega$ such that $|\mathscr{X}| = \omega_1$ and for every $Y \in [\omega_1]^{\omega_1}$ there is $A \in \mathscr{X} \cap [Y]^\omega$; this is exactly the consequence of CH used in the proof of Theorem 1.

Remark 3. Theorem 1 has an easy generalisation to higher cardinals, which we leave to the reader. See Section 8 of this paper for some *positive* results from GCH assumptions which do not follow trivially from the Erdős-Rado theorem.

§3. A positive result for ω_1 from PFA. In this section we will give an alternative proof of a result by Todorčević [17] which implies that $\omega_1 \to^{\text{poly}} (\omega_1)^2_{2-\text{bd}}$ under PFA. The result is more general than this in several ways, to state it we introduce the following definition:

DEFINITION 2. Let κ be a cardinal. A colouring of $[X]^n$ is $(<\kappa)$-*bounded* if and only if each colour is used for fewer than κ many n-tuples.

Of course our previous notion of k-boundedness is just the special case with $\kappa = k + 1$.

THEOREM 2 (Todorčević [17]). (*PFA*) *If f is a $(<\omega)$-bounded colouring of $[\omega_1]^2$ then ω_1 is the union of ω many polychromatic sets.*

The proof will occupy most of this section. We will allow ourselves several digressions.

3.1. Normal colourings. We introduce a special property of colourings (*normality*), and show that we can sometimes decompose a colouring into a small number of normal colourings.

DEFINITION 3. Let X be a set of ordinals. A colouring f of $[X]^2$ is *normal* if and only if for all pairs $a = \{\alpha, \beta\}_<$ and $b = \{\gamma, \delta\}_<$, if $f(a) = f(b)$ then $\beta = \delta$.

LEMMA 1. *Let λ be an infinite cardinal and let $\kappa = \lambda^+$. If f is a $(<\kappa)$-bounded colouring of $[\kappa]^2$ then there exist sets X_j for $j < \lambda$ such that $\kappa = \bigcup_j X_j$ and $f \restriction [X_j]^2$ is normal.*

We prove Lemma 1 using elementary substructures. This is an overkill for the task at hand, but introduces ideas which will be crucial at several points later in the paper.

PROOF. Let θ be a large regular cardinal, let $<_\theta$ be a well-ordering of H_θ and let $\mathscr{A} = (H_\theta, <_\theta)$. Build an increasing continuous chain $\langle N_i : i < \kappa \rangle$ such that

- $\kappa, \lambda, f \in N_0$.
- $N_i \prec \mathscr{A}$ with $|N_i| = \lambda$ and $N_i \cap \kappa \in \kappa$.

Let $\delta_i = N_i \cap \kappa$ so that $\{\delta_i\}$ is club in κ. Let $a = \{\alpha, \beta\}_<$ and let i be such that $\delta_i \leq \beta < \delta_{i+1}$.

Let X be the set of pairs b with $f(b) = f(a)$, then X is definable in $\mathscr{A}$ from f and a so easily $X \in N_{i+1}$. Since f is $(<\kappa)$-bounded, $|X| < \kappa$ and so $\mu =_{def} |X| \in N_{i+1} \cap \kappa$, hence $\mu \subseteq N_{i+1}$ since $N_{i+1} \cap \kappa \in \kappa$.

Let g be the $<_\theta$-least bijection between μ and X, then $g \in N_{i+1}$ so $X = g[\mu] \subseteq N_{i+1}$. That is to say every pair b with $f(b) = f(a)$ lies in $[\delta_{i+1}]^2$.

We may now easily decompose κ as $\bigcup_{j<\lambda} X_j$ where $|X_j \cap [\delta_i, \delta_{i+1})| \leq 1$ for all i, j. It follows immediately from the discussion above that $f \restriction [X_j]^2$ is normal. ⊣

3.2. Proper forcing. We give a brief review of Shelah's theory of proper forcing, referring the reader to the survey by Abraham [1] and Shelah's monograph [15] for the details.

DEFINITION 4. Let $\mathbb{P}$ be a forcing poset and let θ be a regular cardinal so large that $P(\mathbb{P}) \in H_\theta$. Let $N \prec H_\theta$ be countable with $\mathbb{P} \in N$. Then $q \in \mathbb{P}$ is $(N, \mathbb{P})$-*generic* if and only if for every dense subset $D \subseteq \mathbb{P}$ with $D \in N$, $D \cap N$ is predense below q.

This can be motivated as follows: if $\pi : N \simeq \bar{N}$ is the Mostowski collapse isomorphism from N to a transitive set $\bar{N}$, then p is an $(N, \mathbb{P})$-generic condition if and only if it forces that $\pi[G_{\mathbb{P}} \cap N]$ is an $\bar{N}$-generic filter on $\pi(\mathbb{P})$. We note that in general $\pi[G_{\mathbb{P}} \cap N]$ will not lie in V.

Definition 5. Let $\mathbb{P}$ be a forcing poset and let θ be a regular cardinal so large that $P(\mathbb{P}) \in H_\theta$. Let $N \prec H_\theta$ be countable with $\mathbb{P} \in N$. Then $q \in \mathbb{P}$ is $(N, \mathbb{P})$-*strongly generic* if and only if for every dense subset $D \subseteq \mathbb{P}$ with $D \in N$, there is $r \in D \cap N$ with $q \leq r$.

A strongly generic condition is one which not only forces that $\pi[G_{\mathbb{P}} \cap N]$ is $\bar{N}$-generic, but actually determines $\pi[G_{\mathbb{P}} \cap N]$. Strongly generic conditions occur naturally in the study of forcing posets which add no new reals.

We will find several uses for the following standard facts about elementary substructures and generic conditions:

Lemma 2. *Let $N \prec H_\theta$ be countable and let $\delta = N \cap \omega_1$.*
Then

1. *If $C \in N$ is a club subset of ω_1, then $\delta \in C$.*
2. *If $S \in N$ is a subset of ω_1 and $\delta \in S$, then S is stationary.*
3. *Let $\mathbb{P}$ be a forcing poset with $P(\mathbb{P}) \in H_\theta$ and suppose that $\mathbb{P} \in N$. Then*
   (a) *For any $\mathbb{P}$-generic G, $H_\theta[G] = H_\theta^{V[G]}$ and $N[G] =_{\text{def}} \{\dot{\tau}^G : \dot{\tau} \in N\} \prec H_\theta[G]$.*
   (b) *$p \in \mathbb{P}$ is $\mathbb{P}$-generic if and only if it satisfies either of the following equivalent conditions*:
      (i) *$p \Vdash N[\dot{G}] \cap On = N \cap On$.*
      (ii) *$p \Vdash N[\dot{G}] \cap V = N \cap V$.*
   (c) *If $p \in \mathbb{P}$ is $\mathbb{P}$-generic and $\dot{C} \in N$ is a $\mathbb{P}$-name for a club subset of ω_1 then $p \Vdash \delta \in \dot{C}$.*

Definition 6. Let X be uncountable.

1. A set C of countable subsets of X is *club* if and only if every countable subset of X is contained in an element of C, and C is closed under unions of chains of length ω.
2. A set S of countable subsets of X is *stationary* if and only if S meets every club set C.

The following facts are standard:

1. Let S be a set of countable subsets of the uncountable set X.
   (a) S is stationary if and only if for every $F : {}^{<\omega}X \to X$ there is an nonempty set $A \in S$ with $F[{}^{<\omega}A] \subseteq A$.
   (b) S is stationary if and only if for every countable first order language L and every L-structure $\mathscr{X}$ with underlying set X, there is $A \in S$ with $A \prec \mathscr{X}$.
   (c) (Fodor's lemma) If S is stationary and $F : S \to X$ is *regressive* in the sense that $F(A) \in A$ for all $A \in S$, then there is $T \subseteq S$ stationary with $F \restriction T$ constant.
2. If Y and Z are uncountable sets with $Y \subseteq Z$ then
   (a) If S is a stationary set of countable subsets of Y, then the set of countable $B \subseteq Z$ with $B \cap Y \in S$ is a stationary set of countable subsets of Z.

(b) If T is a stationary set of countable subsets of Z, then $\{B \cap Y : B \in T\}$ is a stationary set of countable subsets of Y.

DEFINITION 7. Let $\mathbb{P}$ be a forcing poset. Then $\mathbb{P}$ is *proper* if and only if for every uncountable set X and every stationary set S of countable subsets of X, the stationarity of S is preserved by $\mathbb{P}$.

Properness can be characterised in terms of generic conditions: $\mathbb{P}$ is proper if and only if for some (equivalently all large) θ such that $P(\mathbb{P}) \in H_\theta$, there is a club set of countable $N \prec H_\theta$ such that for all $p \in \mathbb{P} \cap N$ there is $q \leq p$ where q is $(N, \mathbb{P})$-generic.

All ccc and all countably closed forcing posets are proper. The standard posets for adding new reals (Sacks, Laver, Mathias, Miller et cetera) are all proper. Properness is preserved by countable support iterations.

The *Proper Forcing Axiom (PFA)* is the assertion that for every proper $\mathbb{P}$ and every family $\mathscr{F}$ of dense subsets of $\mathbb{P}$ with $|\mathscr{F}| = \omega_1$, there is a filter F on $\mathbb{P}$ such that $F \cap D \neq \emptyset$ for all $D \in \mathscr{F}$.

3.3. Amalgamating finite polychromatic sets. Let f be a normal and $(< \omega)$-bounded colouring of $[\omega_1]^2$. We consider the problem of *amalgamating* finite polychromatic sets, that is finding conditions under which the union of two such sets is polychromatic. This sort of "amalgamation problem" is ubiquitous in forcing, and almost always arises when we are trying to prove that a forcing poset is proper: after all the very definition of generic condition demands that we should amalgamate extensions of the generic condition p with certain conditions lying in the elementary substructure N.

If X is a finite subset of ω_1 we define

$$F(X) =_{\text{def}} \{\alpha : \exists \gamma\ \exists b \in [X]^2\ f(\alpha, \gamma) = f(b)\}$$

Since f is normal, $F(X)$ is the set of α such that there exist $\beta, \gamma \in X$ with $\alpha, \beta < \gamma$ and $f(\alpha, \gamma) = f(\beta, \gamma)$. Clearly $F(X) \subseteq \max(X)$, and since f is $(< \omega)$-bounded we see that $F(X)$ is finite.

When X and Y are finite sets of ordinals we will abuse notation slightly by writing "$X < Y$" as a shorthand for "$\max(X) < \min(Y)$". Similarly when ζ is an ordinal, "$X < \zeta$" and "$\zeta < X$" have the obvious meanings.

LEMMA 3. *Let $X < \alpha < Y$ where $X \cup \{\alpha\}$ and $X \cup Y$ are both polychromatic for f. If $\alpha \notin F(X \cup Y)$ then $X \cup \{\alpha\} \cup Y$ is polychromatic for f.*

PROOF. Since f is normal, the only problem that can occur is that there may exist $\delta, \varepsilon, \zeta \in X \cup \{\alpha\} \cup Y$ with $\delta < \varepsilon < \zeta$ and $f(\delta, \zeta) = f(\varepsilon, \zeta)$.

Since $X \cup \{\alpha\}$ is polychromatic, it must be that $\zeta \in Y$. Since $X \cup Y$ is polychromatic, either δ or ε equals α. But both of these possibilities are ruled out by the assumption that $\alpha \notin F(X \cup Y)$. $\dashv$

Now we generalise this lemma somewhat. Say that a finite partial function from ω_1 to ω is *good for* f if and only if for all n the set $\{i \in \text{dom}(p) : p(i) = n\}$ is polychromatic for f. Good partial functions will eventually be building blocks in a forcing poset for decomposing ω_1 into ω many polychromatic sets.

The following lemma follows immediately from Lemma 3.

LEMMA 4. *Let p, q, r be finite partial functions from ω_1 to ω such that*

1. $\mathrm{dom}(p) < \mathrm{dom}(q) < \mathrm{dom}(r)$.
2. *q is* 1-1.
3. *$p \cup q$ and $p \cup r$ are both good for f.*
4. $\mathrm{dom}(q)$ *is disjoint from* $F(\mathrm{dom}(p) \cup \mathrm{dom}(r))$.

Then $p \cup q \cup r$ is good for f.

3.4. Elementary substructures. We need some standard facts about elementary substructures $M \prec H_\theta$ where M is countable and $\theta > \omega_1$ is regular.

Firstly if $x \in M$ is countable then $x \subseteq M$. In particular $M \cap \omega_1$ is an initial segment of ω_1, so $M \cap \omega_1 = \delta$ for some $\delta \in \omega_1$.

Secondly suppose that $\lambda \in M$ is a regular cardinal. Then $H_\lambda \in M$, and so also a set of Skolem functions for H_λ is in M. It follows that $M \cap H_\lambda \prec H_\lambda$. In particular if λ is definable in H_θ then $\lambda \in M$ for all countable $M \prec H_\theta$, and so $M \cap H_\lambda \prec H_\lambda$ for all such M.

Thirdly let $M, N \prec H_\theta$ be countable with $M \in N$. Then as we saw above $M \subseteq N$, so in particular $M \prec N$. Also M and ω_1 are both in N so $M \cap \omega_1 \in N$, in particular $M \cap \omega_1 < N \cap \omega_1$.

In the following section we will define a forcing poset in which each condition is partially composed of countable elementary substructures of H_{ω_2}. The idea is that these models will act as "side conditions" which will avoid some obstacle to properness; the idea of using models as side conditions in proper forcing is due to Todorčević [18].

As motivation for the use of countable elementary substructures of H_{ω_2}, we prove a lemma about the kind of amalgamation problem discussed in the last section.

Definition 8. A sequence $\vec{p} = \langle p_\alpha : \alpha < \omega_1 \rangle$ of finite partial functions from ω_1 to ω is *increasing* if and only if $\alpha < \beta \Longrightarrow \mathrm{dom}(p_\alpha) < \mathrm{dom}(p_\beta)$ for all α, β.

We note that by an easy induction, if $\vec{p}$ is increasing then $\alpha \leq \min \mathrm{dom}(p_\alpha)$. Also easily every such sequence $\vec{p}$ is a member of H_{ω_2}. We recall that f is a normal $(< \omega)$-bounded colouring of $[\omega_1]^2$ and that for finite $X \subseteq \omega_1$, $F(X)$ is the set of α with $f(\alpha, \gamma) = f(\beta, \gamma)$ for some $\beta, \gamma \in X$. The following technical lemma will be used to show that the main forcing for Theorem 2 is proper.

Lemma 5. *Let $M \prec H_{\omega_2}$ be countable with $f \in M$, and let $\delta = M \cap \omega_1$. Let p and q be finite partial functions from ω_1 to ω such that*

1. $\mathrm{dom}(p) \subseteq \delta$, $\mathrm{dom}(q) \subseteq \omega_1 \setminus \delta$ (*so in particular* $\mathrm{dom}(p) < \mathrm{dom}(q)$).
2. *$p \cup q$ is good for f.*
3. *q is* 1-1.

Let $\vec{r}$ be an increasing ω_1-sequence of partial functions from ω_1 to ω such that

1. $\vec{r} \in M$.
2. *For all $i < \omega_1$, $p \cup r_i$ is good for f.*

Then there exists $S \in M$ a stationary subset of ω_1 such that for all large enough $i \in S$, $p \cup q \cup r_i$ is good for f.

Proof. Consider the function g on limit ordinals $\alpha < \omega_1$ given by

$$g : \alpha \mapsto \max\{\max \mathrm{dom}(p), \max(\alpha \cap F(\mathrm{dom}(p) \cup \mathrm{dom}(r_\alpha)))\}$$

The function g is in M because $f, F, \vec{r} \in M$. By Fodor's lemma and elementarity there exist $S \in M$ stationary and $\eta \in \delta$ such that g is constant on S with value η. We note that $\mathrm{dom}(p) \subseteq \eta$.

If $i \in S$ and $\mathrm{dom}(q) < i$ then we see that

1. $\mathrm{dom}(p) < \mathrm{dom}(q) < \mathrm{dom}(r_i)$.
2. $\eta < \mathrm{dom}(q) < i$.
3. $F(\mathrm{dom}(p) \cup \mathrm{dom}(r_i)) \cap i \subseteq \eta$.

So $\mathrm{dom}(q)$ is disjoint from $F(\mathrm{dom}(p) \cup \mathrm{dom}(r_i))$, and since also q is 1-1 we may apply Lemma 4 to conclude that $p \cup q \cup r_i$ is good for f. ⊣

3.5. The forcing poset $\mathbb{P}_f$. Recall that f is a normal $(< \omega)$-bounded colouring of $[\omega_1]^2$, and that a finite partial function q from ω_1 to ω is good for f if and only if $\{i : q(i) = n\}$ is polychromatic for each n. We define a forcing poset $\mathbb{P}_f$ whose goal is to add a decomposition of ω_1 into ω polychromatic sets. The conditions are pairs $(p, \mathcal{M})$ where

1. p is a finite partial function p from ω_1 to ω, and p is good for f.
2. $\mathcal{M}$ is a nonempty finite set $\{M_0, \dots M_{n-1}\}$ where $M_i \prec H_{\omega_2}$, M_i is countable and $M_i \in M_{i+1}$ for $i + 1 < n$.
3. For all $\alpha < \beta < \omega_1$, if $p(\alpha) = p(\beta)$ then there exists i such that $\alpha < M_i \cap \omega_1 \leq \beta$.

The ordering is by extension, formally $(p, \mathcal{M}) \leq (q, \mathcal{N})$ if and only if $p \supseteq q$ and $\mathcal{M} \supseteq \mathcal{N}$.

We record a couple of remarks about the poset $\mathbb{P}_f$:

1. Every condition is an element of H_{ω_2}.
2. If $s = (p, \mathcal{M})$ is a condition with $\mathcal{M} = \{M_0, \dots M_{n-1}\}$ and we define
$$s \upharpoonright M_i = (p \upharpoonright M_i, \mathcal{M} \cap M_i)$$
then $s \upharpoonright M_i$ is a condition and $s \upharpoonright M_i \in M_i$ and $s \leq s \upharpoonright M_i$.

If $\gamma < \omega_1$ then we say that the condition $(p, \mathcal{M})$ is *below* γ if and only if $\mathrm{dom}(p) < \gamma$ and $M \cap \omega_1 < \gamma$ for all $M \in \mathcal{M}$. If $(p, \mathcal{M}) \in N$ for some countable $N \prec H_{\omega_2}$ with $N \cap \omega_1 = \gamma$ then easily $(p, \mathcal{M})$ is below γ. The converse is however false in general, if $(p, \mathcal{M})$ is below $N \cap \omega_1$ then $p \in N$ but the models in $\mathcal{M}$ may not lie in N.

The following lemma shows that by forcing with $\mathbb{P}_f$ we add a total function from ω_1 to ω.

Lemma 6. *For every $\alpha < \omega_1$, the set of conditions $(p, \mathcal{M})$ with $\alpha \in \mathrm{dom}(p)$ is dense.*

Proof. Suppose that $(p, \mathcal{M})$ is a condition and that $\alpha \notin \mathrm{dom}(p)$. Choose $n \in \omega$ with $n \notin \mathrm{rge}(p)$ and extend p to q with $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{\alpha\}$, $q(\alpha) = n$. Then $(q, \mathcal{M})$ is as required. ⊣

It is also helpful to know that densely many conditions have a model in the "$\mathcal{M}$-part" which contains the whole of the "p-part".

Lemma 7. *If $(p, \mathcal{M})$ is a condition and $N \prec H_{\omega_2}$ is countable with $(p, \mathcal{M}) \in N$, then $(p, \mathcal{M} \cup \{N\})$ is a condition extending $(p, \mathcal{M})$.*

The next two lemmas are the motivation for clause 3 in the definition of the conditions.

Lemma 8. *Let $(p, \mathcal{M})$ be a condition and $\delta_j = M_j \cap \omega_1$ for $M_j \in \mathcal{M}$. Then*

1. $p \restriction \delta_0$ *is* 1-1.
2. $p \restriction [\delta_i, \delta_{i+1})$ *is* 1-1 *for all* i.
3. *If* M_n *is the largest model in* $\mathscr{M}$ *then* $p \restriction \omega_1 \setminus \delta_n$ *is* 1-1.

PROOF. We only prove the second claim, the others are similar. Suppose that $\delta_i \leq \alpha < \beta < \delta_{i+1}$. Then there is no j such that $\alpha < \delta_j \leq \beta$, so that $p(\alpha) \neq p(\beta)$. ⊣

LEMMA 9. *Let* $(p, \mathscr{M})$ *be a condition where* $\mathscr{M} = \{M_0, \dots M_{n-1}\}$ *say. Let* $\delta_j = M_j \cap \omega_1$ *for each* j, *and let* $q = p \restriction M_k$ *for some* $k < n$. *If* $\vec{r} \in M_k$ *is an increasing* ω_1*-sequence of finite partial functions* r_i *such that* $q \cup r_i$ *is good for every* $i < \omega_1$, *then there is a stationary set* $S \in M_{n-1}$ *such that* $p \cup r_i$ *is good for all* $i \in S$.

PROOF. Apply Lemma 5 repeatedly with $p \restriction \delta_j$, $p \restriction [\delta_j, \delta_{j+1})$, and M_j playing the roles of p, q, M for $j = k, k+1, \dots n-1$ ⊣

3.6. Properness of $\mathbb{P}_f$. Now we are ready to argue that $\mathbb{P}_f$ is proper. Let θ be a large regular cardinal, and let $s = (p, \mathscr{M}) \in N \prec H_\theta$ where N is countable. Let $M = N \cap H_{\omega_2}$ and note that $M \prec H_{\omega_2}$, $M \cap \mathbb{P}_f = N \cap \mathbb{P}_f$, and $s \in M$. We define $t = (p, \mathscr{M} \cup \{M\})$, observe that $t \leq s$ and argue that t is $(N, \mathbb{P}_f)$-generic. The genericity of the condition t will follow from the following slightly more general result.

LEMMA 10. *Let* $N \prec H_\theta$ *be countable and let* $t = (p, \mathscr{M})$ *be a condition such that* $N \cap H_{\omega_2} \in \mathscr{M}$. *Then* t *is* $(N, \mathbb{P}_f)$*-generic.*

PROOF. Let $D \in N$ be dense, so that we need to produce a condition in $D \cap N$ compatible with t. We may as well assume that $t \in D$, otherwise extend it to a condition in D and apply the argument below to the extension.

Let $M = N \cap H_{\omega_2}$ and $\delta = M \cap \omega_1 = N \cap \omega_1$. We notice that $\mathbb{P} \cap M = \mathbb{P} \cap N$ and $D \cap M = D \cap N$. As above we define $t \restriction M = (p \restriction \delta, \mathscr{M} \cap M)$, so that $t \restriction M \in M$ and $t \leq t \restriction M$. Let $q = p \restriction (\omega_1 \setminus \delta)$. It will suffice to produce $\bar{t} = (\bar{p}, \bar{\mathscr{M}}) \leq t \restriction M$ such that $\bar{t} \in D \cap M$ and $\bar{p} \cup q$ is good. For if we have such a $\bar{t}$ then easily $(\bar{p} \cup q, \bar{\mathscr{M}} \cup \mathscr{M})$ is a condition, so that t is compatible with $\bar{t}$ which lies in $D \cap N$.

So we suppose for a contradiction that no such condition $\bar{t}$ exists. We define B to be the set of pairs (α, r) such that

1. r is a finite partial function from ω_1 to ω with $|r| = |q|$.
2. $\mathrm{dom}(p \restriction \delta) < \alpha \leq \mathrm{dom}(r)$.
3. $p \restriction \delta \cup r$ is good.
4. There is no condition $(p^*, \mathscr{M}^*)$ such that
   (a) $(p^*, \mathscr{M}^*) \leq t \restriction M$.
   (b) $(p^*, \mathscr{M}^*)$ is below α.
   (c) $p^* \cup r$ is good.
   (d) $(p^*, \mathscr{M}^*) \in D$.

We claim that $(\delta, q) \in B$. Suppose not: then we may find an extension $(p^*, \mathscr{M}^*)$ of $t \restriction M$ which is below δ, lies in D and is such that $p^* \cup r$ is good. As we pointed out above, $p^* \in M$ but $\mathscr{M}^*$ may lie outside M. However if $\mathscr{M}^* = \{M_0^*, \dots M_{k-1}^*\}$ then $\{M_i^* \cap \omega_1 : i < k\}$ *is* in M, and using elementarity we may find a condition $(p^*, \mathscr{M}^{**}) \in M$ which lies in D with $\mathscr{M}^{**} = \{M_0^{**}, \dots M_{k-1}^{**}\}$ and $M_i^* \cap \omega_1 = M_i^{**} \cap \omega_1$ for $i < k$. But this is impossible by our assumption that no $\bar{t}$ as above exists.

Now $B \in N$ because B is defined from parameters in N. Since $B \in H_{\omega_2}$, in fact $B \in M$. It follows that since $(\delta, q) \in B$ the set $\{\alpha : \exists r\ (\alpha, r) \in B\}$ is unbounded in ω_1, else it could be bounded by an ordinal in M. So we may construct in M a sequence of elements $(\delta_i, r_i) \in B$ such that $\delta_i < \mathrm{dom}(r_i) < \delta_{i+1}$.

Appealing to Lemma 9 we get an uncountable subsequence of this sequence consisting of pairs for which $p \cup r_i$ is good. If we now choose such a pair with $p < \delta_i$ then $t \leq t \upharpoonright M$, t is below δ_i, $p \cup r_i$ is good and $t \in D$; this is a contradiction because $(\delta_i, r_i) \in B$ and the last clause in the definition of B rules this out! ⊣

Now that we have showed that $\mathbb{P}_f$ is proper, a routine application of the PFA shows that for any $(< \omega)$-bounded colouring of pairs from ω_1, ω_1 is the union of ω polychromatic sets. This establishes Theorem 2.

In particular this shows that under PFA, $\omega_1 \to^{\mathrm{poly}} (\omega_1)^2_{2-\mathrm{bd}}$. Actually we get something a bit stronger, namely that there is a *stationary* polychromatic set.

Remark 4. As we mentioned in the introduction, Todorčević proved [17] that the conclusion of Theorem 2 is consistent relative to the consistency of ZFC, and also that it follows from PFA. The forcing argument of that paper is rather different from that of this section: it involves a mixed (finite/countable) support iteration of Cohen forcing and Jensen's "fast club forcing", followed by a finite support ccc iteration to add the polychromatic sets. The argument from PFA involves showing that after adding a fast club, the poset whose conditions are finite polychromatic sets separated by points of the fast club has the ccc.

Remark 5. Todorčević [17] points out that his methods give the following ZFC result: for every countable ordinal α, every 2-bounded colouring of pairs from $[\omega_1]^2$ has a closed polychromatic set of order type α. This also follows from our results: we can add a stationary polychromatic set and then shoot a club through that set without collapsing ω_1, then borrow an argument from [4] and observe that the tree of attempts to build a closed polychromatic set of type α must have a branch in V. Yet another proof would be to apply Galvin's dual colouring idea as described in Section 2 to Schipperus "topological Baumgartner-Hajnal theorem" [14].

§4. MA does not suffice for Theorem 2.

Since the relation $\omega_1 \to^{\mathrm{poly}} (\omega_1)^2_{2-\mathrm{bd}}$ holds under PFA but fails under CH, it is natural to ask about its status under Martin's Axiom (MA). It turns out to be independent of MA, in fact we can get the consistency of a "ccc-indestructible bad partition".

Theorem 3. *It is consistent that there is a 2-bounded normal colouring of pairs from ω_1 which has no uncountable polychromatic set in any ccc forcing extension.*

In particular since we can force MA by a ccc forcing extension over the model of Theorem 3, MA does not imply that $\omega_1 \to^{\mathrm{poly}} (\omega_1)^2_{2-\mathrm{bd}}$. This proof uses similar techniques to those in a paper by Abraham and Todorčević [2] about indestructible properties in topology.

4.1. The poset $\mathbb{Q}$. We describe the poset $\mathbb{Q}$ which we will ultimately use to add the "bad colouring". We fix for each $\beta < \omega_1$ a set $W_\beta \subseteq \omega_1$, such that $|W_\beta| = \omega$ and the sets W_β are pairwise disjoint.

Conditions in $\mathbb{Q}$ are finite partial functions q such that

1. $\mathrm{dom}(q) = [F]^2$ for some finite $F \subseteq \omega_1$.

2. $q(\alpha, \beta) \in W_\beta$.
3. q is 2-bounded.

The ordering is extension. Abusing notation a bit we refer to the finite set F as the *domain* of q and write $F = \mathrm{dom}(q)$.

The demand that $q(\alpha, \beta) \in W_\beta$ makes the normality of the generic colouring automatic.

LEMMA 11. *For every $\alpha < \omega_1$ the set of q with $\alpha \in \mathrm{dom}(q)$ is dense.*

PROOF. Let $F = \mathrm{dom}(q)$. Enumerate the pairs in $[F \cup \{\alpha\}]^2 \setminus [F]^2$ as (α_i, β_i) for $i < N$ and then colour them, making sure to give (α_i, β_i) a new colour from W_{β_i}. ⊣

The chain condition argument is also routine but we do it carefully to stress a point about amalgamation of conditions which will be crucial later.

LEMMA 12. *The poset $\mathbb{Q}$ has the ω_1-Knaster property, in particular it is ccc.*

PROOF. Let $\langle q_i : i < \omega_1 \rangle$ be an ω_1-sequence of conditions. By a standard application of the Δ-system lemma we may assume that the sets $F_i =_{\mathrm{def}} \mathrm{dom}(q_i)$ form a "head-tail-tail Δ-system". That is to say there are finite subsets r and s_i (for $i < \omega_1$) of ω_1 such that $r < s_0 < s_1 \dots$ and $F_i = r \cup s_i$.

Since $W =_{\mathrm{def}} \cup_{\beta \in r} W_\beta$ is countable and $q_i : [r]^2 \to W$ for all i, we may further assume that there is a fixed p such that $q_i \restriction [r]^2 = p$ for all i. Now let $i < j < \omega_1$ and consider q_i and q_j. We see that $[F_i]^2 \cap [F_j]^2 = [r]^2$ so that $q_i \cup q_j$ is a function. What is more $q_i \cup q_j$ is 2-bounded, because pairs with top points in s_i assume colours in $\cup_{\beta \in s_i} W_\beta$ while pairs with top points in s_j assume colours in the disjoint set $\cup_{\beta \in s_j} W_\beta$.

Now $[F_i \cup F_j]^2 = [F_i]^2 \cup [F_j]^2 \cup s_i \times s_j$, so to produce a common extension of q_i and q_j we need to assign colours to pairs in $s_i \times s_j$. This is possible because there are infinitely many unused colours in W_β for each $\beta \in s_j$; in particular (and this is the point we wished to stress) we have great freedom in the choice of the colouring on $s_i \times s_j$. ⊣

LEMMA 13. *The generic colouring c of $[\omega_1]^2$ added by $\mathbb{Q}$ is normal, 2-bounded and has no uncountable polychromatic set.*

PROOF. The verification that the colouring is normal and 2-bounded is easy so we concentrate on the last claim. Let $\dot{A}$ name an uncountable set. As usual we may choose q_i and α_i for $i < \omega_1$ such that

1. α_i increases with i.
2. $q_i \Vdash \alpha_i \in \dot{A}$.
3. $\alpha_i \in \mathrm{dom}(q_i)$.
4. The sets $\mathrm{dom}(q_i)$ form a head-tail-tail Δ-system with root r, and $q_i \restriction [r]^2$ is independent of i.

Now we choose $i < j < k$ so that $\alpha_i, \alpha_j, \alpha_k \notin r$ and then (as in the proof of Lemma 12) we may easily find $q \leq q_i, q_j, q_k$ forcing that (α_i, α_k) gets the same colour as (α_j, α_k). ⊣

4.2. Motivation for the main construction. We digress to discuss some considerations which motivate the combinatorics of the main construction. This digression is purely for motivation, the impatient reader can skip everything except Definition 9.

Recall that our ultimate goal is to produce a normal 2-bounded $c : [\omega_1]^2 \to \omega_1$ which has no uncountable polychromatic set in any ccc forcing extension. We are searching for a property which will guarantee this and is consistent. Clearly if we are to have any chance of success then c must start off as a partition with no uncountable polychromatic set. Let us see whether this property is preserved by ccc forcing.

So let us be given a normal 2-bounded $c : [\omega_1]^2 \to \omega_1$ a ccc poset $\mathbb{R}$, and a $\mathbb{R}$-name $\dot{A}$ for an uncountable subset of ω_1. Assume that c has

Porperty 1: c has no uncountable polychromatic set, that is to say for every uncountable $B \subseteq \omega_1$ there exist ordinals $\alpha < \beta < \gamma$ in B with $c(\alpha, \gamma) = c(\beta, \gamma)$.

As usual we may produce r_i and α_i for $i < \omega_1$ so that the α_i are increasing with i and $r_i \Vdash \alpha_i \in \dot{A}$. Let $B = \{\alpha_i : i < \omega_1\}$. Then

1. By our assumption on c there are many triples $i < j < k$ with $c(\alpha_i, \alpha_k) = c(\alpha_j, \alpha_k)$.
2. It follows from the ccc (see Lemma 15 below) that there are many triples $i < j < k$ such that there is some $r \leq r_i, r_j, r_k$.

Unfortunately there is no reason to believe that there is any overlap between these two classes of triples. So a natural thing to try might be to try and make the first set of triples larger by considering

Porperty 2: For every uncountable $B \subseteq \omega_1$ there exists an uncountable set $B' \subseteq B$ such that for every triple $\alpha < \beta < \gamma$ from B' we have $c(\alpha, \gamma) = c(\beta, \gamma)$.

This property is preserved by ccc forcing but unfortunately it is incompatible with the 2-boundedness of c. We modify property 2 as follows.

Definition 9. An *increasing sequence of pairs of ordinals* is a sequence $\langle y_i : i < i^* \rangle$ where each y_i is an (unordered) pair of ordinals and $i < j \Longrightarrow \max y_i < \min y_j$.

Given pairs of countable ordinals $a = \{\alpha, \beta\}$ and $b = \{\gamma, \delta\}$ with $\alpha < \beta < \gamma < \delta$ we say that $G(a, b)$ holds (for c) if and only if $c(\alpha, \zeta) = c(\beta, \zeta)$ for some $\zeta \in b$.

Now consider

Porperty 3: for every increasing sequence of pairs of countable ordinals $\langle y_i : i < \omega_1 \rangle$, there is an uncountable $B \subseteq \omega_1$ such that for every pair $i < j$ from B we have $G(y_i, y_j)$.

Property 3 is preserved by ccc forcing, it implies that c has no uncountable polychromatic set, and it is not obviously inconsistent with 2-boundedness. Better still, it is true in $V^{\mathbb{Q}}$ that for any given sequence $y = \langle y_i : i < \omega_1 \rangle$ there is a ccc forcing $\mathbb{R}_y$ which adds a witness to Property 3; the forcing is easy to describe, conditions are just finite sets $a \subseteq \{y_i : i < \omega_1\}$ such that for any $i < j$, $y_i, y_j \in a \Longrightarrow G(y_i, y_j)$.

To see that $\mathbb{R}_y$ is ccc in $V^{\mathbb{Q}}$ we verify that $\mathbb{Q} * \dot{\mathbb{R}}_y$ is ccc in V. So let $(q_i, \dot{a}_i)$ for $i < \omega$ be conditions in $\mathbb{Q} * \dot{\mathbb{R}}_y$. As usual we may assume (extending q_i and thinning out as necessary) that

1. q_i determines the value of $\dot{a}_i$, say $q_i \Vdash \dot{a}_i = a_i$.
2. $a_i \subseteq \mathrm{dom}(q_i)$.
3. The sets $\mathrm{dom}(q_i)$ form a head-tail-tail Δ-system with root r, and $q_i \restriction [r]^2$ is constant.

It is now easy to see that if $i < j$ then as in the proof of Lemma 12 we may find $q \le q_i, q_j$ such that $q \Vdash a_i \cup a_j \in \dot{\mathbb{R}}_y$. Unfortunately when we try to iterate this forcing we run into trouble with ccc; analysing a two-step iteration $\mathbb{R}_y * \dot{\mathbb{R}}_z$ in the same way, we may have difficulties amalgamating conditions (q_i, a_i, b_i) and (q_j, a_j, b_j). The amalgamation problem can be cured if each pair $z_i \in z$ is "widely spaced" in a sense to be made precise in the next section. So property 3 is not quite what we want, we need to allow "spacing out" of the pairs. It is tempting to try

PORPERTY 4: for every uncountable $A \subseteq \omega_1$ there is an increasing sequence of pairs $\langle y_i \in [A]^2 : i < \omega_1 \rangle$ such that $G(y_i, y_j)$ for all $i < j < \omega_1$.

However property 4 is not obviously preserved by ccc forcing. The property $(*)$ which we will use, and which is defined in Definition 10, is a strengthening of property 4 which *is* easily seen to be preserved by ccc forcing. To add a witness to an instance of $(*)$ we can force with $\mathbb{R}_y$ for a well-chosen y, but need to see that each instance of $\mathbb{R}_y$ which we force with is ccc.

To see that the various instances of $\mathbb{R}_y$ which we are forcing with are ccc we will maintain an inductive hypothesis "there exists a club C with $P(C)$" where $P(C)$ is defined in Section 4.5. $P(C)$ implies the following statement $Q(C)$: "if the pairs in y are each separated by a point of C then any sequence of disjoint conditions in $\mathbb{R}_y$ contains two compatible conditions". Unfortunately the statement $Q(C)$ is not strong enough to propagate forwards by induction through the stages of our iteration: the statement $P(C)$ is a sort of "n-dimensional" version of $Q(C)$ which can be inductively propagated.

4.3. The property $(*)$.

DEFINITION 10. Let $c : [\omega_1]^2 \to \omega_1$. c has property $(*)$ if and only if for every club $C \subseteq \omega_1$, every unbounded set $A \subseteq \omega_1$ and every $f : A \times \omega_1 \to A$ such that $f(\zeta, \eta) \ge \eta$ for all ζ, η there is a sequence $\langle x_i : i < \omega_1 \rangle$ such that

1. $x_i = (\alpha_i, \beta_i, \gamma_i)$ where $\alpha_i < \beta_i \le \gamma_i$ with $\alpha_i, \gamma_i \in A$ and $\beta_i \in C$.
2. $\gamma_i = f(\alpha_i, \beta_i)$
3. If $i < j < \omega_1$ then $\gamma_i < \alpha_j$ and $G(\{\alpha_i, \gamma_i\}, \{\alpha_j, \gamma_j\})$ holds.

LEMMA 14. *If c has $(*)$ then c has no uncountable polychromatic set.*

PROOF. Suppose for a contradiction that A is an uncountable polychromatic set and define $f(\alpha, \beta) = \min(A \setminus \beta)$. Let $C = \omega_1$ and find x_i as in $(*)$. Then for any $i < j$ we get that $\alpha_i < \gamma_i < \alpha_j < \gamma_j$ where all four ordinals are in A and $c(\alpha_i, \zeta) = c(\gamma_i, \zeta)$ for some $\zeta \in \{\alpha_j, \gamma_j\}$. So A is not polychromatic, contradiction! ⊣

We now show that property $(*)$ is ccc indestructible. This uses two easy and well known observations about ccc forcing.

LEMMA 15. *Let $\mathbb{P}$ be ccc and let $\langle p_i : i < \omega_1 \rangle$ be any sequence of conditions in $\mathbb{P}$. Then some condition p_i forces that $\{i : p_i \in G\}$ is uncountable.*

PROOF. Suppose not, and choose for every i a condition $q_i \le p_i$ and an ordinal β_i such that $q_i \Vdash \{j : p_j \in G\} \subseteq \beta_i$. Then $q_i \Vdash p_j \notin G$ for all $j \ge \beta_i$, that is to say that $q_i \perp p_j$ for $j \ge \beta_i$. If we now define a club set $E = \{j : \forall i < j \ \beta_i < j\}$ then easily $\{q_i : i \in E\}$ is an antichain, contradicting the ccc of $\mathbb{P}$. ⊣

Lemma 16. *Let $\mathbb{P}$ be ccc and let $\dot{E}$ be a name for a club subset of ω_1. Then there is a club set D such that $\Vdash \check{D} \subseteq \dot{E}$.*

Proof. Let $\dot{f}$ name the map $\alpha \mapsto \min(\dot{E} \setminus \alpha)$ and let $A_\alpha = \{\beta : \exists p\ p \Vdash \dot{f}(\alpha) = \beta\}$. Then by ccc A_α is countable. If we let $D = \{\beta : \forall \alpha < \beta\ A_\alpha \subseteq \beta\}$ then every element of D is forced to be a limit point of $\dot{E}$, and hence a member of $\dot{E}$. ⊣

We use the preceding lemma to show that we only need consider club sets in V in the next lemma.

Lemma 17. *If $(*)$ holds then it remains true in any ccc forcing extension.*

Proof. Let $(*)$ hold in V and let $\mathbb{P}$ be ccc. Let p force that $\dot{C}$ is a club subset of ω_1, $\dot{A}$ is an unbounded subset of ω_1, and $\dot{f} : \dot{A} \times \omega_1 \to \dot{A}$ is such that $\dot{f}(\zeta, \eta) \geq \eta$ for all ζ, η. Fix $D \subseteq \omega_1$ a club set such that $p \Vdash \check{D} \subseteq \dot{C}$.

In V let $A = \{\alpha : \exists q \leq p\ q \Vdash \alpha \in \dot{A}\}$, and choose for each $\zeta \in A$ and $\eta \in \omega_1$ a condition $p(\zeta, \eta) \leq p$ which forces that $\zeta \in \dot{A}$ and determines the value of $\dot{f}(\zeta, \eta)$ as some ordinal $f(\zeta, \eta)$. Note that $p(\zeta, \eta)$ forces that $f(\zeta, \eta) \in \dot{A}$ so that $f(\zeta, \eta) \in A$ and so $f : A \times \omega_1 \to A$.

Now we appeal to $(*)$ in V to find a suitable sequence of triples $x_i = (\alpha_i, \beta_i, \gamma_i)$ with $\beta_i \in D$. If $p_i = p(\alpha_i, \beta_i)$ then p_i forces that $\alpha_i, \gamma_i \in \dot{A}$ and $\dot{f}(\alpha_i, \beta_i) = \gamma_i$. By Lemma 15 there is p_j which forces that $\{i : p_i \in G\}$ is unbounded. Then $p_j \leq p$ and p_j forces that $\langle (\alpha_i, \beta_i, \gamma_i) : p_i \in G \rangle$ is a suitable witness to the truth of $(*)$ in $V^{\mathbb{P}}$. ⊣

4.4. A combinatorial lemma. At the heart of the proof is a combinatorial lemma from a paper of Abraham and Todorčević [2]. For the reader's convenience we reproduce the proof here.

Definition 11. Let $Y = \langle \{\alpha_i, \beta_i\} : i < i^* \rangle$ be an increasing sequence of pairs of ordinals.

1. An ordinal γ is a *closure point* of Y if and only if $\forall i < i^*\ (\alpha_i < \gamma \Longrightarrow \beta_i < \gamma)$. $C(Y)$ is the class of closure points of Y.
2. The sequence Y is *separated by A* if and only if for every $i < i^*$ there is $\eta \in A$ with $\alpha_i < \eta < \beta_i$, and for every $i < i + 1 < i^*$ there is $\eta \in A$ with $\beta_i < \eta < \alpha_{i+1}$.

Definition 12. 1. A *matrix of pairs* is a finite sequence $(Y_1, \dots Y_n)$ such that each Y_i is a finite increasing sequence of pairs of countable ordinals, and for each i with $1 < i \leq n$ the sequence Y_i is separated by $\bigcap_{j<i} C(Y_j)$. We say that n is the *length* of the matrix, and we refer to the finite sequence Y_j as the j^{th} *column* of the matrix.

2. A matrix of pairs $M = (Y_1, \dots Y_n)$ is *separated by A* if and only if the sequence Y_1 is separated by A.
3. If $M = (Y_1, \dots Y_n)$ is a matrix of pairs then we say a pair of ordinals *appears in M* if it is an entry in some Y_j, and an ordinal *appears in M* if it is a member of some pair appearing in M.
4. If M and N are matrices of pairs then $M < N$ if every ordinal appearing in M is less than every ordinal appearing in N. Similarly $M < \beta$ if every ordinal appearing in M is less than β, and $\beta < M$ if β is less than every ordinal appearing in M.

The key fact about matrices of pairs is the following lemma, which appears as Lemma 5.3 in [2].

Lemma 18. *Let M be a matrix of pairs. Then M has a* 1-1 *choice function, that is to say there is a* 1-1 *function which maps each pair u appearing in M to a member of u.*

Proof. We will establish the stronger claim that for every matrix of pairs $M = (Y_1, \dots Y_n)$ and every ordinal ρ, there is a 1-1 choice function for M which does not take the value ρ. The proof is by induction on n, and for fixed n by induction on $|Y_n|$.

There is no problem for $n = 1$. Suppose that $n > 1$ and that $|Y_n| > 0$, and choose a pair $u = \{\gamma, \delta\}$ appearing in Y_n. Since M is a matrix of pairs there is η such that $\gamma < \eta < \delta$ and $\eta \in C(Y_i)$ for all $i < n$.

The key point is that for every pair $\{\alpha, \beta\}$ appearing in $(Y_1, \dots Y_{n-1})$, either $\alpha < \beta < \eta$ or $\eta \leq \alpha$. We now define new matrices by (informally) "dropping the pair $\{\gamma, \delta\}$ from column n and using η to split the earlier columns".

Formally we define matrices of pairs $M^{\text{low}} = (Y_1^{\text{low}}, \dots Y_n^{\text{low}})$ and $M^{\text{high}} = (Y_1^{\text{high}}, \dots Y_n^{\text{high}})$ as follows:

1. For $1 \leq i < n$, the pairs appearing in Y_i^{low} are the pairs $\{\alpha, \beta\}$ appearing in Y_i with $\alpha < \beta < \eta$.
2. The pairs appearing in Y_n^{low} are the pairs $\{\alpha, \beta\}$ appearing in Y_n with $\beta < \gamma$.
3. For $1 \leq i < n$, the pairs appearing in Y_i^{high} are the pairs $\{\alpha, \beta\}$ appearing in Y_i with $\eta \leq \alpha < \beta$.
4. The pairs appearing in Y_n^{high} are the pairs $\{\alpha, \beta\}$ appearing in Y_n with $\alpha > \delta$.

Note that the induction hypothesis applies to both M^{low} and M^{high} since we made sure to drop a pair from the last column Y_n of the matrix M. Note also that all pairs appearing in M^{low} consist of ordinals which are less than η, and all pairs appearing in M^{high} consist of ordinals which are greater than or equal to η.

Now suppose that the ordinal ρ which we wish to avoid has $\rho < \eta$. By the induction hypothesis we may find a 1-1 choice function for M^{low} avoiding the value ρ and a 1-1 choice function for M^{high} avoiding the value δ. By the remarks in the last paragraph the ranges of these functions are disjoint, and also the union of these functions avoids both the values ρ and δ. To finish we choose from $\{\gamma, \delta\}$ the element δ. Similarly if $\rho \geq \eta$ we take the union of a 1-1 choice function for M^{low} avoiding the value γ and a 1-1 choice function for M^{high} avoiding the value ρ, and choose γ from $\{\gamma, \delta\}$. ⊣

4.5. The property $P(C)$. We are almost ready to describe the main construction. We will start with V a model of GCH and force with $\mathbb{Q}$. Working over $V^{\mathbb{Q}}$, we will do a certain ccc finite support iteration $\mathbb{P}_{\omega_2}$ of length ω_2 (whose details will be given in the next section) to produce a model in which c has $(*)$, where $c : [\omega_1]^2 \to \omega_1$ is the generic colouring added by $\mathbb{Q}$. A key technical point will be that each stage $i < \omega_2$, there is a club set $C_i \subseteq \omega_1$ such that a certain statement $P(C_i)$ holds in $V^{\mathbb{P}_i}$.

Definition 13. Let $C \subseteq \omega_1$ be club and let $c : [\omega_1]^2 \to \omega_1$.

1. Let $M = (Y_1, \dots Y_n)$ and $N = (Z_1, \dots Z_n)$ be matrices of pairs with the same length n. Then $G(M, N)$ *holds* if and only if $M < N$ and for all i with $1 \leq i \leq n$, all pairs u appearing in Y_i and all pairs v appearing in Z_i, $G(u, v)$ holds.

2. A sequence of matrices of pairs $\langle M_i : i < i^* \rangle$ is an *increasing sequence of matrices separated by C* if and only if
(a) Each M_i is separated by C.
(b) For $i < j$ there is $\beta \in C$ such that $M_i < \beta < M_j$.

3. $P(C)$ *holds (for C)* if and only if for every $\langle M_i : i < \omega_1 \rangle$ an increasing sequence of matrices separated by C, there exist $i < j$ such that $G(M_i, M_j)$.

We will inductively construct a sequence of clubs C_i so that $P(C_i)$ holds after i steps of the iteration over $V^{\mathbb{Q}}$. The following lemma says we may take $C_0 = \omega_1$.

LEMMA 19. *$P(\omega_1)$ holds in $V^{\mathbb{Q}}$.*

PROOF. Let $\langle \dot{M}_i : i < \omega_1 \rangle$ be a $\mathbb{Q}$-name for an increasing sequence of matrices of pairs. As usual we may choose conditions q_i and matrices of pairs M_i for $i < \omega_1$ such that

1. $q_i \Vdash \dot{M}_i = M_i$.
2. $\langle M_i : i < \omega_1 \rangle$ is an increasing sequence of matrices of pairs, and all the M_i have the same length n. Let $M_i = (Y_i^1, \dots Y_i^n)$.
3. All ordinals appearing in M_i are in $\mathrm{dom}(q_i)$.
4. The sets $\mathrm{dom}(q_i)$ form a head-tail-tail Δ-system with root r, and $q_i \restriction [r]^2$ is constant.
5. $\sup(r) < M_i$.

Choose i, j with $i < j$ and use Lemma 18 to find a 1-1 choice function F for the pairs in M_j. Enumerate the pairs in M_j as v_k for $k < N$, where v_k appears in $Y_j^{d_k}$, that is column d_k of the matrix M_j. Now for each k in turn we run through the pairs u appearing in column $Y_j^{d_k}$ in increasing order; if $u = \{\zeta, \eta\}$ then we will assign the pairs $\{\zeta, F(v_k)\}$ and $\{\eta, F(v_k)\}$ a legal colour not used before. At the end we extend $q_i \cup q_j$ to force these colour assignments and we have produced $q \leq q_i \cup q_j$ such that $q \Vdash G(M_i, M_j)$. ⊣

The argument that at every stage of the iteration there is a club set $C \subseteq \omega_1$ with $P(C)$ is modelled on the proof that ccc is preserved in a finite support iteration. We will leave the successor step to the next section. For the limit step we need a characterisation of how $P(C)$ comes to hold in a ccc generic extension.

LEMMA 20. *Let $\mathbb{P}$ be ccc, let $p \in \mathbb{P}$, let $C \subseteq \omega_1$ be club and $c : [\omega_1]^2 \to \omega_1$. Then the following are equivalent.*

1. $p \Vdash_{\mathbb{P}} P(C)$.
2. *For every sequence $\langle (p_i, M_i) : i < \omega_1 \rangle$ where $p_i \leq p$ and $\langle M_i : i < \omega_1 \rangle$ is an increasing sequence of matrices of pairs separated by C, there exist $i < j$ such that p_i is compatible with p_j and $G(M_i, M_j)$.*

PROOF. Suppose that $p \Vdash_{\mathbb{P}} P(C)$. By Lemma 15 we may force below some p_i and get a generic filter $G \ni p$ such that $p_i \in G$ for unboundedly many $i < \omega_1$. Now $\langle M_i : p_i \in G \rangle$ is (relabelling) an increasing ω_1-sequence of matrices of pairs which is separated by C, and since $P(C)$ holds in $V[G]$ we may find $p_i, p_j \in G$ with $G(M_i, M_j)$. Of course p_i and p_j are compatible since they lie in the filter G, and the statement $G(M_i, M_j)$ is absolute between V and $V[G]$.

Now suppose that 2 holds and $q \leq p$. Fix $\langle \dot{M}_i : i < \omega \rangle$ a $\mathbb{P}$-name for an increasing ω_1-sequence of matrices of pairs which is separated by C. Choose $q_i \leq q$ and M_i

such that $q_i \Vdash \dot{M}_i = M_i$, thinning out if necessary we may assume that $\langle M_i : i < \omega_1 \rangle$ is an increasing sequence of matrices of pairs separated by C. Appealing to 2 there are $i < j$ so that q_i, q_j are compatible and $G(M_i, M_j)$, so that if $r \leq q_i, q_j$ then $r \Vdash G(\dot{M}_i, \dot{M}_j)$. We are done by the usual density argument. ⊣

Lemma 21. *Let $\delta < \omega_2$ be a limit ordinal, and let $\langle C_i : i < \delta \rangle$ be a sequence of club subsets of ω_1.*

If P_δ is a ccc finite support iteration of limit length $\delta < \omega_2$, such that for every $i < \delta$ the statement $P(C_i)$ holds in $V^{\mathbb{P}_i}$, then there is $C_\delta \subseteq \omega_1$ club such that $P(C_\delta)$ holds in $V^{\mathbb{P}_\delta}$.

Proof. There are two cases.

$cf(\delta) = \omega$. Choose $\langle \delta_n : n < \omega \rangle$ cofinal in δ and set $C_\delta = \bigcap_n C_{\delta_n}$.

We verify the condition from Lemma 20. Let $\langle (p_i, M_i) : i < \omega_1 \rangle$ where $p_i \in \mathbb{P}_\delta$ and $\langle M_i : i < \omega_1 \rangle$ is an increasing sequence of matrices of pairs separated by C_δ. Since the iteration is done with finite support, for every $i < \omega_1$ there is $n < \omega$ such that $\mathrm{supp}(p_i) \subseteq \delta_n$. Thinning out if necessary we may assume that there is a fixed n such that $\mathrm{supp}(p_i) \subseteq \delta_n$ for all i.

By the choice of C_δ, we have $C_\delta \subseteq C_{\delta_n}$. So we have a sequence $\langle (p_i, M_i) : i < \omega_1 \rangle$ where (morally speaking) $p_i \in \mathbb{P}_{\delta_n}$ and $\langle M_i : i < \omega_1 \rangle$ is an increasing sequence of matrices of pairs separated by C_{δ_n}.

Since $P(C_n)$ holds in $V^{\mathbb{P}_{\delta_n}}$, another appeal to Lemma 20 shows that there are $i < j$ so that p_i, p_j are compatible and $G(M_i, M_j)$. We are done.

$cf(\delta) = \omega_1$. Choose $\langle \delta_j : j < \omega_1 \rangle$ cofinal in δ and let C_δ be the diagonal intersection of the C_{δ_j}; by construction for each j we have that $C_\delta \setminus C_{\delta_j}$ is bounded.

Again we verify the condition from Lemma 20. Let $\langle (p_i, M_i) : i < \omega_1 \rangle$ where $p_i \in \mathbb{P}_\delta$ and $\langle M_i : i < \omega_1 \rangle$ is an increasing sequence of matrices of pairs separated by C_δ. Appealing to the Δ-system lemma we may assume, thinning out if necessary, that the finite sets $\mathrm{supp}(p_i)$ form a Δ-system with some root r. We fix i^* such that $r \subseteq \delta_{i^*}$; note that if $i < j$ then p_i, p_j are compatible in $\mathbb{P}_\delta$ if and only if the restrictions $p_i \restriction \delta_{i^*}, p_j \restriction \delta_{i^*}$ are compatible in $\mathbb{P}_{\delta^*}$.

The sequence $\langle M_i : i < \omega_1 \rangle$ is separated by C_δ and $C_\delta \setminus C_{\delta_{i^*}}$ is bounded. So throwing away an initial segment if necessary, we may assume that $\langle M_i : i < \omega_1 \rangle$ is separated by $C_{\delta_{i^*}}$. Recalling that $P(C_{\delta_{i^*}})$ holds in $V^{\mathbb{P}_{\delta^*}}$ and applying Lemma 20 to the sequence $\langle (p_i \restriction \delta_{i^*}, M_i) : i < \omega_1 \rangle$ we obtain $i < j$ such that $p_i \restriction \delta_{i^*}, p_j \restriction \delta_{i^*}$ are compatible and $G(M_i, M_j)$. By the remarks in the last paragraph p_i and p_j are compatible so we are done. ⊣

4.6. Putting it all together. Recall that the plan of the proof is as follows: start with GCH in V, force with $\mathbb{Q}$ to add a colouring c, and then build a ccc iteration $\mathbb{P}_{\omega_2}$ to force that c has $(*)$, constructing along the way a sequence of clubs C_i so that $P(C_i)$ holds in $V^{\mathbb{Q} * \dot{\mathbb{P}}_i}$.

The next lemma is the final piece in the puzzle; it shows that we can go one step in the construction, that is assuming $P(D)$ we may force an instance of $(*)$ in a ccc fashion so that in the extension $P(E)$ holds for some E.

Lemma 22. *Let $c : [\omega_1]^2 \to \omega_1$ and let property $P(D)$ hold for some club set $D \subseteq \omega_1$. Let f, A, C be as in the hypotheses of $(*)$, that is*

1. *A is an unbounded subset of ω_1.*

2. $f : A \times \omega_1 \to A$ *and* $f(\zeta, \eta) \geq \eta$ *for all* ζ, η.
3. C *is a club subset of* ω_1.

Then there are a ccc poset $\mathbb{R}$ *and a club set* $E \subseteq \omega_1$ *such that in the extension by* $\mathbb{R}$

1. *The conclusion of* $(*)$ *holds; that is there is a sequence* $\langle x_i : i < \omega_1 \rangle$ *such that*
   (a) $x_i = (\alpha_i, \beta_i, \gamma_i)$ *where* $\alpha_i < \beta_i \leq \gamma_i$ *with* $\alpha_i, \gamma_i \in A$ *and* $\beta_i \in C$.
   (b) $\gamma_i = f(\alpha_i, \beta_i)$
   (c) *If* $i < j < \omega_1$ *then* $\gamma_i < \alpha_j$ *and* $G(\{\alpha_i, \gamma_i\}, \{\alpha_j, \gamma_j\})$ *holds.*
2. $P(E)$ *holds.*

Proof. We start by fixing a sequence $\langle y_i : i < \omega_1 \rangle$ such that

1. $y_i = (\alpha_i, \beta_i, \gamma_i)$ where $\alpha_i < \beta_i \leq \gamma_i$ with $\alpha_i, \gamma_i \in A$ and $\beta_i \in C \cap D$.
2. $\gamma_i = f(\alpha_i, \beta_i)$
3. If $i < j < \omega_1$ then there is $\eta \in C \cap D$ such that $\gamma_i < \alpha_j$.

Let $z = \langle z_i : i < \omega_1 \rangle$ where $z_i = (\alpha_i, \gamma_i)$, and note that z is an increasing sequence of pairs separated by D. To produce a witness for $(*)$ we need to produce a subsequence such that G holds for each pair from the subsequence; this is precisely what $\mathbb{R}$ will do.

We define $\mathbb{R}$ to be the set of finite sets $a \subseteq \{z_i : i < \omega_1\}$ such that for all $i < j$, $z_i, z_j \in a \implies G(z_i, z_j)$. The ordering is by inclusion.

Claim 1. $\mathbb{R}$ is ccc.

Proof of Claim 1. Let $\langle a_i : i < \omega_1 \rangle$ be conditions in $\mathbb{R}$. As usual we may assume that the sets $\bigcup a_i$ form a head-tail-tail Δ-system with some root r. Accordingly we write $a_i = r \cup b_i$ where each b_i may be regarded as a finite increasing sequence of pairs.

Now the sequence $\langle b_i : i < \omega_1 \rangle$ may be regarded as an increasing sequence of matrices of pairs (each of length 1) and what is more it is separated by D. Appealing to the property $P(D)$ in the $n = 1$ case we get $i < j$ so that $G(b_i, b_j)$, from which it follows easily that $a_i \cup a_j$ is a condition. $\dashv$

Claim 2. If $E = C(z) \cap D$ then E is a club subset of ω_1 and $P(E)$ holds in $V^{\mathbb{R}}$.

Proof of Claim 2. It is routine to check that E is club. To show that $P(E)$ holds in $V^{\mathbb{R}}$, once more we appeal to Lemma 20. Let $\langle (r_i, M_i) : i < \omega_1 \rangle$ be such that $r_i \in \mathbb{R}$ and $\langle M_i : i < \omega_1 \rangle$ is an increasing sequence of matrices of pairs separated by E.

By a Δ-system argument as in the proof that $\mathbb{R}$ is ccc we may assume that $r_i = r \cup s_i$, where $\langle s_i : i < \omega_1 \rangle$ is an increasing sequence of matrices of pairs each of length 1.

Then easily if we define matrices N_i so that $N_{i,1} = s_i$ and $N_{i,j+1} = M_{i,j}$ for $1 \leq j \leq lh(M_i)$, the sequence $\langle N_i : i < \omega_1 \rangle$ is an increasing sequence of matrices of pairs separated by D. Appealing to $P(D)$ in V we get $i < j$ such that $G(N_i, N_j)$, which amounts to saying that $G(s_i, s_j)$ (so that r_i, r_j are compatible) and $G(M_i, M_j)$. By Lemma 20 we showed that $P(E)$ holds in $V^{\mathbb{R}}$. $\dashv$

The proof of Theorem 3 is now completely routine. We force with $\mathbb{Q}$ and then build a ccc iteration $\mathbb{P}_{\omega_2}$ to force $(*)$, along with a sequence in V of clubs C_i such that $P(C_i)$ holds in $V^{\mathbb{Q} * \dot{\mathbb{P}}_i}$. At each stage i we will use Lemma 22 to handle some instance of $(*)$ corresponding to f_i, A_i, C_i; the Lemma only guarantees that there is $C_{i+1} \in V^{\mathbb{P}_i}$ which works but we may appeal to Lemma 16 to make a suitable choice of C_{i+1} in V. The same book-keeping arguments that are used in the consistency proof for Martin's Axiom show that in ω_2 steps we can handle all possible f, A, C so that $(*)$ holds in $V^{\mathbb{Q}*\dot{\mathbb{P}}_{\omega_2}}$. Working over this model we can now force MA (or any other statement which can be forced by ccc forcing) while preserving $(*)$.

After we proved the results of this section we learned of some related work [16] by Soukup, also using similar methods to those of [2]. Soukup builds a very general framework for making combinatorial properties of a certain kind indestructible under ccc forcing; his intended applications are topological, but a proof of Theorem 3 can easily be read off from his results.

More specifically given a set K, we consider a graph G with vertex set $\omega_1 \times K$; the graph is said to be *m-solid* if and only if for every ω_1-sequence of finite partial functions from ω_1 to K where the domains are pairwise disjoint sets of size m, there exist members s and t of the sequence such that $s \times t$ is contained in the edge set of G. G is *strongly solid* if and only if G is m-solid for all m. Soukup shows that if $2^{\omega_1} = \omega_2$, $|K| \leq \omega_2$ and G is strongly solid, then for each m there is a ccc poset of size ω_2 which makes the m-solidity of G become ccc-indestructible.

Soukup uses this result to get certain topological statements consistent with MA. Among these statements are "there is an uncountable first countable space in which every open set is countable or co-countable", "there is an S-group", and "there is an L-group". In rough outline the general idea is to construct a subspace X of 2^{ω_1} and an associated strongly solid graph G_X such that the desired property of X corresponds to the m-solidity of G_X for some m.

§5. A ccc destructible partition. In the light of the results from the preceding section it is natural to ask for a concrete example of a 2-bounded normal colouring of $[\omega_1]^2$ which has no uncountable polychromatic set in V, but acquires one in some ccc generic extension. It is routine to see that a poset with the Knaster property cannot add an uncountable polychromatic set for a colouring with no such set, so the obvious forcing poset to try is an (inverted) Souslin tree. This turns out to work.

THEOREM 4. *Let T be a Souslin tree and let CH hold. Then there is a colouring $f : [T]^2 \to \omega_1$ such that*

1. *f is 2-bounded and has no uncountable polychromatic set.*
2. *For every $L \subseteq T$, if L is linearly ordered by $<_T$ then L is polychromatic.*

In particular an uncountable polychromatic set is added by the ccc forcing $\mathbb{P}_T = (T, \geq_T)$, which adds an uncountable branch in T.

PROOF. Fix a Souslin tree T, where without loss of generality we may assume that the underlying set of the tree is ω_1. We adopt the conventions that

$$T_\alpha = \{t \in T : ht(t) < \alpha\},$$

and

$$Lev_\alpha = \{t \in T : ht(t) = \alpha\}.$$

We may assume without loss of generality that T is *normal*. That is to say

1. T has a unique element on level zero.
2. Every element has at least two immediate successors.

3. Every point on every limit level is determined uniquely by its set of predecessors.
4. For all α, β with $\alpha < \beta < \omega_1$ and all $s \in Lev_\alpha$ there is $t \in Lev_\beta$ with $s <_T t$.

The following property of some subsets of T will be helpful in the construction: say that $X \subseteq T$ is *far from linear* if and only if X is not the union of a linearly ordered set and a finite set.

Fix an enumeration of the countable subsets of ω_1 as X_i for $i < \omega_1$. Fix a decomposition of ω_1 into pairwise disjoint countably infinite sets A_i for $i < \omega_1$. Fix also, for each $\beta < \omega_1$, a decomposition of $A_{2\beta+1}$ as the union of infinite disjoint sets $A^0_{2\beta+1}$ and $A^1_{2\beta+1}$. We will build a function $f : [T]^2 \to \omega_1$ such that

1. For all distinct $s, t \in T$ if $\alpha = ht(s) = ht(t)$ then $f(\{s, t\}) \in A_{2\alpha}$. If $ht(s) < ht(t)$ then $f(\{s, t\}) \in A_{2t+1}$.[1]
2. For every $\alpha < \omega_1$, $f \restriction [Lev_\alpha]^2$ is 1-1.
3. For all $s, t, u \in T$, if $s <_T t <_T u$ then $f(\{s, u\}) \neq f(\{t, u\})$.
4. f is 2-bounded.

We will construct f level by level. More explicitly, given $f \restriction [T_\alpha]^2$ we will describe how to extend it to $f \restriction [T_{\alpha+1}]^2$. Note that pairs in $[T_{\alpha+1}]^2 \setminus [T_\alpha]^2$ are of two kinds, either they are drawn from $[Lev_\alpha]^2$ or they of the form $\{s, t\}$ where $ht(s) < ht(t) = \alpha$.

Suppose that we are given $f \restriction [T_\alpha]^2$. We define $f \restriction [Lev_\alpha]^2$ to be any 1-1 map from $[Lev_\alpha]^2$ to $A_{2\alpha}$.

To colour the pairs $\{s, t\}$ where $ht(s) < ht(t) = \alpha$, we proceed as follows. We first enumerate the set of those X_i with $i < \alpha$ such that

1. $X_i \subseteq T_\alpha$,
2. X_i is far from linear,
3. X_i is polychromatic for $f \restriction [T_\alpha]^2$,

as $\langle Y_j : j < \omega \rangle$. We choose inductively $\alpha_j, \beta_j \in Y_j$ for $j < \omega$ such that

1. α_j and β_j are not comparable in T.
2. $\alpha_j, \beta_j \notin \bigcup_{j' < j} \{\alpha_{j'}, \beta_{j'}\}$

This is possible because each Y_j is far from linear.

Now for each element t of Lev_α we proceed as follows. We set $f(\{\alpha_j, t\}) = f(\{\beta_j, t\}) = c^t_j$ where the c^t_j are pairwise distinct elements of A^0_{2t+1}. We then enumerate the set $T_\alpha \setminus \bigcup_{j<\omega} \{\alpha_j, \beta_j\}$ as γ_k for $k < \omega$, and set $f(\{\gamma_k, t\}) = d^t_k$ where the d^t_k are pairwise distinct elements of A^1_{2t+1}. For use a bit later we note that if s, t are distinct elements of T_α, $u \in Lev_\alpha$ and $f(\{s, u\}) = f(\{t, u\})$, then $\{s, t\} = \{\alpha_j, \beta_j\}$ for some j.

We should check that the various properties of f are preserved. Clauses 1 and 2 are immediate because we chose colours from the appropriate sets, and made sure that f was 1-1 on the set of pairs from the new level Lev_α.

We claim that we have preserved clause 3. We know that $f \restriction [T_\alpha]^2$ satisfies clause 3 so suppose for a contradiction that $s <_T t <_T u$ with $ht(u) = \alpha$ and $f(\{s, u\}) = f(\{t, u\})$. As we pointed out above, this implies that there is some

[1] Remember that t is itself a countable ordinal.

j with $\{s,t\} = \{\alpha_j, \beta_j\}$. This is impossible because α_j, β_j were chosen to be incomparable.

For clause 4 we analyse how two pairs from $T_{\alpha+1}$ may come to have the same colour. Suppose that $f(\{s,u\}) = f(\{t,v\}) = c$ for $s,t,u,v \in T_{\alpha+1}$, where $\{s,u\} \neq \{t,v\}$. Suppose that $c \in A_i$. If i is even then all of s,t,u,v must be on the same level, but that is impossible by clause 2. So i is odd, say $i = 2j+1$. By clause 1 we see that without loss of generality $ht(s) < ht(u)$ and $u = j$, and $ht(t) < ht(v)$ and $v = j$. Summarising the analysis: if $f(\{s,u\}) = f(\{t,v\})$ for $s,t,u,v \in T_{\alpha+1}$ then $u = v$ and $ht(s), ht(t) < ht(u)$.

We claim that we have preserved clause 4. Suppose for a contradiction that $\{s,v\}$, $\{t,w\}$ and $\{u,x\}$ are three pairs in $[T_{\alpha+1}]^2$ such that $f(\{s,v\}) = f(\{t,w\}) = f(\{u,x\})$. Then by the analysis from the last paragraph $v = w = x$, and also we have $ht(s), ht(t), ht(u) < ht(v)$. What is more $f \restriction [T_\alpha]^2$ satisfies clause 4, so that necessarily $ht(x) = \alpha$. But then as in the discussion of 3 all three of s,t,u must be among $\{\alpha_j, \beta_j\}$ for some j, which is impossible. This concludes the proof that we can construct f.

By construction f is 2-bounded, and by clauses 1 and 3 any linearly ordered set is polychromatic.. So to finish we need only show that there is no uncountable polychromatic set. Suppose for a contradiction that X is such a set. Then X is far from linear since T is a Souslin tree, and in particular has no uncountable branches; it is easy to see that there exists $\alpha < \omega_1$ such that $X \cap T_\alpha$ is far from linear. We fix i such that $X \cap T_\alpha = X_i$. Now we find $u \in X$ such that $u \in Lev_\beta$ for $\beta > \alpha, i$. The construction of f guarantees that there are distinct $s,t \in X_i$ with $f(\{s,u\}) = f(\{t,u\})$, a contradiction as X is polychromatic. ⊣

§6. A result from Martin's maximum. In this section we consider 2-bounded colourings of $[\omega_2]^2$. Our main result will use the powerful forcing axiom Martin's Maximum (MM) [9]. We say that a forcing poset $\mathbb{P}$ is *stationary preserving* if and only if $\mathbb{P}$ forces that every stationary subset of ω_1 remains stationary.

Martin's Maximum (MM) is the assertion that for every stationary preserving $\mathbb{P}$ and every family $\mathscr{F}$ of dense subsets of $\mathbb{P}$ with $|\mathscr{F}| = \omega_1$, there is a filter F on $\mathbb{P}$ such that $F \cap D \neq \emptyset$ for all $D \in \mathscr{F}$.

THEOREM 5. *Assuming MM, every normal 2-bounded colouring of $[\omega_2]^2$ has a closed polychromatic set of order type ω_1.*

The main point is to show that if c is such a colouring then we can add a closed polychromatic set of order type ω_1 with a stationary preserving forcing poset.

6.1. The poset $\mathbb{Q}(d,E)$. We begin by describing a poset which aims, given a 2-bounded normal $d : [\omega_1]^2 \to \omega_1$, to add a closed unbounded polychromatic subset of ω_1. This poset will in general collapse ω_1, it will be used as a component of our final construction. In some sense our poset $\mathbb{Q}(d,E)$ is a descendant of Baumgartner's poset [5] for adding a club subset with finite conditions but it also has something in common with the poset $\mathbb{P}_f$ from Section 3.

Let E be a fixed club set of countable elementary substructures of H_{ω_2}; for technical reasons we will define our poset so that only models from E may appear as side conditions. The conditions in $\mathbb{Q}(d,E)$ have the form $p = (\mathscr{M}_p, \mathscr{B}_p)$ where

1. $\mathscr{M}_p$ is a finite set $\{M_0, \dots M_{n-1}\}$ of elements of E, such that $M_0 \in M_1 \dots \in M_{n-1}$. We will denote by X_p the set $\{M_i \cap \omega_1 : i < n\}$.
2. $d \in M_0$.
3. $\mathscr{B}_p$ is a set of half-open intervals $\{(\alpha_0, \beta_0], \dots (\alpha_{m-1}, \beta_{m-1}]\}$ where $\alpha_0 < \beta_0 < \alpha_1 < \dots \alpha_{m-1} < \beta_{m-1} < \omega_1$.
4. The set X_p is polychromatic for d.
5. The set X_p is disjoint from $\bigcup \mathscr{B}_p$.

Conditions are ordered as follows: $p \leq q$ if and only if $\mathscr{M}_q \subseteq \mathscr{M}_p$ and $\mathscr{B}_q \subseteq \mathscr{B}_p$.

The motivation is that p forces the elements of X_p into the generic club set and blocks out the elements of $\bigcup \mathscr{B}_p$. Accordingly if G is $\mathbb{Q}(d, E)$-generic we define $C_G = \bigcup_{p \in G} X_p$. Clearly C_G is polychromatic, but we should verify that it is closed and unbounded in ω_1; since $\mathbb{Q}(d, E)$ may collapse ω_1 it is unclear at this point what will be the order type of C_G.

The following lemma will show that C_G is unbounded.

Lemma 23. *Let $p \in \mathbb{Q}(d, E)$ and let $\gamma < \omega_1$. Then there exists $q \leq p$ with $\gamma < \max(X_q)$.*

Proof. Let $\zeta = \max X_p$ and let $M \in \mathscr{M}_p$ be the model with $M \cap \omega_1 = \zeta$. Notice that $X_p \cap \zeta \in M$. Let

$$S = \{\eta \in \omega_1 : (X_p \cap \zeta) \cup \{\eta\} \text{ is polychromatic } \}$$

Then since $d \in M$ we see that $S \in M$, and so since $\zeta = M \cap \omega_1 \in S$ we have by Lemma 2 that S is stationary.

Now we recall to the reader's attention the function F from Section 3, and in particular Lemma 3 on adding points to polychromatic sets. By elementarity we may find $T \in M$ stationary with $T \subseteq S$ and $\rho \in M$ such that for all $\eta \in T$, $F((X_p \cap \zeta) \cup \{\eta\}) \subseteq \rho$. Now of course $\rho \in M \cap \omega_1 = \zeta$, so for all $\eta \in T$ with $\eta > \zeta$ we can appeal to Lemma 3 to see that $X_p \cup \{\eta\}$ is polychromatic.

Finally we can build a continuous increasing chain $\langle N_j : j < \omega_1 \rangle$ with $N_j \in E$, $p \in N_0$ and $N_j \in N_{j+1}$ for all j. The set $\{N_j \cap \omega_1 : j < \omega_1\}$ is club so we can find j so that $\eta =_{\text{def}} N_j \cap \omega_1 \in T$, $\eta > \gamma$, and η is above the maximum point of the last interval in B_p. It is now easy to see that $q =_{\text{def}} (\mathscr{M}_p \cup \{N_j\}, \mathscr{B}_p)$ is as required. ⊣

The following lemma will show that C_G is closed.

Lemma 24. *Let $p \in \mathbb{Q}(d, E)$, and let $\gamma < \omega_1$ be a limit ordinal such that $p \Vdash \gamma \notin \dot{C}_G$. Then there is $q \leq p$ such that $q \Vdash \sup(\dot{C}_G \cap \gamma) < \gamma$.*

Proof. Since $p \Vdash \gamma \notin \dot{C}_G$, surely $\gamma \notin X_p$. If $\gamma \in \bigcup \mathscr{B}_p$ we are done, so we assume this is not the case. Since γ is limit we see that $X_p \cap \gamma$ and $\bigcup \mathscr{B}_p \cap \gamma$ are both bounded in γ, say by δ. Now it is easy to see that $q =_{\text{def}} (\mathscr{M}_p, \mathscr{B}_p \cup \{(\delta, \gamma]\})$ is as required. ⊣

It will not in general be the case that $\mathbb{Q}(d, E)$ is proper. To see this suppose that $N \prec H_\theta$ for some large regular θ, $p \in \mathbb{Q}(d, E) \cap N$ and $X_p \cup \{N \cap \omega_1\}$ is not polychromatic. Then if we could find an N-generic $q \leq p$, by Lemma 2 q would force that $N \cap \omega_1$ is in C_G; but also q would force $X_p \subseteq C_G$, a contradiction since C_G is polychromatic.

This discussion gives us the clue about how to find generic conditions in $\mathbb{Q}(d, E)$. Before we get down to the details we introduce some convenient notation similar to that we used for $\mathbb{P}_f$ in Section 3.

1. If $p = (\mathscr{M}, \mathscr{B})$ is a condition in $\mathbb{Q}(d, E)$ and $M \in \mathscr{M}$ then we define

$$p \restriction M = (\mathscr{M} \cap M, \mathscr{B} \cap M)$$

Routinely we see that $p \restriction M$ is a condition, $p \restriction M \in M$ and $p \leq p \restriction M$.

2. Given a countable ordinal γ, say that a condition p is *below* γ if and only if $X_p \subseteq \gamma$ and $\bigcup \mathscr{B}_p \subseteq \gamma$. It is true that $p \restriction M$ is below $M \cap \omega_1$, but false in general that every condition below $M \cap \omega_1$ is in M.

Now we can argue that under some circumstances generic conditions do exist in $\mathbb{Q}(d, E)$. The proof is similar in some respects to that of Lemma 10.

Lemma 25. *Let θ be a large regular cardinal. Let $N \prec H_\theta$ be countable with $\mathbb{Q}(d, E) \in N$. If $p \in \mathbb{Q}(d, E)$ is such that $N \cap H_{\omega_2} \in \mathscr{M}_p$ then p is N-generic.*

Proof. Let $M = N \cap H_{\omega_2}$. Since $\mathbb{Q}(d, E) \subseteq H_{\omega_2}$, $N \cap \mathbb{Q}(d, E) = M \cap \mathbb{Q}(d, E)$. Given a dense set $D \in N$ we need to show that $D \cap N$ is predense below p. Since $M \in \mathscr{M}_q$ for any $q \leq p$, we may as well assume that $p \in D$.

Let $\delta = M \cap \omega_1$ and let $q = p \restriction M$. We claim that it will suffice to produce r such that

1. r is below δ.
2. $r \leq q$.
3. $r \in D$.
4. $X_r \cup X_p$ is polychromatic.

For although such an r may not be in M the finite set X_r will be in M, and so by elementarity of N we will be able to find $r^* \in M$ with $r^* \leq q$, $r^* \in D$, $X_{r^*} = X_r$. Then easily $(\mathscr{M}_{r^*} \cup \mathscr{M}_p, \mathscr{B}_{r^*} \cup \mathscr{B}_p)$ is a condition extending both r^* and p. We suppose for a contradiction that there is no r with the four properties listed above.

Now let $X_p \setminus M = \{\alpha_0, \dots \alpha_{t-1}\}$ where α_i is increasing with i. Of course $\alpha_0 = M \cap \omega_1 = \delta$. We define a set $S \in M$ of ordinals "like δ". Specifically we let S be the set of those β such that

1. q is below β.
2. There exist $\beta_1, \dots \beta_{t-1}$ such that
   - (a) $\beta < \beta_1 < \dots \beta_{t-1} < \omega_1$
   - (b) $X_q \cup \{\beta, \beta_1, \dots \beta_{t-1}\}$ is polychromatic.
   - (c) There is no r such that
     - (i) r is below β.
     - (ii) $r \leq q$.
     - (iii) $r \in D$.
     - (iv) $X_r \cup \{\beta, \beta_1, \dots \beta_{t-1}\}$ is polychromatic.

$S \in M$ because S is defined from parameters in N and lies in H_{ω_2}, and since $\delta \in S$ (using $\beta_i = \alpha_i$ for $0 < i < t$ as witnesses) we see that S is stationary.

Now we enumerate the models in $\mathscr{M}_p \setminus M$ as $M_0, \dots M_{t-1}$ where $M_i \cap \omega_1 = \alpha_i$, so in particular $M_0 = M$. Now we proceed exactly as in the proof of Lemma 23 to choose stationary sets $S_0, \dots S_{t-1}$ such that $S_i \in M_i$, $S_0 \subseteq S$, $S_{i+1} \subseteq S_i$ and for every large enough $\beta \in S_i$ there exist $\beta_1, \dots \beta_{t-1}$ such that

1. $\beta < \beta_1 < \dots \beta_{t-1} < \omega_1$.
2. $X_q \cup \{\alpha_0, \dots \alpha_i\} \cup \{\beta, \beta_1, \dots \beta_{t-1}\}$ is polychromatic.
3. There is no r such that
   - (a) r is below β.

(b) $r \leq q$.
(c) $r \in D$.
(d) $X_r \cup \{\beta_1, \dots \beta_{t-1}\}$ is polychromatic.

At the end of the construction choose $\beta \in S_{t-1}$ with p below β and suitable β_i as above. This is a contradiction because

$$X_q \cup \{\alpha_0, \dots \alpha_{t-1}\} \cup \{\beta, \beta_1, \dots \beta_{t-1}\} = X_p \cup \{\beta, \beta_1, \dots \beta_{t-1}\}$$

and is polychromatic, also p is below β, $p \leq q$, $p \in D$. $\dashv$

6.2. The poset $\mathbb{P}$. We define a poset $\mathbb{P}$ to add an increasing, continuous and cofinal map from ω_1 to ω_2, preserving ω_1. The poset $\mathbb{P}$ is just a variation on the standard countably closed Levy collapse $Coll(\omega_1, \omega_2)$. We will assume that $\mathbb{P}$ is defined in a universe V where $2^\omega = \omega_2$.

Conditions in $\mathbb{P}$ are functions f such that

1. $\mathrm{dom}(f)$ is a successor ordinal less than ω_1.
2. $\mathrm{rge}(f)$ is a subset of $\omega_2 \cap \mathrm{cof}(\omega)$.
3. f is increasing and continuous.

The ordering is extension. It is easy to see that $\mathbb{P}$ is countably closed and $|\mathbb{P}| = \omega_2$. It follows that ω_1^V is preserved, ω_2^V is an ordinal of cardinality and cofinality ω_1 in $V[G]$, and $\omega_3^V = \omega_2^{V[G]}$.

To avoid confusion in the discussion that follows we let $\kappa = \omega_3^V$. We note that easily $H_\kappa^{V[G]} = H_\kappa[G]$, because every bounded subset of κ in $V[G]$ has a canonical name in H_κ. We need some analysis of the countable elementary submodels of $H_\kappa[G]$ in $V[G]$.

We start by noticing that if $F \in V$ is a *Skolem function*[2] for H_κ, then in $V[G]$, the set of countable $Y \subseteq H_\kappa[G]$ with $Y \cap H_\kappa$ closed under F is club. So in $V[G]$ the set of countable $N \prec H_\kappa[G]$ such that $N \cap H_\kappa^V \prec H_\kappa^V$ forms a club set E.

Now let $N \in E$ and let $M = N \cap H_\kappa$, so that $M \prec H_\kappa$. Since $\mathbb{P}$ is countably closed, $M \in V$. What is more every element of N is coded by a bounded subset of κ in N, and every such set has a canonical name in $N \cap H_\kappa$ (that is in M), so in fact $N = M[G]$.

It is profitable to discuss this in terms of forcing and generic conditions. Let p force that $\dot{N} \in \dot{E}$. Then arguing as above we can find $q \leq p$ such that $q \Vdash \dot{N} \cap H_\kappa = M$. Automatically we have that $q \Vdash \dot{N} = M[\dot{G}]$, in particular $q \Vdash M[\dot{G}] \cap V = M$ and so q is $(M, \mathbb{P})$-generic.

Note: We are *not* proving that $M[G] \cap V = M$ for all M and G. If we fix M and apply the analysis of the last paragraph to the name $M[\dot{G}]$ all we get is $M[G] \cap V = M^*$ and $M[G] = M^*[G]$ for some $M^* \supseteq M$, and G need only contain an $(M^*, \mathbb{P})$-generic condition.

The following lemma will be useful in the main construction.

LEMMA 26. *Let $M \prec H_\kappa$ be countable, let $\delta = M \cap \omega_1$ and let $f \in \mathbb{P}$ be $(M, \mathbb{P})$-generic. Then $\delta + 1 \subseteq \mathrm{dom}(f)$ and $f(\delta) = \sup(M \cap \omega_2)$. Also f forces that $\sup(M[G] \cap \omega_2) = f(\delta)$.*

PROOF. Let $\alpha < \delta$. Then in particular $\alpha \in \mathrm{dom}(f)$, since otherwise we can extend f to force a bad value at $\alpha + 1$. It is then clear that $f(\alpha) \in M \cap \omega_2$.

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[2]That is to say, if $X \subseteq H_\kappa$ is closed under F then $X \prec H_\kappa$.

Similarly let $\beta \in M \cap \omega_2$ and consider a name for the least $\alpha < \omega_1$ such that the generic function maps α to a point greater than β. This name is in M so f must force that the least such α is in M, which implies there is $\alpha < \delta$ such that $f(\alpha) > \beta$.

So $f \restriction \delta$ maps δ cofinally into $M \cap \omega_2$, and since f must be continuous with domain a successor ordinal we see that $\delta \in \mathrm{dom}(f)$ and $f(\delta) = \sup(M \cap \omega_2)$. The second claim is immediate because f forces that $M[G] \cap V = M \cap V$. $\dashv$

6.3. Preserving stationary sets. We recall that our goal is to use MM to show that every normal 2-bounded colouring c of $[\omega_2]^2$ has a closed polychromatic set of order type ω_1. We will assume that all elementary substructures of H_{ω_3} discussed in the proof contain c.

We will argue that a certain two-step iteration is stationary preserving. The first step is to force with the poset $\mathbb{P}$ from Section 6.2. This poset is countably closed and adds a continuous increasing cofinal map $g : \omega_1^V \to \omega_2^V$. We use g to define a colouring $d \in V[g]$ by the formula $d(\alpha, \beta) = c(g(\alpha), g(\beta))$ so that easily d is a normal 2-bounded colouring of $[\omega_1]^2$ The second step of the iteration is to force over $V[g]$ with $\mathbb{Q}(d, E)$ as defined in Section 6.1, where E is chosen in $V[g]$ to be the club set of countable $N \prec H_\kappa[g]$ such that $N \cap H_\kappa^V \prec H_\kappa^V$.

It is important to note that $\omega_2^{V[g]} = \omega_3^V$ so that $\mathbb{Q}$ is defined using countable elementary substructures of $H_{\omega_3}[G]$. To minimise confusion we carry over from the last section the convention that $\kappa = \omega_3^V$.

Lemma 27. *$\mathbb{P} * \dot{\mathbb{Q}}(d, E)$ is stationary preserving.*

Proof. Let us say that a condition $(f, \dot{\mathcal{N}}, \dot{\mathscr{B}})$ in $\mathbb{P} * \dot{\mathbb{Q}}(d, E)$ is *nice* if and only if

1. f decides the value of $\dot{\mathscr{B}}$.
2. There is a finite set $\{M_0, \dots M_{n-1}\}$ of countable elementary substructures of H_κ such that
   (a) $M_0 \cap \omega_1 < M_1 \cap \omega_1 < \dots M_{n-1} \cap \omega_1$.
   (b) $f \Vdash \dot{\mathcal{N}} = \{M_i[\dot{G}] : i < n\}$.
   (c) f is $(M_i, \mathbb{P})$-generic for all $i < n$.

Note that by the discussion from the last section the set of nice conditions is dense, so we may assume that all conditions are nice.

Now let $S \subseteq \omega_1$ be stationary and let $\dot{C}$ be a $\mathbb{P} * \dot{\mathbb{Q}}(d, E)$-name for a club subset of ω_1. Let $p = (f, \{M_i[G] : i < n\}, \mathscr{B})$ be a nice condition. Let $\delta_i = M_i \cap \omega_1$ for $i < n$, and recall from the last section that $\delta_{n-1} + 1 \subseteq \mathrm{dom}(f)$ and $f(\delta_i) = \sup(M_i \cap \omega_2)$. Let $\eta_i = f(\delta_i)$, $X = \{\delta_i : i < n\}$, $Y = \{\eta_i : i < n\}$.

By the definition of d and $\mathbb{Q}(d, E)$, we see that Y is polychromatic for c. Also we see that $X \cap \delta_{n-1} \in M_{n-1}$ and so easily f forces that $Y \cap \eta_{n-1} \in M_{n-1}[g]$, so that in fact $Y \cap \eta_{n-1} \in M_{n-1}$. Let T be the set of ordinals $\eta \in \omega_2 \cap \mathrm{cof}(\omega)$ such that $(Y \cap \delta_{n-1}) \cup \{\eta\}$ is polychromatic for c, then $T \in M_{n-1}$ and $\delta_{n-1} \in T$ so (by an easy variation on the argument of Lemma 2) T is stationary in ω_2.

Fix a large regular cardinal θ. We may find (using a standard result from the Martin's maximum paper [9]) a countable $N \prec H_\theta$ such that $c, \dot{C}, S, \mathbb{P} \in N$, $\alpha =_{\mathrm{def}} N \cap \omega_1 \in S$, and $\eta =_{\mathrm{def}} \sup(N \cap \omega_2) \in T$. Let $M = N \cap H_\kappa$, so that $M \prec H_\kappa$, and note that $f \in M$ and $M_i \in M$ for all $i < n$.

Enumerating the dense subsets of $\mathbb{P}$ which lie in N in order type ω, and building a decreasing chain of elements of $\mathbb{P} \cap M$ to meet them all, we may readily build

$f^* \leq f$ such that f is $(N, \mathbb{P})$-generic. By the discussion from the last section, we see that $\alpha + 1 \subseteq \text{dom}(f^*)$ and $f^*(\alpha) = \eta$.

It is now routine to check that $q = (f^*, \{M_i[\dot{G}] : i < n\} \cup \{M[\dot{G}]\}, \mathscr{B})$ is a nice condition. What is more f^* is $(N, \mathbb{P})$-generic and $f^* \Vdash N[\dot{G}] \cap H_\kappa[G] = \dot{M}[G]$, so by Lemma 25 f^* forces that $(\{M_i[\dot{G}] : i < n\} \cup \{M[\dot{G}]\}, \mathscr{B})$ is $(N[\dot{G}], \dot{\mathbb{Q}}(d, E))$-generic. It follows that q is $\mathbb{P} * \dot{\mathbb{Q}}(d, E)$-generic, and so $q \Vdash \beta \in \dot{C}$. ⊣

Once we know that $\mathbb{P} * \dot{\mathbb{Q}}(d, E)$ is stationary preserving, a routine application of MM concludes the proof of Theorem 5. We note that actually the same proof would work for $(< \omega)$-bounded colourings.

§7. PFA does not suffice for Theorem 5. We showed in Section 6 that under MM, every 2-bounded colouring of $[\omega_2]^2$ has a closed polychromatic set of order type ω_1. It follows immediately from Theorem 2 that under PFA every such colouring has a polychromatic set of order type ω_1 which is stationary in its supremum. So it is natural to ask whether PFA would have sufficed to show that every 2-bounded colouring of $[\omega_2]^2$ has a closed polychromatic set of order type ω_1. We will show that this is not the case.

We consider the following combinatorial principle:

Principle W: there exists a function g such that

1. The domain of g is $\omega_2 \cap cof(\omega)$.
2. For all $\gamma \in \omega_2 \cap cof(\omega)$, $g(\gamma)$ is a pair of ordinals (α, β) such that $\alpha < \beta < \gamma$.
3. For every $\delta \in \omega_2 \cap cof(\omega_1)$ and every pair (α, β) such that $\alpha < \beta < \delta$, the set $\{\gamma \in \delta \cap cof(\omega) : g(\gamma) = (\alpha, \beta)\}$ is stationary in δ.

The motivation for introducing principle W is of course to use it to build wild colourings.

LEMMA 28. *If principle W holds then there is a 2-bounded colouring of $[\omega_2]^2$ with no polychromatic set which is a closed copy of ω_1.*

PROOF. We build the colouring by induction in such a way that if $g(\gamma) = (\alpha, \beta)$ then the pairs $\{\alpha, \gamma\}$ and $\{\beta, \gamma\}$ get the same colour. If c is a closed set of order type ω_1 and α, β are the first two points of c then by property 3 above there is $\gamma \in c$ with $g(\gamma) = (\alpha, \beta)$. By construction c is not polychromatic. ⊣

A few remarks are in order about the principle W.

1. It would surely be equivalent to assert the existence of a regressive function G defined on $\omega_2 \cap cof(\omega)$, such that G assumes every value less than δ stationarily often in every $\delta \in \omega_2 \cap cof(\omega)$. Principle W is formulated as it is for use in the proof of Lemma 28.

2. The principle W evolved in the following way. We first constructed a colouring as in Lemma 28 assuming $\square_{\omega_1}$, which is inconsistent with PFA (for example from $\square_{\omega_1}$ we may build a special ω_2-tree, while PFA implies there are no ω_2-Aronszajn trees). We then began to look for a consequence of $\square_{\omega_1}$ strong enough to give the conclusion of Lemma 28, yet weak enough to be consistent with PFA.

3. If g is a witness to principle W then actually for every $\alpha < \beta < \omega_2$ the set $\{\gamma \in \omega_2 \cap cof(\omega) : g(\gamma) = (\alpha, \beta)\}$ is stationary in ω_2. To see this let C be club in

ω_2, let $\delta \in \lim(C) \cap cof(\omega_1)$, and use property 3 to see that $g(\gamma) = (\alpha, \beta)$ for some $\gamma \in C \cap \delta$.

4. Principle W is consistently false, for example it fails under MM. This follows from Lemma 28 and Theorem 5, but we may also give a direct proof. A well-known consequence of MM is that every stationary subset of $\omega_2 \cap cof(\omega)$ contains a closed copy of ω_1, but this contradicts property 3.

We describe a forcing poset $\mathbb{P}$ to add g witnessing principle W. We add g by forcing with initial segments. Explicitly conditions are functions p such that

1. The domain of p is a proper initial segment of $\omega_2 \cap cof(\omega)$.
2. Conditions 2 and 3 are satisfied for all $\gamma \in \mathrm{dom}(p)$ and all $\delta \leq \sup(\mathrm{dom}(p))$ with $cf(\delta) = \omega_1$.

The ordering is extension. We will ultimately prove if PFA holds in V, the PFA still holds after forcing with $\mathbb{P}$.

It is clear that $\mathbb{P}$ is countably closed. To show that $\mathbb{P}$ adds no ω_1-sequences of ordinals we use the idea of *strategic closure*. For more about strategic closure we refer the reader to Foreman's paper about games on Boolean algebras [8].

A forcing poset $\mathbb{P}$ is said to be η-strategically closed for some ordinal η if and only if player II has a winning strategy in the following game of perfect information. Players I and II collaborate to build a (non-strictly) decreasing chain in $\mathbb{P}$ with I playing at odd stages and II at nonzero even stages (including all limit stages). Play continues for at most η steps, the first player who does not move at their turn loses, and II wins if and only if play continues for η steps.

We will show that $\mathbb{P}$ is $(\omega_1 + 1)$-strategically closed, and implicit in the proof will be the argument that any condition can be extended to any prescribed length. In particular forcing with $\mathbb{P}$ adds no subsets of ω_1, so that stationary subsets of ω_1 are preserved, as is the cardinal ω_2. It follows that the generic function added by $\mathbb{P}$ is a witness to principle W.

We fix in V a decomposition of ω_1 as the disjoint union of a family of stationary sets $\{T_i : i < \omega_1\}$. Let p_i be the move at stage i of the strategic closure game and let $\mathrm{dom}(p_i) = \lambda_i \cap cof(\omega)$. As play proceeds II builds an enumeration of all the pairs (α, β) with $\alpha < \beta < \sup_{i<\omega_1} \lambda_i$ as $\{(\alpha_j, \beta_j) : j < \omega_1\}$. For every limit ordinal $\gamma \leq \omega_1$ she forms p_γ as follows:

1. $\lambda_\gamma = \mu + 1$ where $\mu = \sup_{i<\gamma} \lambda_i$.
2. $p_\gamma \restriction \mu = \bigcup_{i<\gamma} p_i$.
3. $p_\gamma(\mu) = (\alpha_j, \beta_j)$ for the unique $j < \omega_1$ with $\gamma \in T_j$, IF (α_j, β_j) has already been defined (so that in particular $\alpha_j < \beta_j < \mu$).

Clearly this is a winning strategy.

We now argue that principle W is consistent with PFA. The argument is similar to the proof by Beaudoin [6] that PFA is consistent with the existence of a non-reflecting stationary set in ω_2, and also owes something to the coding arguments from the Martin's Maximum paper [9].

The obvious strategy for proving that PFA is consistent with W would be to start with a model of PFA, force with $\mathbb{P}$ as above and then try to argue that PFA is preserved as follows. Given $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a proper poset and $\dot{D}_i$ for $i < \omega_1$ which are $\mathbb{P}$-names for dense subsets of $\dot{\mathbb{Q}}$, apply the PFA in V to the proper poset $\mathbb{P} * \dot{\mathbb{Q}}$ and the dense sets $E_i = \{(p, \dot{q}) : p \Vdash \dot{q} \in \dot{D}_i\}$ to get a filter F on $\mathbb{P} * \dot{\mathbb{Q}}$ which meets the E_i. Then take a lower bound $\bar{p}$ for $\{p : \exists \dot{q}\ (p, \dot{q}) \in F\}$, so that $\bar{p}$ forces that $\{\dot{q}^G : \exists p\ (p, \dot{q}) \in F\}$ generates a suitable filter.

There is an equally obvious problem with this strategy, namely that no such lower bound $\bar{p}$ may exist. The idea of the proof, which we owe to Beaudoin [6], is to define an auxiliary forcing $\mathbb{R}$ so that $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper, and that when we apply PFA to get a filter on $\mathbb{P} * \dot{\mathbb{Q}} * \mathbb{R}$ the "$\mathbb{R}$-part of F" will let us find a lower bound for the conditions in the "$\mathbb{P}$-part of F".

We will fix in V a regressive function r on ω_1 such that $\{\alpha : r(\alpha) = \beta\}$ is stationary for all $\beta < \omega_1$. Let $\dot{\mathbb{Q}} \in V^{\mathbb{P}}$ be a name for a proper forcing poset and let $\dot{D}_i$ for $i < \omega_1$ be $\mathbb{P}$-names for dense open sets in $\dot{\mathbb{Q}}$. We describe an auxiliary forcing $\mathbb{R}$ in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$. $\mathbb{R}$ is a two step iteration $\mathbb{R}_0 * \dot{\mathbb{R}}_1$. $\mathbb{R}_0$ is a version of the standard Lévy collapse $Coll(\omega_1, \omega_2^V)$, explicitly conditions are functions p with $\mathrm{dom}(p) \in \omega_1$ and $\mathrm{rge}(p) \subseteq [\omega_2^V]^2$, ordered by extension.

We stress that ω_2^V is the second uncountable cardinal as defined in V, which may be collapsed when we force with $\mathbb{Q}$. There is no need to write ω_1^V since the iteration of proper forcing posets is again proper and hence forcing with $\mathbb{P} * \dot{\mathbb{Q}}$ preserves ω_1.

Conditions in $\mathbb{R}_1$ are sets d such that

1. d is countable.
2. d is a closed and bounded subset of ω_2, in particular d has a largest element.
3. Every element of d has cofinality ω.
4. For all $\alpha \in d$, $g(\alpha) = P(j)$ where g is the witness to principle W added by $\mathbb{P}$, P is the surjection from ω_1 to $[\omega_2^V]^2$ added by $\mathbb{R}_0$, and $j = r(\mathrm{ot}(d \cap \alpha))$.

The ordering on $\mathbb{R}_1$ is end-extension. We note that since $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_0$ is proper, an ordinal has cofinality ω in the extension by this poset if and only if it has cofinality ω in V.

Intuitively $\mathbb{R}_1$ is intended to add a club set in ω_2^V of order type ω_1 such that for every pair $(\alpha, \beta) \in [\omega_2^V]^2$ there are stationarily many points in that club set which are mapped by g to the pair (α, β).

We need to know that there are sufficiently many conditions in $\mathbb{R}_1$. Accordingly we prove that for every $d \in \mathbb{R}_1$, $\eta < \omega_1$ and $\zeta < \omega_2^V$, if $\mathrm{ot}(r) \leq \eta < \omega_1$ then there is $e \leq d$ such that $\max(e) > \zeta$ and $\mathrm{ot}(e) = \eta + 1$. We do this by induction on η for all d and ζ. Suppose we have established it for all $\eta' < \eta$. We assume that η is limit, the successor case is similar and easier.

We will work in the extension by $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_0$. Let $j = r(\eta)$, and $P(j) = (\alpha, \beta)$. Find a countable elementary substructure M of some large H_θ such that $d, \eta, \zeta \in M$ and $\gamma = \sup(M \cap \omega_2^V)$ is such that $g(\gamma) = (\alpha, \beta)$ (this is possible because we are in a proper extension of V and g assumes each value on a stationary set in ω_2^V). Now build (using the induction hypothesis) a decreasing sequence of conditions d_n such that their union d^* has order type η and supremum γ, and observe that by construction $e = d^* \cup \{\gamma\}$ is a condition with order type $\eta + 1$.

Lemma 29. *$\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper.*

Proof. We work in V. Let $N \prec H_\theta$ be a countable elementary substructure containing everything relevant and let $(p, \dot{q}, \dot{r}_0, \dot{r}_1)$ be a condition in N. Let $\delta = N \cap \omega_1$ and let $\gamma = \sup(N \cap \omega_2)$.

Let $j = r(\delta)$ so that $j < \delta$ and in particular $j \in N$. Extending $(p, \dot{q}, \dot{r}_0, \dot{r}_1)$ to a stronger condition in N we may assume without loss of generality that $(p, \dot{q})$

forces that $j \in \mathrm{dom}(\dot{r}_0)$ and determines that $\dot{r}_0(j) = (\alpha, \beta)$ for some α and β in $N \cap \omega_2$ (so in particular $\alpha < \beta < \gamma$). Since $\mathbb{P}$ is countably closed we may extend p to a condition p^* such that p^* is strongly $(N, \mathbb{P})$-generic, the domain of p^* is $(\gamma + 1) \cap cof(\omega)$ and $p^*(\gamma) = (\alpha, \beta)$.

We then choose $(\dot{q}^*, \dot{r}_0^*)$ a name for an $(N[\dot{G}_{\mathbb{P}}], \mathbb{Q}*\dot{\mathbb{R}}_0)$-generic condition extending $(\dot{q}, \dot{r}_0)$, so that easily $(p^*, \dot{q}^*, \dot{r}_0^*)$ is an $(N, \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_0)$-generic condition extending $(p, \dot{q}, \dot{r}_0)$. We force below this condition to get a generic filter $G * H * I$, where we know that $N[G * H * I] \cap ON = N \cap ON$ by the genericity of the condition.

We now build a decreasing sequence of length ω in $N[G * H * I] \cap \mathbb{R}_1$ which meets each dense set in $N[G*H*I]$. Let r_1^* be the union of this sequence, where it is routine to check that $\sup(r_1^*) = \gamma$ and $\mathrm{ot}(r_1^*) = \delta$. We claim that $r_1^* \cup \{\gamma\}$ is a condition in $\mathbb{R}_1$. To see this we merely observe that $r(\delta) = j$ and $P(j) = (\alpha, \beta) = g(\gamma)$. ⊣

Lemma 30. *PFA holds in $V^{\mathbb{P}}$.*

Proof. We apply the PFA to $\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{R}}$ to get a suitably generic filter F. Taking the union of the "$\mathbb{P}$-parts" of the conditions in F we get a certain function $\bar{p}$. Arranging that F meets suitable dense sets we can use the union of the "$\mathbb{R}$-parts" to produce a witness that $\bar{p}$ is a condition, and this is all we need. ⊣

The upshot of this discussion is that we have proved the following theorem. Here the consistency is relative to that of PFA. PFA is known to be consistent relative to the existence of a supercompact cardinal.

Theorem 6. *It is consistent that PFA holds and there is a 2-bounded colouring of $[\omega_2]^2$ such that there is no closed polychromatic set of order type ω_1.*

§8. A positive result from GCH.

We saw in Section 2 that under CH there is a 2-bounded colouring of pairs from ω_1 with no uncountable polychromatic set. A similar argument easily gives that if $\kappa^{<\kappa} = \kappa$ for some uncountable cardinal κ, then there is a 2-bounded colouring of $[\kappa]^2$ with no polychromatic set of size κ. We now prove a positive result about $(< \kappa)$-bounded colourings of $[\kappa^+]^2$ from the same assumption that $\kappa^{<\kappa} = \kappa$.

Theorem 7. *Let κ be an infinite cardinal and assume that $\kappa^{<\kappa} = \kappa$. Then for every $(< \kappa)$-bounded colouring of $[\kappa^+]^2$ and every ordinal $\eta < \kappa^+$ there is a polychromatic set of order type η.*

Proof. Fix f a $(< \kappa)$-bounded colouring of $[\kappa^+]^2$. We may may assume without loss of generality that f is normal. Fix also $\eta < \kappa^+$, where without loss of generality we may assume that $\eta \geq \kappa$.

We will generalise an idea used earlier in the paper. If $X \subseteq \kappa^+$ we define

$$F(X) =_{\mathrm{def}} \{\alpha : \exists \gamma \ \exists b \in [X]^2 \ f(\alpha, \gamma) = f(b)\}$$

Since f is normal, $F(X)$ is the set of α such that there exist $\beta, \gamma \in X$ with $\alpha, \beta < \gamma$ and $f(\alpha, \gamma) = f(\beta, \gamma)$. Clearly $F(X) \subseteq \max(X)$. Since κ is regular we see that if $|X| < \kappa$, then $|F(X)| < \kappa$.

The proof of Lemma 3 goes through in this setting. so for all $X, Y \subseteq \kappa^+$ and $\alpha < \kappa^+$, if $X < \alpha < Y$, $X \cup Y$ and $X \cup \{\alpha\}$ are polychromatic and $\alpha \notin F(X \cup Y)$, then $X \cup \{\alpha\} \cup Y$ is polychromatic.

Now we will build a polychromatic set of order type η; actually for technical reasons we will end up building the set with type $\eta + 1$. We start by fixing a large regular θ and building a continuous increasing chain $\langle M_i : i \leq \eta \rangle$ such that $M_i \prec H_\theta$, $|M_i| = \kappa$, $M_i \cap \kappa^+ \in \kappa^+$, $M_i \cap \kappa^+ < M_{i+1} \cap \kappa^+$ for $i < \eta$, ${}^{<\kappa} M_i \subseteq M_i$ for i successor.

Let $\beta_i = M_i \cap \kappa^+$ for $i \leq \eta$, we note that $i \mapsto \beta_i$ is strictly increasing and continuous and also that $cf(\beta_i) = \kappa$ for i successor. Fix $\alpha > \beta_\eta$, and fix also a bijection $g : \kappa \simeq \eta$. we will construct by induction ordinals α_j for $j < \kappa$ such that

1. $\beta_{g(j)} \leq \alpha_j < \beta_{g(j)+1}$.
2. $\{\alpha_k : k < j\} \cup \{\alpha\}$ is polychromatic.

Suppose that we have chosen α_k for $k < j$. Let $X = \{\alpha_k : k < j, g(k) < g(j)\}$ and $Y = \{\alpha_k : k < j, g(k) > g(j)\} \cup \{\alpha\}$. Note that by construction $X \subseteq \beta_{g(j)}$ and $Y \subseteq \kappa^+ \setminus \beta_{g(j)+1}$.

By the choice of the M_i we see that $X \in M_{g(j)+1}$. Since $\alpha > \beta_{g(j)+1}$ and $X \cup \{\alpha\}$ is polychromatic, the usual elementarity argument shows that the set of $\bar{\alpha}$ with $X \cup \{\bar{\alpha}\}$ polychromatic is unbounded in $\beta_{g(j)+1}$. On the other hand $|F(X \cup Y)| < \kappa$ and $cf(\beta_{g(j)+1}) = \kappa$, so that $F(X \cup Y)$ is bounded in $\beta_{g(j)+1}$.

So we may choose α_j such that

1. $\beta_{g(j)} \leq \alpha_j < \beta_{g(j)+1}$, in particular $X < \alpha_j < Y$.
2. $X \cup \{\alpha_j\}$ is polychromatic
3. $\alpha_j \notin F(X \cup Y)$.

Now we conclude that $X \cup \{\alpha_j\} \cup Y$ is polychromatic, so that the construction can continue. ⊣

In particular under GCH the situation is now rather clear for κ^+ with κ regular and uncountable: we have seen that if $\kappa^{<\kappa} = \kappa$ then $\kappa^+ \rightarrow^{\text{poly}} (\alpha)^2_{<\kappa-\text{bd}}$ for all $\alpha < \kappa^+$, while if $2^\kappa = \kappa^+$ then $\kappa^+ \not\rightarrow^{\text{poly}} (\kappa^+)^2_{2-\text{bd}}$.

REFERENCES

[1] U. Abraham, *Proper forcing*, To appear in the handbook of set theory.

[2] U. Abraham and S. Todorčević, *Martin's axiom and first-countable S- and L-spaces*, ***Handbook of set-theoretic topology***, North-Holland, Amsterdam, 1984, pp. 327–346.

[3] B. Alspach, M. Gerson, G. Hahn, and P. Hell, *On sub-Ramsey numbers*, ***Ars Combinatoria***, vol. 22 (1986), pp. 199–206.

[4] J. Baumgartner and A. Hajnal, *A proof (involving Martin's axiom) of a partition relation*, ***Fundamenta Mathematicae***, vol. 78 (1973), no. 3, pp. 193–203.

[5] J. E. Baumgartner, *Applications of the proper forcing axiom*, ***Handbook of set-theoretic topology***, North-Holland, Amsterdam, 1984, pp. 913–959.

[6] Robert E. Beaudoin, *The proper forcing axiom and stationary set reflection*, ***Pacific Journal of Mathematics***, vol. 149 (1991), no. 1, pp. 13–24.

[7] P. Erdős, J. Nešetřil, and V. Rödl, *On some problems related to partitions of edges of a graph*, ***Graphs and other combinatorial topics (Prague, 1982)***, Teubner-Texte Math., vol. 59, Teubner, Leipzig, 1983, pp. 54–63.

[8] M. Foreman, *Games played on Boolean algebras*, this Journal, vol. 48 (1983), no. 3, pp. 714–723.

[9] M. Foreman, M. Magidor, and S. Shelah, *Martin's maximum, saturated ideals, and nonregular ultrafilters. I*, ***Annals of Mathematics. Second Series***, vol. 127 (1988), no. 1, pp. 1–47.

[10] F. Galvin, *Letters to S. Todorcevic*, cited in [17].

[11] F. Galvin, *Advanced problem number 6034*, ***The American Mathematical Monthly***, vol. 82 (1975), p. 529.

[12] P. Hell and J. J. Montellano-Ballesteros, *Polychromatic cliques*, ***Discrete Mathematics***, vol. 285 (2004), no. 1-3, pp. 319–322.

[13] H. J. PRÖMEL and B. VOIGT, *Canonical forms of Borel-measurable mappings* $\Delta\colon [\omega]^{\omega} \to \mathbf{R}$, ***Journal of Combinatorial Theory. Series A***, vol. 40 (1985), no. 2, pp. 409–417.

[14] R. SCHIPPERUS, *The topological Baumgartner-Hajnal theorem*, To appear.

[15] S. SHELAH, ***Proper and improper forcing***, second ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.

[16] L. SOUKUP, *Indestructible properties of S- and L-spaces*, ***Topology and its Applications***, vol. 112 (2001), no. 3, pp. 245–257.

[17] STEVO TODORČEVIĆ, *Forcing positive partition relations*, ***Transactions of the American Mathematical Society***, vol. 280 (1983), no. 2, pp. 703–720.

[18] ———, *A note on the proper forcing axiom*, ***Axiomatic set theory (boulder, colorado, 1983)***, Contemporary Mathematics, vol. 31, American Mathematical Society, Providence, RI, 1984, pp. 209–218.

[19] ———, *Partitioning pairs of countable ordinals*, ***Acta Mathematica***, vol. 159 (1987), no. 3-4, pp. 261–294.

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