COLLAPSING SUCCESSORS OF SINGULARS

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Abstract. Let κ be a singular cardinal in V, and let $W \supseteq V$ be a model such that $\kappa^+_V = \lambda^+_W$ for some W-cardinal λ with $W \models \mathrm{cf}(\kappa) \neq \mathrm{cf}(\lambda)$. We apply Shelah's pcf theory to study this situation, and prove the following results.
1) W is not a κ^+-c.c generic extension of V.
2) There is no "good scale for κ" in V, so in particular weak forms of square must fail at κ.
3) If $V \models \mathrm{cf}(\kappa) = \aleph_0$ then $V \models$ "κ is strong limit $\implies 2^\kappa = \kappa^+$", and also ${}^\omega\kappa \cap W \supsetneq {}^\omega\kappa \cap V$.
4) If $\kappa = \aleph^V_\omega$ then $\lambda \leq (2^{\aleph_0})_V$.

1. Introduction

The results in this paper were motivated by the following natural question.

Question 1: Can there exist a pair of inner models of ZFC, V and W say, such that $V \subseteq W$ and $\aleph^V_{\omega+1} = \aleph^W_2$?

This question was raised by Bukovský and Copláková-Hartová [1] who observed (Theorem 1.7 of [1]) that if we have V and W as above then $W \models 2^{\aleph_0} \geq \aleph_2$, because $V \models \aleph_\omega^{\aleph_0} \geq \aleph_{\omega+1}$. They also note that if Question 1 has a positive answer then it is possible to violate CH by adding a real.

We can ask more general questions about the possibilities for collapsing successors of singulars; for example

Question 2: Let $V \models$ "κ is singular". Can there be $W \supseteq V$ such that

$$W \models \text{"}\kappa^+_V \text{ is a cardinal and } \mathrm{cf}(\kappa) \neq \mathrm{cf}(|\kappa|)\text{"}$$

The restriction "$\mathrm{cf}(\kappa) \neq \mathrm{cf}(|\kappa|)$" is not unreasonable, because for example it is easy to collapse $\aleph^V_{\omega.2+1}$ to $\aleph^{V[G]}_{\omega+1}$ by forcing with $Coll(\omega, \aleph_\omega)$.

Work of Shelah [10] shows that a positive solution to Question 1 or Question 2 would require the use of large cardinals.

Definition 1.1. Let κ be a cardinal, then AD_κ *holds* iff there exists a sequence $\langle A_\alpha : \alpha < \kappa^+\rangle$ such that

1. For each $\alpha < \kappa^+$, A_α is an unbounded subset of κ.
2. For each $\beta < \kappa^+$ there exists $g : \beta \longrightarrow \kappa$ such that $\langle A_\alpha \setminus g(\alpha) : \alpha < \beta\rangle$ is a sequence of mutually disjoint sets.

Shelah proved ([10], page 440) that

1. If κ is regular, or if κ is singular and $\square_\kappa$ holds, then AD_κ holds.
2. If $V \models \mathrm{AD}_\kappa$, $W \supseteq V$ and $W \models$ "κ^+_V is a cardinal", then $W \models \mathrm{cf}(\kappa) = \mathrm{cf}(|\kappa|)$.

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In particular, if $V \models \mathrm{AD}_\kappa + \mathrm{cf}(\kappa) = \omega$ then $W \models \mathrm{cf}(|\kappa|) = \omega$, so that in W the cardinal κ^+_V must be $\aleph_1^W$ or the successor of a cardinal of cofinality ω.

Remark 1.2. For κ singular the weak square principle $\Box^*_\kappa$ is enough to derive AD_κ. See [3] for a proof.

The connection with large cardinals arises because for κ singular the principle $\Box^*_\kappa$ holds unless there exist inner models of some very strong large cardinal axioms (at the level of Woodin cardinals). See [6], [7] and [8].

The results in this paper are more general than Shelah's in the sense that they can be applied in some situations where $V \models \neg\mathrm{AD}_\kappa$. See [4] and [3] for a discussion of the implications between AD_κ, squares, scales and reflection principles.

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2. Pcf theory, good points and approachability

In this section we review some facts about pcf theory that will be used later. For a detailed development of pcf theory see [2] or [11]. All the facts quoted here are due to Shelah unless otherwise stated.

Definition 2.1. Let κ be a singular cardinal. A *scale for κ* (of length κ^+) is $(\vec\kappa, \vec f)$ where

1. $\vec\kappa = \langle \kappa_i : i < \mathrm{cf}(\kappa)\rangle$ is a strictly increasing sequence of regular cardinals with $\sup_i \kappa_i = \kappa$.
2. $\vec f = \langle f_\alpha : \alpha < \kappa^+\rangle$ is a sequence such that
   (a) $f_\alpha \in \prod_i \kappa_i$.
   (b) $\vec f$ is strictly increasing and cofinal in $(\prod_i \kappa_i, <^*)$ where $<^*$ is the eventual domination relation on $\prod \kappa_i$.

Fact 2.2. If κ is singular then there exists a scale of length κ^+ for κ.

Definition 2.3. Let $(\vec\kappa, \vec f)$ be a scale of length κ^+ for κ. $\alpha < \kappa^+$ is *good (for $\vec f$)* iff there exists $A \subseteq \alpha$ unbounded in α and $i < \mathrm{cf}(\kappa)$ such that whenever $\beta, \gamma \in A$ with $\beta < \gamma$ and $j > i$ then $f_\beta(j) < f_\gamma(j)$.

Remark 2.4. If $\mathrm{cf}(\alpha) < \mathrm{cf}(\kappa)$ then it is easy to see that α is good.

Remark 2.5. If $\mathrm{cf}(\alpha) > \mathrm{cf}(\kappa)$, α being good is equivalent to $\langle f_\beta : \beta < \alpha\rangle$ having a least upper bound g such that $cf(g(i)) = \mathrm{cf}(\alpha)$ for all large i.

Remark 2.6. The definition of *good point* actually makes sense for any $\vec\kappa$ and any increasing sequence from $(\prod_i \kappa_i, <^*)$.

Fact 2.7. If $\rho = \mathrm{cf}(\rho) < \kappa$ and $(\vec\kappa, \vec f)$ is a scale of length κ^+ for κ then

$$\{\, \alpha < \kappa^+ \;:\; \mathrm{cf}(\alpha) = \rho \text{ and } \alpha \text{ is good} \,\}$$

is stationary in κ^+.

Since we will use some of the ideas later, we sketch a proof of this fact.

Proof. We prove it first when $\mathrm{cf}(\kappa) < \rho$. Let $C \subseteq \kappa^+$ be club. Fix θ some large regular cardinal. Build a increasing and continuous sequence $\langle N_i : i < \rho\rangle$ such that

1. $N_i \prec (H_\theta, \in, <_\theta)$ where $<_\theta$ is some wellordering of H_θ.
2. $C, \vec{\kappa}, \vec{f} \in N_0$ and $\mathrm{cf}(\kappa) \subseteq N_0$.
3. $|N_i| < \rho$.
4. $\langle N_i : i \leq j \rangle \in N_{j+1}$ for all $j < \rho$ (it will follow that $i \subseteq N_i$ and hence that $i < j \implies N_i \in N_j$).

Notice that if $i < j$ then $\sup(N_i \cap \kappa^+) \in N_j$, so that $\langle \sup(N_i \cap \kappa^+) : i < \rho \rangle$ is increasing and continuous. The limit of this sequence is $\sup(\bigcup_i N_i \cap \kappa^+)$. It follows from elementarity that C is unbounded in $N_i \cap \kappa^+$, so that $\sup(N_i \cap \kappa^+) \in C$.

Define $\chi_i \in \prod_n \kappa_n$ by setting $\chi_i(n) = 0$ for $\kappa_n < \rho$, $\chi_i(n) = \sup(N_i \cap \kappa_n)$ for $\kappa_n \geq \rho$. Since $N_i \in N_{i+1}$ we see that $\chi_i \in N_{i+1}$, so that there exists $\alpha_i \in N_{i+1} \cap \kappa^+$ such that $\alpha_i > \sup(N_i \cap \kappa^+)$ and $\chi_i <^* f_{\alpha_i}$. Since $\mathrm{cf}(\kappa) \subseteq N_{i+1}$ we see that for all $n < \mathrm{cf}(\kappa)$ we have $f_{\alpha_i}(\kappa_n) \in N_{i+1} \cap \kappa_n$, in particular we have $f_{\alpha_i} <^* \chi_{i+1}$.

Now for every $i < \rho$ we may choose $j < \mathrm{cf}(\kappa)$ such that

$$k > j \implies \chi_i(k) < f_{\alpha_i}(k) < \chi_{i+1}(k).$$

Since $\rho > \mathrm{cf}(\kappa)$ we may fix $A \subseteq \rho$ unbounded and $j < \mathrm{cf}(\kappa)$ such that

$$\forall i \in A\ \forall k > j\ \chi_i(k) < f_{\alpha_i}(k) < \chi_{i+1}(k).$$

It follows that if $i_0, i_1 \in A$ with $i_0 < i_1$ then for $k > j$ we have

$$f_{\alpha_{i_0}}(k) < \chi_{i_0+1}(k) \leq \chi_{i_1}(k) < f_{\alpha_{i_1}}(k).$$

Now we know that $\sup(N_i \cap \kappa^+) < \alpha_i < \sup(N_{i+1} \cap \kappa^+)$, so this shows that $\sup_i \alpha_i = \sup_i \sup(N_i \cap \kappa^+)$ is a good point. This point is also a point of C with cofinality ρ, so we have shown the set of good points of cofinality ρ to be stationary. This concludes the proof when $\rho > \mathrm{cf}(\kappa)$.

To finish we observe that if A witnesses α to be a good point for $\vec{f}$, and β is a limit point of A, then $A \cap \beta$ witnesses that β is good for $\vec{f}$. Given $\rho = \mathrm{cf}(\rho) \leq \mathrm{cf}(\kappa)$ and a club set C, we first use the argument above to find $\alpha \in \lim(C)$ with $\mathrm{cf}(\alpha) = \mathrm{cf}(\kappa)^+$ and α good and choose A witnessing this; then we choose $\beta \in C$ a limit point of A with cofinality ρ, so that $A \cap \beta$ witnesses β is good. □

Remark 2.8. It is easy to see that if $(\vec{\kappa}, \vec{f})$ and $(\vec{\kappa}, \vec{g})$ are two scales of length κ^+ for κ in the same product $\prod_i \kappa_i$, then the set of good points for $\vec{f}$ is equal modulo clubs to the set of good points for $\vec{g}$.

Definition 2.9. A *good scale for* κ is a scale for κ in which, modulo the club filter, almost every point of cofinality greater than $\mathrm{cf}(\kappa)$ is good.

Now we look at a form of weak square (the approachability square AP_κ) introduced by Shelah in [9]. AP_κ is substantially weaker than $\square^*_\kappa$.

Definition 2.10. AP_κ holds iff there exists $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that

1. C_α is a club subset of α of order type $\mathrm{cf}(\alpha)$ for each limit α.
2. $C_{\gamma+1}$ is a subset of κ^+ for each successor ordinal $\gamma + 1$.
3. For a club of $\alpha < \kappa^+$, $\forall \beta < \alpha\ \exists \gamma < \alpha\ C_\alpha \cap \beta = C_{\gamma+1}$.

We are interested in this principle mainly because of the following fact, which is Claim 4.4 in [4].

Fact 2.11. If AP_κ holds and $(\vec{\kappa}, \vec{f})$ is a scale then almost every point α with $\mathrm{cf}(\alpha) > \mathrm{cf}(\kappa)$ is good for $\vec{f}$. That is, every scale is good.

We also collect here a few more facts about good points.

Fact 2.12. Let $\mathrm{cf}(\kappa) = \omega$, and let κ be strong limit with $2^\kappa > \kappa^+$. Then there exists a scale $(\vec{\kappa}, \vec{f})$ for κ such that every point of uncountable cofinality is good for $\vec{f}$.

Fact 2.13. If $(\vec{\kappa}, \vec{f})$ is a scale for $\aleph_\omega$ and $\alpha < \aleph_{\omega+1}$, $\mathrm{cf}(\alpha) > 2^{\aleph_0}$ then α is good for $\vec{f}$.

3. The Main Lemma

In this section we will prove the key technical lemma of the paper, which connects the notion of "good point" with Questions 1 and 2 from the Introduction. We originally proved the lemma for points of cofinality λ, with the additional assumption that λ is regular in W; Burke pointed out that it holds in the more general form given here.

Lemma 3.1. Let V, W be inner models of set theory with $V \subseteq W$. Let κ be singular in V, and fix $(\vec{\kappa}, \vec{f})$ a scale of length κ^+ for κ. Suppose that $\kappa^+_V = \lambda^+_W$ where $\lambda > \aleph_0$ and $W \models \mathrm{cf}(\lambda) \neq \mathrm{cf}(\kappa)$. Then there exists $\eta < \lambda$ such that for every $\delta \in REG^W \cap (\eta, \lambda]$

$$\{ \gamma < \kappa^+_V \ : \ W \models \text{“}\mathrm{cf}(\gamma) = \delta \text{ and } \gamma \text{ is good for } \vec{f}\text{”} \}$$

is non-stationary in W.

Proof. We work in W. Since $\kappa^+_V = \kappa^+_W$ we will just denote this cardinal by "κ^+". Observe that $\mu =_{\mathrm{def}} \mathrm{cf}_W(\kappa) < \lambda$ because λ is the largest cardinal less than κ^+. $\mathrm{cf}_W(\mathrm{cf}_V(\kappa)) = \mu$, so we choose $C \subseteq \mathrm{cf}_V(\kappa)$ a cofinal set of order type μ. $|\kappa| = \lambda$ so we may write $\kappa = \bigcup_{i<\lambda} X_i$ where $\vec{X}$ is an increasing sequence of sets with $|X_i| < \lambda$ for all $i < \lambda$.

Now for all $\alpha < \kappa^+$, $\mathrm{rge}(f_\alpha \restriction C) \subseteq \kappa = \bigcup_i X_i$. Since $\mu \neq \mathrm{cf}(\lambda)$, for every $\alpha < \kappa^+$ there is $i < \lambda$ such that $\{ n \in C \ : \ f_\alpha(n) \in X_i \}$ is unbounded in μ. Hence we can fix $i < \lambda$ such that the set

$$B =_{\mathrm{def}} \{ \alpha < \kappa^+ \ : \ \{ n \in C \ : \ f_\alpha(n) \in X_i \} \text{ is unbounded in } \mu \}$$

is unbounded in κ^+.

Now let $\eta = \max\{|X_i|, \mu\}$, and fix $\delta \in REG^W$ with $\eta < \delta \leq \lambda$. Since B is unbounded it has a club set of limit points; we claim that every limit point of B with cofinality δ is not good for $\vec{f}$. To see this, suppose for a contradiction that γ is a limit point of B of cofinality δ and that γ is good. This means that there is $A \subseteq \gamma$ unbounded and $m < \mathrm{cf}_V(\kappa)$ such that if $\alpha, \beta \in A$, $\alpha < \beta$ and $m < n < \mathrm{cf}_V(\kappa)$ then $f_\alpha(n) < f_\beta(n)$.

Now we build an increasing sequence of ordinals $\langle \gamma_j : j < \delta \rangle$ such that $\gamma_{2j} \in A$ and $\gamma_{2j+1} \in B$. Notice that $f_{\gamma_{2j}} <^* f_{\gamma_{2j+1}} <^* f_{\gamma_{2j+2}}$ for each $j < \delta$. For each j let us choose $n_j > m$ such that $n_j \in C$,

$$f_{\gamma_{2j}}(n_j) < f_{\gamma_{2j+1}}(n_j) < f_{\gamma_{2j+2}}(n_j),$$

and $f_{\gamma_{2j+1}}(n_j) \in X_i$.

Since $\delta > \mu = |C|$ we can find $Z \subseteq \delta$ unbounded and a fixed $n \in C$ such that for all $j \in Z$

$$f_{\gamma_{2j}}(n) < f_{\gamma_{2j+1}}(n) < f_{\gamma_{2j+2}}(n)$$

and $f_{\gamma_{2j+1}}(n) \in X_i$. Now if j, k are two elements of Z with $j < k$ then

$$f_{\gamma_{2j+1}}(n) < f_{\gamma_{2j+2}}(n) \leq f_{\gamma_{2k}}(n) < f_{\gamma_{2k+1}}(n).$$

This implies that $|\{ f_{\gamma_{2j+1}}(n) : j \in Z \}| = |Z| = \delta$. However for each $j \in Z$ we have $f_{\gamma_{2j+1}}(n) \in X_i$, and $|X_i| < \delta$, which is a contradiction. So almost every point of cofinality δ is not good and we have proved the claim of the Lemma. □

Remark 3.2. $REG^W \cap (\eta, \lambda]$ is always non-empty. If λ is regular in W then λ will do, and λ is singular then λ is limit and we are guaranteed many suitable δ.

4. Exploiting the main lemma

Throughout this section our running assumptions are that $V \subseteq W$, $\kappa^+_V = \lambda^+_W$, $V \models$ "κ is a singular cardinal", and $W \models \mathrm{cf}(\kappa) \neq \mathrm{cf}(\lambda)$.

Theorem 1. *W is not a κ^+-c.c. extension of V.*

Proof. Fix $(\vec{\kappa}, \vec{f})$ a scale for κ. Applying Lemma 3.1, let $\delta \in REG^W$ be such that in W the set of good points with cofinality δ is non-stationary.

Let us define

$$S = \{ \gamma < \kappa^+ : V \models \text{“cf}(\gamma) = \delta \text{ and } \gamma \text{ is good for } \vec{f}\text{”} \}.$$

By Lemma 2.7, $V \models$ "S is stationary". Since δ is regular in W we know that

$$V \models \text{“cf}(\gamma) = \delta \text{ and } \gamma \text{ is good”} \implies W \models \text{“cf}(\gamma) = \delta \text{ and } \gamma \text{ is good”},$$

and so by the choice of δ we have $W \models$ "S is nonstationary". Since any κ^+-c.c. forcing preserves stationary subsets of κ^+, W is not a κ^+-c.c. generic extension of V. □

We originally proved the following result under the additional assumptions that λ is regular and $V \models \mathrm{cf}(\kappa) < \lambda$; Burke pointed out that it is true in the general form given here.

Theorem 2. *$V \models$ "There is no good scale of length κ^+ for κ".*

Proof. Let $(\vec{\kappa}, \vec{f})$ be a good scale for κ in V, and let C be a club in κ^+ such that

$$V \models \text{“every point of } C \text{ with cofinality greater than cf}(\kappa) \text{ is good”}.$$

Let B be the unbounded set constructed in the proof of Lemma 3.1, and let $\delta \in REG^W$ be such that every limit point of B with W-cofinality δ is not good and $\delta > \mathrm{cf}_W(\kappa)$.

Choose $\gamma \in C \cap \mathrm{acc}(B)$ such that $W \models \mathrm{cf}(\gamma) = \delta$. By the choice of B and δ, γ cannot be good. On the other hand $W \models \mathrm{cf}(\gamma) = \delta > \mathrm{cf}(\kappa)$ so that $V \models \mathrm{cf}(\gamma) \neq \mathrm{cf}(\kappa)$; if $V \models \mathrm{cf}(\gamma) < \mathrm{cf}(\kappa)$ then γ is good by Remark 2.4 and if $V \models \mathrm{cf}(\gamma) > \mathrm{cf}(\kappa)$ then γ is good by the choice of C. Contradiction! □

Theorem 3. *Let $V \models \mathrm{cf}(\kappa) = \omega$. Then*

1. *$V \models$ "κ is strong limit $\implies 2^\kappa = \kappa^+$".*
2. *If $\lambda = \aleph_1^W$ then ${}^\omega\kappa \cap W \supsetneq {}^\omega\kappa \cap V$.*

Proof. The first claim follows immediately from Fact 2.12 and Theorem 2. For the second claim, suppose for a contradiction that ${}^\omega\kappa \cap W = {}^\omega\kappa \cap V$ and fix (in V) a scale $(\vec{\kappa}, \vec{f})$ of length κ^+ for κ.

Now $W \models \mathrm{cf}(\kappa_n) > \omega$ for all $n < \omega$ and $W \models$ "$\vec{f}$ is cofinal in $\prod_n \kappa_n$". This is enough for us to imitate the proof of Lemma 2.7 and to prove (by building in W an $\aleph_1$-chain of countable structures) that

$$W \models \text{"}\{\ \gamma < \kappa^+ \ :\ \mathrm{cf}(\gamma) = \aleph_1 \text{ and } \gamma \text{ is good }\} \text{ is stationary".}$$

This contradicts Lemma 3.1. □

Theorem 4. If $\kappa = \aleph_\omega^V$ then $\lambda \leq (2^{\aleph_0})_V$.

Proof. Fix $(\vec{\kappa}, \vec{f})$ a scale of length κ^+ for κ. Recall that by Fact 2.13 all points of cofinality greater than $2^{\aleph_0}$ are good for $\vec{f}$.

Suppose for a contradiction that $\lambda > (2^{\aleph_0})_V$. Using Lemma 3.1 find γ such that $W \models \mathrm{cf}(\gamma) > (2^{\aleph_0})_V$, and γ is not good for $\vec{f}$. Now $V \models \mathrm{cf}(\gamma) > (2^{\aleph_0})_V$, contradicting 2.13. □

5. Concluding remarks

The problems discussed in this paper are connected with some other open problems in set theory.

5.1. **Chang's conjecture.** The consistency of "$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$" is open. If this holds then $S =_{\mathrm{def}} \{\ X \subseteq \aleph_{\omega+1} \ :\ \mathrm{ot}(X) = \aleph_2\ \}$ is stationary, and supposing that there exists a Woodin cardinal δ we can force with Woodin's stationary tower forcing [12] below the condition S. It is easy to see that $S \Vdash \aleph_{\omega+1}^V = \aleph_2^{V[G]}$.

5.2. **Good points.** Let $(\vec{\kappa}, \vec{f})$ be a scale of length $\aleph_{\omega+1}$ for $\aleph_\omega$. It is open whether the set of points $\{\ \alpha < \aleph_{\omega+1} \ :\ \mathrm{cf}(\alpha) > \aleph_1 \text{ and } \alpha \text{ is not good }\}$ can ever be stationary. If this set cannot be stationary then Lemma 3.1 implies that $\aleph_{\omega+1}$ can only be collapsed to $\aleph_1$ or $\aleph_2$.

This problem is also connected with "$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$", because if this holds then by Claim 4.3 in [4] there is a stationary set of cofinality $\aleph_2$ points which are not good. It is known [4] that if MM holds or $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ then the set $\{\ \alpha < \aleph_{\omega+1} \ :\ \mathrm{cf}(\alpha) = \aleph_1 \text{ and } \alpha \text{ is not good }\}$ is stationary. The statement "$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$" is known [5] to be consistent.

5.3. **Saturated ideals.** Burke and Matsubara have pointed out that the results in this paper can be used to show that the nonstationary ideal on $\mathcal{P}_\kappa \lambda$ is not saturated for certain values of κ and λ.

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