Coherent sequences versus Radin sequences

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Abstract

We attempt to make a connection between the sequences of measures used to define Radin forcing and the coherent sequences of extenders which are the basis of modern inner model theory. We show that in certain circumstances we can read off sequences of measures as defined by Radin from coherent sequences of extenders, and that we can define Radin forcing directly from a coherent extender sequence and a sequence of ordinals; this generalises Mitchell's construction of Radin forcing from a coherent sequence of measures.

1 Introduction

This paper was inspired by the following questions:

1. What is the relationship between Radin's version of Radin forcing, as described in [5], and Mitchell's as described in [3] ?

2. Let $V = L[\vec{E}]$ where $\vec{E}$ is a coherent non-overlapping sequence of extenders, let $j : V \longrightarrow M$ be the embedding arising from $\vec{E}(\kappa, \beta)$, and following Radin let a sequence u be defined by setting

$$u(0) = \kappa,$$

and then

$$u(\alpha) = \{ X \subseteq V_\kappa \mid u \restriction \alpha \in j(X) \},$$

for as long as $u \restriction \alpha \in M$.

If $\alpha > 0$ then $u(\alpha)$ is a κ-complete ultrafilter on V_κ. What is the relationship between the measures $u(\alpha)$ and the extenders on the sequence $\vec{E}$?

We will give a partial answer to question 2 in the case that $\beta = \kappa^{++}$, and the form that the answer takes will give us the clue to a partial answer for question 1. We give a brief review of the facts about extenders and $L[\vec{E}]$-models that we shall use.

Let κ be a cardinal, λ an ordinal. E is said to be a (κ, λ)-*extender* if $E = \langle E_a : a \in [\lambda]^{<\omega} \rangle$ where

1. E_a is a κ-complete ultrafilter on $[\kappa]^{|a|}$.

2. The ultrafilters E_a have a certain coherence property, enabling us to form a limit ultrapower

$$j_E : V \longrightarrow Ult(V, E).$$

3. $Ult(V, E)$ is well-founded, so we may identify it with a transitive class M_E.

4. j_E has critical point κ.

5. $\lambda < j_E(\kappa)$, and for $a \in [\lambda]^{<\omega}$ and $X \subseteq [\kappa]^{|a|}$

$$X \in E_a \iff a \in j_E(X).$$

The reader is referred to [1] for a full treatment of extenders. Our main interest here is in coherent non-overlapping extender sequences (hereafter referred to simply as "extender sequences") and the associated inner models.

$\vec{E}$ is an *extender sequence* if $\vec{E}$ is a function with $\mathrm{dom}(\vec{E}) \subseteq On \times On$ such that

1. There is some function $o : On \longrightarrow On$ such that $o(\alpha) \geq \alpha + 1$ and

$$\mathrm{dom}(\vec{E}) = \{ \ (\kappa, \lambda) \mid \kappa < \lambda < o(\kappa) \ \}.$$

2. If $(\kappa, \lambda) \in \mathrm{dom}(\vec{E})$ then $\vec{E}(\kappa, \lambda)$ is a (κ, λ)-extender.

3. Let $\prec$ be the lexicographic order on $On \times On$, so that

$$(\kappa_0, \lambda_0) \prec (\kappa_1, \lambda_1) \iff (\kappa_0 < \kappa_1) \vee (\kappa_0 = \kappa_1 \wedge \lambda_0 < \lambda_1).$$

   If $(\kappa, \lambda) \in \mathrm{dom}(\vec{E})$ and $E = \vec{E}(\kappa, \lambda)$ then

   - $j_E(o)(\kappa) = \lambda$, $j_E(o) \restriction \kappa = o \restriction \kappa$.
   - If $(\alpha, \beta) \prec (\kappa, \lambda)$ then $\vec{E}(\alpha, \beta) = j_E(\vec{E})(\alpha, \beta)$.

4. If $\kappa_0 < \kappa_1$ and $o(\kappa_1) > \kappa_1 + 1$ then $o(\kappa_0) < \kappa_1$.

The motivation for this definition is that if $\vec{E}$ is an extender sequence then $L[\vec{E}]$ is a good inner model, having many properties in common with Kunen's $L[\mu]$ and Mitchell's $L[\vec{U}]$, but capable of accommodating larger cardinals. In fact $L[\vec{E}]$ models are the canonical inner models for cardinals below a strong cardinal, the theory of these models being due to Mitchell, Dodd and Jensen.

The key to the good properties of $L[\vec{E}]$ is the Comparison Lemma, due to Mitchell, of which the following is a special case. Let $\vec{E}$ be a coherent sequence in $L[\vec{E}]$ [1] and let $\vec{F}$ be coherent in $L[\vec{F}]$. Then we can find normal iterations of the two models, say $i : L[\vec{E}] \longrightarrow M = L[\vec{E}_M]$ and $j : L[\vec{F}] \longrightarrow N = L[\vec{F}_N]$, such that one of the sequences $\vec{E}_M$ and $\vec{F}_N$ is an initial segment of the other as far as sets in $M \cap N$ are concerned.

[1]With the definition we have adopted, if $\vec{E}$ is coherent then $\vec{E} \cap L[\vec{E}]$ is coherent in $L[\vec{E}]$.

For a proof of the Comparison Lemma in the case when $\vec{E}$ and $\vec{F}$ are sequences of measures, see [2]. Using the Comparison Lemma (in a version involving iterable set models) it may be shown that $L[\vec{E}]$ enjoys the GCH, among other pleasant "L-like" properties.

2 Identifying the Radin measures

Let $V = L[\vec{E}]$ where $\vec{E}$ is a coherent sequence. Let κ be a cardinal such that $(\kappa, \kappa^{++}) \in \mathrm{dom}(\vec{E})$, and let $E = \vec{E}(\kappa, \kappa^{++})$. Mitchell showed in [4] that if $j_E : V \longrightarrow M_E$ is the ultrapower by E then $V_{\kappa+2} \subseteq M_E$. This will mean that if we define a "Radin" sequence by the recursion

$$\begin{aligned} u(0) &= \kappa, \\ u(\alpha) &= \{ X \subseteq V_\kappa \mid u \restriction \alpha \in j_E(X) \}, \end{aligned}$$

then the recursion runs for at least κ^{++} steps. This is significant because for some $\alpha < \kappa^{++}$ the measure $u(\alpha)$ will be a so-called *weak repeat point,* that is every set in $u(\alpha)$ has already been given measure one by some $u(\beta)$ with $\beta < \alpha$; if we let $v = u \restriction \alpha$ for such an α then the associated Radin forcing $\mathbb{R}_v$ will preserve the measurability of κ.

We will characterise the measures $u(\alpha)$ in terms of the extenders on $\vec{E}$, for α belonging to a long initial segment of κ^{++}. We will use an idea from Mitchell's paper [2], where in order to classify a normal measure $V \in L[\vec{U}]$ he applies the Comparison Lemma to $L[\vec{U}]$ and its ultrapower by V.

We could have used core model theory to do this, but that is not necessary here (and in any case the ideas in the core model proofs are very similar).

We start by identifying the measures $u(1)$ and $u(2)$, which will illustrate the ideas we need to do the general case.

2.1 The measure $u(1)$

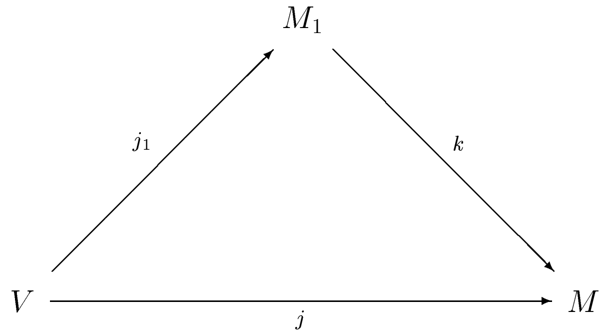
Recall that $u(1)$ is the ultrafilter on V_κ given by

$$X \in u(1) \iff \langle \kappa \rangle \in j(X).$$

If $M_1 = Ult(V, u(1))$, then it is a standard fact (easily checked) that M_1 is the transitive collapse of the subclass of M given by

$$\{ j(F)(\kappa) \mid \mathrm{dom}(F) = V_\kappa \},$$

and that if $j_1 : V \longrightarrow M_1$ is the ultrapower map then we have a commutative triangle



with k being the inverse of the collapsing map.

We need to know the critical point of k. Since as usual $\mathcal{P}\kappa \subseteq M_1$ we have $\kappa^+ = (\kappa^+)_{M_1}$, and since GCH holds in V we have by the usual arguments that $(\kappa^{++})_{M_1} < j_1(\kappa) < \kappa^{++}$. But $V_{\kappa+2} \subseteq M$, so that $(\kappa^{++})_M = \kappa^{++}$. The upshot of all this is that the critical point of k is $(\kappa^{++})_{M_1}$, an ordinal which we will denote by β_1.

Claim 1: For $X \subseteq V_\kappa$, $X \in u(1)$ if and only if

$$\langle \kappa \rangle \in j_{\vec{E}(\kappa,\beta_1)}(X).$$

Proof: By elementarity $M_1 = L[\vec{E}_1]$, where $\vec{E}_1$ is a coherent sequence in M_1.

Let $\phi(\alpha, \vec{F})$ abbreviate the conjunction of the propositions:

1. $\vec{F}$ is a coherent sequence in $L[\vec{F}]$.
2. In the sense of the model $L[\vec{F}]$:
   (a) $(\alpha, \alpha^{++}) \in \mathrm{dom}(\vec{F})$.

(b) If $i : V \longrightarrow N$ is the ultrapower of V by $\vec{F}(\alpha, \alpha^{++})$, $i_1 : V \longrightarrow N_1$ the ultrapower by $\{ X \mid \langle \alpha \rangle \in i(X) \}$, and $\beta = (\alpha^{++})_{N_1}$, then for some $X \subseteq V_\alpha$ we have $\langle \alpha \rangle \in i(X)$ and $\langle \alpha \rangle \notin j_{\vec{F}(\alpha,\beta)}(X)$.

The claim is proven if $\phi(\kappa, \vec{E})$ is false, so assume for a contradiction that it is true.

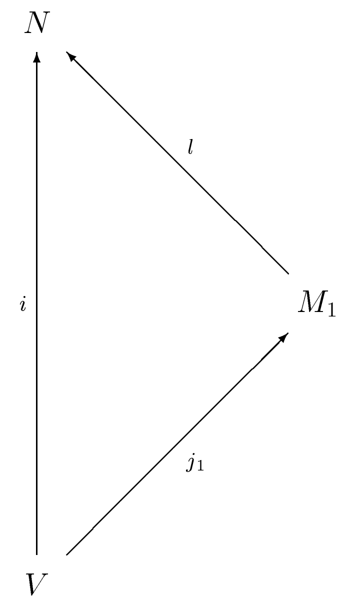
We may assume without loss of generality that

- If $\vec{G}$ is a proper initial segment of $\vec{E}$, then $\phi(\alpha, \vec{G})$ is false for all α.
- $\phi(\alpha, \vec{E})$ is false for all $\alpha < \kappa$.

In Mitchell's terminology from [2], $\vec{E}$ is ϕ-minimal. Also, κ is definable from $\vec{E}$ as the least α such that $\phi(\alpha, \vec{E})$, a fact which we exploit later.

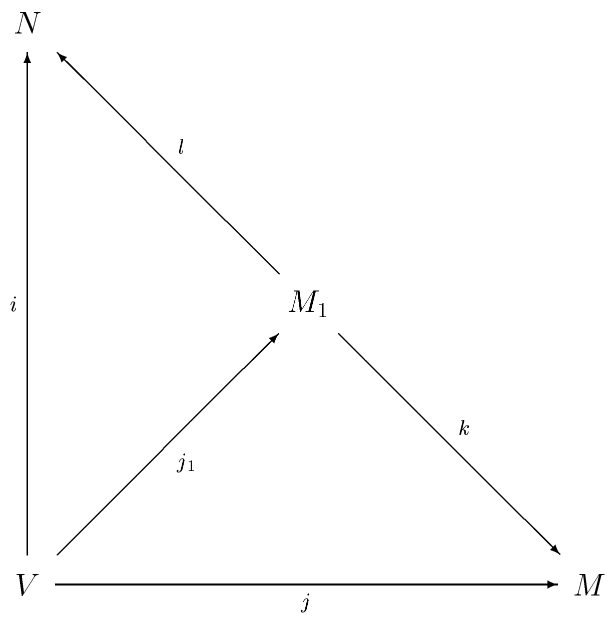
We co-iterate $V = L[\vec{E}]$ and $M_1 = L[\vec{E}_1]$; exactly as in [2] the ϕ-minimal choice of $\vec{E}$ guarantees that the co-iteration terminates with the same model on both sides, because it is impossible that one sequence be aligned with a proper initial segment of the other. Let N be the common iterate, with iteration maps $i : V \longrightarrow N$ and $l : M_1 \longrightarrow N$.

We claim that the diagram



is commutative. Again we use an idea from [2]. Suppose it fails to commute, and let x be the minimal element (in the standard well-order of $L[\vec{E}]$) such that $i(x) \neq l(j_1(x))$. The maps j_1, l and i are all definable from κ and $\vec{E}$, so since κ is definable from $\vec{E}$ we may conclude that x is definable from $\vec{E}$. If $N = L[\vec{G}]$ and y denotes the result of working that definition in N from parameter $\vec{G}$, then clearly $y = i(x) = l(j_1(x))$ and we have a contradiction.

So we have a commutative diagram



We inspect the first step of the comparison process; $k(\vec{E}_1) = j(\vec{E})$, and in M we have $o^{j(\vec{E})}(\kappa) = \kappa^{++}$, because $\vec{E}$ is coherent and M was had by applying $\vec{E}(\kappa, \kappa^{++})$ to V. Therefore $o^{\vec{E}_1}(\kappa) = (\kappa^{++})_{M_1} = \beta_1$. Since β_1 is the critical point of k, the extenders $\vec{E}(\kappa, \lambda)$ and $\vec{E}_1(\kappa, \lambda)$ must coincide for $\kappa < \lambda < \beta_1$, so that (since we certainly have agreement on critical points less than κ) (κ, β_1) is the least point of disagreement between $\vec{E}$ and $\vec{E}_1$.

Thus, the first step in the comparison process is to apply $E(\kappa, \beta_1)$ to V and do nothing to M_1. This handles the disagreement at κ, so that subsequent steps in the comparison will involve applying extenders with critical points greater than κ.

Suppose that X is a witness to the truth of $\phi(\kappa, \vec{E})$, which is to say that $\langle \kappa \rangle \in j(X)$ and $\langle \kappa \rangle \notin j_{\vec{E}(\kappa,\beta_1)}(X)$. $\langle \kappa \rangle \in j_1(X)$ because $\mathrm{crit}(k) = \beta_1 > \kappa$, so $\langle \kappa \rangle \in i(X)$ since we just argued that $\mathrm{crit}(l) > \kappa$. But the analysis of the iteration i shows that $\langle \kappa \rangle \in j_{\vec{E}(\kappa,\beta_1)}(X)$, which is a contradiction. This finishes the proof of the claim.

♦

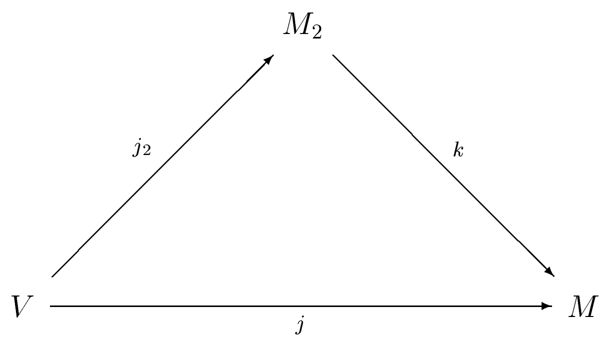
2.2 The measure u(2)

The analysis of $u(2)$ is a little trickier, because while $u(1)$ is essentially just a measure on ordinals, $u(2)$ is essentially a measure on measures.

Recall that $u(2)$ is the measure on V_κ defined by

$$X \in u(2) \iff \langle \kappa, u(1) \rangle \in j(X).$$

Just as in the analysis of $u(1)$, we may factor j through the ultrapower by $u(2)$ to get a diagram



where the critical point of k is now $\beta_2 = (\kappa^{++})_{M_2}$, and where $M_2 = L[\vec{E}_2]$.

We need a little analysis of this situation. If we let

$$X = \{ \ j(F)(\kappa, u(1)) \mid F \in V \ \}$$

then M_2 is the transitive collapse of X and k is the inverse of the collapsing map. Since $\mathcal{P}\kappa \cup \{u(1)\} \subseteq X$ it is easy to see that $u(1) \in M_2$. We claim that $\beta_1 < \beta_2$.

To see this observe that V and M_2 agree to rank $\kappa + 1$, so that their ultrapowers by $u(1)$ agree to rank $j_1(\kappa) + 1$. In particular β_1 is the κ^{++} of $Ult(M_2, u(1))$, so that by the standard arguments applied inside the model M_2 we have $\beta_1 < \kappa^{++}_{M_2} = \beta_2$.

It is convenient to have a notation for reading off measures from coherent sequences.

Definition 1: If $\vec{F}$ is a coherent sequence and $(\gamma, \delta) \in \mathrm{dom}(\vec{F})$, $\mu_1(\vec{F}, \gamma, \delta)$ is the measure on V_γ given by

$$\mu_1(\vec{F}, \gamma, \delta) = \{ \ X \subseteq V_\gamma \mid \langle \gamma \rangle \in j_{\vec{F}(\gamma,\delta)}(X) \ \}.$$

Claim 2: For $X \subseteq V_\kappa$,

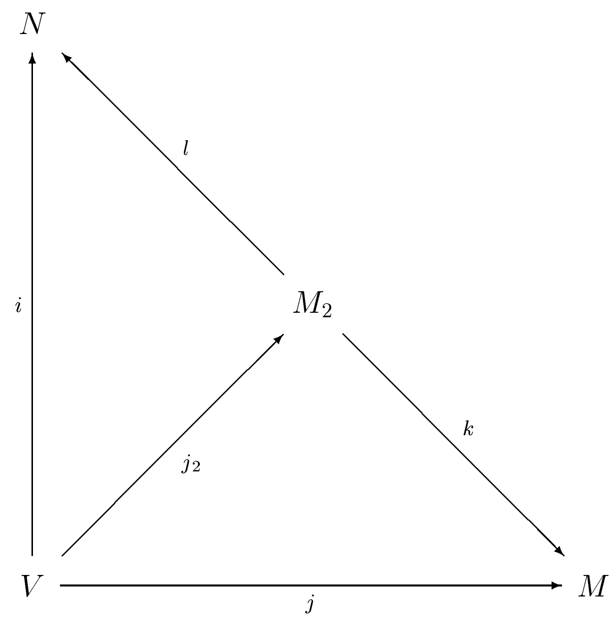
$$X \in u(2) \iff \langle \kappa, \beta_1 \rangle \in j_{\vec{E}(\kappa,\beta_2)}(X^*),$$

where X^* is defined from X by

$$X^* = \{ \, (\gamma, \delta) \mid \langle \gamma, \mu_1(\vec{E}, \gamma, \delta) \rangle \in X \, \}.$$

Proof: As in the case of $u(1)$, we really need to proceed by contradiction, but in this section we will ignore that aspect of the proof and simply show that if $i : V \longrightarrow N$ and $l : M_2 \longrightarrow N$ are the co-iteration maps for the comparison of V and M_2, and $i = l \circ j_2$, then the claim holds. The reader who is disturbed by this is referred to the next subsection, in which we give a complete treatment for the general case.

We are now assuming that the diagram



commutes, where i and l are the co-iteration maps. As before we may argue that $o^{\vec{E}_2}(\kappa) = \beta_2$, so that the initial step of the co-iteration is to apply $\vec{E}(\kappa, \beta_2)$ to V while applying nothing to M_2. Notice that, by the non-overlapping nature of the coherent sequences, all extenders subsequently applied during the comparison process have critical points greater than β_2. In particular $\text{crit}(l) > \beta_2$.

In the notation we just introduced, we showed in the last section that $u(1) = \mu_1(\vec{E}, \kappa, \beta_1)$. By coherence

$$\vec{E}(\kappa, \beta_1) = j(\vec{E})(\kappa, \beta_1),$$

and then as $\mathrm{crit}(k) = \beta_2 > \beta_1$ we also have

$$j(\vec{E})(\kappa, \beta_1) = \vec{E}_2(\kappa, \beta_1).$$

The models $Ult(V, \vec{E}(\kappa, \beta_1))$ and $Ult(M_2, \vec{E}(\kappa, \beta_1))$ agree to a high rank, so that

$$M_2 \models u(1) = \mu_1(\vec{E}_2, \kappa, \beta_1).$$

Let $X \in u(2)$. This means that $\langle \kappa, u(1) \rangle \in j(X)$, which is equivalent to $\langle \kappa, u(1) \rangle \in j_2(X)$. That is to say

$$M_2 \models \langle \kappa, \mu_1(j_2(\vec{E}), \kappa, \beta_1) \rangle \in j_2(X).$$

By the same kind of diagram-chasing as we did for $u(1)$, we may show that

$$Ult(V, \vec{E}(\kappa, \beta_2)) \models \langle \kappa, \mu_1(j_{\vec{E}(\kappa,\beta_2)}(\vec{E}), \kappa, \beta_1) \rangle \in j_{\vec{E}(\kappa,\beta_2)}(X).$$

Equivalently,

$$\langle \kappa, \beta_1 \rangle \in j_{\vec{E}(\kappa,\beta_2)}(\{ \langle \gamma, \delta \rangle \mid \langle \gamma, \mu_1(\vec{E}, \gamma, \delta) \rangle \in X \}),$$

which is to say

$$\langle \kappa, \beta_1 \rangle \in j_{\vec{E}(\kappa,\beta_2)}(X^*).$$

♦

2.3 The measure $u(\alpha)$

Now we prove a general statement relating the measures $u(\alpha)$ to the sequence $\vec{E}$.

Definition 2: α is *nice* if the critical point of the elementary embedding

$$k : Ult(V, u(\alpha)) \longrightarrow M$$

given by $k : [F] \longmapsto j(F)(u \restriction \alpha)$ is greater than α.

Observe that if α is nice then $u \restriction \alpha \in Ult(V, u(\alpha))$, and $k(u \restriction \alpha) = u \restriction \alpha$. All this follows from the standard analysis of $Ult(V, u(\alpha))$ as the transitive collapse of $\{ \, j(F)(u \restriction \alpha) \mid F \in V \, \}$, and of k as the inverse of the collapsing map.

We show that a reasonably long initial segment of κ^{++} consists of nice α. We use some information about so-called "repeat points" which will be familiar to devotees of Radin forcing.

Definition 3: α is a *repeat point* of u if $u(\alpha) \subseteq \bigcup_{\beta<\alpha} u(\beta)$.

Lemma 1: There is no repeat point α less than κ^+, and there are κ^{++} many in $[\kappa^+, \kappa^{++})$.

Proof: Let $\alpha < \kappa^+$, suppose that α is the order type of some ordering relation $X \subseteq \kappa \times \kappa$. Define

$$A = \{ \, v \in V_\kappa \mid \mathrm{lh}(v) = \mathrm{o.t.}(X \cap (v(0) \times v(0))) \, \}.$$

Then it is easy to see that $A \in u(\alpha)$, and $A \notin u(\beta)$ for $\beta < \alpha$.

For the second part of the claim, observe that if α is not a repeat we may choose X_α to witness this. By GCH there are only κ^+ choices for X_α, so there must be κ^{++} repeat points below κ^{++}.

♦

The following result could certainly be pushed further, but indicates that a long initial segment of κ^{++} consists of nice α.

Lemma 2: If $\{ \gamma < \alpha \mid \gamma \text{ is a repeat} \}$ has order type less than κ^+, then α is nice.

Proof: We show that all $\beta \leq \alpha$ are of the form $j(F)(u \restriction \alpha)$. Defining $F(v) = \mathrm{lh}(v)$ handles α, so let $\beta < \alpha$.

If β is not a repeat, let $X \in u(\beta)$ witness this. Define $F(v)$ to be the least $\eta < \mathrm{lh}(v)$ such that $X \cap V_{v(0)} \in v(\eta)$, then $j(F)(u \restriction \alpha) = \beta$.

If β is a repeat, let it be the γ^{th} repeat where $\gamma < \kappa^+$, and find $X \subseteq \kappa \times \kappa$ such that γ is the order type of X. Define $F(v)$ to be the η^{th} repeat point of v, where η is the order type of $X \cap (v(0) \times v(0))$.

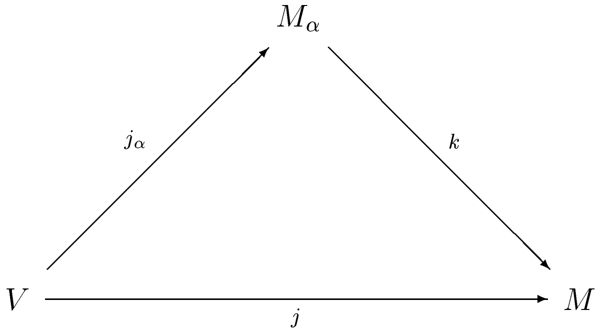
♦

Claim 3: If $\{ \gamma < \alpha \mid \gamma \text{ is a repeat} \}$ has order type less than κ^+, then for $X \subseteq V_\kappa$

$$X \in u(\alpha) \iff u \restriction \alpha \in j_{\vec{E}(\kappa,\beta_\alpha)}(X),$$

where $\beta_\alpha = (\kappa^{++})_{Ult(V,u(\alpha))}$.

Proof: α is nice, so that if we factor j through the ultrapower by $u(\alpha)$ then we get a commutative triangle



where (by niceness) $u \restriction \alpha \in M_\alpha$, and $k(u \restriction \alpha) = u \restriction \alpha$. As usual we may show $\mathrm{crit}(k) = (\kappa^{++})_{M_\alpha}$, and if we denote this by β_α then we know that $\beta_\alpha > \alpha$. It is interesting to notice that if $\nu < \alpha$ then $u(\nu) \in M_\alpha$, so that we get $\beta_\nu < \beta_\alpha$.

As in the treatment of $u(1)$ we define a formula ϕ, this time a slightly more complex version; $\phi(\gamma, \delta, \vec{F})$ means that

1. $\vec{F}$ is coherent in $L[\vec{F}]$.

2. In the model $L[\vec{F}]$

   (a) $(\gamma, \gamma^{++}) \in \mathrm{dom}(\vec{F})$.

   (b) Let $F = \vec{F}(\gamma, \gamma^{++})$, let $j_F : V \longrightarrow M_F$ be the ultrapower by F, and define a sequence v from F by the recursion $v(0) = \gamma$, $v(\eta) = \{ X \subseteq V_\gamma \mid v \restriction \eta \in j_F(X) \}$. Then there is $X \in v(\delta)$ such that $v \restriction \delta \notin j_{\vec{F}(\gamma,\epsilon)}(X)$, where $\epsilon = (\gamma^{++})_{Ult(V,v(\delta))}$.

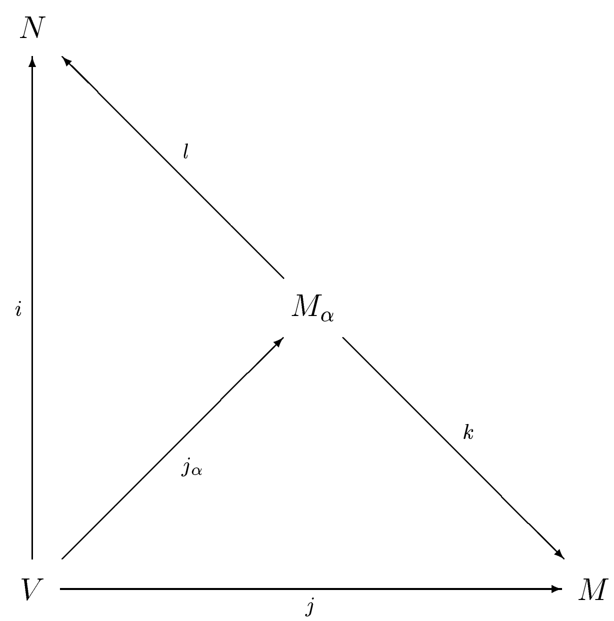
For a contradiction, assume that $\phi(\kappa, \alpha, \vec{E})$. We may take it, without loss of generality, that

1. $\phi(\gamma, \delta, \vec{G})$ is false for all γ, δ, and $\vec{G}$ any proper initial segment of $\vec{E}$.

2. κ is minimal such that $\exists\delta\ \phi(\kappa, \delta, \vec{E})$.

3. α is minimal such that $\phi(\kappa, \alpha, \epsilon)$.

Exactly as in the analysis of $u(1)$, we may now argue that we have a commutative diagram



where $i : V \longrightarrow N$ and $l : M_\alpha \longrightarrow N$ are the co-iteration maps. We also know that in the comparison iteration the first step is to apply $\vec{E}(\kappa, \beta_\alpha)$ to V and nothing to M_α, and that all subsequently applied extenders have critical points greater than β_α.

Now let $X \in u(\alpha)$. Then $u \restriction \alpha \in j(X)$, and since $k(u \restriction \alpha) = u \restriction \alpha$ we have $u \restriction \alpha \in j_\alpha(X)$. l has critical point greater than β_α, which in turn is greater than α, so $u \restriction \alpha \in i(X)$. Finally, the analysis of i gives us $u \restriction \alpha \in j_{\vec{E}(\kappa,\beta_\alpha)}(X)$.

The claim is proved.

♦

3 A direct definition

In this section we will construct Radin forcing directly from a coherent sequence $\vec{E}$ and a sequence of ordinals $\vec{\beta}$. We start with a motivating example, then do a general construction.

3.1 An example

Retain from the last section the assumptions that

1. $\vec{E}$ is a coherent sequence, $V = L[\vec{E}]$.
2. $(\kappa, \kappa^{++}) \in \mathrm{dom}(\vec{E})$.
3. u is the Radin sequence generated from the embedding $j_E : V \longrightarrow M_E$, where E is the extender $\vec{E}(\kappa, \kappa^{++})$.

Let $v = u \restriction 3$, then we will use what we learned in the last section to analyse the Radin forcing $\mathbb{R}_v$ in terms of the coherent sequence $\vec{E}$ and the sequence of ordinals $\langle \kappa, \beta_1, \beta_2 \rangle$.

Definition 4: The Radin forcing $\mathbb{R}_v$ is defined in the following way.

1. A condition is a finite sequence $\langle (v_0, A_0), \ldots, (v_n, A_n) \rangle$ where

(a) $v_n = v$, and $A_n \in v(1) \cap v(2)$. Let $\beta_n = \kappa$.

(b) If $i < n$ then either $v_i = \langle \beta_i \rangle$ for some $\beta_i < \kappa$ and $A_i = \emptyset$, or $v_i = \langle \beta_i, V_i \rangle$ with V_i a measure on $[\beta_i]^1$ and $A_i \in V_i$.

(c) If $i < n$ then $v_i \in V_{\beta_{i+1}}$.

2. If p and q are conditions then $p \leq q$ means that

(a) Whenever a pair (s, A) appears in q, then a pair (s, B) appears in p with $B \subseteq A$.

(b) If a pair (s, B) appears in p then either a pair of form (s, A) appears in q with $B \subseteq A$, or for (w, A) the first pair appearing in p with $w(0) > s(0)$ we have $s \in A$ and $B \subseteq A$.

What does it mean to say that $A \in v(1) \cap v(2)$? $A \in v(1)$ means

$$\langle \kappa \rangle \in j_{\vec{E}(\kappa,\beta_1)}(A),$$

that is there are many $\langle \alpha \rangle$ for $\alpha < \kappa$ in A. $A \in v(2)$ means

$$\langle \kappa, \beta_1 \rangle \in j_{\vec{E}(\kappa,\beta_2)}(A^*),$$

which is to say that for many pairs of ordinals $(\gamma, \delta) \in \mathrm{dom}(\vec{E})$ we have $\langle \gamma, \mu_1(\vec{E}, \gamma, \delta) \rangle \in A$.

Now observe that from the sequence $\langle \kappa, u(1) \rangle$ we may recover β_1, because β_1 is $(\kappa^{++})_{Ult(V,u(1))}$. This recovery can be done inside $Ult(V, \vec{E}(\kappa, \beta_2))$, so that $v(2)$ concentrates on sequences $\langle \gamma, W \rangle$ with

$$W = \mu_1(\vec{E}, \gamma, \nu)$$

for $\nu = (\gamma^{++})_{Ult(V,W)}$.

The upshot of all this is that (below a suitable condition $\langle (v, A) \rangle$) $\mathbb{R}_v$ is equivalent to the following forcing notion $\mathbb{R}(\vec{E}, \langle \kappa, \beta_1, \beta_2 \rangle)$.

Definition 5: The forcing $\mathbb{R}(\vec{E}, \langle \kappa, \beta_1, \beta_2 \rangle)$ is defined in the following way.

1. Let $\mu_1 = \mu_1(\vec{E}, \kappa, \beta_1)$, that is

$$\{ X \subseteq V_\kappa \mid \langle \kappa \rangle \in j_{\vec{E}(\kappa,\beta_1)}(X) \}.$$

2. Let μ_2 be

$$\{ X \subseteq V_\kappa \mid \langle \kappa, \beta_1 \rangle \in j_{\vec{E}(\kappa,\beta_2)}(X) \}.$$

3. A condition is a finite sequence $\langle (\vec{\gamma}_i, A_0), \ldots, (\vec{\gamma}_n, A_n) \rangle$ where
   (a) $\gamma_n = \langle \kappa, \beta_1, \beta_2 \rangle$, and $A_n \in \mu_1 \cap \mu_2$.
   (b) If $i < n$ then either $\vec{\gamma}_i = \langle \beta_i \rangle$ for some $\beta_i < \kappa$ and $A_i = \emptyset$, or $\vec{\gamma}_i = \langle \beta_i, \delta_i \rangle$ with $(\beta_i, \delta_i) \in \mathrm{dom}(\vec{E})$ and $A_i \in \mu_1(\vec{E}, \beta_i, \delta_i)$.
   (c) If $i < n$ then $\vec{\gamma}_i \in V_{\beta_{i+1}}$.
4. If p and q are conditions then $p \leq q$ means that
   (a) Whenever a pair (s, A) appears in q, then a pair (s, B) appears in p with $B \subseteq A$.
   (b) If a pair (s, B) appears in p then either a pair of form (s, A) appears in q with $B \subseteq A$, or for (w, A) the first pair appearing in p with $w(0) > s(0)$ we have $s \in A$ and $B \subseteq A$.

The difference here is simply that in $\mathbb{R}_v$ we had sequences of measures, each measure concentrating on sequences of measures, where now we have sequences of ordinals, each of which codes (via $\vec{E}$) a sequence of measures on sequences of ordinals.

This example motivates the general definition which follows.

3.2 The general definition

We will define a forcing notion $\mathbb{R}(\vec{F}, \vec{\gamma})$ from a coherent sequence $\vec{F}$ and a suitable sequence of ordinals γ.

Definition 6: Let $\vec{F}$ be a coherent sequence. $\vec{\gamma}$ is a *good sequence for* $\vec{F}$ if

1. $\vec{\gamma}$ is an increasing sequence of ordinals.
2. If $0 < \delta < \mathrm{lh}(\vec{\gamma})$ then
   (a) $(\gamma_0, \gamma_\delta) \in \mathrm{dom}(\vec{F})$.

(b) $\vec{\gamma} \restriction \delta \in Ult(V, \vec{F}(\gamma_0, \gamma_\delta))$.

Definition 7: If $\vec{F}$ is coherent, $\vec{\gamma}$ is good for $\vec{F}$, and $0 < \delta < \mathrm{lh}(\vec{\gamma})$ then define

$$\mu_\delta(\vec{F}, \vec{\gamma}) = \{ X \subseteq V_{\gamma_0} \mid \vec{\gamma} \restriction \delta \in j_{\vec{F}(\gamma_0, \gamma_\delta)}(X) \}.$$

Definition 8: If $\vec{F}$ is coherent and $\vec{\gamma}$ is good for $\vec{F}$, with $\mathrm{lh}(\vec{\gamma}) \geq 2$, then

$$\mathcal{F}(\vec{F}, \vec{\gamma}) = \{ X \subseteq V_{\gamma_0} \mid 0 < \delta < \mathrm{lh}(\vec{\gamma}) \Longrightarrow X \in \mu_\delta(\vec{F}, \vec{\gamma}) \}.$$

Lemma 3: $\mathcal{F}(\vec{F}, \vec{\gamma})$ is a γ_0-complete filter on V_{γ_0}, and concentrates on the set of sequences which are good for $\vec{F}$.

Proof: $\mathcal{F}(\vec{F}, \vec{\gamma})$ is the intersection of γ_0-complete ultrafilters, so is certainly a γ_0-complete filter. For the other claim we need to show that if $0 < \delta < \mathrm{lh}(\vec{\gamma})$ and we define $j_\delta : V \longrightarrow M_\delta$ to be the ultrapower by $\vec{F}(\gamma_0, \gamma_\delta)$ then $\vec{\gamma} \restriction \delta$ is good for the sequence $j_\delta(\vec{F})$ in the model $M_\delta = Ult(V, \vec{F}(\gamma_0, \gamma_\delta))$.

To show this let $0 < \beta < \delta = \mathrm{lh}(\vec{\gamma} \restriction \delta)$. By coherence we get

$$(\gamma_0, \gamma_\beta) \in \mathrm{dom}(j_\delta(\vec{F})),$$

and moreover

$$j_\delta(\vec{F})(\gamma_0, \gamma_\beta) = \vec{F}(\gamma_0, \gamma_\beta).$$

Let F denote this extender. $Ult(V, F)$ and $Ult(M_\delta, F)$ agree past rank γ_β, $\vec{\gamma} \restriction \beta \in Ult(V, F)$ because $\vec{\gamma}$ is good for $\vec{F}$, so $\vec{\gamma} \restriction \beta \in Ult(M_\delta, F)$.

♦

The definition of $\mathbb{R}(\vec{F}, \vec{\gamma})$ now follows a familiar pattern.

Definition 9: Let $\vec{\gamma}$ be good for the coherent sequence $\vec{F}$. The forcing $\mathbb{R}(\vec{F}, \vec{\gamma})$ is defined in the following way.

1. A condition is a finite sequence $\langle (\vec{\gamma}_i, A_0), \ldots, (\vec{\gamma}_n, A_n) \rangle$ where

(a) $\vec{\gamma}_n = \vec{\gamma}$, and each $\vec{\gamma}_i$ is an increasing sequence of ordinals which is good for $\vec{F}$.

(b) Let $\beta_i = \vec{\gamma}_i(0)$. For $i \leq n$ either $\vec{\gamma}_i = \langle \beta_i \rangle$ and $A_i = \emptyset$, or $\mathrm{lh}(\gamma_i) > 1$ and $A_i \in \mathcal{F}(\vec{F}, \vec{\gamma}_i)$.

(c) If $i < n$ then $\vec{\gamma}_i \in V_{\beta_{i+1}}$.

2. If p and q are conditions then $p \leq q$ means that

(a) Whenever a pair (s, A) appears in q, then a pair (s, B) appears in p with $B \subseteq A$.

(b) If a pair (s, B) appears in p then either a pair of form (s, A) appears in q with $B \subseteq A$, or for (w, A) the first pair appearing in p with $w(0) > s(0)$ we have $s \in A$ and $B \subseteq A$.

We will not give a complete analysis of this forcing, as the results and proofs are so similar to those for the Radin forcing $\mathbb{R}_v$ as expounded in [5]. We will however prove a few key facts, to give the flavour of the proofs; the reader will notice how the coherence of the sequence $\vec{F}$ is used here at points where, in the situation of [5], we would be using the uniform generation of the measures via a single embedding.

Lemma 4 (The addability lemma): Let $\vec{\gamma}$ be good for $\vec{F}$, and suppose that $A \in \mathcal{F}(\vec{F}, \vec{\gamma})$. Then if

$$B = \{ \vec{\delta} \in V_{\gamma_0} \mid A \cap V_{\delta_0} \in \mathcal{F}(\vec{F}, \vec{\delta}) \}$$

we have $B \in \mathcal{F}(\vec{F}, \vec{\gamma})$.

Proof: If $0 < \nu < \mathrm{lh}(\gamma)$ then we need to show that, defining $j_\nu : V \longrightarrow M_\nu$ to be the ultrapower by $\vec{F}(\gamma_0, \gamma_\nu)$, $\vec{\gamma} \upharpoonright \nu \in j_\nu(B)$. This amounts to showing that in $M_\nu = Ult(V, \vec{F}(\gamma_0, \gamma_\nu))$ we have

$$A \in \mathcal{F}(j_\nu(\vec{F}), \gamma \upharpoonright \nu).$$

If $0 < \beta < \nu$ then $j_\nu(\vec{F})(\gamma_0, \gamma_\beta) = \vec{F}(\gamma_0, \gamma_\beta)$, and we denote this extender by F. Let

$$\begin{aligned} j_F^V &: V \longrightarrow Ult(V, F) \\ j_F^{M_\nu} &: M_\nu \longrightarrow Ult(M_\nu, F) \end{aligned}$$

be the ultrapowers by F as computed in V, M_ν. There is agreement between $Ult(V,F)$ and $Ult(M_\nu,F)$ to a rank greater than γ_β. We also have $j^V_F(A) = j^{M_\nu}_F(A)$. Now $A \in \mathcal{F}(\vec{F},\vec{\gamma})$, so certainly $A \in \mu_\beta(\vec{\mathcal{F}},\vec{\gamma})$, that is $\vec{\gamma} \restriction \beta \in j^V_F(A) = j^{M_\nu}_F(A)$, which is what we needed to prove.

♦

The addability lemma shows that it is possible to extend a condition in a non-trivial fashion.

Definition 10: Let $p = \langle(\vec{\gamma}_0, A_0), \ldots, (\vec{\gamma}_n, A_n)\rangle$ be a condition, then the *lower part* of p is the sequence $\langle(\vec{\gamma}_0, A_0), \ldots, (\vec{\gamma}_{n-1}, A_{n-1})\rangle$. Let LP be the set of lower parts.

Lemma 5 (The diagonal intersection lemma): Let $\langle A_x : x \in LP\rangle$ be a family of subsets of V_{γ_0} such that $A_x \in \mathcal{F}(\vec{F},\vec{\gamma})$ for all x. Then there is $A \in \mathcal{F}(\vec{F},\vec{\gamma})$ such that

$$x \frown (\vec{\gamma}, A) \leq x \frown (\vec{\gamma}, A_x)$$

for all x.

Proof: If we define

$$A = \{\ \vec{\delta} \mid x \in LP \cap V_{\delta_0} \Longrightarrow \vec{\delta} \in A_x\ \},$$

then it suffices to show $A \in \mathcal{F}(\vec{F},\vec{\gamma})$. Let $0 < \beta < \mathrm{lh}(\gamma)$, and define $j_\beta : V \longrightarrow M_\beta$ to be the ultrapower by $\vec{F}(\gamma_0, \gamma_\beta)$. Then we want

$$\gamma \restriction \beta \in j_\beta(A).$$

$j_\beta(LP) \cap V_{\gamma_0} = LP$, and if $x \in LP$ then $A_x \in \mu_\beta(\vec{F},\vec{\gamma})$, so that $\vec{\gamma} \restriction \beta \in j_\beta(A)_x$ and we are done.

The set A defined here is called the *diagonal intersection* of the function $x \longmapsto A_x$.

♦

4 The Prikry lemma

In this section we will sketch a proof that $\mathbb{R}(\vec{\mathcal{F}}, \vec{\gamma})$ has a certain property, namely that questions about the forcing extension can be decided by "direct" extensions. It is this property, which is characteristic of Prikry forcing and its generalisations, that legitimates $\mathbb{R}(\vec{F}, \vec{\gamma})$ as a kind of Radin forcing.

Fix for the rest of this section a coherent sequence $\vec{F}$ and a sequence $\vec{\gamma}$ which is good for F, and define $\mathbb{R} = \mathbb{R}(\vec{F}, \vec{\gamma})$.

Definition 11: Let p and q be conditions in $\mathbb{R}$. p is a *direct extension* of q (we write $p \leq_d q$) if and only if $p \leq q$ and $\mathrm{lh}(p) = \mathrm{lh}(q)$.

The direct extensions of q are those conditions that can be obtained from p by shrinking the measure one sets which appear as the second entries in the pairs comprising q. The idea is that a direct extension restricts the commitments we may make, while making no (positive) commitments itself.

Theorem 1: Let $p \in \mathbb{R}$, let $\mathbf{b} \in RO(\mathbb{R})$. Then there exists q such that $q \leq_d p$ and $q \parallel \mathbf{b}$.

Proof: Suppose for simplicity that p is of the form $\langle(\vec{\gamma}, C)\rangle$. The general case is an easy variation on this one.

By taking an appropriate diagonal intersection, we may assume that for all lower parts x

$$(\exists D\ x \frown (\vec{\gamma}, D) \parallel \mathbf{b}) \iff x \frown (\vec{\gamma}, C) \parallel \mathbf{b}.$$

For each x such that $x \frown (u, C) \not\parallel \mathbf{b}$, define

$$\begin{aligned} C_1^x &= \{\ \vec{\delta} \in C \mid \exists D\ x \frown (\vec{\delta}, D) \frown (\vec{\gamma}, C) \Vdash \mathbf{b}\ \}, \\ C_2^x &= \{\ \vec{\delta} \in C \mid \exists D\ x \frown (\vec{\delta}, D) \frown (\vec{\gamma}, C) \Vdash \neg\mathbf{b}\ \}, \\ C_3^x &= C \setminus (C_1^x \cup C_2^x). \end{aligned}$$

For α with $0 < \alpha < \mathrm{lh}(\vec{\gamma})$ let $C^x(\alpha) = C_i^x$ for the unique i such that $C_i^x \in \mu_\alpha(\vec{F}, \vec{\gamma})$. Define

$$C^x = \bigcup_{0<\alpha<\mathrm{lh}(\vec{\gamma})} C^x(\alpha),$$

and let E be the diagonal intersection of the function $x \mapsto C_x$. We claim that $(\vec{\gamma}, E) \parallel \mathbf{b}$.

Suppose (towards a contradiction) that this is not so, and let

$$\langle (\vec{\gamma}^0, A^0), \ldots, (\vec{\gamma}^n, A^n), (\vec{\gamma}, E) \rangle$$

be an extension of $\langle (\vec{\gamma}, E) \rangle$ deciding $\mathbf{b}$ and having minimal length. Without loss of generality this condition forces $\mathbf{b}$ (else we may replace $\mathbf{b}$ by $\neg\mathbf{b}$). Let $x = \langle (\vec{\gamma}^0, A^0), \ldots, (\vec{\gamma}^{n-1}, A^{n-1}) \rangle$. $\vec{\gamma}^n \in E$, so by construction $\vec{\gamma}^n \in C^x$, that is $\vec{\gamma}^n \in C^x(\alpha)$ for some $\alpha < \mathrm{lh}(\vec{\gamma})$. By the construction of C, $x \frown (\vec{\gamma}^n, A_n) \frown (\vec{\gamma}, C) \Vdash \mathbf{b}$, so $\vec{\gamma}^n \in C^x_1$ and $C^x(\alpha)$ must have been chosen as C^x_1. Therefore $\forall \vec{\delta} \in C^x(\alpha)\ \exists D\ x \frown (\vec{\delta}, D) \frown (\vec{\gamma}, C) \Vdash \mathbf{b}$. Fix some choice function h which chooses appropriate D, so that we have

$$\forall \vec{\delta} \in C^x(\alpha)\ x \frown (\vec{\delta}, h(\vec{\delta})) \frown (\vec{\gamma}, C) \Vdash \mathbf{b}.$$

Notice that $h(\vec{\delta}) \in \mathcal{F}(\vec{F}, \vec{\delta})$ for all $\vec{\delta}$.

The contradiction will be almost immediate once we have the following lemma.

Lemma 6: There is $B \in \mathcal{F}(\vec{F}, \vec{\gamma})$ such that if $p \leq \langle (\vec{\gamma}, B) \rangle$, then p is compatible with $\langle (\vec{\delta}, h(\vec{\delta})), (\vec{\gamma}, B) \rangle$ for some $\vec{\delta} \in \mathrm{dom}(h)$.

Proof: We construct B as $B_1 \cup B_2 \cup B_3$, where

$$\begin{aligned} B_1 &\in \bigcap \{\, \mu_\beta(\vec{F}, \vec{\gamma}) \mid 0 < \beta < \alpha \,\}, \\ B_2 &\in \mu_\alpha(\vec{F}, \vec{\gamma}), \\ B_3 &\in \bigcap \{\, \mu_\beta(\vec{F}, \vec{\gamma}) \mid \alpha < \beta < \mathrm{lh}(\gamma) \,\}. \end{aligned}$$

The reader who finds the definitions of the B_i puzzling is encouraged to look ahead to the end of the proof and see how they are used. For each ν let

$$j_\nu : V \longrightarrow M_\nu = Ult(V, \vec{F}(\gamma_0, \gamma_\nu))$$

be the standard ultrapower map, and let $\mu_\nu = \mu_\nu(\vec{F}, \vec{\gamma})$.

Define

$$B^0_1 = \{\, \vec{\delta} \mid \{\, \vec{\epsilon} \in \mathrm{dom}(h) \mid \vec{\delta} \in h(\vec{\epsilon}) \,\} \in \mu_\alpha \,\}$$

We claim that $B_1^0 \in \mu_\beta$ for $0 < \beta < \alpha$. Fix δ for the moment, and consider

$$A = \{ \vec{\epsilon} \in \mathrm{dom}(h) \mid \vec{\delta} \in h(\vec{\epsilon}) \}.$$

By the definition of μ_α and the elementarity of j_α,

$$A \in \mu_\alpha(\vec{F}, \vec{\gamma}) \iff \vec{\gamma} \restriction \alpha \in j_\alpha(A) \iff \vec{\delta} \in j_\alpha(h)(\vec{\gamma} \restriction \alpha).$$

We have shown that $B_1^0 = j_\alpha(h)(\vec{\gamma} \restriction \alpha)$, so that by elementarity we have $B_1^0 \in \mathcal{F}(j_\alpha(\vec{F}), \vec{\gamma} \restriction \alpha)$. But by the coherence of $\vec{F}$ we see that

$$\mathcal{F}(j_\alpha(\vec{F}), \vec{\gamma} \restriction \alpha) = \mathcal{F}(\vec{F}, \vec{\gamma} \restriction \alpha),$$

and since $\mu_\beta(\vec{F}, \vec{\gamma}) = \mu_\beta(\vec{F}, \vec{\gamma} \restriction \alpha)$ for $\beta < \alpha$ we are done.

Now define

$$B_1^1 = \{ \vec{\delta} \mid \{ \vec{\epsilon} \in \mathrm{dom}(h) \mid h(\vec{\epsilon}) \cap V_{\delta_0} \in \mathcal{F}(\vec{F}, \vec{\delta}) \} \in \mu_\alpha \}.$$

By an argument very similar to that in the last paragraph for B_1^0,

$$B_1^1 = \{ \vec{\delta} \mid B_1^0 \cap V_{\delta_0} \in \mathcal{F}(\vec{F}, \vec{\delta}) \}.$$

We claim that $B_1^1 \in \mu_\beta$ for $0 < \beta < \alpha$. To show that $B_1^1 \in \mu_\beta$ is to show $\vec{\gamma} \restriction \beta \in j_\beta(B_1^1)$, which is to say that

$$j_\beta(B_1^0) \cap V_{\gamma_0} \in \mathcal{F}(j_\beta(\vec{F}), \vec{\gamma} \restriction \beta).$$

This is immediate since $j_\beta(B_1^0) \cap V_{\gamma_0} = B_1^0$, and

$$\mathcal{F}(j_\beta(\vec{F}), \vec{\gamma} \restriction \beta) = \mathcal{F}(\vec{F}, \vec{\gamma} \restriction \beta)$$

using coherence again.

Having defined B_1^0 and B_1^1, let

$$B_1 = B_1^0 \cap B_1^1.$$

B_2 is now defined by a kind of diagonal intersection. For $\vec{\delta} \in B_1$ define

$$g(\vec{\delta}) = \{ \vec{\epsilon} \in \mathrm{dom}(h) \mid \vec{\delta} \in h(\vec{\epsilon}),\ h(\vec{\epsilon}) \cap V_{\delta_0} \in \mathcal{F}(\vec{F}, \vec{\delta}) \}.$$

By the definition of B_1, $g(\vec{\delta}) \in \mu_\alpha$. Let

$$B_2 = \{ \vec{\epsilon} \in \mathrm{dom}(h) \mid \vec{\delta} \in B_1 \cap V_{\epsilon_0} \Longrightarrow \vec{\epsilon} \in g(\vec{\delta}) \}.$$

We need to show $B_2 \in \mu_\alpha$, that is to say that $\vec{\gamma} \restriction \alpha \in j_\alpha(B_2)$. Now $j_\alpha(B_1) \cap V_{\gamma_0} = B_1$, and for $\vec{\delta} \in B_1$ we know $g(\vec{\delta}) \in \mu_\alpha$, hence $\vec{\gamma} \restriction \alpha \in j_\alpha(g)(\vec{\delta})$, using again the fact that for $\vec{\delta} \in V_{\gamma_0}$ we have $j_\alpha(\vec{\delta}) = \vec{\delta}$. This confirms that $B_2 \in \mu_\alpha$.

Finally let

$$B_3 = \{ \vec{\delta} \mid \exists \nu < \mathrm{lh}(\vec{\delta}) \, \forall C \in \mathcal{F}(\vec{F}, \vec{\delta}) \, \{ \vec{\epsilon} \in B_2 \cap C \mid h(\vec{\epsilon}) \cap C \in \mathcal{F}(\vec{F}, \vec{\epsilon}) \} \in \mu_\nu(\vec{F}, \vec{\delta}) \}.$$

We need to show that $B_3 \in \mu_\beta$ for every β with $\alpha < \beta < \mathrm{lh}(\vec{\gamma})$. That is, we need to show that for all such β we have $\vec{\gamma} \restriction \beta \in j_\beta(B_3)$. We will use α as the $\nu < \beta$ demanded by the definition and show that, for each $C \in \mathcal{F}(j_\beta(\vec{F}), \vec{\gamma} \restriction \beta)$,

$$\{ \vec{\epsilon} \in j_\beta(B_2) \cap C \mid j_\beta(h)(\vec{\epsilon}) \cap C \in \mathcal{F}(j_\beta(\vec{F}), \vec{\epsilon}) \} \in \mu_\alpha(j_\beta(\vec{F}), \vec{\gamma} \restriction \beta)$$

As usual we know that $C \in \mathcal{F}(\vec{F}, \vec{\gamma} \restriction \beta)$ and that $\mu_\alpha(j_\beta(\vec{F}), \vec{\gamma} \restriction \beta) = \mu_\alpha$. $C \subseteq V_{\gamma_0}$, so $j_\beta(B_2) \cap C = B_2 \cap C$. For $\vec{\epsilon} \in B_2 \cap C$ we know $\vec{\epsilon}$ and $h(\vec{\epsilon})$ are in V_{γ_0}, hence $h(\vec{\epsilon}) = j_\beta(h(\vec{\epsilon})) = j_\beta(h)(\vec{\epsilon})$. What is more

$$\mathcal{F}(j_\beta(\vec{F}), \vec{\epsilon}) = j_\beta(\mathcal{F}(\vec{F}, \vec{\epsilon})) = \mathcal{F}(\vec{F}, \vec{\epsilon}).$$

So we are required to show that

$$\{ \vec{\epsilon} \in B_2 \cap C \mid h(\vec{\epsilon}) \cap C \in \mathcal{F}(\vec{F}, \vec{\epsilon}) \} \in \mu_\alpha.$$

$B_2 \cap C \in \mu_\alpha$ because $\beta > \alpha$. All that remains to be seen is that

$$j_\alpha(h)(\vec{\gamma} \restriction \alpha) \cap j(C) \in \mathcal{F}(j_\alpha(\vec{F}), \vec{\gamma} \restriction \alpha).$$

This is easy as $j_\alpha(h)(\vec{\gamma} \restriction \alpha) = B_1^0$, and $B_1^0 \cap j_\alpha(C) = B_1^0 \cap C$ which is clearly a member of $\mathcal{F}(\vec{F}, \vec{\gamma} \restriction \alpha)$.

This concludes the construction of the set B.

We now show that $B = B_1 \cup B_2 \cup B_3$ has the property desired. Let

$$p = \langle (\vec{\gamma}^0, A^0), \ldots, (\vec{\gamma}^n, A^n), (\vec{\gamma}, \bar{B}) \rangle \leq \langle (\vec{\gamma}, B) \rangle.$$

For each $i \leq n$ there is $j < 3$ with $\vec{\gamma}^i \in B_j$. Let $\vec{\delta}$ be the first sequence occurring in p with $\vec{\delta} \notin B_1$. Then $\vec{\delta} \in B_2$, $\vec{\delta} \in B_3$, or $\vec{\delta} = \vec{\gamma}$; we take each case in turn.

1. $\vec{\delta} = \vec{\gamma}^j \in B_2$. By construction $\vec{\gamma}^j \in \mathrm{dom}(h)$. By the definition of B_2 and the fact that for $i < j$ we have $\vec{\gamma}^i \in B_1$, we have for all $i < j$ that $\vec{\gamma}^i \in h(\vec{\gamma}^j)$ and $h(\vec{\gamma}^j) \cap V_{\gamma_0^i} \in \mathcal{F}(\vec{F}, \vec{\gamma}^i)$.

   Therefore

$$\langle (\vec{\gamma}^0, A^0 \cap h(\vec{\gamma}^j)), \ldots, (\vec{\gamma}^j, A^j \cap h(\vec{\gamma}^j)), (\vec{\gamma}^{j+1}, A^{j+1}), \ldots, (\vec{\gamma}, \bar{B}) \rangle$$

   is a common extension of p and $\langle (\vec{\gamma}^j, h(\vec{\gamma}^j)), (\vec{\gamma}, B) \rangle$.

2. $\vec{\delta} = \vec{\gamma}^j \in B_3$. By construction it is possible to find $\vec{\epsilon}$ such that $\epsilon_0 > \gamma_0^{j-1}$, $\vec{\epsilon} \in A^j \cap B^2$, $h(\vec{\epsilon}) \cap A^j \in \mathcal{F}(\vec{F}, \vec{\epsilon})$. As before, if $i < j$ then $\vec{\gamma}^i \in h(\vec{\epsilon})$ and $f(\vec{\epsilon}) \cap V_{\kappa(\vec{\gamma}^i)} \in \mathcal{F}(\vec{F}, \vec{\gamma}^i)$, so

$$\langle (\vec{\gamma}^1, A^1 \cap h(\vec{\epsilon})), \ldots, (\vec{\gamma}^{j-1}, A^{j-1} \cap h(\vec{\epsilon})), (\vec{\epsilon}, A^j \cap h(\vec{\epsilon})), (\vec{\gamma}^j, A^j) \ldots, (\vec{\gamma}, \bar{B}) \rangle$$

   is a common extension of p and $\langle (\vec{\epsilon}, h(\vec{\epsilon})), (\vec{\gamma}, B) \rangle$.

3. $\vec{\delta} = \vec{\gamma}$. The proof is similar to that of Case 2. We claim that

$$\{ \, \vec{\epsilon} \in B_2 \cap \bar{B} \mid h(\vec{\epsilon}) \cap \bar{B} \in \mathcal{F}(\vec{F}, \vec{\epsilon}) \, \} \in \mu_\alpha.$$

   To see this, observe that

$$j_\alpha(h)(\vec{\gamma} \restriction \alpha) \cap j_\alpha(\bar{B}) = B_1^0 \cap \bar{B} \in \mathcal{F}(\vec{F}, \vec{\gamma} \restriction \alpha).$$

   Using this we may find $\vec{\epsilon} \in \bar{B} \cap B_2$ with $\epsilon_0 > \gamma_0^n$, we have $\vec{\gamma}^i \in B_1$ for all $i \leq n$, and we may take as our common extension

$$\langle (\vec{\gamma}^1, A^1 \cap h(\vec{\epsilon})), \ldots, (\vec{\gamma}^n, A^n \cap h(\vec{\epsilon})), (\vec{\epsilon}, h(\vec{\epsilon}) \cap \bar{B}), (\vec{\gamma}, \bar{B}) \rangle.$$

This ends the proof of the lemma.

♦

To finish the proof of the theorem, apply the lemma we just proved. This produces $E^* \in \mathcal{F}(\vec{F}, \vec{\gamma})$, $E^* \subseteq E$, such that every extension of the condition $\langle (\vec{\gamma}, E^*) \rangle$ is compatible with some condition $\langle (\vec{\epsilon}, h(\vec{\epsilon})), (\vec{\gamma}, E^*) \rangle$. In fact more is true; by the uniform definition of the forcing, every extension of $x \frown (\vec{\gamma}, E^*)$ is compatible with some $x \frown (\vec{\epsilon}, h(\vec{\epsilon})) \frown (\vec{\gamma}, E^*)$, that is to say with a condition that forces $\mathbf{b}$. So $x \frown (\vec{\gamma}, E^*) \Vdash \mathbf{b}$, contradicting the assumption that $x \frown (\vec{\gamma}^n, A^n) \frown (\vec{\gamma}, E)$ had minimal length.

It remains for us to extend the result to general conditions. Let

$$p = \langle (\vec{\gamma}_0, A_0), \ldots, (\vec{\gamma}_n, A_n), (\vec{\gamma}, A) \rangle$$

be an arbitrary condition. Construct E as above and refine p to

$$\bar{p} = \langle (\vec{\gamma}_0, A_0), \ldots, (\vec{\gamma}_n, A_n), (\vec{\gamma}, A \cap E) \rangle$$

The argument used above shows that if q is an extension of $\bar{p}$ deciding $\mathbf{b}$ and having minimal length then q is of the form $\bar{x} \frown (\vec{\gamma}, Z)$ with $\bar{x} \in \mathbb{R}(\vec{F}, \vec{\gamma}_n)$, $\bar{x} \leq \langle (\vec{\gamma}_1, A_1), \ldots, (\vec{\gamma}_n, A_n) \rangle$. Now an easy induction on length of conditions gives the result.

♦

Using similar ideas, we can show that $\mathbb{R}(\vec{F}, \vec{\gamma})$ shares all the basic properties of Radin forcing as set out in [5] and [3].

5 Conclusion

In the light of what we have proved, we can see that Radin's version of Radin forcing in [5] corresponds to $\mathbb{R}(\vec{F}, \vec{\gamma})$ where $\vec{F}$ has some long extenders and $\vec{\gamma}$ grows rapidly. Mitchell's version in [3] corresponds to the situation in which $\vec{F}$ has only short extenders (measures) and $\vec{\gamma}$ grows slowly. We hope that this formulation may be helpful in understanding the exact strength of the assumptions that are needed to make various applications of Radin forcing in cardinal arithmetic and choiceless set theory.

We may also speculate that there is some similarly direct way of extracting interesting forcing notions from overlapping extender sequences, such as are used in the inner model theory for large cardinals past a strong cardinal.

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