Possible behaviours for the Mitchell ordering II

James Cummings
Institute of Mathematics
Hebrew University
Givat Ram
91904 Jerusalem
ISRAEL
cummings@math.huji.ac.il

February 25, 1998

Abstract

We analyse the Mitchell ordering in a model where κ is $\mathcal{P}_2\kappa$-hypermeasurable and $2^{2^\kappa} > (2^\kappa)^+$.

1 Introduction and preliminaries

The Mitchell ordering on normal measures was introduced by Mitchell [7], in his work on inner models for cardinals carrying many measures.

Definition 1: Let κ be measurable, let U_0 and U_1 be normal measures on a cardinal κ. Then $U_0 \lhd U_1$ if and only if $U_0 \in Ult(V, U_1)$, the ultrapower of V by U_1.

The following facts are standard.

- $\lhd$ is transitive.
- $\lhd$ is well-founded.

- $\lhd$ is strict.
- An ultrafilter has at most 2^κ ancestors in the ordering $\lhd$.

We make the remark that there are certain limitations on the partial ordering $\lhd$; there can be at most 2^{2^κ} normal measures on κ, and the ordering can have height at most $(2^\kappa)^+$.

If κ is supercompact then it is shown in [9] that these maximal values are achieved; in fact a much weaker hypothesis on κ will suffice [10].

Theorem 1 (Solovay): Let $j : V \longrightarrow M$ be an elementary embedding where $\kappa = \text{crit}(j)$ and $V_{\kappa+2} \subseteq M$, ${}^\kappa M \subseteq M$. Then for every $X \in V_{\kappa+2}$ there is a normal measure U on κ such that $X \in Ult(V, U)$.

Proof: An easy adaptation of the proof in [9].

♦

A $\mathcal{P}_2\kappa$*-hypermeasurable cardinal* is one which satisfies the hypothesis of this theorem.

We spell out the consequences for the Mitchell ordering.

1. Observe that $|V_{\kappa+2} \cap Ult(V, U)| = 2^\kappa$, since any element of $V_{\kappa+2}$ in $Ult(V, U)$ is represented by a function from κ to V_κ. Since $2^{2^\kappa} > 2^\kappa$, there must be 2^{2^κ} normal measures U.
2. Any 2^κ measures may be coded by an element of $V_{\kappa+2}$. So any family of 2^κ measures has a upper bound in the Mitchell ordering. In particular the height of the ordering has its maximal value, namely $(2^\kappa)^+$.

Using inner model theory (of which more anon) we may get models where GCH holds, and the Mitchell ordering at κ is linear of ordertype κ^{++}. On the other hand if κ is a $\mathcal{P}_2\kappa$-hypermeasurable cardinal) and $2^{2^\kappa} > (2^\kappa)^+$ then the ordering is necessarily non-linear; in this paper we explore the ordering in a particular model where these circumstances prevail, and prove a result (theorem 13) which goes some way towards characterising it.

We have taken a fairly digressive approach to the proof. Some arguments needed here are so similar to those of [3] that we have just sketched them here.

2 Inner model theory

We sketch what we need from the theory of inner models for large cardinals below a strong cardinal. The reader is referred to [8] and [5] for more details. The theory is due to Mitchell, Jensen, Dodd and Koepke.

Definition 2: $\vec{E}$ is a *(coherent, non-overlapping) extender sequence* if $\vec{E}$ is a function with $\mathrm{dom}(\vec{E}) \subseteq On \times On$ such that

1. There is some function $o : On \longrightarrow On$ such that $o(\alpha) \geq \alpha + 1$ and
$$\mathrm{dom}(\vec{E}) = \{ (\kappa, \lambda) \mid \kappa < \lambda < o(\kappa) \}.$$
2. If $(\kappa, \lambda) \in \mathrm{dom}(\vec{E})$ then $\vec{E}(\kappa, \lambda)$ is a (κ, λ)-extender.
3. (coherence) Let $\prec$ be the lexicographic order on $On \times On$, so that
$$(\kappa_0, \lambda_0) \prec (\kappa_1, \lambda_1) \iff (\kappa_0 < \kappa_1) \vee (\kappa_0 = \kappa_1 \wedge \lambda_0 < \lambda_1).$$
If $(\kappa, \lambda) \in \mathrm{dom}(\vec{E})$ and $E = \vec{E}(\kappa, \lambda)$ then
    - $j_E(o)(\kappa) = \lambda$, $j_E(o) \restriction \kappa = o \restriction \kappa$.
    - If $(\alpha, \beta) \prec (\kappa, \lambda)$ then $\vec{E}(\alpha, \beta) = j_E(\vec{E})(\alpha, \beta)$.
4. (non-overlap) If $\kappa_0 < \kappa_1$ and $o(\kappa_1) > \kappa_1 + 1$ then $o(\kappa_0) < \kappa_1$.

The ideology of inner model theory is roughly this; if κ is a large cardinal in V, then we can build $\vec{E}$ reflecting this, and $L[\vec{E}]$ is a well-behaved "L-like" model in which κ is still large. As a more concrete example, let κ be $\mathcal{P}_2\kappa$-hypermeasurable. Then it may be shown that

1. There is an extender sequence $\vec{E}$ such that $o(\kappa) > \kappa^{++}$.
2. In $L[\vec{E}]$, the embedding associated with the extender $\vec{E}(\kappa, \kappa^{++})$ witnesses that κ is $\mathcal{P}_2\kappa$-hypermeasurable.
3. $L[\vec{E}]$ is a model of GCH, has a Δ^1_3 wellorder of the reals, and has a reasonable fine-structure theory.

The key to the good properties of $L[\vec{E}]$ is the Comparison Lemma.

Lemma 1 (Comparison): If $(M, \vec{F})$ and $(N, \vec{G})$ are such that

$$\begin{aligned} M &\models \vec{F} \text{ is an extender sequence} \\ N &\models \vec{G} \text{ is an extender sequence} \end{aligned}$$

and M and N are sufficiently iterable, then we may **compare** the models M and N. That is, we may iterate M by $\vec{F}$ and N by $\vec{G}$ so as to get $(M^*, \vec{F}^*)$ and $(N^*, \vec{G}^*)$, such that one of the sequences F^*, G^* is an initial segment of the other as far as sets in $M^* \cap N^*$ are concerned.

For the benefit of the reader, we collect some information about $L[\vec{E}]$, with (very sketchy) proofs. The theory of $L[\vec{E}]$ was worked out by Mitchell in [8], using the language of "hypermeasures"; here we just translate into the language of extenders. Assume till the end of this section that $V = L[\vec{E}]$, where $\vec{E}$ is an arbitrary extender sequence whose domain is a set.

Lemma 2 (Uniqueness): $\vec{E}(\kappa, \lambda)$ is the only (κ, λ)-extender which has the "coherence" properties demanded in clause 3 of definition 2.

Proof:[Sketch] Let $\vec{E}$ be a minimal counterexample, in the sense that no proper initial segment of $\vec{E}$ falsifies the lemma. Let (κ, λ) be the first place on $\vec{E}$ where the lemma is false.

Let F be an extender differing from $\vec{E}(\kappa, \lambda)$, but having the same coherence property. Compare the ultrapowers $Ult(V, F)$ and $Ult(V, \vec{E}(\kappa, \lambda))$, and argue as in [7] that they iterate to a common model N and that the diagram

$$\begin{tikzcd}
 & N & \\
Ult(V, \vec{E}(\kappa, \lambda)) \arrow[ur, "k"] & & Ult(V, F) \arrow[ul, "l"'] \\
 & V \arrow[ul, "j_{\vec{E}(\kappa,\lambda)}"] \arrow[ur, "j_F"'] &
\end{tikzcd}$$

commutes, where k and l are the maps from the comparison iteration.

Since F has the coherence property, the least point of disagreement between the extender sequences of the two ultrapowers is greater than λ, so that k and l have critical points greater than λ. Using this, we may argue that in fact $F = \vec{E}(\kappa, \lambda)$.

♦

Lemma 3 (Condensation): For θ some large regular cardinal, let $X \prec H_\theta$ be a structure with $V_{\kappa+1} \subseteq X$ and $\vec{E} \in X$. Let X collapse to M and let $\vec{E}$ collapse to $\vec{F}$. Then for all $\eta < o^{\vec{F}}(\kappa)$ we have $\vec{F}(\kappa, \eta) = \vec{E}(\kappa, \eta)$.

Proof: V and M agree to rank $\kappa + 1$, and below critical point κ their sequences agree. Use this to argue (by induction on η) that $\vec{F}(\kappa, \eta)$ is an extender with the same coherence properties as $\vec{E}(\kappa, \eta)$, so that by uniqueness the two are equal.

♦

Lemma 4: GCH holds.

Proof: By induction on the cardinal λ. Suppose we have established GCH for cardinals less than λ, and notice that $|H_\lambda| = \lambda$. Now there are two cases.

1. For all $\alpha < \lambda$ we have $o(\alpha) < \lambda^+$.

   Then any subset of λ can be built into an iterable $M \supseteq H_\lambda$ of cardinality λ, whose extender sequence agrees with $\vec{E}$ at critical points less than λ. If two such models M and N are compared then the iterations on each side fix the powerset of λ, hence one of $\mathcal{P}\lambda \cap M$ and $\mathcal{P}\lambda \cap N$ is a subset of the other. So the powerset of λ is the union of a chain of sets each of cardinality λ, so $2^\lambda = \lambda^+$.

2. For some $\alpha < \lambda$ we have $o(\alpha) \geq \lambda^+$.

   We claim that if $A \subseteq \lambda$, then $A \in Ult(V, \vec{E}(\alpha, \beta))$ for some $\beta < \lambda^+$.

   To see this, let us build A into $X \prec \theta$ (θ large and regular) such that $H_\lambda \subseteq X$, $\vec{E} \in X$, $|X| = \lambda$. Collapsing we get an iterable $M = L_\mu[\vec{F}]$ where $\vec{F}$ agrees with $\vec{E}$ at critical points below α, and (by condensation) $\vec{F}$ is an initial segment of $\vec{E}$ at critical point α. Let $\beta = o^{\vec{F}}(\alpha)$, then $\lambda < \beta < \lambda^+$.

   Now compare M and $Ult(V, \vec{E}(\alpha, \beta))$. Let the iteration maps be

   $$\begin{aligned} i &: M \longrightarrow M^* \\ j &: Ult(V, \vec{E}(\alpha, \beta)) \longrightarrow N \end{aligned}$$

   Both have critical points greater than λ. It may be argued that M^* is an initial segment of N, for otherwise we could iterate M^* to get indiscernibles for N, and hence for V. M can be recovered from M^* so $M \in N$, then by agreement $M \in Ult(V, \vec{E}(\alpha, \beta))$.

   This proves the claim. Now we are done, because in $Ult(V, \vec{E}(\alpha, \beta))$ there are at worst $2^\alpha . \lambda = \lambda$ subsets of λ.

♦

The claim in case 2 of the last proof is also the key to the hypermeasurability of κ in $L[\vec{E}]$.

Lemma 5: Let $o(\kappa) > \kappa^{++}$. Let $E = \vec{E}(\kappa, \kappa^{++})$. Let $j_E : V \longrightarrow M$ be the ultrapower by E. Then $V_{\kappa+2} \subseteq M$.

Proof: By GCH, it is enough to show $\mathcal{P}\kappa^+ \subseteq M$. If $A \subseteq \kappa^+$, the claim from the last lemma shows that for some $\beta < \kappa^{++}$ we have $A \in Ult(V, \vec{E}(\kappa, \beta))$. By coherence $\vec{E}(\kappa, \beta) = j(\vec{E})(\kappa, \beta)$, and we also have that $Ult(V, \vec{E}(\kappa, \beta))$ and $Ult(M, \vec{E}(\kappa, \beta))$ agree to a high rank. So $A \in Ult(M, \vec{E}(\kappa, \beta)) \subseteq M$.

♦

3 Core models

We look briefly at the theory of (non-overlapping) core models, which was developed in tandem with the inner model theory of the last chapter by the same authors (Mitchell, Jensen, Dodd, Koepke). The reader is referred to [5] for details. Our goal is to show that if $V = L[\vec{E}]$ then the construction of the core model and maximal sequence as in [5] will recover V and $\vec{E}$. Combined with some facts about core models, this will let us do an analysis of elementary embeddings (in particular ultrapower maps) definable in certain generic extensions of $L[\vec{E}]$.

The reader who is prepared to take our use of core models on trust should skip ahead to the results at the end of the section, which are all that will be used subsequently.

Claim 1: If $V = L[\vec{E}]$, then there is no inner model of a strong cardinal.

Proof: [Sketch] If there is such an inner model, then there is an inner model of the form $L[\vec{F}]$, where $\vec{F}$ is a sequence with similar properties to those in definition 2, but such that for some κ we have $o^{\vec{F}}(\kappa) = \infty$. That is to say, κ is the largest critical point on $\vec{F}$ and $\vec{F}$ has (κ, λ)-extenders for λ arbitrarily large.

It can be shown that the comparison of this model with V will terminate. This must lead to a situation in which an iterate M of V is a proper initial segment of an iterate N of $L[\vec{F}]$, because the extender sequence $\vec{E}$ does not have unboundedly large extenders on any critical point. But were this situation to arise, we could iterate N to generate indiscernibles for M, so that V would be able to compute a set of indiscernibles for V. Contradiction.

$\blacklozenge$

Theorem 2: Let $V = L[\vec{E}]$. If we perform the construction of the canonical maximal sequence $\vec{F}$ and core model $K[\vec{F}]$ as in [5] then $\vec{F} = \vec{E}$ and $K = V$.

Proof:

It is enough to show that $\vec{F} = \vec{E}$, for then

$$V \supseteq K = K[\vec{F}] \supseteq L[\vec{F}] = V.$$

We will proceed by induction. Suppose that $(\kappa, \nu) \in \mathrm{dom}(\vec{E})$ and we have established that

$$\vec{E} \restriction (\kappa, \nu) = \vec{F} \restriction (\kappa, \nu).$$

Claim 2: $\mathcal{P}\kappa \subseteq K[\vec{F} \restriction (\kappa, \nu)]$.

To see this let $X \subseteq \kappa$. Then $X \in Ult(V, \vec{E}(\kappa, \nu))$, and this shows that X is in the lower part of an iterable premouse over $\vec{E} \restriction (\kappa, \nu) = \vec{F} \restriction (\kappa, \nu)$. Hence $X \in K[\vec{F} \restriction (\kappa, \nu)]$ as claimed.

The model $K[\vec{F} \restriction (\kappa, \nu)]$ agrees with V to rank $\kappa + 1$, their extender sequences are the same, so by the Uniqueness Lemma we are forced to choose $\vec{F}(\kappa, \nu) = \vec{E}(\kappa, \nu)$.

To complete an inductive proof that $\vec{F} = \vec{E}$ we need to show that the construction of $\vec{F}$ cannot go wrong in either of the following ways; by putting an extender at $(\kappa, o^{\vec{E}}(\kappa))$, or by putting an extender on a critical point between κ and the next λ with $o^{\vec{E}}(\lambda) > \lambda + 1$.

Claim 3: If $\nu = o(\kappa)$ then there is no (κ, ν)-extender in $L[\vec{E}]$ which coheres $E \restriction (\kappa, \nu)$.

The proof of the claim is very similar to that of the Uniqueness Lemma. If it fails, let $\vec{E}$ be a minimal counterexample. Let E be the first (κ, ν) extender providing a counterexample, and compare the models V and $Ult(V, E)$. We get a diagram

$$\begin{tikzcd}
 & N & \\
V \arrow[ur] \arrow[rr] & & Ult(V, E) \arrow[ul]
\end{tikzcd}$$

which commutes exactly as in the Uniqueness Lemma. But this cannot be because V and $Ult(V, E)$ agree to such an extent that the comparison maps have critical point larger than κ, while the map from V to $Ult(V, E)$ has critical point κ.

It follows from this that $\vec{F}$ cannot put an extender at $(\kappa, o^{\vec{E}}(\kappa))$. The point is that V agrees with $K[\vec{F} \upharpoonright (\kappa, o^{\vec{E}}(\kappa))]$ to rank $\kappa + 1$, the extender sequences agree, so that such an extender would be a counterexample to the previous claim.

The last ingredient in the proof that $\vec{F} = \vec{E}$ is the following claim.

Claim 4: If $\nu = o^{\vec{E}}(\kappa)$ then there does not exist an extender $E \in L[\vec{E}]$ which coheres $\vec{E} \upharpoonright (\kappa, \nu)$ and has critical point between κ and the next λ with $o^{\vec{E}}(\lambda) > \lambda + 1$.

The proof is very similar to that of the last claim. Assume that $\vec{E}$ is a minimal counterexample, with E the first extender witnessing this. Co-iterate as in the previous lemma, to get a commutative triangle exactly as before. Again there is a problem with critical points.

The proof is now routine, an induction on $\mathrm{dom}(\vec{E})$ shows that we have $\mathrm{dom}(\vec{F}) = \mathrm{dom}(\vec{E})$ and $\vec{F} = \vec{E}$. Hence $V = L[\vec{E}] = K[\vec{F}]$ and we are done.

♦

We now recall some results about the non-overlapping core model, which are true under the assumption there is no inner model of a strong cardinal.

Fact 1: Let $\vec{F}$ be the canonical maximal sequence, and $K[\vec{F}]$ the associated core model.

1. If $\pi : K[\vec{F}] \longrightarrow W$ is an elementary embedding into a transitive class W then W is a normal iteration of $K[\vec{F}]$ by $\vec{F}$.
2. If G is set-generic, then $\vec{F}$ and $K[\vec{F}]$ as defined in $V[G]$ coincide with $\vec{F}$ and $K[\vec{F}]$ as defined in V.

Putting everything together, we can finally get the result that we need to analyse measures in generic extensions of $L[\vec{E}]$.

Theorem 3: Let $V = L[\vec{E}]$, where $\vec{E}$ is an extender sequence. Let G be set-generic over V and let $U \in V[G]$ be some normal measure on some cardinal κ. If $j_U : V[G] \longrightarrow Ult(V[G], U)$ is the ultrapower map associated with U, then $j_U \restriction V : V \longrightarrow j_U(V)$ is a normal iteration of V by $\vec{E}$.

Proof: The core of $V[G]$ equals the core of V equals V.

♦

4 The model

Theorem 4: Let GCH hold, and let $j : V \longrightarrow M$ be such that $V_{\kappa+3} \subseteq M$, where $\kappa = \mathrm{crit}(j)$. Suppose [1] that j is generated by a (κ, κ^{+++})-extender.

Let $\mathbb{P}$ be the Reverse Easton iteration of length $\kappa+1$ in which we force with $Add(\alpha, \alpha^+) \times Add(\alpha^+, \alpha^{+++})$ (as defined in $V[G_\alpha]$) at each strong inaccessible $\alpha \leq \kappa$. Let G be $\mathbb{P}$-generic over V.

Then in the generic extension κ is $\mathcal{P}_2\kappa$-hypermeasurable, and we have $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \kappa^{+++}$.

Proof: We will use a number of forcing tricks without much explanation; the reader is referred to [2] for the details, and to [1] for general information about Reverse Easton forcing. Several similar arguments may also be found in [3] and [4].

[1] If such a j exists, then there is such a j which arises from such an extender.

We follow the standard strategy of building in $V[G]$ an embedding which extends j, and which witnesses that κ has the desired property in $V[G]$.

We shall adopt the convention that $Add(\alpha, \beta)$ adds β generic functions from α to α. Often we think of the generic object as a single function from $\alpha \times \beta$ to α.

We let G_α denote the generic object for the forcing up to stage α, and g_α the generic for the forcing at stage α. We break up g_α as $g^0_\alpha \times g^1_\alpha$ where g^0_α is generic for $Add(\alpha, \alpha^+)$ and g^1_α is generic for $Add(\alpha^+, \alpha^{+++})$. So $G = G_\kappa * g_\kappa = G_\kappa * g^0_\kappa * g^1_\kappa$.

As an aid to constructing generic objects, we factor j through the ultrapower by a normal measure in the standard way. This gives a diagram

$$\begin{tikzcd}
V \arrow[rr, "j"] \arrow[dr, "i"'] & & M \\
& M_0 \arrow[ur, "k"'] &
\end{tikzcd}$$

where as usual $U = \{ \ X \subseteq \kappa \mid \kappa \in j(X) \ \}$, i is the ultrapower map from V to $M_0 = Ult(V, U)$, and $k : [F]_U \longmapsto j(F)(\kappa)$.

Let $\lambda = (\kappa^{+++})_{M_0}$. We note for later use that

$$\begin{aligned}
M &= \{ \ j(F)(a) \mid F \in V, \mathrm{dom}(F) = [\kappa]^{<\omega}, a \in [\kappa^{+++}]^{<\omega} \ \} \\
&= \{ \ k(G)(b) \mid G \in M_0, \mathrm{dom}(G) = [\lambda]^{<\omega}, a \in [\kappa^{+++}]^{<\omega} \ \}
\end{aligned}$$

As usual, in $V[G_\kappa]$ the models $M[G_\kappa]$ and $M_0[G_\kappa]$ are closed under κ-sequences. As a corollary we may use g_κ as the generic at stage κ in the iteration $j(P)$. In $i(P)$ we may use

$$g^*_\kappa = g^0_\kappa \times (g^1_\kappa \restriction \kappa^+ \times \lambda).$$

In $M_0[G_\kappa]$ we define a certain term poset. The terms are names in the forcing $Add(\kappa, \kappa^+) \times Add(\kappa^+, \lambda)$ for elements of the forcing

$$R_{\kappa+1,i(\kappa)} * Add(i(\kappa), i(\kappa^+))).$$

By the standard chain condition and closure arguments we may find $X \in V[G_\kappa]$ generic for this term forcing. $k : M_0 \longrightarrow M$ extends easily to a map $k : M_0[G_\kappa] \longrightarrow M[G_\kappa]$, and we may transfer X along k to get a generic X^+ for the term forcing whose elements are names in $Add(\kappa, \kappa^+) \times Add(\kappa^+, \kappa^{+++})$ for elements of $R_{\kappa+1,j(\kappa)} * Add(j(\kappa), j(\kappa^+))).$

Now we interpret this term generic using g_κ. This leaves us with

$$H * a \in V[G_\kappa][g_\kappa]$$

which is generic over $M[G_\kappa][g_\kappa]$ for $R_{\kappa+1,j(\kappa)} * Add(j(\kappa), j(\kappa^+)))$. It is now easy to build a map

$$j : V[G_\kappa] \longrightarrow M[G_\kappa][g_\kappa][H].$$

Now we alter a to get a new generic a^* such that $j``g^0_\kappa \subseteq a^*$. This works because $M[G_\kappa][g_\kappa]$ is closed under κ-sequences inside $V[G_\kappa][g_\kappa]$, and we only need alter any one condition at κ many places.

This enables us to build

$$j : V[G_\kappa][g^0_\kappa] \longrightarrow M[G_\kappa][g_\kappa][H][a^*].$$

Now we may finish by transferring g^1_κ along j, which is legitimate because g^1_κ is generic over $V[G_\kappa][g^0_\kappa]$ for κ^+-dense forcing. This gives an embedding

$$j : V[G] \longrightarrow M[j(G)]$$

defined in $V[G]$.

As the cardinal arithmetic claim is trivial, we need only verify that this embedding witnesses $\mathcal{P}_2\kappa$-hypermeasurability. As $2^\kappa = \kappa^+$ it suffices to check that

$$\mathcal{P}\kappa^+ \cap V[G] \subseteq M[j(G)].$$

But $\mathcal{P}\kappa^+ \cap V[G] \subseteq M[G]$ by a standard chain condition argument, and $M[G] \subseteq M[j(G)]$ by construction.

♦

5 Building measures

In this section we show how to construct measures in the generic extension. The methods used here are very similar to those of [3].

Theorem 5: Let $i : V \longrightarrow N$ be an elementary embedding such that

1. $\kappa = \mathrm{crit}(i)$.
2. $i(\kappa^+) < \kappa^{++}$.
3. $N = \{ \ i(f)(a) \mid f \in V, \mathrm{dom}(f) = [\kappa]^{<\omega}, a \in [i(\kappa)]^{<\omega} \ \}$.
4. ${}^\kappa N \subseteq N$.

Let $\mathbb{P}$ be the forcing iteration defined in the last section, and let G be $\mathbb{P}$-generic.

Then in $V[G]$ there are κ^{+++} ultrafilters U with the property that the ultrapower map $j_U : V[G] \longrightarrow Ult(V[G], U)$ is an extension of i.

Proof:

Let G_α, g_α, g^i_α be as in the proof of theorem 4. The iterations $\mathbb{P}$ and $i(\mathbb{P})$ agree up to κ, and $V[G_\kappa] \models {}^\kappa N[G_\kappa] \subseteq N[G_\kappa]$. At κ, the iteration $i(\mathbb{P})$ requires a generic over $N[G_\kappa]$ for

$$Q = Add(\kappa, \kappa^+) \times Add(\kappa^+, \eta)$$

where $\eta = (\kappa^{+++})_N$, and by closure we can see Q as an initial segment of the forcing which $\mathbb{P}$ does at κ.

Claim 5: In $V[G]$ there are exactly κ^{+++} generics h for Q over $N[G_\kappa]$ with the property that

$$V[G] \models {}^\kappa N[G_\kappa][h] \subseteq N[G_\kappa][h].$$

Proof: Q has cardinality κ^+ and $2^{\kappa^+} = \kappa^{+++}$ in $V[G]$, so there are at most κ^{+++} such generics. We know that (in ordinal arithmetic) $\kappa^{+++} = \eta.\kappa^{+++}$, so we can break up g^1_κ into blocks of length η; let h^1_α be the α'th block. It is easily checked that $h = g^0_\kappa * h^1_\alpha$ is generic (by Easton's lemma) and has the desired property (as all κ-sequences from $V[G]$ are actually in $V[g_\kappa][g^0_\kappa]$).

The claim is proved

$\blacklozenge$

Now fix a generic h as above. Using the closure of $N[G_\kappa][h]$ in $V[G]$, we may prove by the standard methods the following claim.

Claim 6: In $V[G]$ there are exactly κ^{+++} generics H for the forcing $\mathbb{R}_{\kappa+1,i(\kappa)}$ over the model $N[G_\kappa][h]$.

The key point here is that $i(\kappa)$ has cardinality κ^+, so that we may enumerate the dense sets we must meet in order type κ^+; the closure hypothesis then enables us to meet them, and we have so much latitude in the construction that we may do it in $2^{\kappa^+} = \kappa^{+++}$ different ways.

Fix such an H. It is easy to extend i to an embedding

$$i : V[G_\kappa] \longrightarrow N[G_\kappa][h][H].$$

Notice also that $V[G] \models {}^\kappa N[G_\kappa][h][H] \subseteq N[G_\kappa][h][H]$.

Claim 7: In $V[G]$ there are exactly κ^{+++} generics a for the notion of forcing $Add(i(\kappa), i(\kappa^+))$ over the model $N[G_\kappa][h][H]$, with the following properties.

1. $i``g^0_\kappa \subseteq a$.
2. Let $i : V[G_\kappa][g^0_\kappa] \longrightarrow N[G_\kappa][h][H][a]$ be the natural extension of i. Then for each $\beta < i(\kappa)$ there is $g : \kappa \longrightarrow \kappa$ in $V[G_\kappa][g^0_\kappa]$ such that $i(g)(\kappa) = \beta$.

Proof: Just as in the construction of H above, we may use closure to build many generics a, using the fact that $i(\kappa^+) < \kappa^{++}$. We shall now describe how to alter such a generic a so as to get the properties demanded.

Fix an enumeration of $i(\kappa)$ in order type β, say $\langle f(\beta) : \beta < \kappa^+ \rangle$. Now define a^* by altering a on the set $(\kappa + 1) \times i``\kappa^+$, while leaving it unchanged elsewhere. For all $\beta < \kappa^+$ let

$$a^*(\alpha, i(\beta)) = g^0_\kappa(\alpha, \beta)$$

if $\alpha < \kappa$, and

$$a^*(\kappa, i(\beta)) = f(\beta).$$

a^* gives rise to a generic filter; the key points are that each condition need only be altered at κ many places, and that the forcing is so highly closed that small perturbations do not kill genericity. We identify a^* with the associated filter.

It is clear from the construction that $i``g^0_\kappa \subseteq a^*$, so that as usual we may build

$$i : V[G_\kappa][g^0_\kappa] \longrightarrow N[G_\kappa][h][H][a^*].$$

To see that a^* has the other property demanded of it, define $g_\beta \in V[G_\kappa][g^0_\kappa]$ by $g_\beta(\alpha) = g^0_\kappa(\alpha, \beta)$. Now $i(g_\beta)(\kappa) = a^*(\kappa, i(\beta)) = f(\beta)$.

It is easy to check that we may do this construction in κ^{+++} many ways. The claim is proved.

♦

Fix a as in the previous claim. It is now routine to transfer g^1_κ along i, to get at last a map

$$i : V[G] \longrightarrow N[G_\kappa][h][H][a][i(g^1_\kappa)].$$

We will be done once we have the following claim.

Claim 8: The map $i : V[G] \longrightarrow N[G_\kappa][h][H][a][i(g^1_\kappa)]$ arises as the ultrapower map associated with an ultrafilter U. Moreover, different ultrafilters arise from different choices for (h, H, a).

Proof: It is easy to see that

$$N[G_\kappa][h][H][a][i(g^1_\kappa)] = \{\ i(f)(a) \mid f \in V[G], a \in [i(\kappa)]^{<\omega}\ \}.$$

Each $\beta < i(\kappa)$ has form $i(g)(\kappa)$ for some $g \in V[G]$, so actually

$$N[G_\kappa][h][H][a][i(g^1_\kappa)] = \{\ i(f)(\kappa) \mid f \in V[G], \mathrm{dom}(f) = \kappa\ \}.$$

It follows that i arises from an ultrafilter U. For the other claim, simply notice that we can rebuild (h, H, a) from U in $V[G]$ because

$$G_\kappa * h * H * a = j_U(G_\kappa * g^0_\kappa).$$

♦

The theorem is proved.

♦

For our later convenience, we introduce some notation for the ultrafilters that arise in this way. Let V and G be as in the last result.

Definition 3: In $V[G]$, suppose that

1. $i : V \longrightarrow N$ is an embedding with $\text{crit}(i) = \kappa$.
2. h is a generic over $V[G_\kappa]$ for Q, and $V[G] \models {}^\kappa N[G_\kappa][h] \subseteq N[G_\kappa][h]$.
3. H is generic over $N[G_\kappa][h]$ for $\mathbb{R}$.
4. a is generic over $N[G_\kappa][h][H]$ for $Add(i(\kappa, \kappa^{++}))$ and $i``g^0_\kappa \subseteq a$.
5. The unique map

$$i : V[G] \longrightarrow N[G_\kappa][h][H][a][i(g^1_\kappa)]$$

extending $i : V \longrightarrow N$ and having $i(G) = G_\kappa * h * H * a * i(g^1_\kappa)$ is the ultrapower by a normal measure.

Then $U(i, h, H * a)$ is the normal measure alluded to in the last clause.

6 Closure

In the construction of theorem 5 for generating ultrafilters, it was crucial that the target model of the embedding i be closed under κ-sequences, where $\kappa = \text{crit}(i)$. In this section we prove some results about the closure of the target model of an elementary embedding. We then prove a theorem which says essentially that we need not worry about embeddings with bad closure, when we come to classifying the measures on κ in our final model.

Theorem 6: Let $V = L[\vec{E}]$, where $\vec{E}$ is an extender sequence. Suppose κ is such that $(\kappa, \kappa^{++}) \in \text{dom}(\vec{E})$. Then there are many $\eta < \kappa^{++}$ such that the ultrapower of V by $\vec{E}(\kappa, \eta)$ is not closed under ω-sequences.

Proof: Let $E = \vec{E}(\kappa, \kappa^{++})$. Since $V_{\kappa+2} \subseteq Ult(V, E)$ and GCH holds, we have that for every $\beta < \kappa^{++}$

$$E \restriction [\beta]^{<\omega} \in Ult(V, E).$$

Fix some large θ. We build a chain $\langle X_\alpha : \alpha < \kappa^{++} \rangle$ of elementary substructures $X_\alpha \prec H_\theta$ with the properties

1. $\vec{E}, \kappa, \kappa^{++} \in X_0$.
2. $V_{\kappa+1} \subseteq X_0$.
3. $X_\alpha \subseteq X_{\alpha+1}$ for all α, $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$ for limit λ.
4. $|X_\alpha| = \kappa^+$ for all α.
5. $X_\alpha \cap \kappa^{++} \in \kappa^{++}$.
6. $X_\alpha \cap \kappa^{++} < X_{\alpha+1} \cap \kappa^{++}$, for all α.

Now let $\eta_\alpha = X_\alpha \cap \kappa^{++}$. If X_α is collapsed to a transitive structure M_α, then κ^{++} collapses to $\eta_\alpha = (\kappa^{++})_{M_\alpha}$. What is more the Condensation Lemma implies that E collapses to $E_\alpha = \vec{E}(\kappa, \eta_\alpha)$.

By elementarity,

$$M_\alpha \models \forall \beta < \eta_\alpha \; E_\alpha \restriction [\beta]^{<\omega} \in Ult(M_\alpha, E_\alpha).$$

As $V_{\kappa+1} \subseteq M_\alpha$ this implies that in fact

$$\forall \beta < \eta_\alpha \; E_\alpha \restriction [\beta]^{<\omega} \in Ult(V, E_\alpha).$$

Now let α be any ordinal in κ^{++} of cofinality ω. By construction the sequence $\vec{\eta}$ is continuous, so η_α has cofinality ω. We claim that $Ult(V, E_\alpha)$ is not closed under ω-sequences. For were it so, we would have $E_\alpha \in Ult(V, E_\alpha)$; but this cannot be, as an extender can never be in its own ultrapower [6].

♦

Next we prove a result providing target models which **are** closed under κ-sequences. We need a preliminary lemma.

Lemma 6: Let E be a (κ, η) extender, let $M = Ult(V, E)$. Then ${}^{\kappa}M \subseteq M$ if and only if ${}^{\kappa}\eta \subseteq M$.

Proof: One direction is trivial. For the other, suppose that ${}^{\kappa}\eta \subseteq M$. Let $\langle j_E(F_\alpha)(a_\alpha) : \alpha < \kappa \rangle$ be some κ-sequence of elements of M, where for each $\alpha < \kappa$ we have $\mathrm{dom}(F_\alpha) = [\kappa]^{<\omega}$ and $a_\alpha \in [\eta]^{<\omega}$. Let $F(\alpha, x) = F_\alpha(x)$, and observe that $j_E(F) \in M$ and by hypothesis $\langle a_\alpha : \alpha < \kappa \rangle \in M$. Now if $\alpha < \kappa$

$$j_E(F)(\alpha, a_\alpha) = j_E(F_\alpha)(a_\alpha),$$

so that the κ-sequence $\langle j_E(F_\alpha)(a_\alpha) : \alpha < \kappa \rangle$ is a member of M.

♦

We notice some useful corollaries.

Corollary 1: If $V = L[\vec{E}]$ where $\vec{E}$ is an extender sequence, $\alpha < \beta < o(\kappa)$, then V and $Ult(V, \vec{E}(\kappa, \beta))$ agree as to whether the ultrapower by $\vec{E}(\kappa, \alpha)$ is closed under κ-sequences.

Proof: $Ult(V, \vec{E}(\kappa, \alpha))$ and $Ult(Ult(V, \vec{E}(\kappa, \beta)), \vec{E}(\kappa, \alpha))$ agree to rank $\alpha+1$, so one model contains ${}^{\kappa}\alpha$ exactly when the other model does.

♦

Corollary 2: If i is a normal iteration of V, E is a (κ, η)-extender which is the first extender applied in i, and the target model of i is closed under κ-sequences, then $Ult(V, E)$ is closed under κ-sequences.

Proof: If not, then ${}^{\kappa}\eta$ is not contained in $Ult(V, E)$. But $Ult(V, E)$ and the target model of i agree to rank $\eta + 1$, so this is absurd.

♦

Theorem 7: Let $V = L[\vec{E}]$, where $\vec{E}$ is an extender sequence. Suppose κ is such that $(\kappa, \kappa^{++}) \in \mathrm{dom}(\vec{E})$. Then there are cofinally many $\eta < \kappa^{++}$ such that the ultrapower of V by $\vec{E}(\kappa, \eta)$ is closed under κ-sequences.

Proof: Let X_α, M_α, E_α be as in the proof of the last theorem, with the additional proviso that ${}^\kappa X_\alpha \subseteq X_\alpha$ for successor ordinals α. Let $\alpha < \kappa^+$ be a point of cofinality κ^+. It is easy to see that ${}^\kappa\eta_\alpha \subseteq M_\alpha$. By elementarity

$$M_\alpha \models {}^\kappa\eta_\alpha \subseteq Ult(M_\alpha, E_\alpha),$$

so by the same argument as was used in the last theorem ${}^\kappa\eta_\alpha \subseteq Ult(V, E_\alpha)$. So ${}^\kappa Ult(V, E_\alpha) \subseteq Ult(V, E_\alpha)$.

♦

The next result will be used to show that we need not be concerned with these non-closed ultrapowers when it comes to classifying the measures in our final model.

Theorem 8: Let κ be some measurable cardinal, let $\mathbb{P}$ be the forcing iteration of section 4.

Let $j : V \longrightarrow M$ be an embedding into an inner model M, and suppose $\kappa = \text{crit}(j)$. Suppose that G is $\mathbb{P}$-generic, and in $V[G]$ there is a measure U on κ such that $j_U^{V[G]} \restriction V = j$.

Then $V \models {}^\kappa M \subseteq M$.

Proof: If $f \in V$, $f : \kappa \longrightarrow \kappa$, then $f = j(f) \restriction \kappa$ so that $f \in M$.

Now fix $g \in V$, $g : \kappa \longrightarrow ON$. $g \in V[G]$, so by the standard closure facts about ultrapowers $g \in Ult(V[G], U)$. $Ult(V[G], U)$ is $M[H]$, where H is generic for $j(\mathbb{P})$. By the usual arguments $g \in M[h]$, where h is the initial segment of H generic for a κ^+-c.c. initial segment of $j(\mathbb{P})$ (consisting of the part which adds subsets to α for $\alpha \leq \kappa$).

Now using the chain condition in the standard way, we may find in M a function $G : \kappa \times \kappa \longrightarrow ON$ such that

$$\forall\alpha\, \exists\beta\; g(\alpha) = G(\alpha, \beta).$$

As $g \in V$ and $M \subseteq V$ we may find in V a function $f : \kappa \longrightarrow \kappa$ such that $g(\alpha) = G(\alpha, f(\alpha))$ for all α. By the remark at the beginning of the proof $f \in M$, and hence $g \in M$.

♦

7 Classification and analysis of measures

With the machinery of the preceding sections in place, we are finally ready to classify the normal measures on κ and the Mitchell ordering between them in a certain model where κ is $\mathcal{P}_2\kappa$-hypermeasurable and $2^{2^\kappa} > (2^\kappa)^+$.

Theorem 9: Let $V = L[\vec{E}]$, where for some κ we have $(\kappa, \kappa^{++}) \in \mathrm{dom}(\vec{E})$. Let $\mathbb{P}$ be the forcing iteration of length $\kappa + 1$ defined in section 4, and let G be $\mathbb{P}$-generic over V.

Then in the model $V[G]$

1. κ is $\mathcal{P}_2\kappa$-hypermeasurable.
2. $2^\kappa = \kappa^+$, $2^{\kappa^+} = \kappa^{+++}$.
3. Every normal measure on κ has form $U(i, h, H * a)$, where $i : V \longrightarrow N$ is a finite normal iteration of V by $\vec{E}$, $V \models {}^\kappa N \subseteq N$, and $i(\kappa) < \kappa^{++}$.

Proof: This is mostly a summary of results already proved. The third claim is the only one with any novelty.

Let $U \in V[G]$ be a measure on κ in $V[G]$, giving rise to an embedding $j_U : V[G] \longrightarrow Ult(V[G], U)$. Let $i = j \restriction V$, and let

$$j_U(G) = G_\kappa * h * H * a * j_U(g^1_\kappa).$$

As $V = L[\vec{E}]$, the results from the section on core model theory show that i is an iteration of V by $\vec{E}$.

Claim 9: i is a finite iteration.

Proof: If not, then it is easy to see that the sequence $\vec{x}$ consisting of the first ω critical points is not a member of the target model N. By closure of the forcing, $\vec{x}$ cannot be in $N[G_\kappa][h]$. But i is defined in $V[G]$ and $V[G] \models {}^\kappa N[G_\kappa][h] \subseteq N[G_\kappa][h]$, so this is absurd and the claim is proved.

♦

It now follows from the section on closure of models that $V \models {}^{\kappa}N \subseteq N$. j_U is an ultrapower map so $i(\kappa) = j_U(\kappa) < \kappa^{++}$. j_U is an extension of i and $j_U(G) = G_\kappa * h * H * a$, so it must be the case that $U = U(i, h, H * a)$.

♦

Notice that whenever i is an iteration as above and (h, H, a) are appropriate generics then $U(i, h, H * a)$ is a normal measure in $V[G]$. So we have a complete classification of the measures in $V[G]$.

It remains to analyse the ordering $\lhd$ on these measures. Unfortunately the analysis here is not as complete as was achieved in [3]

We start with a general analysis of subsets of κ^+ in $V[G]$. Observe that

- An ultrafilter on κ in $V[G]$ may be coded as a subset of κ^+, since $2^\kappa = \kappa^+$ in $V[G]$.
- κ is $\mathcal{P}_2\kappa$-hypermeasurable in $V[G]$, so we should expect that each subset of κ^+ should appear in the ultrapower of $V[G]$ by some normal measure on κ.

If $x \subseteq \kappa^+$ with $x \in V[G]$, then by the usual chain condition arguments $x \in V[G_\kappa][g^0_\kappa][h]$, where h is $Add(\kappa^+, 1)$-generic and for some $\delta < \kappa^{+++}$ we have $h \in V[G_\kappa][g^0_\kappa][g^1_\kappa \restriction \delta \times \kappa]$. What is more $x = \dot{\tau}^{G_\kappa * g^0_\kappa * h}$, where $\dot{\tau}$ can be construed as a subset of κ^+.

By the analysis we did in section 2, $\dot{\tau}$ is in $Ult(V, \vec{E}(\kappa, \eta))$ for all sufficiently large $\eta < \kappa^{++}$. From the results in section 6, we may as well take it that $\dot{\tau} \in N = Ult(V, \vec{E}(\kappa, \eta))$ where $V \models {}^{\kappa}N \subseteq N$. That is $x \in N[G_\kappa][g^0_\kappa][h]$.

We may now use the methods of of section 5 to expand this model to one of the form $Ult(V[G], U)$ for some measure U.

This analysis leaves some questions open:

1. Given $\zeta < \kappa^{++}$, is there some x which only appears in extensions of $Ult(V, \vec{E}(\kappa, \eta))$ as above for $\eta > \zeta$?
2. if x appears in some such extension, what can we say about the set of extensions in which it appears?

Some information about these questions will come from a more specific analysis of measures.

In what follows, let i and j be two finite normal iterations of V by E, with target models closed under κ-sequences, and with $i(\kappa)$ and $j(\kappa)$ both less than κ^{++}. Let $j_{01} : V \longrightarrow N_1$ be the first step in the iteration j.

Let

$$\begin{aligned} U_0 &= U(i, h_0, H_0 * a_0) \\ U_1 &= U(j, h_1, H_1 * a_1) \end{aligned}$$

for appropriate generics (h_0, H_0, a_0) and (h_1, H_1, a_1).

Lemma 7: $V[G] \models U_0 \lhd U_1$ iff $U_0 \in N_1[G_\kappa][h_1]$.

Proof: N_1 and $j(V)$ agree to a high rank, $G_\kappa * h_1$ is generic over $j(V)$ for small forcing, so it is generic over N_1 also and the extensions $j(V)[G_\kappa][h_1]$ and $N_1[G_\kappa][h_1]$ agree to a high rank. The usual closure arguments show that $j(V)[G_\kappa][h_1]$ and $Ult(V[G], U_1)$ agree to a high rank, so that the models $Ult(V[G], U_1)$ and $N_1[G_\kappa][h_1]$ agree to a high rank, in particular they contain the same measures on κ.

♦

Theorem 10: If $U_0 \lhd U_1$ then

1. $h_0, H_0 \in N_1[G_\kappa][h_1]$.
2. $i \restriction N_1$ is an iteration of N_1 by $j_{01}(\vec{E})$.

Proof: Let $N^* = N_1[G_\kappa][h_1]$. We know that $U_0 \in N^*$ and $V[G] \models {}^\kappa N^* \subseteq N^*$, so that $j_{U_0}^{V[G]} \restriction N^* = j_{U_0}^{N^*}$.

For the first claim observe that $j_{U_0}(G_\kappa) = G_\kappa * h_0 * H_0$, so that N^* can reconstruct h_0 and H_0.

For the second claim, we know that N_1 is the core of N^* and that N^* can define $j_{U_0} \restriction N^*$, so $j_{U_0} \restriction N_1$ must be an iteration of N_1. But $N_1 \subseteq V$ and $j_{U_0} \restriction V$ is i, so $i \restriction N_1$ is an iteration of N_1.

♦

The following result is proved in [3] in the context of measures, and the proof works equally well for iterations of extenders in the context of $L[\vec{E}]$.

Theorem 11: Let $j_{01} : V \longrightarrow N_1$ be the ultrapower map associated with the extender $\vec{E}(\kappa, \beta)$. Let i be a normal iteration with critical point κ, in which $i_{0n}(\vec{E})(\kappa_n, \lambda_n)$ is applied to $M_n = i_{0n}(V)$ at stage n. Then $i \restriction N_1$ is an internal iteration of N_1 if and only if for all n either $\kappa_n < i_{0n}(\kappa)$, or $\kappa_n = i_{0n}(\kappa)$ and $\lambda_n < i_{0n}(\beta)$.

This answers one of the questions we posed about subsets of κ^+ earlier on. If an ultrafilter U arises from an iteration which starts off with $E(\kappa, \alpha)$, and $U \lhd V$, then the iteration associated with V must start with $E(\kappa, \beta)$ for $\beta > \alpha$.

We can also say something about the measures on κ which occur in models of the form $N_1[G_\kappa][h_1]$

Theorem 12: Suppose $N = Ult(V, \vec{E}(\kappa, \eta))$ for some $\eta < \kappa^{++}$, ${}^\kappa N \subseteq N$, and h is generic over $N[G_\kappa]$ for $Add(\kappa, \kappa^+) \times Add(\kappa^+, \kappa^{+++})$ as computed in $N[G_\kappa]$. If i is a finite internal iteration of N which has a target closed under κ-sequences, and $i(\kappa) < (\kappa^{++})_N$, then there are κ^+ measures $U \in N[G_\kappa][h]$ such that $j_u \restriction N = i$.

Proof: The analysis we have done for measures in the model V works equally well for the model N. There are κ^+ measures for each i because $|\kappa_N^{+++}| = \kappa^+$ in V.

♦

We can finally derive some consequences for the structure of the Mitchell ordering in $V[G]$.

Definition 4: Let $U \in V[G]$ be a normal measure on κ. Then the **level** of U is α, where $\vec{E}(\kappa, \alpha)$ is the first extender applied in the iteration of V induced by U. The **block** of U is h, where h is the generic object at κ in the forcing iteration $j_U(G_\kappa)$.

Notice that if U has level α and block h, the $\lhd$-ancestors of U are precisely the measures in $Ult(V, \vec{E}(\kappa, \alpha))[G_\kappa][h]$.

The following theorem is just a summary of what has been proved.

Theorem 13: In the model $V[G]$

1. If $U \lhd V$, the level of U is less than the level of V.
2. If $U \lhd V$, $U \lhd W$ for all W with the same level and block as V.
3. There are precisely κ^+ measures with a prescribed block and level.
4. There are κ^{++} possible levels.
5. For each level, there are κ^{+++} possible blocks.
6. The ultrafilters of a prescribed level and block have at most κ^+ ancestors.
7. For each U, there is a final segment of levels containing a block which has U as ancestor.

References

[1] J. E. Baumgartner, Iterated forcing. In *Surveys in set theory,* pp 1–55, Cambridge, 1983. Cambridge University Press.

[2] J. Cummings, A model in which GCH holds at successors but fails at limits. *Transactions of the AMS*, 329:1–39, 1992.

[3] J. Cummings, Possible behaviours for the Mitchell ordering I. To appear in *Annals of Pure and Applied Logic.*

[4] J. Cummings, Strong ultrapowers and long core models. *Journal of Symbolic Logic* 58, 1993.

[5] P. Koepke, An introduction to extenders and core models for extender sequences. In *Logic Colloquium '87,* pp 137–182, Amsterdam, 1989. North-Holland.

[6] D. Martin and J. Steel, Iteration trees. To appear.

[7] W. Mitchell, Sets constructible from sequences of ultrafilters. *Journal of Symbolic Logic,* 39:57–66, 1974.

[8] W. Mitchell, Hypermeasurable cardinals. In *Logic Colloquium '78,* pp 303–316, Amsterdam, 1979. North-Holland.

[9] Solovay, Reinhardt and Kanamori, Strong axioms of infinity and elementary embeddings. *Annals of Mathematical Logic,* 13:73–116, 1978.

[10] Solovay, Berkeley course notes.