Possible behaviours for the Mitchell ordering

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Abstract

We use a mixture of forcing and inner models techniques to get some results on the possible behaviours of the Mitchell ordering at a measurable κ.

1 Introduction

The Mitchell ordering on normal measures was invented by Mitchell [3] as a tool in his study of inner models for large cardinals.

Definition 1: Let κ be measurable, let U_0 and U_1 be normal measures on κ. Then $U_0 \lhd U_1$ if and only if $U_0 \in Ult(V, U_1)$, the ultrapower of V by U_1.

The following facts are standard.

- $\lhd$ is transitive.
- $\lhd$ is well-founded.
- $\lhd$ is strict.
- An ultrafilter has at most 2^κ ancestors in the ordering $\lhd$.

Definition 2: $o(\kappa)$ is the height of the well-founded relation $\lhd$.

Notice that we must have $o(\kappa) \leq (2^\kappa)^+$.

Much is known about the possible behaviours of $\lhd$. For example

- Mitchell has shown [3] that in a highly structured inner model we can have GCH holding and $o(\kappa) = \kappa^{++}$, with $\lhd$ being a linear ordering.

- Baldwin has shown [6] that from suitable hypotheses we can have models in which $\lhd$ is a given prewellordering of cardinality less than κ.

- If κ is the critical point of $j : V \longrightarrow M$ such that $V_{\kappa+2} \subseteq M$, then we may show that every element of $V_{\kappa+2}$ is in $Ult(V, U)$ for some U on κ. In particular any 2^κ measures on κ will have an upper bound in the ordering $\lhd$. What is more, for any particular U there will only be 2^κ elements of $V_{\kappa+2}$ in $Ult(V, U)$, so that there must be 2^{2^κ} measures on κ. If it happens that $2^{2^\kappa} > (2^\kappa)^+$ then $\lhd$ cannot be linear, and it is not clear what the structure of $\lhd$ will be.

  This question is addressed in [1].

In this paper we will produce a model in which κ is measurable, and all measures on κ may be divided into "blocks" in the following way:

1. For each $\alpha < o(\kappa)$ and $\beta \in (\alpha, o(\kappa)) \cup \infty$ there is a block $M(\alpha, \beta)$.

2. All the measures in $M(\alpha, \beta)$ have height α in the Mitchell ordering.

3. $M(\alpha, \beta)$ has cardinality κ^+ if $\beta \in (\alpha, o(\kappa))$, and cardinality κ^{++} if $\beta = \infty$.

4. For $U \in M(\alpha, \beta)$ and $V \in M(\gamma, \delta)$, $U \lhd V$ iff $\beta \leq \gamma$ (with the convention that ∞ is bigger than any ordinal).

2 Preliminaries

In this paper we will use large cardinals and forcing to produce some models where the Mitchell ordering is rather complex. In the interests of clarity and self-containedness we have collected various key facts in this section, facts which we will use repeatedly in the sequel. None of them are due to us; in many cases we are unsure to whom they should be attributed.

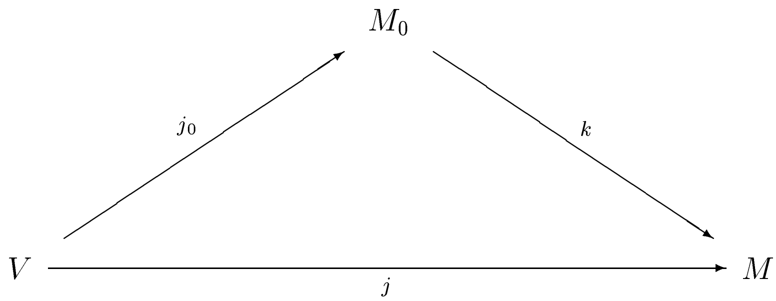
We start with a remark about Cohen forcing. The forcing for adding a single Cohen subset to a regular cardinal κ can be regarded as having conditions which are functions $p : \alpha \longrightarrow \kappa$ for $\alpha < \kappa$ (rather than the more standard functions from $\alpha < \kappa$ to $\{0, 1\}$). In this form we can consider the forcing as adding a generic function from κ to κ.

We will be interested in elementary embeddings $k : M \longrightarrow N$ between inner models of ZFC. In general it will not be the case that k is a class of M or that $N \subseteq M$ (notice that the former implies the latter, as $N = \bigcup_\alpha k(V_\alpha^M)$).

If a model M believes that U (with $U \in M$) is a measure on κ, we will denote the natural embedding from M into $Ult(M, U)$ by j_U^M.

Lemma 1: Let $j : V \longrightarrow M$ be an elementary embedding with j a class of V, $\kappa = \text{crit}(j)$, such that every element of M is $j(F)(\kappa)$ for some function $F \in V$. Then j is the ultrapower by the normal measure $U = \{ X \mid \kappa \in j(X) \}$.

Proof: Factor j through the ultrapower of V by U,



by defining $k : [f] \longmapsto j(f)(\kappa)$. k is a surjection, and M_0 is the transitive collapse of the range of k, so $M_0 = M$ and $j_0 = j$.

♦

Lemma 1 will prove useful in identifying certain embeddings as ultrapowers.

Lemma 2: Let M and N be inner models of ZFC such that

- $M \subseteq N$.
- $N \models {}^{\kappa}M \subseteq M$.
- $M \models U$ is a normal measure on κ.

Then U is a normal measure in N and $j_U^N \upharpoonright M = j_U^M$.

Proof: It follows immediately from the closure of M that U is a normal measure in N. Let $x \in M$. $j_U^N(x)$ is the transitive collapse of the structure (F, E_U) where

$$F = \{ f : \kappa \longrightarrow x \mid f \in N \},$$

and

$$f E_U g \iff \{ \alpha \mid f(\alpha) \in g(\alpha) \} \in U.$$

By the closure of M inside N we have

$$F = \{ f : \kappa \longrightarrow x \mid f \in M \},$$

which is the set of functions whose collapse is $j_U^M(x)$, so by the absoluteness of the collapsing construction $j_U^N(x) = j_U^M(x)$.

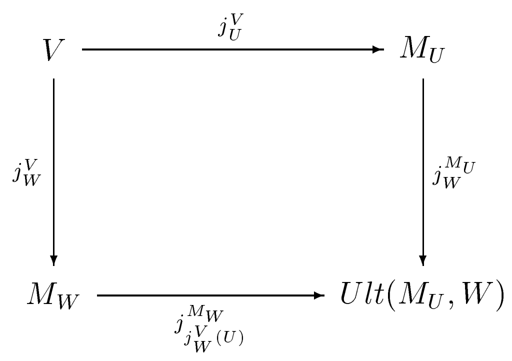
♦

Lemma 2 will be useful in understanding restrictions of ultrapower maps, as for example in the proof of the following lemma.

Lemma 3: Let U be a measure on κ, W a measure on $\lambda \leq \kappa$ and suppose that $W \in Ult(V, U)$. Let M_U be the ultrapower of V by U, M_W the ultrapower of V by W. Then

$$Ult(M_U, W) = Ult(M_W, j_W^V(U))$$

and the following diagram commutes.



Proof: Let $x \in V$.

$$j^{M_W}_{j^V_W(U)}(j^V_W(x)) = j^V_W(j^V_U(x)),$$

by elementarity. $W \in M_U$ and (as $\lambda \leq \kappa$) ${}^\lambda M_U \subseteq M_U$, so that

$$j^V_W \restriction M_U = j^{M_U}_W.$$

In particular

$$j^V_W(j^V_U(x)) = j^{M_U}_W(j^V_U(x)).$$

From this we can deduce that the two ultrapowers are equal (let $x = V_\alpha$), and that the diagram commutes.

♦

We will use lemma 3 to analyse restrictions of iterated ultrapowers.

Lemma 4: Let $k : M \longrightarrow N$ be an elementary embedding between inner models of ZFC. Let $\mathbb{P} \in M$ be a forcing notion, let G be $\mathbb{P}$-generic over M and let H be $k(\mathbb{P})$-generic over N. Suppose that

$$p \in G \Longrightarrow k(p) \in H.$$

Then

1. There is a unique extension of k to a map $k^* : M[G] \longrightarrow N[H]$ such that $k^* : G \longmapsto H$.
2. If Λ is a set of ordinals such that

   $$N = \{ \ k(F)(a) \mid F \in M, a \in [\Lambda]^{<\omega} \ \},$$

   then

   $$N[H] = \{ \ k^*(F)(a) \mid F \in M[G], a \in [\Lambda]^{<\omega} \ \}.$$

Proof: For the first claim, it is clear that if k^* exists it must be given by

$$k^* : \dot{\tau}^G \longmapsto k(\dot{\tau})^H,$$

where $\dot{\tau}^G$ denotes the interpretation of the term τ by the generic G.

We check that this is well-defined. Let $\dot{\tau}^G = \dot{\sigma}^G$, then there is $p \in G$ such that $p \Vdash_{\mathbb{P}}^M \dot{\tau} = \dot{\sigma}$. By elementarity $k(p) \Vdash_{k(\mathbb{P})}^N k(\dot{\tau}) = k(\dot{\sigma})$. By assumption $k(p) \in H$, so that $k(\dot{\tau})^H = k(\dot{\sigma})^H$. The proof that k^* is elementary is entirely similar.

For the second claim, let $\dot{\tau}^H \in N[H]$. Then $\dot{\tau} = k(F)(a)$ for some $F \in M$ and $a \in [\Lambda]^{<\omega}$, and we may take it that for all x in the domain $F(x)$ is a $\mathbb{P}$-term. In $M[G]$ we may define a function $F_1 : x \longmapsto F(x)^G$, and then

$$k^*(F_1)(a) = k(F)(a)^{k^*(G)} = \dot{\tau}^H.$$

♦

Lemma 4 will be used to take elementary embeddings (usually finitely iterated ultrapowers) and extend them onto certain generic extensions of V. The second claim will play a key rôle in understanding the nature of the extended embedding. The next lemmas goes into more detail about the extensions that we will make. We start with a technical result about equivalence between generics.

Lemma 5: Let $\mathbb{P}$ be the forcing notion given by a Reverse Easton iteration of length $\kappa + 1$, in which one Cohen subset of α is added at each strong inaccessible $\alpha \leq \kappa$. Let G_1 and G_2 be $\mathbb{P}$-generics over V, with the property that $V[G_1] = V[G_2]$. Then for any model V^* agreeing with V to rank $\kappa + 1$, G_1 and G_2 are $\mathbb{P}$-generic over V^* and $V^*[G_1] = V^*[G_2]$.

Proof: By the agreement $\mathbb{P} \in V^*$ and (since $|\mathbb{P}| = \kappa$) both models compute the same maximal antichains, so G_1 and G_2 are generic over V^* for $\mathbb{P}$. G_1 is the interpretation under G_2 of some term $\dot{\tau}$, and by the agreement again we may take it that $\dot{\tau} \in V^*$. So $G_1 \in V^*[G_2]$ and *vice versa,* so that $V^*[G_1] = V^*[G_2]$.

♦

Next we give the lemma that will be used to generate measures.

Lemma 6: Let GCH hold, and let $j : V \longrightarrow M$ be an embedding which is a class in V, such that $\kappa = \mathrm{crit}(j)$ and ${}^\kappa M \subseteq M$. Suppose also that the ordinal $j(\kappa^+)$ has cardinality κ^+ in V. Let $\mathbb{P}$ be as in lemma 5, and observe that $\mathbb{P}$ can be factored as $\mathbb{P}_\kappa$ followed by $Add(\kappa, 1)$ as computed by $V^{\mathbb{P}_\kappa}$.

Let $G = G_\kappa * g$ be $\mathbb{P}$-generic, and suppose that there is $G_1 = G_\kappa * g_1$ with $V[G] = V[G_1]$.

Then in $V[G]$ there are κ^{++} many H such that $G_1 * H$ is $j(\mathbb{P})$-generic over M and j extends to $j^* : V[G] \longrightarrow M[G_1][H]$.

Proof: By lemma 5 $M[G] = M[G_1]$. In $M[G_1]$ the factor iteration $j(\mathbb{P})/G_1$ is highly-closed and has $j(\kappa^+)$ many antichains. As $\mathbb{P}$ has the κ^+-chain condition and $M[G] = M[G_1]$ we have $V[G] \models {}^\kappa M[G_1] \subseteq M[G_1]$. Hence in $V[G]$ the forcing $j(\mathbb{P})/G_1$ is κ^+-closed, and the set of its maximal antichains which lie in $M[G_1]$ has cardinality κ^+.

We wish to build generics which are compatible with G. Working in $M[G_1]$, define a function q with domain the M-inaccessibles η such that $\kappa < \eta \leq j(\kappa)$, by $q(\eta) = \emptyset$ for $\eta < j(\kappa)$ and $q(j(\kappa)) = g$. q is a condition in $j(\mathbb{P})/G_1$. We build in $V[G]$ a binary tree of height κ^+ such that

- The top node is q.
- Any path is a descending sequence in $j(\mathbb{P})/G_1$, meeting each antichain in $M[G_1]$.
- Every element has incompatible immediate successors.

The construction proceeds for the requisite κ^+ steps, because $j(\mathbb{P})/G_1$ is κ^+-closed in $V[G]$. This construction will give us κ^{++} distinct generic filters H, each with the property that $j``G \subseteq G_1 * H$. We can use these to build extensions j^* of j such that $j^*(G) = G_1 * H$.

♦

This last construction was a "master condition" argument a la Silver; notice that any extension of q in $j(\mathbb{P})/G$ would have done equally well as the top node of the tree.

We will make heavy use of Mitchell's theory of core models for sequences of measures; nowadays this should be seen as a special case of the core model theory for non-overlapping extenders (due to Mitchell, Dodd, Jensen and Koepke) in which every extender happens to be equivalent to a measure. The reader is referred to Mitchell's paper [4] for proofs.

Definition 3: $\vec{U}$ is a *coherent sequence of measures* if and only if

- $\vec{U}$ is a function, with $\mathrm{dom}(\vec{U}) \subseteq On \times On$.
- For some function $o^{\vec{U}} : On \longrightarrow On$,

$$\mathrm{dom}(\vec{U}) = \{ (\kappa, \eta) \mid 0 \leq \eta < o^{\vec{U}}(\kappa) \}.$$

- If $(\kappa, \eta) \in \mathrm{dom}(\vec{U})$ then $\vec{U}(\kappa, \eta)$ is a normal measure on κ.
- If $(\kappa, \eta) \in \mathrm{dom}(\vec{U})$, and $j : V \longrightarrow M$ is the ultrapower of V by the measure $\vec{U}(\kappa, \eta)$ then
  - For all $\alpha \leq \kappa$, $(\alpha, \beta) \in \mathrm{dom}(j(\vec{U}))$ if and only if $\alpha \leq \kappa$ or $\alpha = \kappa$ and $\beta < \eta$.
  - If $\alpha \leq \kappa$ and $(\alpha, \beta) \in \mathrm{dom}(j(\vec{U}))$ then

$$j(\vec{U})(\alpha, \beta) = \vec{U}(\alpha, \beta).$$

Definition 4: Let M be an inner model of ZFC, let

$$M \models \vec{U} \text{ is a coherent sequence of measures.}$$

A *normal iteration of M by $\vec{U}$, of length* η is a pair

$$(\langle M_\alpha : \alpha < \eta \rangle, \langle j_{\alpha\beta} : \alpha \leq \beta < \eta \rangle)$$

where

- $M_0 = M$.
- M_α is an inner model of ZFC for each $\alpha < \eta$.
- For $\alpha \leq \beta < \eta$, $j_{\alpha\beta} : M_\alpha \longrightarrow M_\beta$ is an elementary embedding.
- $j_{\alpha\alpha} = id$, and for $\alpha \leq \beta \leq \gamma$, $j_{\alpha\gamma} = j_{\beta\gamma} \circ j_{\alpha\beta}$.

- $M_{\alpha+1} = Ult(M_\alpha, j_{0\alpha}(\vec{U})(\kappa_\alpha, \eta_\alpha))$, and $j_{\alpha\alpha+1} : M_\alpha \longrightarrow M_{\alpha+1}$ is the associated ultrapower map, if $\alpha + 1 < \eta$.
- If $\lambda < \eta$, λ is limit, then M_λ and $j_{\alpha\lambda}$ are had by taking a direct limit in the natural way.
- The sequence $\langle \kappa_\alpha : \alpha + 1 < \eta \rangle$ is strictly increasing.

The following structural fact is easy, by induction on $\alpha < \eta$.

Lemma 7: If $(\vec{M}, \vec{\jmath}\,)$ is a normal iteration of M by $\vec{U}$ in length η then for every $\alpha < \eta$

$$M_\alpha = \{\ j_{0\alpha}(F)(a) \mid F \in M, a \in [\Lambda]^{<\omega}\ \},$$

where $\Lambda = \{\ \kappa_\nu \mid \nu < \alpha\ \}$.

We will denote by K Mitchell's core model $K[\vec{U}_{max}]$, which exists under the assumption that there is no inner model in which $\exists\kappa\ o(\kappa) = \kappa^{++}$. We will use the following facts about K (see section 2 of [5]).

Lemma 8 (Mitchell): Suppose that $\neg\exists\kappa\ o(\kappa) = \kappa^{++}$ in any inner model of ZFC. Then

- K is a uniformly definable inner model of ZFC+GCH.
- $K \models V = K$.
- $K \models \vec{U}_{max}$ is a coherent sequence of measures.
- K is invariant under set forcing.
- If $i : K \longrightarrow M$ is an elementary embedding into an inner model M then i arises from a normal iteration of K by $\vec{U}_{max}$.

It is worth making the following easy observations about K and $\vec{U}_{max}$.

Lemma 9: If K, $\vec{U}_{max}$ are as above then

- All measures in K appear on the sequence $\vec{U}_{max}$.

- If $\alpha < \beta < o^{\vec{U}_{max}}(\kappa)$ then $\vec{U}_{max}(\kappa, \alpha) \neq \vec{U}_{max}(\kappa, \beta)$.
- $K \models \vec{U}_{max}(\kappa, \alpha) \lhd \vec{U}_{max}(\kappa, \beta)$ iff $\alpha < \beta$.

We will be particularly interested in finite normal iterations of K, in the case when there is a largest measurable on $\vec{U}_{max}$.

Lemma 10: Suppose that κ is the largest ordinal with $o^{\vec{U}_{max}}(\kappa) > 0$. Let $n+1 < \omega$, let $(\vec{M}, \vec{\jmath}\,)$ be a normal iteration of K by $\vec{U}_{max}$ of length $n+1$, with j_{01} the ultrapower of K by $\vec{U}_{max}(\kappa, \eta)$ for some η. Then

1. $M_n \subseteq K$, and $K \models {}^{\kappa}M_n \subseteq M_n$.
2. For each $i < n$, $\kappa_i < j_{0n}(\kappa)$.
3. In M_0, the ordinal $j_{0n}(\kappa^+)$ has cardinality κ^+

Proof:

1. The critical points are increasing and each model is closed inside the previous one.
2. $\kappa_i \leq j_{0i}(\kappa)$, as κ is the largest measurable on $\vec{U}_{max}$. If $\kappa_i < j_{0i}(\kappa)$ then we are done as $j_{0n}(\kappa) = j_{in}(j_{0i}(\kappa)) \geq j_{0i}(\kappa)$; if $\kappa_i = j_{0i}(\kappa)$ then this is the critical point of j_{in} so $\kappa_i < j_{in}(j_{0i}(\kappa)) = j_{0n}(\kappa)$.
3. The ordinals less than $j_{0n}(\kappa^+)$ all have the form

   $$j_{0n}(F)(\kappa_0, \ldots, \kappa_{n-1}),$$

   where $F : [\kappa]^n \longrightarrow \kappa^+$. By GCH there are κ^+ such functions F.

♦

The next result puts some limits on the possible closure of the models in a normal iteration of infinite length.

Lemma 11: If $(\vec{M}, \vec{\jmath}\,)$ is a normal iteration of M by $\vec{U}$, of length $\eta \geq \omega$, then the sequence of ordinals $\vec{\kappa} = \langle \kappa_n : n < \omega \rangle$ is not a member of M_α for $\omega \leq \alpha < \eta$.

Proof: The model M_α agrees with M_ω to rank $\kappa_\omega + 1$, so it is enough to show that $\vec{\kappa} \notin M_\omega$. M_ω was constructed as a direct limit, so if $\vec{\kappa} \in M_\omega$ then $\vec{\kappa} = j_{n\omega}(\vec{\lambda})$ for some $\vec{\lambda} \in M_n$; in particular $\kappa_n = j_{n\omega}(\lambda_n)$. But $\text{crit}(j_{n\omega}) = \kappa_n$ as we are in a normal iteration, so that $\kappa_n \notin \text{rge}(j_{n\omega})$.

♦

This completes the preliminaries. We make the remark that in what follows we assume that the ground model is of form $K[\vec{U}_{max}]$, but could have taken it in the form $L[\vec{U}]$ because for suitable $\vec{U}$ we have $L[\vec{U}] \models V = K[\vec{U}_{max}]$.

3 Classifying measures

In this section we will take the core model $K[\vec{U}_{max}]$ discussed in the last section, in the case when there is a largest measurable on $\vec{U}_{max}$, and force over it with an iteration $\mathbb{P}$ as in lemma 6. We will then classify completely the measures on κ in $K[G]$, and will describe the Mitchell ordering on these measures.

For the rest of this section let $V = K$, and suppose that there is κ maximal with $o^{\vec{U}_{max}}(\kappa) > 0$. Fix G which is $\mathbb{P}$-generic over K, where $\mathbb{P}$ is the Reverse Easton iteration in which a Cohen subset is added to each inaccessible $\alpha \leq \kappa$, as computed in K. As in lemma 6 we may factor $\mathbb{P}$ as $\mathbb{P}_\kappa * Add(\kappa, 1)$, and correspondingly we may factor G as $G_\kappa * g$.

Lemma 12: Let U be a measure on κ in the model $K[G]$. Let

$$i : K[G] \longrightarrow N$$

be the ultrapower of $K[G]$ by U. Let

$$j : K \longrightarrow K^* = i(K)$$

be the restriction of i to K. Then

1. $i(G) = G_\kappa * g_1 * H$, where g_1 is $Add(\kappa, 1)$-generic over $K^*[G_\kappa]$ and H is $j(\mathbb{P})/G_\kappa * g_1$-generic over $K^*[G_\kappa][g_1]$.

2. If $G_1 = G_\kappa * g_1$ then $K[G_1] = K[G]$.

3. $N = K^*[i(G)]$.

4. $j``G \subseteq i(G)$.

5. $j : K \longrightarrow K^*$ is a finite normal iteration of K by $\vec{U}_{max}$, with the first step being an ultrapower map with critical point κ.

Proof:

By elementarity $N = K^*[i(G)]$, where K^* is $K[\vec{U}_{max}]$ as computed in the sense of N. $i(G)$ is generic over K^* for $i(\mathbb{P})$, which equals $j(\mathbb{P})$ since $\mathbb{P} \in K$.

$j : K \longrightarrow K^*$ must be a normal iteration with first step an ultrapower by a measure on κ, because K is still $K[\vec{U}_{max}]$ in $K[G]$. In particular K and K^* agree to rank $\kappa + 1$.

$i(G) = G_\kappa * g_1 * H$, where g_1 is generic for $Add(\kappa, 1)$ as computed in $K^*[G_\kappa]$ and H is generic for $j(\mathbb{P})/G_\kappa * g_1$. $K[G_\kappa]$ and $K^*[G_\kappa]$ agree to rank $\kappa + 1$, so g_1 is actually $K[G_\kappa]$ generic for $Add(\kappa, 1)$. Also $K[G_1]$ and $K^*[G_1]$ agree to rank $\kappa + 1$.

As N is an ultrapower, $K[G] \models {}^\kappa N \subseteq N$. As H is generic for highly closed forcing, $K[G] \models {}^\kappa K^*[G_1] \subseteq K^*[G_1]$. In particular $g \in K^*[G_1]$, so that by the last paragraph $g \in K[G_1]$. Hence $K[G] = K[G_1]$.

If j is not a finite iteration, then lemma 11 implies that there is an ω-sequence of ordinals $\vec{\kappa} \in K[G]$ such that $\vec{\kappa} \notin K^*$. But $\mathbb{P}$ is ω_1-closed, and so $\vec{\kappa} \notin K^*[G]$, in contradiction to what we just proved about the closure of $K^*[G]$.

♦

Definition 5: $U \in K[G]$ is an *n-step extension* of $\vec{U}_{max}(\kappa, \eta)$ if, when we define j as in the last lemma, j has length $n + 1$ and the first step in j is the application of $\vec{U}_{max}(\kappa, \eta)$ to K.

Notice that this is reasonable terminology, as when U is an n-step extension of $\vec{U}_{max}(\kappa, \eta)$ we certainly have $\vec{U}_{max}(\kappa, \eta) \subseteq U$. The one-step extensions are the easiest ones to understand.

Lemma 13: Let $\eta < o^{\vec{U}_{max}}(\kappa)$, and let $j_\eta : K \longrightarrow M_\eta$ be the ultrapower of K by $\vec{U}_{max}(\kappa, \eta)$. Then in $K[G]$ the set of $H_1 = g_1 * H$ such that (setting $G_1 = G_\kappa * g_1$)

- G_1 is $\mathbb{P}$-generic over K.
- $K[G] = K[G_1]$.
- H is $j_\eta(P)/G_1$-generic over $M_\eta[G_1]$.
- j_η"$G \subseteq G_\kappa * H_1$.

has cardinality κ^{++}, and each one gives rise to a distinct one-step extension U_{H_1} of $\vec{U}_{max}(\kappa, \eta)$.

Proof: There are κ^+ generics g_1 such that $K[G] = K[G_\kappa][g_1]$. Fix one such, and observe that by lemma 5 $M_\eta[G] = M_\eta[G_1]$. By lemma 6 we may build κ^{++} many appropriate generics H, and by cardinality considerations there can be at most κ^{++} many.

Let H be one such, and consider the unique map

$$j^*_\eta : K[G] \longrightarrow M_\eta[G_1][H]$$

such that j^*_η extends j_η and $j^*_\eta(G) = G_1 * H$. By lemma 4,

$$M_\eta[G_1][H] = \{ \ j^*_\eta(F)(\kappa) \mid F \in K[G] \ \},$$

so lemma 1 tells us that j^*_η is the ultrapower of $K[G]$ by the measure

$$U_H = \{ \ X \subseteq \kappa \mid \kappa \in j^*_\eta(X) \ \}.$$

Distinct generics H_1 give distinct one-step extensions, because given U_{H_1} we may recover H_1 by computing $j^{K[G]}_{U_{H_1}}(G) = G_\kappa * H_1$.

♦

This last lemma gives a complete description of the one-step extensions of measures $\vec{U}_{max}(\kappa, \eta)$. We need to do a bit more work to produce n-step extensions; the point will be to guarantee that each critical point we use can be defined from κ in a certain way.

Lemma 14: Let $j : K \longrightarrow K^*$ be a normal iteration of K by $\vec{U}_{max}$ of length $n+1$, with $j_{0i}(\vec{U}_{max})(\kappa_i, \eta_i)$ being applied at stage i in the iteration, and $\kappa_0 = \kappa$. Then in $K[G]$ there are κ^{++} many $H_1 = g_1 * H$ such that (setting $G_1 = G_\kappa * g_1$)

- G_1 is $\mathbb{P}$-generic over K.
- $K[G] = K[G_1]$.
- H is $j(P)/G_1$-generic over $K^*[G_1]$.
- $j``G \subseteq G_\kappa * H_1$.
- If

  $$j^* : K[G] \longrightarrow K^*[G_\kappa][H_1]$$

  is the unique extension of j with $j^*(G) = G_\kappa * H_1$, then

  $$K^*[G_\kappa][H_1] = \{ \, j^*(F)(\kappa) \mid F \in K[G] \, \}.$$

Proof: As before there are κ^+ appropriate g_1, and we will fix one. Then we know that $K^*[G] = K^*[G_1]$.

We will define a "master condition" for $j(P)/G_1$, much as in lemma 6. As there the condition q will have value $\emptyset$ at M-inaccessible η with $\kappa < \eta < j(\kappa)$, but $q(j(\kappa))$ will be slightly bigger than in lemma 6. Define $q(j(\kappa))$ by

- $\mathrm{dom}(q(j(\kappa))) = \kappa + n$.
- $q(j(\kappa)) \restriction \kappa = g$.
- $q(j(\kappa) + i) = \kappa_i$, for $i < n$.

Just as in lemma 6 we may build κ^{++} many H with q as a member, and argue that H is generic and that $j``G \subseteq G_1 * H$. It will suffice to show that for every $i < n$ the ordinal κ_i has the form $j^*(F)(\kappa)$, as lemma 7 then shows that every element of $K^*[G_\kappa][H_1]$ may be written in this form. Now fix $i < n$, and define a function F in $K[G]$ by

$$F(\alpha) = g(\alpha + i).$$

We have

$$j^*(F)(\kappa) = j^*(g)(\kappa + i) = H(j(\kappa))(\kappa + i) = q(\kappa)(\kappa + i) = \kappa_i,$$

so the lemma is proved.

♦

This result classifies the n-step extensions of measures on κ in K. It remains to determine when the relation $\lhd$ holds between two such extension measures. As one might expect, the situation is simplest when considering one-step extensions.

Lemma 15: Let U, V be two measures on κ in $K[G]$. Suppose further that U is a 1-step extension of $U_0 = \vec{U}_{max}(\kappa, \alpha)$, using some generic $H^1_U = g_U * H_U$, and that V is a 1-step extension of $V_0 = \vec{U}_{max}(\kappa, \beta)$ using some generic $H^1_V = g_V * H_V$. Set $G_U = G_\kappa * g_U$, $G_V = G_\kappa * g_V$.

Then $K[G] \models U \lhd V$ if and only if

- $\alpha < \beta$.
- $H^1_U \in Ult(K, V_0)[G]$.

Proof: Let $M = Ult(K, U_0)$, let $N = Ult(K, V_0)$.

- First suppose that $K[G] \models U \lhd V$. This means that

  $$U \in Ult(K[G], V) = N[G_V][H_V].$$

  As $K[G] = K[G_V]$ we know that $N[G] = N[G_V]$. H_V is generic for highly closed forcing, so this will imply that $U \in N[G]$. Since $K[G] \models {}^\kappa N[G] \subseteq N[G]$, $K[G]$ and $N[G]$ agree to rank $\kappa + 1$, so that there is agreement between $j_U^{K[G]}$ and $j_U^{N[G]}$ to that rank. In particular

  $$G_U * H_U = j_U^{K[G]}(G) = j_U^{N[G]}(G),$$

  so that $H^1_U \in N[G]$.

  To show that $\alpha < \beta$, observe that $N \subseteq K \subseteq K[G]$. Also

  $$j_U^{K[G]} \restriction N[G] = j_U^{N[G]},$$

so that the restriction of $j_U^{K[G]}$ to N is an embedding definable in $N[G]$, from N to some well-founded model. It must therefore be a normal iteration of N, since N is the core model of $N[G]$. But $j_U^{K[G]} \restriction K = j_{U_0}^K$, so that $j_U^{K[G]} \restriction N = j_{U_0}^K \restriction N$. It is easy to see that the first step in the iteration of N induced by this restriction is to take the ultrapower by

$$U_0 = \{ X \subseteq \kappa \mid X \in N, \kappa \in j_{U_0}^K(X) \},$$

so that $U_0 \in N$. Hence $U_0 \lhd V_0$, and $\alpha < \beta$.

- For the other direction, suppose that $H_U^1 \in N[G]$ and $\alpha < \beta$, that is $K \models U_0 \lhd V_0$ and so $U_0 \in N$.

  We will show that $N[G]$ can reconstruct U from H_U^1. $K[G]$ and $N[G]$ (which equals $N[G_V]$) agree to rank $\kappa + 1$, and $j_{U_0}^K \restriction N = j_{U_0}^N$, what is more N contains all canonical P-names for subsets of κ. So if $\dot\tau$ is such a name then $N[G]$ can compute

  $$j_U^{K[G]}(\dot\tau^G) = j_{U_0}^K(\dot\tau)^{G_U * H_U} = j_{U_0}^N(\dot\tau)^{G_U * H_U},$$

  and hence $N[G]$ can compute U, so

  $$U \in N[G_V] \subseteq N[G_V][H_V] = Ult(K[G], V).$$

  Hence $K[G] \models U \lhd V$ and we are done.

♦

At this point we are almost ready to describe the ordering $\lhd$ of one-step extensions. What we still need is some idea of how many generics on $j_{U_0}^V(P)/G$ are constructed by models of the form $Ult(K, V_0)[G]$ as V_0 runs through the measures on κ with $U_0 \lhd V_0$. The next lemma will provide us with this information.

Lemma 16: Let $\alpha < \beta < \gamma < o^{\vec{U}_{max}}(\kappa)$. Let us define $U = \vec{U}_{max}(\kappa, \alpha)$, $V = \vec{U}_{max}(\kappa, \beta)$, and finally $W = \vec{U}_{max}(\kappa, \gamma)$. Then the $Ult(K, U)[G]$-generics on $j_U^K(P)/G$ constructed in the model $Ult(K, V)[G]$ form a proper subset of those constructed in the model $Ult(K, W)[G]$, and the same is true if we restrict to those generics H such that j_U"$G \subseteq G * H$.

Proof: Let $M_U = Ult(K, U)$ and define M_V, M_W similarly. K and M_W agree to rank $\kappa + 1$, so that M_V and $N = Ult(M_W, V)$ agree to rank $j_V(\kappa) + 1$. As P is relatively small, $M_V[G]$ and $N[G]$ also agree to this level, which is much greater than $j_U(\kappa)$. So $M_V[G]$ and $N[G]$ construct the same generics H for the forcing $j_U(P)/G$.

But now by the same arguments as in lemma 6, $M_W[G]$ believes that it can construct κ^{++} many generics, but that the inner model $N[G]$ can only build κ^+ many. This proves the lemma.

♦

We use this to get a picture of the ordering on one-step extensions in the case when $o^{\vec{U}_{max}}(\kappa) = 3$. This is fairly representative of the general case.

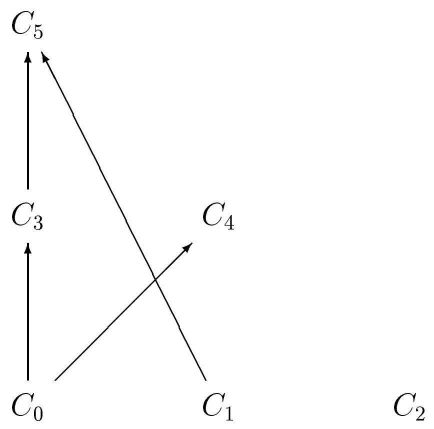
Lemma 17: Let $o^{\vec{U}_{max}}(\kappa) = 3$, with $U = \vec{U}_{max}(\kappa, 0)$, $V = \vec{U}_{max}(\kappa, 1)$, $W = \vec{U}_{max}(\kappa, 2)$. Let M_U, M_V, M_W denote the ultrapowers of K by these measures. Work in $K[G]$. Then we may divide the one-step extensions of these measures into classes

- C_0: extensions of U via generics in $M_V[G]$. $|C_0| = \kappa^+$
- C_1: extensions of U via generics in $M_W[G] \setminus M_V[G]$. $|C_1| = \kappa^+$.
- C_2: extensions of U via generics in $K[G] \setminus M_W[G]$. $|C_2| = \kappa^{++}$.
- C_3: extensions of V via generics in $M_W[G]$. $|C_1| = \kappa^+$.
- C_4: extensions of V via generics in $K[G] \setminus M_W[G]$. $|C_4| = \kappa^{++}$.
- C_5: extensions of W. $|C_5| = \kappa^{++}$.

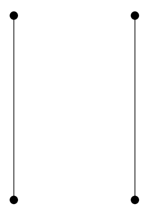
A measure from C_i is below a measure from C_j in the Mitchell ordering if and only if

- $i = 0$ and $j \in \{3, 4, 5\}$ OR
- $i = 1$ and $j = 5$ OR
- $i = 3$ and $j = 5$.

The proof is immediate. We give a picture which may make the shape of the partial ordering clearer.

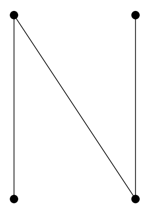


If instead of $o(\kappa) = 3$ we take $o(\kappa) = \omega$, we get an infinite partial ordering P with an interesting universal property; if Q is the four-element poset



then P does not embed Q, and P embeds every finite poset which does not embed Q. This was pointed out to me by Andrew Jergens [2].

Baldwin speculated that the methods of [6] might extend to all well-founded posets which embed neither Q nor the poset R given by



We observe that P does embed R.

Now we consider the general case of the Mitchell ordering between n-step extensions. This problem is not quite as hard as one might expect, largely because the question whether $U \lhd V$ is controlled by the first step in the iteration associated with V.

Theorem 1: Let U be an $m+1$-step extension of U_0, via a generic object $H^1_U = g_U * H_U$ and an iteration $(\vec{M}, \vec{\jmath})$ of length $m+1$, with the ultrafilter $j_{0i}(\vec{U}_{max})(\kappa_i, \lambda_i)$ being applied to M_i at stage i. Let V be an $n+1$-step extension of V_0, via a generic $H^1_V = g_V * H_V$ and an iteration $(\vec{N}, \vec{k})$ of length $n+1$, with the ultrafilter $k_{0i}(\vec{U}_{max})(\mu_i, \nu_i)$ being applied to N_i at stage i.

Then $K[G] \models U \lhd V$ if and only if

- $H_U \in Ult(K, V_0)[G]$.
- $j_{0m} \upharpoonright Ult(K, V_0)$ is a finite normal iteration of $Ult(K, V_0)$ by $k_{01}(\vec{U}_{max})$.

Proof: Notice that $N_1 = Ult(K, V_0)$. As before we let $G_U = G_\kappa * g_U$ and $G_V = G_\kappa * g_V$.

- Suppose that $K[G] \models U \lhd V$. Then

  $$U \in Ult(K[G], V) = N_n[G_V][H_V],$$

  so as in lemma 15 $U \in N_n[G]$. N_1 and N_n agree to rank $\kappa_1 + 1$, so by an easy chain condition argument the models $N_1[G]$ and $N_n[G]$ also agree to this rank, hence $U \in N_1[G]$.

  As in lemma 15 $N_1[G]$ can reconstruct H^1_U, so that $H^1_U \in N_1[G]$.

  For the second part just observe that $j_U^{K[G]} \upharpoonright N_1[G] = j_U^{N_1[G]}$, so that $j_U^{K[G]} \upharpoonright N_1$ must give rise to a normal iteration of N_1 by its version of $\vec{U}_{max}$, which is $k_{01}(\vec{U}_{max})$. But $N_1 \subseteq K$ and $j_U^{K[G]} \upharpoonright K = j_{0m}$, so this amounts to saying that $j_{0m} \upharpoonright N_1$ is a normal iteration of N_1 by $k_{01}(\vec{U}_{max})$.

  This iteration must be finite, as usual, because otherwise the first ω critical points will give a sequence which is in $N_1[G]$ but not in $Ult(N_1[G], U)$.

- Suppose that $H^1_U \in N_1[G]$, and that $j_{0m} \restriction N_1$ can be written as an iteration $(\vec{N}^*, \vec{\jmath}^*)$ of length $s+1$, so that $N^*_s = j_{0m}(N_1)$ and $j_{0m} \restriction N_1 = j^*_{0s}$.

  We will show that $N_1[G]$ can compute U; the proof is precisely parallel to that in lemma 15. $K[G]$ and $N_1[G]$ agree to rank $\kappa + 1$, K and N_1 agree on the set of canonical names for subsets of κ. If $\dot\tau$ is such a name then (since j^*_{0s} is a class in N_1)) $N_1[G]$ can compute

  $$j_U^{K[G]}(\dot\tau^G) = j_{0m}(\dot\tau)^{G_U * H_U} = j^*_{0s}(\dot\tau)^{G_U * H_U}.$$

  Just as in lemma 15 this gives $U \in N_1[G]$, and by the same arguments as we used in the first part of the proof this implies that $U \in N_n[G]$, hence that $K[G] \models U \lhd V$.

◆

Our next task is to explore the circumstances under which an iterated ultrapower of K restricted to a one-step ultrapower N gives rise to a map which is an iterated ultrapower of N.

The following lemma resolves the question about the restriction of a finite iteration to a one-step ultrapower model.

Lemma 18: Let M be a model of ZFC, and assume

$$M \models \vec{U} \text{ is a coherent sequence.}$$

Let κ be the largest critical point on $\vec{U}$. Let j be a finite normal iteration of M, in which a measure

$$U_m = j_{0m}(\vec{U})(\kappa_m, \mu_m)$$

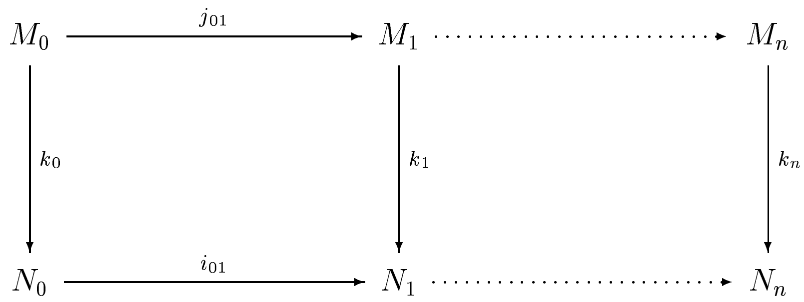
is applied to M_m to get $j_{mm+1} : M_m \longrightarrow M_{m+1}$ for each $m < n$. Let $\kappa_0 = \kappa$, $\mu_0 = \alpha$. Let $N = Ult(M, \vec{U}(\kappa, \beta))$ for some β, and suppose that $i = j \restriction N : N \longrightarrow j(N)$ is a finite normal iteration of N.

Then

1. For each $m < n$, $U_m \in N_m$.

2. i has length n, and step m in the iteration i is the application of U_m to N_m.

3. The diagram

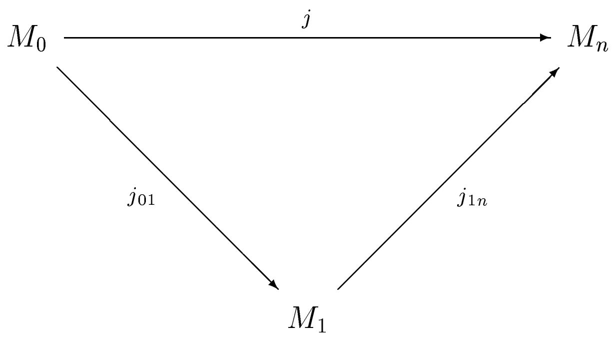


commutes, where $k_i : M_i \longrightarrow N_i$ is the ultrapower map arising from the measure $j_{0i}(\vec{U}(\kappa, \beta))$.

Proof: M_0 can recover U_0 by computing

$$U_0 = \{ X \in \mathcal{P}\kappa \cap M_0 \mid \kappa \in j(X) \}.$$

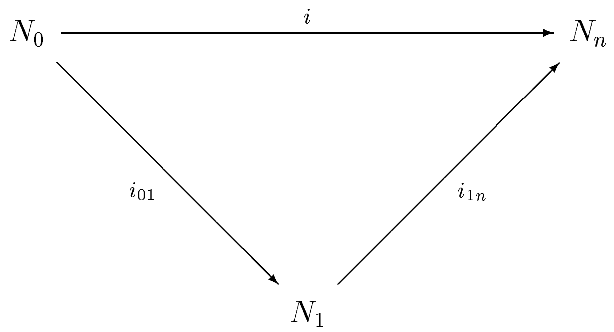
We can then build a commutative triangle



Since $\mathcal{P}\kappa \cap M_0 = \mathcal{P}\kappa \cap N_0$ and $i = j \restriction N$ we have

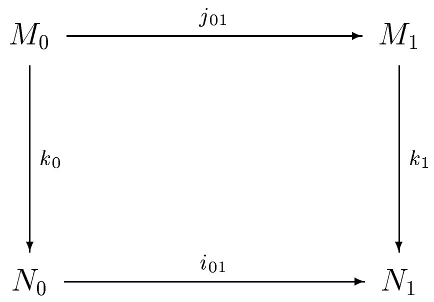
$$U_0 = \{ X \in \mathcal{P}\kappa \cap N_0 \mid \kappa \in i(X) \},$$

and a commutative triangle



So $U_0 \in N_0$. We make the easy observation that $\alpha < \beta$, because N_0 is the ultrapower of M_0 by the measure $\vec{U}(\kappa, \beta)$ on the coherent sequence $\vec{U}$.

Applying lemma 3 the square



commutes.

We now attempt to argue that i_{1n} and j_{1n} resemble each other. Let $\lambda_1 = j_{01}(\kappa)$.

Claim 1: In the situation described above

1. $\lambda_1 = i_{01}(\kappa)$.
2. $V^{M_1}_{\lambda_1+1} = V^{N_1}_{\lambda_1+1}$.
3. $j_{1n} \upharpoonright V^{M_1}_{\lambda_1+1} = i_{1n} \upharpoonright V^{N_1}_{\lambda_1+1}$.

Proof: M_0 and N_0 agree to rank $\kappa + 1$, so by standard arguments

$$i_{01} \upharpoonright V^{N_0}_{\kappa+1} = j_{01} \upharpoonright V^{M_0}_{\kappa+1}$$

and

$$V^{M_1}_{\lambda_1+1} = i_{01}(V^{M_0}_{\kappa+1}) = j_{01}(V^{N_0}_{\kappa+1}) = V^{N_1}_{\lambda_1+1}.$$

The key point is that both models compute the same set of functions from κ to $V_{\kappa+1}$.

If $x \in V^{M_1}_{\lambda_1+1}$ then $x = j_{01}(F)(\kappa) = i_{01}(F)(\kappa)$ for some such function, and so

$$j_{1n}(x) = j(F)(\kappa) = i(F)(\kappa) = i_{1n}(x)$$

by the normality of the iterations.

♦

Since κ is the largest measurable on $\vec{U}$, λ_1 is the largest on $j_{01}(\vec{U})$ and hence $\kappa_1 \leq \lambda_1$. We know that $\kappa_1 = \mathrm{crit}(j_{1n})$, so also $\kappa_1 = \mathrm{crit}(i_{1n})$. What is more

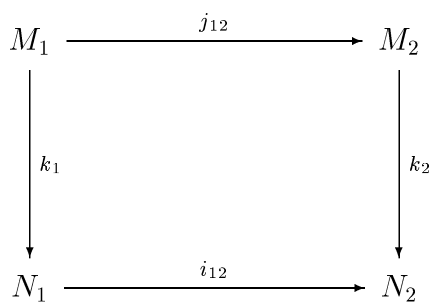
$$\begin{aligned} U_1 &= \{\, X \in \mathcal{P}\kappa_1 \cap M_1 \mid \kappa_1 \in j_{1n}(X) \,\} \\ &= \{\, X \in \mathcal{P}\kappa_1 \cap N_1 \mid \kappa_1 \in i_{1n}(X) \,\}. \end{aligned}$$

Hence U_1 is in N_1 and i_{12} is the ultrapower of N_1 by U_1.

At this point we observe that since N_1 is the ultrapower of M_1 by the measure $j_{01}(\vec{U}(\kappa,\beta))$, there is a certain agreement between the measure sequences in these models: namely these sequences agree below λ_1, and at λ_1 the model N_1 has the same measures as M_1 up to the point $j_{01}(\beta)$.

As a consequence we see that either $\kappa_1 < \lambda_1$ or $\kappa_1 = \lambda_1$ and $\mu_1 < j_{01}(\beta)$.

By lemma 3, we see that the diagram



commutes.

To finish the proof we just repeat these arguments, showing step by step that the diagrams commute and the models N_n construct the measures U_n.

♦

The following corollary can be derived by a close inspection of the proof of the preceding lemma.

Corollary 1: Given an iteration j of M and a model N as described above, it is necessary and sufficient for $j \upharpoonright N$ to be an iteration of N that for all $m < n$ either $\kappa_m < j_{0m}(\kappa)$ or $\kappa_m = j_{0m}(\kappa)$ and $\mu_m < j_{0m}(\beta)$.

We observe that as a consequence, if j_{0n} induces an internal iteration of $Ult(K, \vec{U}_{max}(\kappa, \beta))$, then it induces such an iteration of $Ult(K, \vec{U}_{max}(\kappa, \gamma))$ for any $\gamma > \beta$.

We can finally undertake the general analysis of the ordering between n-step extensions in $K[G]$.

Definition 6: Let $\alpha < o(\kappa)$, and let $\beta \in (\alpha, o(\kappa)) \cup \{\infty\}$.

For $\beta \in (\alpha, o(\kappa))$ let $M(\alpha, \beta)$ be the set of extensions U of $\vec{U}_{max}(\kappa, \alpha)$ such that β is the least γ with the following two properties:

1. The constructing generic H_U is in $Ult(K, \vec{U}_{max}(\kappa, \gamma))[G]$.
2. $j_U^{K[G]}$ induces an internal iteration of $Ult(K, \vec{U}_{max}(\kappa, \gamma))$.

For $\beta = \infty$ let $M(\alpha, \beta)$ be the set of those U such that no γ as described above exists.

The description of the ordering is given by the following result, whose proof follows immediately from the work above.

Theorem 2: Every measure on κ in $K[G]$ is in a unique $M(\alpha, \beta)$. $M(\alpha, \beta)$ has cardinality κ^+ if $\beta \in (\alpha, o(\kappa))$ and cardinality κ^{++} if $\beta = \infty$.

If $U \in M(\alpha, \beta)$ and $V \in M(\gamma, \delta)$, then $U \lhd V$ if and only if $\beta \leq \gamma$.

References

[1] J. Cummings, Possible behaviours for the Mitchell ordering II. In preparation.

[2] A. Jergens, Private communication.

[3] W. Mitchell, Sets constructible from sequences of ultrafilters. Journal of Symbolic Logic **39** (1974) 57–66.

[4] W. Mitchell, The core model for sequences of measures. I. Math. Proc. Camb. Phil. Soc. **95** (1984) 229–260.

[5] W. Mitchell, Indiscernibles, skies and ideals. In Contemporary Mathematics **31** (1984) 161–182.

[6] S. Baldwin, The $\triangleleft$-ordering on normal ultrafilters. Journal of Symbolic Logic **51** (1985) 936–952.