Strong ultrapowers and long core models

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In his paper [4] Steel asked whether there can exist a normal measure U on a cardinal κ such that

$$\mathcal{P}\kappa^+ \subseteq Ult(V, U).$$

In part one we use Reverse Easton forcing to show that this is consistent from a $\mathcal{P}_2\kappa$-hypermeasure; in part two we show that the result of part one is sharp, using the core model for non-overlapping coherent extender sequences.

The proof in part one uses forcing technology due to Woodin.

1 Strong ultrapowers

It is a standard fact that if κ is $\mathcal{P}_2\kappa$-hypermeasurable then there exists $\vec{E}$ a coherent non-overlapping sequence of extenders such that

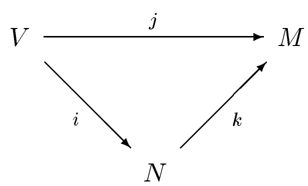
$$L[\vec{E}] \models \text{GCH} + \text{there exists a } (\kappa, \kappa^{++})\text{-extender } E \text{ with } V_{\kappa+2} \subset Ult(V, E).$$

We will take advantage of this added structure in the proof.

Theorem 1: If in V GCH holds and there is a (κ, κ^{++})-extender E such that $V_{\kappa+2} \subset Ult(V, E)$, then there is a generic extension $\bar{V}$ of V in which κ carries a normal measure U such that

$$\mathcal{P}\kappa^+ \subseteq Ult(\bar{V}, U).$$

Proof: Let j be the ultrapower map $j : V \longrightarrow Ult(V, E)$. We factor j through the ultrapower by $\{ X \mid \kappa \in j(X) \}$ to get a commutative triangle



Let $\lambda = \mathrm{crit}(k)$. Then by GCH and standard facts about ultrapowers

$$\lambda = (\kappa^{++})_N < i(\kappa) < i(\kappa^{++}) = \kappa^{++} = (\kappa^{++})_M < j(\kappa)$$

Moreover we may describe the relation between M, N and V in the following terms;

$$\begin{aligned} M &= \{\ j(F)(a) \mid F \in V, a \in [\kappa^{++}]^{<\omega}, \mathrm{dom}(F) = [\kappa]^{|a|}\ \} \\ &= \{\ k(G)(a) \mid G \in N, a \in [\kappa^{++}]^{<\omega}, \mathrm{dom}(G) = [\lambda]^{|a|}\ \}. \end{aligned}$$

This last fact will enable us to transfer generics for sufficiently dense forcings along maps extending k (see [2] for detailed proofs of the necessary facts).

The construction will be in two steps, with the first step providing a generic object for use in the second step; each step is a "Reverse Easton" iteration of length $\kappa + 1$. For a treatment of this style of forcing see Baumgartner's paper [1], from which we will quote facts about Reverse Easton forcing as we need them.

The first step

Our aim in this step is to force so as to produce a model V^* with the following properties.

1. GCH holds in V^*.
2. In V^* there is a (κ, κ^{++})-extender such that if $j^* : V^* \longrightarrow M^*$ is the ultrapower by that extender then $V^*_{\kappa+2} \subseteq M^*$.
3. If $i^* : V^* \longrightarrow N^*$ is the ultrapower of V^* by the canonical normal measure $\{\ X \mid \kappa \in j^*(X)\ \}$, and Q is the Cohen forcing $Add(\kappa, \kappa^{++})$ as computed in V^*, then there is $F \in V^*$ which is $i^*(Q)$-generic over N^*.

We iterate the Cohen forcing $Add(\alpha^+, \alpha^{++})$ at inaccessible $\alpha \leq \kappa$ in a Reverse Easton fashion. That is to say we define by induction

1. $P_0 = 0$.
2. $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, where $\dot{Q}_\alpha$ names $Add(\alpha^+, \alpha^{++})_{V[G_\alpha]}$ if α is inaccessible and 0 otherwise.
3. For limit λ the forcing P_λ is the direct limit of $\langle P_\alpha : \alpha < \lambda \rangle$ if λ is inaccessible, and the inverse limit of that sequence otherwise.

This inductive definition gives us finally a poset $P_{\kappa+1}$. From the results in [1] it follows that $P_{\kappa+1}$ is κ^{++}-c.c. and preserves cardinals and the GCH. We observe also that $P_{\kappa+1} \subseteq H_{\kappa^{++}}$.

Let G_κ denote a P_κ-generic over V, let P be $Add(\kappa^+, \kappa^{++})$ as computed in $V[G_\kappa]$, and let g be P-generic over $V[G_\kappa]$. Factor P as

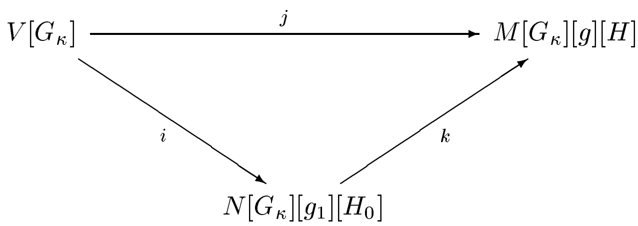
$$P_1 \times P_2 = Add(\kappa^+, \lambda) \times Add(\kappa^+, \kappa^{++} \setminus \lambda)$$

and split up g as $g_1 \times g_2$ accordingly. It may easily be argued that k can be extended to a new map (which we also call k to avoid a plague of sub- and super-scripts)

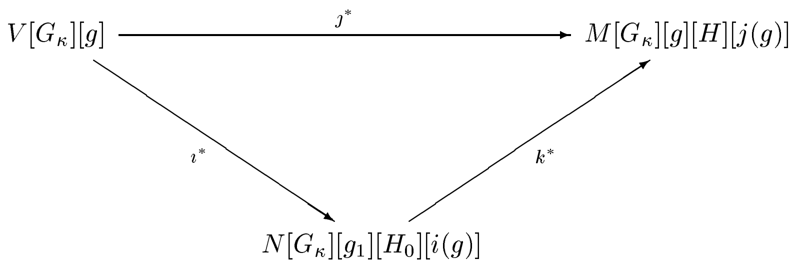
$$N[G_\kappa][g_1] \xrightarrow{k} M[G_\kappa][g]$$

In the model $N[G_\kappa][g_1]$ define a forcing $R = i(P_\kappa)/G_\kappa * g_1$. Using results in [1] again we have that in $N[G_\kappa][g_1]$ the forcing R is λ^+-closed, is of cardinality $i(\kappa)$, and has the $i(\kappa)$-c.c. In particular that model believes R to have $i(\kappa)$ maximal antichains.

Now $V[G_\kappa][g_1] \models {}^\kappa N[G_\kappa][g_1] \subseteq N[G_\kappa][g_1]$ and by GCH the V-cardinality of $i(\kappa)$ is κ^+, so that in $V[G_\kappa][g_1]$ we may construct H_0 which is R-generic over $N[G_\kappa][g_1]$. By the closure of R in $N[G_\kappa][g_1]$ we may transfer H_0 along k to get $H \in V[G_\kappa][g]$ which is R-generic over $M[G_\kappa][g]$, and maps



Transferring g successively along i, k we can build in $V[G_\kappa][g]$ a commutative triangle



We will let

$$\begin{aligned} V^* &= V[G_\kappa][g], \\ M^* &= M[G_\kappa][g][H][j(g)], \\ N^* &= N[G_\kappa][g_1][H_0][i(g)]. \end{aligned}$$

We need to check that M^* is the ultrapower of V^* by a (κ, κ^{++})-extender, that $V^*_{\kappa+2} \subseteq M^*$, and that $\imath^* : V^* \longrightarrow N^*$ is in fact the ultrapower of V^* by $\{ X \mid \kappa \in \jmath^*(X) \}$.

The first and third of these points are immediate by results in [2]. For the second it suffices to remark that

$$V \models [\kappa^+ \times P_{\kappa+1}]^{\kappa^+} \subseteq M,$$

that elements of $V^*_{\kappa+2}$ may be coded by subsets of κ^+ in V^*, and that $P_{\kappa+1}$ has the κ^{++}-c.c. For this shows that every subset of κ^+ in $V[G_\kappa][g]$ has a canonical name in $[\kappa^+ \times P_{\kappa+1}]^{\kappa^+}$, so lies in $M[G_\kappa][g]$ which is a submodel of M^*.

To show that we can find F, return to that stage of the construction where we had defined a map in $V[G_\kappa][g_1]$

$$V[G_\kappa] \xrightarrow{\quad i \quad} N[G_\kappa][g_1][H_0]$$

Observe that we may transfer g_1 along i to get an **internal** ultrapower of $V[G_\kappa][g_1]$

$$V[G_\kappa][g_1] \xrightarrow{\quad \imath^\dagger \quad} N[G_\kappa][g_1][H_0][\imath^\dagger(g_1)]$$

Let Q be $Add(\kappa, \kappa^{++})$ as computed in $V[G_\kappa][g_1]$. A result of Woodin (see [2] for the proof) tells us that, since $V[G_\kappa][g_1]$ is a model with $2^\kappa = \kappa^+$, there is an isomorphism in $V[G_\kappa][g_1]$ between $Add(\kappa^+, \kappa^{++})$ as defined in that model and $\imath^\dagger(Q)$. Hence we may rearrange g_2 as F which is $\imath^\dagger(Q)$-generic over $N[G_\kappa][g_1][H_0][\imath^\dagger(g_1)]$.

Finally, observe that Q is the $Add(\kappa, \kappa^{++})$ of V^*, that $\imath^\dagger(Q) = \imath^*(Q)$, and that F is still generic over N^* because N^* is had from $N[G_\kappa][g_1][H_0][\imath^\dagger(g_1)]$ by adding a generic (namely $i(g_2)$) for forcing so closed that it adds no new antichains in $\imath^*(Q)$.

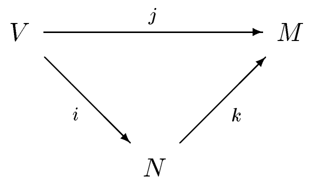
We have found V^* and F as desired, so the first step of the construction is complete.

The second step

To avoid having unwanted asterisks decorating every model and embedding we re-initialise our notation; our new starting assumptions are

1. GCH holds in V.
2. There is a (κ, κ^{++})-extender such that if $j : V \longrightarrow M$ is the ultrapower by that extender then $V_{\kappa+2} \subseteq M$.
3. If $i : V \longrightarrow N$ is the ultrapower of V by the normal measure on κ defined as $\{ X \mid \kappa \in j(X) \}$, and Q is the Cohen forcing $Add(\kappa, \kappa^{++})$, then there is $F \in V$ which is $i(Q)$-generic over N.

Exactly as before we have a triangle



λ is still the critical point of k and

$$\lambda = (\kappa^{++})_N < i(\kappa) < i(\kappa^{++}) = \kappa^{++} = (\kappa^{++})_M < j(\kappa).$$

We now iterate $Add(\alpha, \alpha^{++})$ at every inaccessible $\alpha \leq \kappa$. Arguing as in step one $P_{\kappa+1}$ is κ^+-c.c. and preserves cardinals, also $P_{\kappa+1} \subset H_{\kappa^{++}}$. $P_{\kappa+1}$ adds κ^{++} generic subsets of κ.

Let G_κ be P_κ-generic over V, let P be $Add(\kappa, \kappa^{++})$ as computed in $V[G_\kappa]$, let g be P-generic over $V[G_\kappa]$. Factor P as

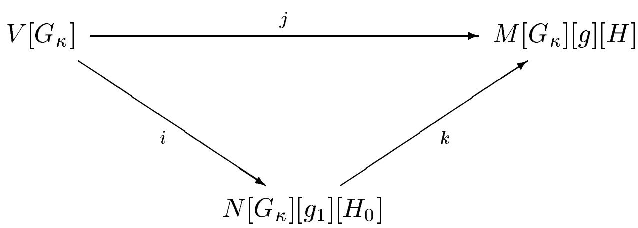
$$P_1 \times P_2 = Add(\kappa^+, \lambda) \times Add(\kappa, \kappa^{++} \setminus \lambda)$$

and split up g as $g_1 \times g_2$.

Just as in the first step we can easily extend k to get

$$N[G_\kappa][g_1] \xrightarrow{k} M[G_\kappa][g]$$

If we let $R = i(P_\kappa)/G_\kappa * g_1$, then as in step one we may build $H_0 \in V[G_\kappa][g_1]$ which is R-generic over $N[G_\kappa][g_1]$ and transfer it along k. We then have in $V[G_\kappa][g]$ a triangle



It is at this point that we need to do something new, as P is not closed enough to permit transfer of g along j, i. What saves us is the generic object $F \in V$ for the forcing $i(Q)$, where $Q = Add(\kappa, \kappa^{++})_V$.

Let us define in V a partial ordering $\dot{P}/P_\kappa$. The members of this partial ordering are canonical terms in V^{P_κ} for members of P, and they are ordered by

$$\dot{\sigma} \leq \dot{\tau} \iff \Vdash_{P_\kappa} \dot{\sigma} \leq_{\dot{P}} \dot{\tau}.$$

We refer the reader to [2] for proofs of the following facts.

1. $\dot{P}/P_\kappa$ is isomorphic in V to Q.

2. If G is P_κ-generic over V and H is $\dot{P}/P_\kappa$-generic over V then

$$H^G = \{ \dot{\sigma}^G \mid \dot{\sigma} \in H \}$$

is P-generic over $V[G]$.

We now apply the elementary embedding i to get a corresponding set of statements true in N. This shows that we may rearrange F as F^* which is $i(\dot{P}/P_\kappa)$-generic over N. We may then compute $X_0 = F^{*(G_\kappa * g_1 * H_0)}$ and conclude that X_0 is $i(P)$-generic over $N[G_\kappa][g_1][H_0]$. Since $\lambda < i(\kappa)$ we may transfer X_0 along k to get $X \in V[G_\kappa][g]$ which is $j(P)$-generic over $M[G_\kappa][g][H]$.

The last hurdle to be overcome is that, since X was obtained by these underhand means, there is no guarantee that $j``g \subseteq X$. Hence we may not be able to lift the embedding j to get a map from $V[G_\kappa][g]$ to $M[G_\kappa][g][H][X]$.

The cure for this is to notice that for each condition $p \in X$

$$|\mathrm{dom}(p) \cap \kappa \times j``\kappa^{++}| \leq \kappa$$

in $V[G_\kappa][g]$. As in [2] this enables us to alter p to conform with j, g using the fact that

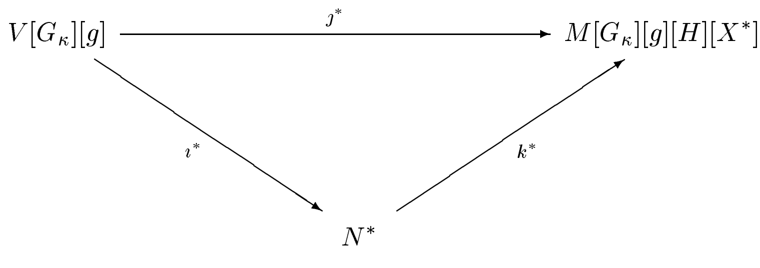
$$V[G_\kappa][g] \models {}^\kappa M[G_\kappa][g][H] \subseteq M[G_\kappa][g][H].$$

Each p has fewer than $j(\kappa)$ possible alterations in $M[G_\kappa][g][H]$ so it can be argued that the altered generic X^* is still $j(P)$-generic over the model $M[G_\kappa][g][H]$. Using this we may now build in $V[G_\kappa][g]$ a map

$$V[G_\kappa][g] \xrightarrow{\jmath^*} M[G_\kappa][g][H][X^*]$$

It remains only to be seen that in $V[G_\kappa][g]$ this map $\jmath^*$ is the ultrapower by $U = \{ X \mid \kappa \in \jmath^*(X) \}$, and that $\mathcal{P}\kappa^+ \subset Ult(V, U)$.

For the first statement factor $\jmath^*$ through the ultrapower by U, to get a triangle



Since the powerset of κ is contained in N^* and has size κ^{++} it is immediate that $\mathrm{crit}(k) > \kappa^{++}$. But $\jmath^*$ arises from a (κ, κ^{++})-extender so $\jmath^* = \imath^*$ and $M^* = N^*$.

Finally consider subsets of κ^+ in $V[G_\kappa][g]$. As before we know that

$$V \models [\kappa^+ \times P_{\kappa+1}]^{\kappa^+} \subseteq M.$$

This time round we may use this to conclude that (since $P_{\kappa+1}$ has the κ^+-c.c)

$$V^* \models \mathcal{P}\kappa^+ \subset N^*.$$

The proof of theorem 1 is complete.

♦

2 Long core models

We use the theory of core models for non-overlapping extender sequences; we refer the reader to [3] for an overview of this theory. We work under the blanket assumption that there is no inner model of a strong cardinal; this is harmless, as we are aiming to get an inner model of a much weaker hypothesis.

We will use the following facts about inner models and core models. We use "extender sequence" as shorthand for "non-overlapping coherent extender sequence".

Fact 1: If $\vec{E}$ is such that

$$L[\vec{E}] \models \vec{E} \text{ is a extender sequence and } (\kappa, \kappa^{++}) \in \mathrm{dom}(\vec{E})$$

then

$$L[\vec{E}] \models \kappa \text{ is } \mathcal{P}_2\kappa\text{-hypermeasurable.}$$

Recall from [3] that a predicate $\vec{E}$ is *strong* if

$$K[\vec{E}] \models \vec{E} \text{ is a extender sequence.}$$

Fact 2: If $\vec{E}$ is strong then

$$L[\vec{E}] \models \vec{E} \text{ is a extender sequence.}$$

Fact 3: There is a class $\vec{F}_{can}$ such that

1. $\vec{F}_{can}$ is strong.
2. If $\pi : K[\vec{F}_{can}] \longrightarrow W$ is an elementary embedding into a transitive class W then W is a normal iteration of $K[\vec{F}_{can}]$.
3. If $\lambda = \mathrm{cf}(\lambda) > \omega$ then $\vec{F}_{can} \cap H_\lambda$ and $K[\vec{F}_{can}] \cap H_\lambda$ are uniformly definable over H_λ.

Theorem 2: Let κ be measurable, with U a normal measure on κ such that

$$\mathcal{P}\kappa^{+} \subseteq Ult(V, U).$$

Then there is an inner model in which κ is $\mathcal{P}_2\kappa$-hypermeasurable.

Proof: Let $i : V \longrightarrow N = Ult(V, U)$ be the ultrapower by U. Certainly $2^{\kappa} > \kappa^{+}$ in V. By forcing to add a Cohen subset of κ^{++} we change nothing essential, so may assume that in V we have

$$2^{\kappa} = 2^{\kappa^{+}} = \kappa^{++}.$$

Notice also that $\kappa^{++} = (\kappa^{++})_N$, and that $H_{\kappa^{++}} \subset N$.

Let $\vec{F} = \vec{F}_{can}$. Define $\vec{G}$ by

$$\vec{G} = i(\vec{F}) = (\vec{F}_{can})_N.$$

By the agreement between V and N we know that

$$K[\vec{F}] \cap H_{\kappa^{++}} = K^N[\vec{G}] \cap H_{\kappa^{++}}.$$

Also we know that we have an elementary embedding

$$i \restriction K[\vec{F}] : K[\vec{F}] \longrightarrow K^N[\vec{G}].$$

Claim 1: $(\kappa, \kappa^{++}) \in \mathrm{dom}(\vec{F})$.

Proof: We know that $i \restriction K[\vec{F}]$ is a normal iteration of $K[\vec{F}]$. The first extender to be applied in that iteration must have critical point κ because $\kappa = \mathrm{crit}(i)$. Suppose that the first extender applied is $\vec{F}(\kappa, \eta)$, and suppose towards a contradiction that $\eta < \kappa^{++}$. The coherence property gives us that

$$(\kappa, \eta) \notin \mathrm{dom}(\vec{G})$$

which contradicts the agreement between $K[\vec{F}]$ and $K^N[\vec{G}]$. Hence $\eta \geq \kappa^{++}$, so $(\kappa, \kappa^{++}) \in \mathrm{dom}(\vec{G})$.

♦

It follows from this and the facts we quoted above that the cardinal κ is $\mathcal{P}_2\kappa$-hypermeasurable in the model $L[\vec{F}]$.

♦

The author would like to thank Alessandro Andretta for drawing this problem to his attention.

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