Some independence results on reflection

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Abstract

We prove that there is a certain degree of independence between stationary reflection phenomena at different cofinalities.

1 Introduction

Recall that a stationary subset S of a regular cardinal κ is said to *reflect* at $\alpha < \kappa$ if $\mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α. Stationary reflection phenomena have been extensively studied by set theorists, see for example [7].

Definition 1.1 Let $\kappa = \mathrm{cf}(\kappa) < \lambda = \mathrm{cf}(\lambda)$. $T^\lambda_\kappa =_{\mathrm{def}} \{ \alpha < \lambda \mid \mathrm{cf}(\alpha) = \kappa \}$. If $m < n < \omega$ then $S^n_m =_{\mathrm{def}} \{ \alpha < \aleph_n \mid \mathrm{cf}(\alpha) = \aleph_m \}$.

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Baumgartner proved in [1] that if κ is weakly compact, GCH holds, and $\omega < \delta = \mathrm{cf}(\delta) < \kappa$ then forcing with the Levy collapse $Coll(\delta, <\kappa)$ gives a model where for all $\rho < \delta$ and all stationary $T \subseteq T^\kappa_\rho$ the stationarity of T reflects to some $\alpha \in T^\kappa_\delta$. In this last result all the cofinalities $\rho < \delta$ are on the same footing; we will build models where reflection holds for some cofinalities but fails badly for others.

We introduce a more compact terminology for talking about reflection.

Definition 1.2 Let $\kappa = \mathrm{cf}(\kappa) < \lambda = \mathrm{cf}(\lambda) < \mu = \mathrm{cf}(\mu)$.

1. $Ref(\mu, \lambda, \kappa)$ holds iff for every stationary $S \subseteq T^\mu_\kappa$ there is a $\alpha \in T^\mu_\lambda$ with $S \cap \alpha$ stationary in α.
2. $Dnr(\mu, \lambda, \kappa)$ (Dense Non-Reflection) holds iff for every stationary $S \subseteq T^\mu_\kappa$ there is a stationary $T \subseteq S$ such that for no $\alpha \in T^\mu_\lambda$ is $T \cap \alpha$ stationary in α.

We will use a variation on an idea from Dzamonja and Shelah's paper [4].

Definition 1.3 Let $\kappa = \mathrm{cf}(\kappa) < \lambda = \mathrm{cf}(\lambda) < \mu = \mathrm{cf}(\mu)$. $Snr(\mu, \lambda, \kappa)$ (Strong Non-Reflection) holds iff there is $F : T^\mu_\kappa \longrightarrow \lambda$ such that for all $\alpha \in T^\mu_\lambda$ there is $C \subseteq \alpha$ closed and unbounded in α with $F \restriction C \cap T^\mu_\kappa$ strictly increasing.

As the name suggests, $Snr(\mu, \lambda, \kappa)$ is a strong failure of reflection. It is easy to see that if Jensen's Global $\Box$ principle holds then $Snr(\mu, \lambda, \kappa)$ holds for all $\kappa < \lambda < \mu$; in some sense the strong non-reflection principle captures exactly that part of $\Box$ which is useful for building non-reflecting stationary sets.

Lemma 1.4 $Snr(\mu, \lambda, \kappa) \Longrightarrow Dnr(\mu, \lambda, \kappa)$.

Proof: Let $S \subseteq T^\mu_\kappa$ be stationary, and let $F : T^\mu_\kappa \longrightarrow \lambda$ witness the strong non-reflection. Let $T \subseteq S$ be stationary such that $F \restriction T$ is constant. Let $\alpha \in T^\mu_\lambda$ and let C be a club in α on which F is strictly increasing, then C meets T at most once and hence T is non-stationary in α.

♦

We will prove the following results in the course of this paper.

Theorem 3.5: If the existence of a weakly compact cardinal is consistent, then $Ref(\aleph_3, \aleph_2, \aleph_0) + Snr(\aleph_3, \aleph_2, \aleph_1)$ is consistent.

Theorem 3.6: If the existence of a measurable cardinal is consistent, then $Ref(\aleph_3, \aleph_2, \aleph_0) + Snr(\aleph_3, \aleph_2, \aleph_1) + Snr(\aleph_3, \aleph_1, \aleph_0)$ is consistent.

Theorem 3.7: If the existence of a supercompact cardinal with a measurable above is consistent, then $Ref(\aleph_3, \aleph_2, \aleph_0) + Snr(\aleph_3, \aleph_2, \aleph_1) + Ref(\aleph_3, \aleph_1, \aleph_0)$ is consistent.

Theorem 4.1: If $Snr(\mu, \lambda, \kappa)$ and $\kappa < \kappa^* < \lambda$ then $Snr(\mu, \lambda, \kappa^*)$.

Theorem 4.2: Let $\kappa < \lambda < \mu < \nu$. Then $Ref(\nu, \mu, \lambda) + Ref(\nu, \lambda, \kappa) \Longrightarrow Ref(\nu, \mu, \kappa)$ and $Ref(\nu, \mu, \kappa) + Ref(\mu, \lambda, \kappa) \Longrightarrow Ref(\nu, \lambda, \kappa)$.

Theorem 5.10: If the existence of a weakly compact cardinal is consistent, then $Ref(\aleph_3, \aleph_2, \aleph_1) + Dnr(\aleph_3, \aleph_2, \aleph_0)$ is consistent.

Theorems 3.5, 3.6 and 3.7 were proved by the first author. Theorems 4.1 and 5.10 were proved by the second author, answering questions put to him by the first author. Theorem 4.2 was noticed by the first author (but has probably been observed many times). We would like to thank the anonymous referee for their very thorough reading of the first version of this paper.

2 Preliminaries

We will use the idea of *strategic closure* of a partial ordering (introduced by Gray in [5]).

Definition 2.1 Let $\mathbb{P}$ be a partial ordering, and let η be an ordinal.

1. The game $G(\mathbb{P}, \eta)$ is played by two players I and II, who take turns to play elements p_α of $\mathbb{P}$ for $0 < \alpha < \eta$, with player I playing at odd stages and player II at even stages (NB limit ordinals are even).

   The rules of the game are that the sequence that is played must be decreasing (not necessarily strictly decreasing), the first player who cannot make a move loses, and player II wins if play proceeds for η stages.

2. $\mathbb{P}$ is *η-strategically closed* iff player II has a winning strategy in $G(\mathbb{P}, \eta)$.

3. $\mathbb{P}$ is *$< \eta$-strategically closed* iff for all $\zeta < \eta$ $\mathbb{P}$ is ζ-strategically closed.

Strategic closure has some of the nice features of the standard notion of closure. For example a $(\delta + 1)$-strategically closed partial ordering will add no δ-sequences, and the property of being $(\delta + 1)$-strategically closed is preserved by forcing with $\leq \delta$-support (see [3] for more information on this subject). We will need to know that under some circumstances we can preserve a stationary set by forcing with a poset that has a sufficient degree of strategic closure.

The following is well-known.

Lemma 2.2 Let GCH hold, let $\lambda = \mathrm{cf}(\lambda)$ and $\kappa = \lambda^+$. Let $\delta = \mathrm{cf}(\delta) \leq \lambda$, and suppose that $\mathbb{P}$ is $(\delta+1)$-strategically closed and S is a stationary subset of T^κ_δ. Then S is still stationary in $V^{\mathbb{P}}$.

Proof: Let $p \Vdash$ "$\dot{C}$ is club in κ". Build $\langle X_\alpha : \alpha < \kappa \rangle$ a continuous increasing chain of elementary substructures of some large H_θ such that everything relevant is in X_0, $|X_\alpha| = \lambda$, ${}^{<\lambda}X_\alpha \subseteq X_{\alpha+1}$. We make the remark here that this would not be possible for λ singular, and indeed the theorem can fail in that case (see [8] for details).

Now find some limit γ such that $X_\gamma \cap \kappa \in S$, clearly $\mathrm{cf}(\gamma) = \delta$ and so ${}^{<\delta}X_\gamma \subseteq X_\gamma$. Let $\beta =_{\mathrm{def}} X_\gamma \cap \kappa$ and fix $\langle \beta_i : i < \delta \rangle$ cofinal in β. Since $\mathbb{P}, \delta \in X_0 \subseteq X_\gamma$ we can find in X_γ a winning strategy σ for the game $G(\mathbb{P}, \delta + 1)$.

Now we build a sequence $\langle p_\alpha : \alpha \leq \delta \rangle$ such that

1. For each even β, $p_\beta = \sigma(\vec{p} \restriction \beta)$.
2. For $\beta < \delta$, $p_\beta \in \mathbb{P} \cap X_\gamma$.
3. For each $i < \delta$, there is η such that $\beta_i < \eta < \beta$ and $p_{2i+1} \Vdash \hat{\eta} \in \dot{C}$.

We can keep going because $X_\gamma \prec H_\theta$, ${}^{<\delta}X_\gamma \subseteq X_\gamma$ and σ is a winning strategy. At the end of the construction p_δ is a refinement of p_0 which forces that β is a limit point of $\dot{C}$, and we are done.

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3 Some consistency results

In this section we prove (starting from a weakly compact cardinal) the consistency of $ZFC + Ref(\aleph_3, \aleph_2, \aleph_0) + Snr(\aleph_3, \aleph_2, \aleph_1)$. We also show that together with this we can have either $Ref(\aleph_3, \aleph_1, \aleph_0)$ or $Snr(\aleph_3, \aleph_1, \aleph_0)$.

We begin by defining a forcing $\mathbb{P}_{\mathrm{Snr}}$ to enforce $Snr(\aleph_3, \aleph_2, \aleph_1)$.

Definition 3.1 p is a condition in $\mathbb{P}_{\mathrm{Snr}}$ iff p is a function from a bounded subset of S^3_1 to $\aleph_2$, and for every $\gamma \in S^3_2$ with $\gamma \leq \sup(\mathrm{dom}(p))$ there is C club in γ such that $p \restriction C \cap S^3_1$ is strictly increasing. $\mathbb{P}_{\mathrm{Snr}}$ is ordered by extension.

It is easy to see that $\mathbb{P}_{\mathrm{Snr}}$ is ω_2-closed, and in fact that it is ω_2-directed closed.

Lemma 3.2 $\mathbb{P}_{\mathrm{Snr}}$ is $(\omega_2 + 1)$-strategically closed.

Proof: We describe a winning strategy for player II in $G(\mathbb{P}_{\mathrm{Snr}}, \omega_2 + 1)$. Suppose that p_α is the condition played at move α. Let β be an even ordinal, then at stage β II will play as follows.

Define $q_\beta =_{\mathrm{def}} \bigcup_{\alpha<\beta} p_\alpha$, $\rho_\beta =_{\mathrm{def}} \mathrm{dom}(q_\beta)$, and then let II play as follows: $p_\beta = q_\beta$ unless β is limit and $\mathrm{cf}(\beta) = \aleph_1$, in which case $p_\beta = q_\beta \cup \{(\rho_\beta, \beta)\}$.

The strategy succeeds because when play reaches stage ω_2, $\{ \rho_\beta \mid \beta < \omega_2 \}$ is a club witnessing that p_{ω_2} is a condition.

♦

This shows that $\mathbb{P}_{\mathrm{Snr}}$ preserves cardinals and cofinalities up to $\aleph_3$, from which it follows that $V^{\mathbb{P}_{\mathrm{Snr}}} \models Snr(\aleph_3, \aleph_2, \aleph_1)$. If GCH holds then $|\mathbb{P}_{\mathrm{Snr}}| = \aleph_3$, so $\mathbb{P}_{\mathrm{Snr}}$ has the $\aleph_4$-c.c. and all cardinals are preserved.

Now we define in $V^{\mathbb{P}_{\mathrm{Snr}}}$ a forcing $\mathbb{Q}$. This will enable us to embed $\mathbb{P}_{\mathrm{Snr}}$ into the Levy collapse $Coll(\omega_2, \omega_3)$ in a particularly nice way.

Definition 3.3 In $V^{\mathbb{P}_{\mathrm{Snr}}}$ let $F : S^3_1 \longrightarrow \aleph_2$ be the function added by $\mathbb{P}_{\mathrm{Snr}}$. $q \in \mathbb{Q}$ iff q is a closed bounded subset of $\aleph_3$, the order type of q is less than $\aleph_2$, and $F \restriction \lim(q) \cap S^3_1$ is strictly increasing.

The aim of $\mathbb{Q}$ is to add a club of order type ω_2 on which F is increasing. It is clear that $\mathbb{Q}$ is countably closed and collapses ω_3.

Lemma 3.4 If GCH holds, $\mathbb{P}_{\mathrm{Snr}} * \dot{\mathbb{Q}}$ is equivalent to $Coll(\aleph_2, \aleph_3)$.

Proof: Since $\mathbb{P}_{\mathrm{Snr}} * \dot{\mathbb{Q}}$ has cardinality $\aleph_3$ and collapses $\aleph_3$, it will suffice to show that it has an $\aleph_2$-closed dense subset. To see this look at those conditions (p, c) where $c \in V$, and $\max(c) = \sup(\mathrm{dom}(p))$. It is easy to see that this set is dense and $\aleph_2$-closed.

♦

Theorem 3.5 Let κ be weakly compact, let GCH hold. Define a two-step iteration by $\mathbb{P}_0 =_{\mathrm{def}} Coll(\omega_2, <\kappa)$ and $\mathbb{P}_1 =_{\mathrm{def}} (\mathbb{P}_{\mathrm{Snr}})_{V^{\mathbb{P}_0}}$. Then $V^{\mathbb{P}_0 * \mathbb{P}_1} \models Ref(\aleph_3, \aleph_2, \aleph_0) + Snr(\aleph_3, \aleph_2, \aleph_1)$.

Proof: We will first give the proof for the case when κ is measurable and then show how to modify it for the case when κ is just weakly compact. Assuming that κ is measurable, let $j : V \longrightarrow M$ be an elementary embedding into a transitive inner model with critical point κ, where ${}^{\kappa}M \subseteq M$. Notice that by elementarity and the closure of M, $j(\mathbb{P}_0) = Coll(\omega_2, < j(\kappa))_M = Coll(\omega_2, < j(\kappa))_V$.

Let G be $\mathbb{P}_0$-generic over V and let H be $\mathbb{P}_1$-generic over $V[G]$. We already know that $V[G * H] \models Snr(\aleph_3, \aleph_2, \aleph_1)$, so let us assume that in $V[G * H]$ we have S a stationary subset of S^3_0. To prove that S reflects we will build a generic embedding with domain $V[G * H]$ extending j. Notice that since $|\mathbb{P}_0 * \mathbb{P}_1| = \kappa$ we can prove (by looking at canonical names) that $V[G * H] \models {}^{\kappa}M[G * H] \subseteq M[G * H]$, so in particular we have $S \in M[G * H]$.

We start by forcing with $\mathbb{Q}$ over $V[G * H]$ to get a generic object I. $H * I$ is generic over $V[G]$ for $(\mathbb{P} * \mathbb{Q})_{V[G]}$ which is equivalent to $Coll(\omega_2, \kappa)$, so we can regard $G * H * I$ as being generic for $Coll(\omega_2, \leq \kappa)$. Now let J be $Coll(\omega_2, [\kappa, j(\kappa)))$-generic over $V[G * H * I]$, then $G * H * I * J$ is $j(\mathbb{P}_0)$-generic over V (so a fortiori over M) and $j``G \subseteq G * H * I * J$ so that we can lift to get $j : V[G] \longrightarrow M[G * H * I * J]$.

It remains to lift j onto $V[G * H]$, for which we need to force a generic K for $j(\mathbb{P}_1)$ with the property that $j``H \subseteq K$. We will get $K$ by constructing a *master condition* in $j(\mathbb{P}_0)$ (that is, a condition refining all the conditions in $j``H$) and forcing below that master condition. A natural candidate for a master condition is $F =_{\text{def}} \bigcup j``H$, where it is easily seen (since $\text{crit}(j) = \kappa$ and $j \restriction \mathbb{P}_1 = id$) that F is the generic function from κ to $\aleph_2$ added by H.

The models $V[G]$ and $M[G * H * I * J]$ agree in their computations of $T^\kappa_{\aleph_1}$ and $T^\kappa_{\aleph_2}$, so F is increasing on a club at all the relevant points below κ. Now $\kappa = (\aleph_3)_{V[G]}$ is an ordinal of cofinality $\aleph_2$ in $M[G * H * I * J]$, but there is no problem here because I has introduced a club in κ on which F is increasing. Hence F is a condition in $j(\mathbb{P}_1)$ and we can force to get $K \supseteq j``H$ as desired.

We claim that S is still stationary in $M[G * H * I * J * K]$. I is generic for countably closed forcing, J is generic for $\aleph_2$-closed forcing and K is generic for $\aleph_2$-closed forcing so that S (being a set of cofinality ω ordinals) remains stationary. Now we argue as usual that since $j(S) \cap \kappa = S$ and $\mathrm{cf}(\kappa) = \omega_2$ in $M[G * H * I * J * K]$, there must exist $\alpha < \kappa$ in $V[G * H]$ such that $\mathrm{cf}(\alpha) = \omega_2$ and $S \cap \alpha$ is stationary in α.

We promised at the start of this proof that we would show how to weaken the assumptions on κ from measurability to weak compactness. We will actually sketch two arguments, based on two well-known characterisations of weak compactness. See Hauser's paper [6] for detailed accounts of some similar arguments.

1. κ is weakly compact iff for every $A \subseteq V_\kappa$ and every Π^1_1 formula ϕ, $V_\kappa \models \phi(A) \Longrightarrow \exists \alpha\ V_\alpha \models \phi(A \cap V_\alpha)$.
2. κ is weakly compact iff κ is strongly inaccessible and for every transitive M such that $|M| = \kappa$, ${}^{<\kappa}M \subseteq M$ and M models enough set theory there is a transitive set N and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$.

Argument 1: $\mathbb{P}_0 * \mathbb{P}_1 \subseteq V_\kappa$, and so if $\dot{S}$ is a name for a stationary subset of κ

we can represent it by $S^* = \{ (p, \alpha) \mid p \Vdash \hat{\alpha} \in \dot{S} \} \subseteq V_\kappa$. The fact that S is forced to be stationary can be written as a Π^1_1 sentence (the universal second-order quantification is over names for clubs). Using the first characterisation given above we can find inaccessible $\alpha < \kappa$ such that $S^* \cap V_\alpha$ is a $Coll(\omega_2, < \alpha) * (\mathbb{P}_{Snr})_{V^{Coll(\omega_2, <\alpha)}}$ name for a stationary set.

Now the argument is just like the one from a measurable, only with α playing the role of κ and κ replacing $j(\kappa)$. We see that S has an initial segment $S \cap \alpha$ which is stationary in a certain intermediate generic extension, and just need to check that $S \cap \alpha$ remains stationary and that α becomes a point of cofinality $\aleph_2$. This is routine.

Argument 2: Given a name S for a stationary subset of S^3_0, build S and $\mathbb{P}_0 * \mathbb{P}_1$ into an appropriate model M of size κ. Get j as in the second characterisation as above. Repeat (mutatis mutandis) the argument from a measurable.

♦

Having proved Theorem 3.5, it is natural to ask whether there is any connection between reflection to points of cofinality ω_2 and reflection to points of cofinality ω_1. The following results provide a partial (negative) answer.

Theorem 3.6 Con($Ref(\aleph_3, \aleph_2, \aleph_0) + Snr(\aleph_3, \aleph_2, \aleph_1) + Snr(\aleph_3, \aleph_1, \aleph_0)$) follows from the consistency of a measurable cardinal.

Proof: Let κ be measurable. Without loss of generality GCH holds (as we can move to the inner model $L[\mu]$). We will sketch a proof that we can force to get $Snr(\kappa, \aleph_1, \aleph_0)$ without destroying the measurability of κ. A more detailed argument for a very similar result is given in [3] (alternatively one can argue that because $\square$ holds in $L[\mu]$, $Snr(\kappa, \aleph_1, \aleph_0)$ is already true in that model). Let $j : V \longrightarrow M$ be the ultrapower map associated with some normal measure U on M.

We will do a reverse Easton iteration of length $\kappa + 2$, forcing at every regular cardinal $\alpha \leq \kappa^+$ with $\mathbb{Q}_\alpha$, where $\mathbb{Q}_\alpha$ is the natural forcing to add a witness to $Snr(\alpha, \aleph_1, \aleph_0)$ by initial segments. An easy induction shows that $\mathbb{Q}_\alpha$ is $< \alpha$-strategically closed, the point being that the witnesses added below α can be used to produce strategies in the game played on $\mathbb{Q}_\alpha$. The argument is exactly parallel to the proof of Lemma 6 in [3].

Let us break up the generic as $G * g * h$, where G is $\mathbb{P}_\kappa$-generic, g is $\mathbb{Q}_\kappa$-generic and h is $\mathbb{Q}_{\kappa^+}$-generic. The key point is that $j(\mathbb{P}_\kappa)/G * g * h$ is κ^+-strategically closed in $V[G * g * h]$, so that by GCH we can build $H \in V[G * g * h]$ which is $j(\mathbb{P}_\kappa)/G * g * h$-generic over $M[G * g * h]$. Since $j``G \subseteq G * g * h * H$, we can lift $j$ to get $j : V[G] \longrightarrow M[G * g * h * H]$. It is easy to see that $\bigcup j``g (= \bigcup g)$ will serve as a master condition, so using GCH again we may find $g^+ \in V[G * g * h]$ such that $j``g \subseteq g^+$ and $g^+$ is $j(\mathbb{P}_\kappa)$-generic over $M[G * g * h * H]$. Finally we claim that $j``h$ generates a $j(\mathbb{P}_{\kappa^+})$-generic filter over $M[G * g * h * H * g^+]$, because $\mathbb{P}_{\kappa^+}$ is distributive enough and $M[G * g * h * H * g^+] = \{ \, j(F)(\kappa) \mid \mathrm{dom}(F) = \kappa, F \in V[G * g] \, \}$. Hence in $V[G * g * h]$ we can lift j onto $V[G * g * h]$, so the measurability of κ is preserved.

Now we just repeat the construction (from a measurable) of Theorem 3.5, and claim that in the final model $Snr(\aleph_3, \aleph_1, \aleph_0)$ holds. The point is that if

$F : T^{\kappa}_{\aleph_0} \longrightarrow \omega_1$ witnesses the truth of $Snr(\kappa, \aleph_1, \aleph_0)$, then in the final model F witnesses $Snr(\aleph_3, \aleph_1, \aleph_0)$ because the forcing from Theorem 1 does not change $T^{\kappa}_{\aleph_0}$ or $T^{\kappa}_{\aleph_1}$.

♦

We can also go to the opposite extreme.

Theorem 3.7 Let κ be λ-supercompact, where $\lambda > \kappa$ and λ is measurable. Let GCH hold. Then $Ref(\aleph_3, \aleph_2, \aleph_0) + Snr(\aleph_3, \aleph_2, \aleph_1) + Ref(\aleph_3, \aleph_1, \aleph_0)$ holds in some forcing extension.

Proof: We will start by forcing with $\mathbb{P} =_{\rm def} Coll(\omega_1, < \kappa)$, after which κ is ω_2 and λ is still measurable. Then we will do the construction of Theorem 3.5, that is we force with $(Coll(\kappa, < \lambda) * \mathbb{P}_{\rm Snr})_{V^{\mathbb{P}}}$. Let $\mathbb{P}_0 = Coll(\kappa, < \lambda)_{V^{\mathbb{P}}}$ and $\mathbb{P}_1 = (\mathbb{P}_{Snr})_{V^{\mathbb{P} * \mathbb{P}_0}}$.

We need to check that $Ref(\aleph_3, \aleph_1, \aleph_0)$ holds in the final model. To see this fix $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$. Let G, H and I be the generics for $\mathbb{P}$, $\mathbb{P}_0$ and $\mathbb{P}_1$ respectively.

Since $\mathbb{P}_0 * \mathbb{P}_1$ is countably closed and has size λ, we may find an embedding $i : \mathbb{P} * \mathbb{P}_0 * \mathbb{P}_1 \longrightarrow j(\mathbb{P})$ such that $i \restriction \mathbb{P} = id$ and $j(\mathbb{P}) / i``(\mathbb{P} * \mathbb{P}_0 * \mathbb{P}_1)$ is countably closed. Let $J$ be $j(\mathbb{P}) / i``(\mathbb{P} * \mathbb{P}_0 * \mathbb{P}_1)$-generic, then we can lift j in the usual way to get $j : V[G] \longrightarrow M[G * H * I * J]$. Since $\mathbb{P}_0 * \mathbb{P}_1$ is a κ-directed-closed forcing notion of size λ, we may find a lower bound for $j``(H * I)$ in $j(\mathbb{P}_0 * \mathbb{P}_1)$ and use it as a master condition, forcing $K$ such that $j``(H * I) \subseteq K$ and lifting j to $j : V[G * H * I] \longrightarrow M[G * H * I * J * K]$.

Now suppose that in $V[G * H * I]$ we have $T \subseteq S^3_0$ a stationary set. If $\mu = \bigcup j``\lambda$ then we may argue as usual that $T \in M[G * H * I]$ and that $M[G{*}H{*}I] \models j(T) \cap \mu$ is stationary in μ. Since $J{*}K$ is generic for countably closed forcing and $j(T) \cap \mu \subseteq T^{\mu}_{\aleph_0}$ this will still be true in $M[G * H * I * J * K]$, and since $\mathrm{cf}(\mu) = \aleph_1$ in this last model we have by elementarity that T reflects to some point in S^3_1.

♦

4 Some ZFC results

In the light of the results from the last section it is natural to ask about the consistency of $Ref(\aleph_3, \aleph_2, \aleph_1) + Snr(\aleph_3, \aleph_2, \aleph_0)$. The first author showed by a rather indirect proof that this is impossible, and the second author observed that there is a simple reason for this.

Theorem 4.1 If $Snr(\mu, \lambda, \kappa)$ and $\kappa < \kappa^* = \mathrm{cf}(\kappa^*) < \lambda$ then $Snr(\mu, \lambda, \kappa^*)$.

Proof: Let $F : T^{\mu}_{\kappa} \longrightarrow \lambda$ witness $Snr(\mu, \lambda, \kappa)$. Define $F^* : T^{\mu}_{\kappa^*} \longrightarrow \lambda$ by

$$F^* : \sigma \longmapsto \min \{ \bigcup_{\alpha \in C \cap T^{\mu}_{\kappa}} F(\alpha) \mid C \text{ club in } \sigma \}.$$

We claim that F^* witnesses $Snr(\mu, \lambda, \kappa^*)$. To see this, let $\delta \in T^{\mu}_{\lambda}$ and fix D club in δ such that $F \restriction D \cap T^{\mu}_{\kappa}$ is strictly increasing. Let $\sigma \in \lim(D) \cap T^{\mu}_{\kappa^*}$, and observe that $F^*(\sigma) = \bigcup_{\alpha \in D \cap T^{\mu}_{\kappa}} F(\alpha)$, because for any club C in σ we have

$$\bigcup_{\alpha \in C \cap T^{\mu}_{\kappa}} F(\alpha) \geq \bigcup_{\alpha \in C \cap D \cap T^{\mu}_{\kappa}} F(\alpha) = \bigcup_{\alpha \in D \cap T^{\mu}_{\kappa}} F(\alpha).$$

It follows that $F^* \restriction \lim(D) \cap T^\mu_{\kappa^*}$ is strictly increasing, because if $\sigma_0, \sigma_1 \in \lim(D) \cap T^\mu_{\kappa^*}$ with $\sigma_0 < \sigma_1$ and β is any point in $\lim(D) \cap T^\mu_\kappa \cap (\sigma_0, \sigma_1)$ then $F^*(\sigma_0) \leq F(\beta) < F^*(\sigma_1)$.

♦

We also take the opportunity to record some other easy remarks, which put limits on the extent of the independence between different forms of reflection.

Theorem 4.2 Let $\kappa < \lambda < \mu < \nu$ be regular. Then

1. $Ref(\nu, \mu, \lambda) + Ref(\nu, \lambda, \kappa) \Longrightarrow Ref(\nu, \mu, \kappa)$.

2. $Ref(\nu, \mu, \kappa) + Ref(\mu, \lambda, \kappa) \Longrightarrow Ref(\nu, \lambda, \kappa)$.

Proof: For the first claim: let $S \subseteq T^\nu_\kappa$ be stationary, and let us define $T = \{ \alpha \in T^\nu_\lambda \mid S \cap \alpha \text{ is stationary} \}$. We claim that T is stationary in ν. To see this suppose C is club in ν and disjoint from T, and consider $C \cap S$; this set must reflect at some $\gamma \in T^\nu_\lambda$, but then on the one hand $S \cap \gamma$ is stationary (so γ is in T) while on the other hand C is unbounded in γ (so $\gamma \in C$), contradicting the assumption that C and T are disjoint.

Now let $\delta \in T^\nu_\mu$ be such that $T \cap \delta$ is stationary in δ. We claim that S reflects at δ. For if D is club in δ then there is $\gamma \in T \cap \lim(D)$ by the stationarity of $T \cap \delta$, and now since $D \cap \gamma$ is club in γ and $S \cap \gamma$ is stationary in γ there is $\beta \in D \cap S$. This proves the first claim.

For the second claim: $S \subseteq T^\nu_\kappa$ be stationary, and let $\delta \in T^\nu_\mu$ be such that $S \cap \delta$ is stationary in δ. Let $f : \mu \longrightarrow \delta$ be continuous increasing and cofinal in δ, and let $S^* = \{ \alpha < \mu \mid f(\alpha) \in S \}$. Then S^* is stationary in μ, and we can find $\beta \in T^\mu_\lambda$ such that $S^* \cap \beta$ is stationary in β. Now $f(\beta) \in T^\nu_\lambda$ and $S \cap f(\beta)$ is stationary in $f(\beta)$. This proves the second claim.

♦

5 More consistency results

From the results in the previous section we saw in particular that we cannot have $Ref(\aleph_3, \aleph_2, \aleph_1) + Snr(\aleph_3, \aleph_2, \aleph_0)$. In this section we will see that $Ref(\aleph_3, \aleph_2, \aleph_1) + Dnr(\aleph_3, \aleph_2, \aleph_0)$ is consistent.

We will need some technical definitions and facts before we can start the main proof.

Definition 5.1 Let S be a stationary subset of $\aleph_3$. We define notions of forcing $\mathbb{P}(S)$, $\mathbb{Q}(S)$ and $\mathbb{R}(S)$.

1. c is a condition in $\mathbb{P}(S)$ iff c is a closed bounded subset of $\aleph_3$ such that $c \cap S = \emptyset$.

2. d is a condition in $\mathbb{Q}(S)$ iff d is a function with $\mathrm{dom}(d) < \aleph_3$, $d : \mathrm{dom}(d) \longrightarrow 2$, $d(\gamma) = 1 \Longrightarrow \gamma \in S$, and for all $\alpha \leq \mathrm{dom}(d)$ if $\mathrm{cf}(\alpha) > \omega$ then there is $C \subseteq \alpha$ closed unbounded in α such that $\gamma \in C \Longrightarrow d(\gamma) = 0$.

3. e is a condition in $\mathbb{R}(S)$ iff e is a closed bounded subset of $\aleph_3$ such that for every point $\alpha \in \lim(e)$ with $\mathrm{cf}(\alpha) > \omega$ the set $S \cap \alpha$ is non-stationary in α.

In each case the conditions are ordered by end-extension.

The aims of these various forcings are respectively to kill the stationarity of S ($\mathbb{P}(S)$), to add a non-reflecting stationary subset of S ($\mathbb{Q}(S)$), and to make S non-reflecting on a closed unbounded set of points ($\mathbb{R}(S)$). Notice that for some choices of S the definitions of $\mathbb{P}(S)$ and $\mathbb{R}(S)$ may not behave very well, for example if $S = \omega_3$ then $\mathbb{P}(S)$ is empty and $\mathbb{R}(S)$ only contains conditions of countable order type. Notice also that $\mathbb{Q}(S)$ and $\mathbb{R}(S)$ are countably closed.

Lemma 5.2 Let GCH hold. Let $S \subseteq S^3_0$ and suppose that there is a club C of ω_3 such that $S \cap \alpha$ is non-stationary for all $\alpha \in C$ with $\mathrm{cf}(\alpha) > \omega$. Let T be a stationary subset of S^3_1. Then $\mathbb{P}(S)$ adds no ω_2-sequences of ordinals, and also preserves the stationarity of T.

Proof: First we prove that $\mathbb{P}(S)$ is ω_2-distributive. Let $\langle D_\alpha : \alpha < \omega_2 \rangle$ be a sequence of dense sets in $\mathbb{P}(S)$, and let $c \in \mathbb{P}(S)$ be a condition. Fix some large regular cardinal θ and let $c, C, \vec{D}, S \in X \prec H_\theta$ where $|X| = \omega_2$, ${}^{\omega_1}X \subseteq X$. Let γ be the ordinal $X \cap \omega_3$, then $\mathrm{cf}(\gamma) = \omega_2$ by the closure of X. By elementarity it follows that C is unbounded in γ, so that $\gamma \in C$ and $S \cap \gamma$ is non-stationary.

Fix $B \subseteq \gamma$ closed unbounded in γ such that $B \cap S = \emptyset$ and B has order type ω_2. Now we build a chain of conditions $c_\alpha \in \mathbb{P}(S) \cap X$ for $\alpha < \omega_2$ such that $c_0 \leq c$, $c_{2\beta+1} \in D_\beta$ and $\max(c_{2\beta}) \in B$; we can continue at each limit stage because B is disjoint from S and X is sufficiently closed. Finally we let $d =_{def} \bigcup_{\alpha<\omega_2} c_\alpha \cup \{\gamma\}$, then $d \leq c$ and $d \in D_\beta$ for all $\beta < \omega_2$. This shows that $\mathbb{P}(S)$ is ω_2-distributive.

The argument for the preservation of stationarity is similar. Let $\dot{E}$ be a $\mathbb{P}(S)$-name for a closed unbounded subset of ω_3 and let $c \in \mathbb{P}(S)$ be a condition. This time build X such that $c, C, \dot{E}, S \in X \prec H_\theta$ where $|X| = \omega_1$, ${}^{\omega}X \subseteq X$ and $\delta =_{def} \sup(X \cap \omega_3) \in T$. Again $\delta \in C$, so $S \cap \delta$ is nonstationary. Choose $B \subseteq \delta$ closed unbounded of order type ω_1 with $B \cap S = \emptyset$ and build a chain of conditions $c_\alpha \in \mathbb{P}(S) \cap X$ such that $c_0 \leq c$, $\max(c_{2\beta}) \in B$ and $c_{2\beta+1}$ forces that $\dot{E} \cap (\max(c_{2\beta}), \max(c_{2\beta+1})) \neq \emptyset$. Finally if $d =_{def} \bigcup_{\alpha<\omega_1} c_\alpha \cup \{\delta\}$ then $d \leq c$ and $d \Vdash \hat{\delta} \in \hat{T} \cap \dot{E}$.

♦

Now we describe a certain kind of forcing iteration. It will transpire that all iterations of this type are ω_4-c.c. and (ω_2+1)-strategically closed, so that in particular all cardinalities and cofinalities are preserved.

Definition 5.3 Fix $F : \omega_4 \times \omega_3 \longrightarrow \omega_4$ such that for all $\beta < \omega_4$ the map $i \longmapsto F(\beta, i)$ is a surjection from ω_3 onto β.

$\mathbb{P}_\beta$ is a *nice iteration* iff

1. $\beta \leq \omega_4$.
2. $\mathbb{P}_\beta$ is an iteration of length β with $\leq \omega_2$-supports.
3. $\mathbb{Q}_0 = \{0\}$.
4. $\mathbb{Q}_{2\gamma+1}$ is $\mathbb{Q}(\dot{S}_\gamma)_{V^{P_{2\gamma+1}}}$ where $\dot{S}_\gamma$ is some $\mathbb{P}_{2\gamma+1}$-name for a stationary subset of S^3_0. Let S^*_γ be the non-reflecting stationary subset of S_γ

which is added by $\mathbb{Q}_{2\gamma+1}$.

5. $\mathbb{Q}_{2\gamma}$ is $\mathbb{R}(\dot{R}_\gamma)$ where R_γ is the diagonal union of $\langle S^*_{F(\gamma,i)} : i < \omega_3 \rangle$. That is $R_\gamma = \{ \delta \in S^3_0 \mid \exists i < \delta \; \delta \in S^*_{F(\gamma,i)} \}$.

It is clear that an initial segment of a nice iteration is nice. Also every final segment of a nice iteration is countably closed, so that all the sets S^*_γ remain stationary throughout the iteration. The following remark will be useful later.

Lemma 5.4 Let $\gamma < \delta < \omega_4$. Then $R_\gamma - R_\delta$ is non-stationary.

Proof: Let C be the closed and unbounded set of $i < \omega_3$ such that

$$\{ F(\gamma, j) \mid j < i \} = \{ F(\delta, j) \mid j < i \} \cap \gamma.$$

Let $i \in C \cap R_\gamma$. Then for some $j < i$ we have $i \in S^*_{F(\gamma,j)}$, and by the definition of C there is $k < i$ such that $F(\gamma, j) = F(\delta, k)$. So $i \in S^*_{F(\delta,k)}$, $i \in R_\delta$, and we have proved that $C \cap R_\gamma \subseteq R_\delta$.

♦

We define a certain subset of $\mathbb{P}_\beta$, which we call $\mathbb{P}^*_\beta$.

Definition 5.5 If $\mathbb{P}_\beta$ is a nice iteration then $\mathbb{P}^*_\beta$ is the set of conditions $p \in \mathbb{P}_\beta$ such that

1. $p(\gamma) \in \hat{V}$ (that is, $p(\gamma)$ is a canonical name for an object in V) for all $\gamma \in \mathrm{dom}(p)$.

2. There is an ordinal $\rho(p)$ such that

(a) $2\delta \in \mathrm{dom}(p) \Longrightarrow \max(p(2\delta)) = \rho(p)$.

(b) $2\delta + 1 \in \mathrm{dom}(p) \Longrightarrow \mathrm{dom}(p(2\delta + 1)) = \rho(p) + 1$.

(c) $2\delta + 1 \in \mathrm{dom}(p) \Longrightarrow p(2\delta + 1)(\rho(p)) = 0$.

3. If $2\delta \in \mathrm{dom}(p)$ then $\forall i < \rho(p)$ $2F(\delta, i) + 1 \in \mathrm{dom}(p)$.

4. If $2\delta + 1 \in \mathrm{dom}(p)$ then $p \restriction 2\delta + 1$ decides $S_\delta \cap \rho(p)$.

Lemma 5.6 If $p \in \mathbb{P}^*_\beta$ and $2\delta \in \mathrm{dom}(p)$ then $p \restriction 2\delta$ forces that $\rho(p) \notin R_\delta$.

Proof: Let $\rho = \rho(p)$. By the definition of $\mathbb{P}^*_\beta$, we see that $2F(\delta, i) + 1 \in \mathrm{dom}(p)$ and $p(2F(\delta, i) + 1)(\rho) = 0$ for all $i < \rho$. This means that $p \Vdash \rho \notin \dot{S}^*_{F(\delta,i)}$ for all $i < \rho$, which is precisely to say $p \restriction 2\delta \Vdash \rho \notin \dot{R}_\delta$.

♦

Lemma 5.7 Let $\mathbb{P}_\beta$ be a nice iteration. Let $\delta \leq \omega_2$ be a limit ordinal and let $\langle p_\gamma : \gamma < \delta \rangle$ be a decreasing sequence of conditions from $\mathbb{P}^*_\beta$ such that $\langle \rho(p_\gamma) : \gamma < \delta \rangle$ is continuous and increasing. Define q by setting $\mathrm{dom}(q) = \bigcup_{\gamma<\delta} \mathrm{dom}(p_\gamma)$, $\rho = \bigcup_{\gamma<\delta} \rho(q_\gamma)$, $q(2\epsilon) = \bigcup \{ \, p_\gamma(2\epsilon) \mid 2\epsilon \in \mathrm{dom}(p_\gamma) \, \} \cup \{\rho\}$, $q(2\epsilon + 1) = \bigcup \{ \, p_\gamma(2\epsilon + 1) \mid 2\epsilon + 1 \in \mathrm{dom}(p_\gamma) \, \} \cup \{(\rho, 0)\}$.

Then $q \in \mathbb{P}^*_\beta$ and $\rho(q) = \rho$.

Proof: Clearly it is enough to show that $q \in \mathbb{P}_\beta$. Most of this is routine; the key points are that $q \restriction 2\epsilon$ forces that $R_\epsilon \cap \rho$ is non-stationary, and that $q \restriction 2\epsilon + 1$ forces that $S^*_\epsilon \cap \rho$ is non-stationary.

For the first point, observe that for all sufficiently large $\gamma < \delta$ we have $2\epsilon \in \mathrm{dom}(p_\gamma)$, so that by the last lemma $p_\gamma \restriction 2\epsilon \Vdash \rho(p_\gamma) \notin R_\epsilon$; since

$\langle \rho(p_\gamma) : \gamma < \delta \rangle$ is continuous and $q \restriction 2\epsilon$ refines $p_\gamma \restriction 2\epsilon$, this implies that $q \restriction 2\epsilon$ forces that R_ϵ is not stationary in ρ.

Similarly, $2\epsilon + 1 \in \mathrm{dom}(p_\gamma)$ for all large γ, so that $p_\gamma(2\epsilon+1)(\rho(p_\gamma)) = 0$ for all large γ. It follows immediately that $q \restriction (2\epsilon+1)$ forces that $S^*_\epsilon \cap \rho$ is non-stationary.

♦

Lemma 5.8 $\mathbb{P}^*_\beta$ is dense in $\mathbb{P}_\beta$, and $\mathbb{P}_\beta$ is (ω_2+1)-strategically closed.

Proof: The proof is by induction on β. We prove first that $\mathbb{P}^*_\beta$ is dense.

$\beta = 0$: there is nothing to do.

$\beta = 2\alpha+2$: fix $p \in \mathbb{P}_\beta$. Since $\mathbb{P}_{2\alpha+1}$ is strategically closed and $\mathbb{P}^*_{2\alpha+1}$ is dense we may find $q_1 \in \mathbb{P}^*_{2\alpha+1}$ such that $q_1 \leq p \restriction (2\alpha+1)$, q_1 decides $p(2\alpha+1)$, and $\rho(q_1) > \mathrm{dom}(p(2\alpha+1))$.

Now we build a decreasing ω-sequence $\vec{q}$ of elements of $\mathbb{P}^*_{2\alpha+1}$ such that $\rho(q_n)$ is increasing and q_{n+1} decides $\dot{S}_\alpha \cap \rho(q_n)$; at each stage we use the strategic closure of $\mathbb{P}_{2\alpha+1}$ and the fact that $\mathbb{P}^*_{2\alpha+1}$ is a dense subset. After ω steps we define q as follows; let $\rho = \bigcup \rho(q_n)$, $\mathrm{dom}(q) = \bigcup_n \mathrm{dom}(q_n) \cup \{2\alpha+1\}$, and

1. For $\gamma < \alpha$, $q(2\gamma+1) = \bigcup_n q_n(2\gamma+1) \cup \{(\rho, 0)\}$.

2. For $\gamma \leq \alpha$, $q(2\gamma) = \bigcup_n q_n(2\gamma) \cup \{\rho\}$.

3. $q(2\alpha+1)(i) = p(2\alpha+1)(i)$ if $i \in \mathrm{dom}(p(2\alpha+1))$, and 0 otherwise.

By the last lemma, $q \restriction 2\alpha + 1 \in \mathbb{P}^*_{2\alpha+1}$. It is routine to check that $q \in \mathbb{P}^*_{2\alpha+2}$ and $\rho(q) = \rho$.

$\beta = 2\alpha + 1$: this is exactly like the last case, except that now we demand $\forall i < \rho(q_n)\ 2F(\alpha, i) + 1 \in \mathrm{dom}(q_{n+1})$.

β is limit, $\mathrm{cf}(\beta) = \omega_1$: Fix $\langle \beta_i : i < \omega_1 \rangle$ which is continuous increasing and cofinal in β. Let $p \in \mathbb{P}_\beta$. Find $q_0 \in \mathbb{P}^*_{\beta_0}$ such that $q_0 \leq p \restriction \beta_0$ and set $p_0 = q_0 \frown p \restriction [\beta_0, \beta)$.

Now we define q_i and p_i by induction for $i \leq \omega_1$.

1. Choose $q_{i+1} \leq p_i \restriction \beta_{i+1}$ with $q_{i+1} \in \mathbb{P}^*_{\beta_{i+1}}$, and then define $p_{i+1} = q_{i+1} \frown p \restriction [\beta_{i+1}, \beta)$.

2. For i limit let $\rho_i = \bigcup_{j<i} \rho(q_j)$, $q_i(2\gamma + 1) = \bigcup_{j<i} q_j(2\gamma + 1) \cup \{(\rho_i, 0)\}$, $q_i(2\gamma) = \bigcup_{j<i} q_j(2\gamma) \cup \{\rho_i\}$. Then let $p_i = q_i \frown p \restriction [\beta_i, \beta_{i+1})$.

For $i < \omega_1$ it is easy to see that p_i, q_i are conditions. We claim that $q = q_{\omega_1} \in \mathbb{P}^*_\beta$. The only subtle point is to see that $q \in \mathbb{P}_\beta$. Let $2\delta + 1 \in \mathrm{dom}(q)$. Then for all large i we know $2\delta + 1 < \beta_i$, $2\delta + 1 \in \mathrm{dom}(q_i)$, so that in particular $q_i(2\delta + 1)(\rho_i) = 0$ for all large i. This means that $q(2\delta + 1)$ is the characteristic function of a set which does not reflect at ρ, so is a legitimate condition in $\mathbb{Q}_{2\delta+1}$. Similarly if $2\delta \in \mathrm{dom}(q)$ then for all large i we see that $q_i \restriction 2\delta \Vdash \rho_i \notin R_\delta$, so that $q \restriction 2\delta$ forces that the stationarity of R_δ does not reflect at ρ.

β is a limit, $\mathrm{cf}(\beta) = \omega$ or ω_2: similar to the cofinality ω_1 case.

$\mathrm{cf}(\beta) = \omega_3$: easy because $\mathbb{P}_\beta$ is the direct limit of the sequence $\langle \mathbb{P}_\gamma : \gamma < \beta \rangle$.

This concludes the proof that $\mathbb{P}^*_\beta$ is dense. It is now easy to see that $\mathbb{P}_\beta$ is $(\omega_2 + 1)$-strategically closed; the strategy for player II is simply to play into the dense set $\mathbb{P}^*_\beta$ at every successor stage, and to play a lower bound constructed as in Lemma 5.7 at each limit stage.

♦

Lemma 5.9 Let $\mathbb{P}_{2\gamma}$ be a nice iteration of length less than ω_4. Then $\mathbb{P}_{2\gamma} * \mathbb{P}(\dot{R}_\gamma)$ is $(\omega_2 + 1)$-strategically closed.

Proof: This is just like the last lemma.

♦

Notice that the effect of forcing with $\mathbb{P}(R_\gamma)$ is to destroy the stationarity of all the sets S^*_δ for $\delta < \gamma$. We are now ready to prove the main result of this section.

Theorem 5.10 If the existence of a weakly compact cardinal is consistent, then $Ref(\aleph_3, \aleph_2, \aleph_1) + Dnr(\aleph_3, \aleph_2, \aleph_0)$ is consistent.

Proof: As in the proof of Theorem 3.5, we will first give a proof assuming the consistency of a measurable cardinal and then show how to weaken the assumption to the consistency of a weakly compact cardinal. We will need a form of "diamond" principle.

Lemma 5.11 If κ is measurable and GCH holds, then in some forcing extension

1. κ is measurable.

2. There exists a sequence $\langle S_\alpha : \alpha < \kappa \rangle$ such that $S_\alpha \subseteq \alpha$ for all α, and for all $S \subseteq \kappa$ there is a normal measure U on κ such that if $j_U : V \longrightarrow M_U \simeq Ult(V, U)$ is the associated elementary embedding then $j_U(\vec{S})_\kappa = S$ (or equivalently, $\{ \alpha \mid S \cap \alpha = S_\alpha \} \in U$.

Proof:[Lemma 5.11] The proof is quite standard. For a similar construction given in more detail see [2].

Fix $j : V \longrightarrow M$ the ultrapower map associated with some normal measure on κ. Let $\mathbb{Q}_\alpha$ be the forcing whose conditions are sequences $\langle T_\beta : \beta < \gamma \rangle$ where $\gamma < \alpha$ and $T_\beta \subseteq \beta$ for all β, ordered by end-extension. (This is really the same as the Cohen forcing $Add(\alpha, 1)$). Let $\mathbb{P}_{\kappa+1}$ be a Reverse Easton iteration of length $\kappa + 1$, where we force with $(\mathbb{Q}_\alpha)_{V^{\mathbb{P}_\alpha}}$ at each inaccessible $\alpha \leq \kappa$.

Let G_κ be $\mathbb{P}_\kappa$-generic over V and let g be $\mathbb{Q}_\kappa$-generic over $V[G_\kappa]$. We will prove that the sequence given by $S_\alpha = g(\alpha)$ will work, by producing an appropriate U for each $S \in V[G_\kappa][g]$ with $S \subseteq \kappa$. Let us fix such an S.

By GCH and the fact that $j(\mathbb{P}_\kappa)/G_\kappa * g$ is κ^+-closed in $V[G_\kappa][g]$, we may build $H \in V[G_\kappa][g]$ which is $j(\mathbb{P}_\kappa)/G_\kappa * g$-generic over $M[G_\kappa][g]$. Now for the key point: we define a condition $q \in \mathbb{Q}_{j(\kappa)}$ by setting $q(\alpha) = g(\alpha)$ for $\alpha < \kappa$ and $q(\kappa) = S$. Then we build $h \ni q$ which is $\mathbb{Q}_{j(\kappa)}$-generic over $M[G_\kappa][g][H]$, using GCH and the κ^+-closure of $\mathbb{Q}_{j(\kappa)}$ in $V[G_\kappa][g]$.

To finish we define $j : V[G_\kappa][g] \longrightarrow M[G_\kappa][g][H][h]$ by $j : \dot{\tau}^{G_\kappa * g} \longmapsto j(\dot{\tau})^{G_\kappa * g * H * h}$, where this map is well-defined and elementary because $j``(G_\kappa * g) \subseteq G_\kappa * g * H * h$. The extended j is still an ultrapower by a normal measure (U say) because $M[G_\kappa][g][H][h] = \{ j(F)(\kappa) \mid F \in V[G_\kappa][g] \}$. It is clear from the definition of this map j that $j(g)(\kappa) = h(\kappa) = q(\kappa) = S$, so the model $V[G_\kappa][g]$ is as required.

This concludes the proof of Lemma 5.11.

♦

Fixing some reasonable coding of members of H_{α^+} by subsets of α, we may write the diamond property in the following equivalent form: there is a sequence $\langle x_\alpha : \alpha < \kappa \rangle$ such that for every $x \in H_{\kappa^+}$ there exists U such that $j_U(\vec{x})_\kappa = x$. Henceforth we will assume that we have fixed a sequence $\vec{x}$ with this property.

Now we describe a certain Reverse Easton forcing iteration of length $\kappa+1$. It will be clear after the iteration is defined that it is $\aleph_2$-strategically closed, so that in particular $\aleph_1$ and $\aleph_2$ are preserved. At stage $\alpha < \kappa$ we will force with $\mathbb{Q}_\alpha$, where $\mathbb{Q}_\alpha$ is trivial forcing unless

1. α is inaccessible.

2. $V^{\mathbb{P}_\alpha} \models \alpha = \aleph_3, \alpha^+_V = \aleph_4$

3. x_α is a $\mathbb{P}_\alpha$-name for a nice iteration $\mathbb{P}^\alpha_{2\delta+1}$ of some length $2\delta + 1 < \alpha^+$.

In this last case $\mathbb{Q}_\alpha$ is defined to be $\mathbb{P}^\alpha_{2\delta+1} * \mathbb{P}(R^\alpha_\delta) * Coll(\aleph_2, \alpha)$.

At stage κ (which will be $(\aleph_3)_{V^{\mathbb{P}_\kappa}}$), we will do a nice iteration $\mathbb{Q}_\kappa$ of length κ^+, with some book-keeping designed to guarantee that for every stationary $S \subseteq S^3_0$ in the final model there exists $T \subseteq S$ a non-reflecting stationary subset. So by design $Dnr(\aleph_3, \aleph_2, \aleph_0)$ holds in the final model (and in fact so does $Dnr(\aleph_3, \aleph_1, \aleph_0)$). It remains to be seen that $Ref(\aleph_3, \aleph_2, \aleph_1)$ is true.

Let $T \subseteq S^3_1$ be a stationary subset of S^3_1 in the final model. Since $\mathbb{Q}_\kappa$ has the κ^+-c.c. we may assume that T is the generic extension by $\mathbb{P}_\kappa * (\mathbb{Q}_\kappa \restriction (2\delta+1))$ for some $\delta < \kappa^+$. Let $\overline{\mathbb{Q}} = \mathbb{Q}_\kappa \restriction (2\delta+1)$. Using the diamond property of $\vec{x}$ and the definition of the forcing iteration we may find U such that

$$j_U(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \overline{\mathbb{Q}} * \mathbb{P}(R_\delta) * \mathbb{R}_{\kappa+1, j_U(\kappa)}$$

where $\mathbb{R}_{\kappa+1, j_U(\kappa)}$ is the iteration above κ. To save on notation, denote j_U by j.

Now if $M \simeq Ult(V, U)$ is the target model of j, then we may assume by the usual arguments that $T \in M^{\mathbb{P}_\kappa * \overline{\mathbb{Q}}}$. Notice that the last step in the iteration $\overline{\mathbb{Q}}$ was to force with $\mathbb{R}(R_\delta)$, that is to add a club of points at which the stationarity of R_δ fails to reflect. Applying Lemma 5.2 we see that T is still stationary in the extension by $\mathbb{P}_\kappa * \overline{\mathbb{Q}} * \mathbb{P}(R_\delta)$. Since $\mathbb{R}_{\kappa+1, j(\kappa)}$ is $\aleph_2$-strategically closed, T will remain stationary in the extension by $j(\mathbb{P}_\kappa)$ (although of course κ will collapse to become some ordinal of cofinality $\aleph_2$).

To finish the proof we will build a generic embedding from $V^{\mathbb{P}_\kappa * \overline{\mathbb{Q}}}$ to $M^{j(\mathbb{P}_\kappa * \overline{\mathbb{Q}})}$. It is easy to get $j : V^{\mathbb{P}_\kappa} \longrightarrow M^{j(\mathbb{P}_\kappa)}$, what is needed is a master condition for $\overline{\mathbb{Q}}$ and j. Since κ is $\aleph_3$ in $V^{\mathbb{P}_\kappa}$ and $\mathbb{Q}_\kappa$ is an iteration with $\leq \aleph_2$-supports, it is clear what the condition should be, we just need to check that it works.

Definition 5.12 Define q by setting $\mathrm{dom}(q) = j``(2\delta+1)$, $q(j(2\gamma+1)) = f_\gamma \cup \{(\kappa, 0)\}$, $q(2\gamma) = C_\gamma \cup \{\kappa\}$, where $f_\gamma : \kappa \longrightarrow 2$ is the function added by $\overline{\mathbb{Q}}$ at stage $2\gamma+1$ and C_γ is the club added at stage 2γ.

We claim that q is a condition in $j(\overline{\mathbb{Q}})$. To see this we should first check that f_γ is the characteristic function of a non-stationary subset of κ; this holds because at stage κ in the forcing $j(\mathbb{P}_\kappa)$ we forced with $\mathbb{P}(R_\delta)$ and made S^*_γ non-stationary for all $\gamma < \delta$. We should also check that R_γ is non-stationary in γ, and again this is easy by Lemma 5.4 and the fact that R_δ has been made non-stationary.

Forcing with $j(\overline{\mathbb{Q}})$ adds no bounded subsets of $j(\kappa)$, so that clearly T is still stationary and $\mathrm{cf}(\kappa)$ is still $\aleph_2$ in the model $M^{j(\mathbb{P}_\kappa * \overline{\mathbb{Q}})}$. By the familiar reflection argument, there exists $\alpha \in S^3_2$ such that $T \cap \alpha$ is stationary in the model $V^{\mathbb{P}_\kappa * \overline{\mathbb{Q}}}$. $T \cap \alpha$ will still be stationary in $V^{\mathbb{P}_\kappa * \mathbb{Q}_\kappa}$, because the rest of the iteration $\mathbb{Q}_\kappa$ does not add any bounded subsets of $\aleph_3$. We have proved that $Ref(\aleph_3, \aleph_2, \aleph_1)$ holds in $V^{\mathbb{P}_\kappa * \mathbb{Q}_\kappa}$, which finishes the proof of Theorem 5.10 using a measurable cardinal.

It remains to be seen that we can replace the measurable cardinal by a weakly compact cardinal. To do this we will use the following result of Jensen.

Fact 5.13 Let $V = L$ and let κ be weakly compact. Then there exists a sequence $\langle S_\alpha : \alpha < \kappa \rangle$ with $S_\alpha \subseteq \alpha$ for all α, such that for all $S \subseteq \kappa$ and all Π^1_1 formulae $\phi(X)$ with one free second-order variable

$$V_\kappa \models \phi(S) \Longrightarrow (\exists \alpha < \kappa\ S_\alpha = S \cap V_\alpha,\ V_\alpha \models \phi(S_\alpha)).$$

Using this fact we can argue exactly as in Argument 1 at the end of the proof of Theorem 3.5. This concludes the proof of Theorem 5.10.

♦

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