Identity Crises and Strong Compactness II: Strong Cardinals

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Abstract

¿From a proper class of supercompact cardinals, we force and obtain a model in which the proper classes of strongly compact and strong cardinals precisely coincide. In this model, it is the case that no strongly compact cardinal κ is $2^\kappa = \kappa^+$ supercompact.

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1 Introduction and Preliminaries

The fact that the notion of strong compactness is a singularity in the large cardinal hierarchy is well-known. There is, of course, the fundamental work of Magidor [19], showing that the least strongly compact cardinal κ can be either the least supercompact cardinal or the least measurable cardinal (in which case κ isn't even 2^κ supercompact). A generalization of this work by Kimchi and Magidor [16] shows that the (possibly proper) classes of supercompact and strongly compact cardinals can coincide except at measurable limit points, where a result of Menas [21] shows they can't. Magidor has also shown (in unpublished work that doesn't even appear in [16]) that it is consistent, relative to $n \in \omega$ supercompact cardinals, for the first n strongly compact cardinals to be the first n measurable cardinals.

Although Magidor's work was groundbreaking and established the general field of "identity crisis studies", there has been additional, extensive research done in this area. We mention three such results along these lines. One is the work of [7], in which, roughly speaking, a model with a level by level correspondence between degrees of strong compactness and supercompactness is provided. Another is the work of [6], where, using the just mentioned unpublished techniques of Magidor and techniques from [8], relative to $n \in \omega$ supercompact cardinals, a model in which the first n measurable cardinals $\kappa_1, \ldots, \kappa_n$ are both the first n strongly compact cardinals and are so that each κ_i is κ_i^+ supercompact is constructed. In the model of [6], $2^{\kappa_i} = \kappa_i^{++}$

for $i = 1, \ldots, n$. A third is the work of [2], in which it is shown, roughly speaking, that the supercompact and non-supercompact strongly compact cardinals can in a generic extension conform to any pattern prescribed by a fixed ground model function.

The purpose of this paper is to add to the litany of confusion by showing, again using among other techniques the aforementioned unpublished ideas of Magidor, that the class of strongly compact cardinals can assume yet another identity. Specifically, we prove the following.

Theorem 1 *Con(ZFC + There is a proper class of supercompact cardinals)* $\Longrightarrow$ *Con(ZFC + There is a proper class of strongly compact cardinals + No strongly compact cardinal κ is $2^\kappa = \kappa^+$ supercompact + $\forall \kappa$[κ is strongly compact iff κ is strong]*.

Unlike Magidor's result that the first n (for $n \in \omega$) strongly compact cardinals can be the first n measurable cardinals and the result of [6], there is no barrier to proving Theorem 1 for more than finitely many strongly compact cardinals. In fact, while these results require severe restrictions on the large cardinal structure of the ground model, the large cardinal structure for the ground model of Theorem 1, modulo a proper class of supercompact cardinals, can otherwise be completely arbitrary. We will comment on this more at the end of Section 2.

The structure of this paper is as follows. Section 1 contains our introductory comments and preliminary remarks concerning notation, terminology, etc. Section 2 contains a proof of Theorem 1 for one cardinal, i.e., a construction of a model, relative to a supercompact cardinal, in which the least strongly compact cardinal κ is the least strong cardinal and isn't $2^\kappa = \kappa^+$ supercompact. Section 3 contains a proof of Theorem 1 in the general case. Section 4 discusses some possible generalizations of Theorem 1 and contains our concluding remarks.

Before giving the proof of Theorem 1, we briefly mention some preliminary information. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. For $\alpha < \beta$ ordinals, $[\alpha, \beta], [\alpha, \beta), (\alpha, \beta]$, and (α, β) are as in standard interval notation.

When forcing, $q \geq p$ will mean that q is stronger than p. If G is V-generic over $\mathbb{P}$, we will use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. If we also have that κ is inaccessible and $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle : \alpha < \kappa\rangle$ is an Easton support iteration of length κ so that at stage α, a non-trivial forcing is done based on the ordinal δ_α, then we will say that δ_α is in the field of $\mathbb{P}$. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$, especially when x is some variant of the generic set G, or x is in the ground model V.

If $\kappa < \lambda$ are regular cardinals, then $\mathrm{Add}(\kappa, \lambda)$ is the standard partial ordering for adding λ Cohen subsets to κ. If κ is a regular cardinal and $\mathbb{P}$ is a partial ordering, $\mathbb{P}$ is κ-closed if for every sequence $\langle p_\alpha : \alpha < \kappa\rangle$ of elements of $\mathbb{P}$ so that $\beta < \gamma < \kappa$ implies $p_\beta \leq p_\gamma$ (an increasing chain of length κ), there is some $p \in \mathbb{P}$ (an upper bound to this chain) so that $p_\alpha \le p$ for all $\alpha < \kappa$. $\mathbb{P}$ is $< \kappa$-closed if $\mathbb{P}$ is δ-closed for all cardinals $\delta < \kappa$. $\mathbb{P}$ is κ-directed closed if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha : \alpha < \delta \rangle$ of elements of $\mathbb{P}$ (where $\langle p_\alpha : \alpha < \delta \rangle$ is directed if for every two distinct elements $p_\rho, p_\nu \in \langle p_\alpha : \alpha < \delta \rangle$, p_ρ and p_ν have a common upper bound of the form p_σ) there is an upper bound $p \in \mathbb{P}$. $\mathbb{P}$ is κ-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha \le \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages (choosing the trivial condition at stage 0), then player II has a strategy which ensures the game can always be continued. Note that if $\mathbb{P}$ is κ-strategically closed and $f : \kappa \to V$ is a function in $V^{\mathbb{P}}$, then $f \in V$. $\mathbb{P}$ is $< \kappa$-strategically closed if $\mathbb{P}$ is δ-strategically closed for all cardinals $\delta < \kappa$. $\mathbb{P}$ is $\prec \kappa$-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, then player II has a strategy which ensures the game can always be continued. Note that trivially, if $\mathbb{P}$ is $< \kappa$-closed, then $\mathbb{P}$ is $< \kappa$-strategically closed and $\prec \kappa$-strategically closed. The converse of both of these facts is false.

Suppose as in the preceding paragraph that $\kappa < \lambda$ are regular cardinals. A partial ordering $\mathbb{P}$ that will be used throughout the course of this paper is the partial ordering for adding a non-reflecting stationary set of ordinals of cofinality κ to λ. Specifically, $\mathbb{P}$ is defined as $\{p :$ For some $\alpha < \lambda$, $p : \alpha \to \{0, 1\}$ is a characteristic function of S_p, a subset of α not stationary at its supremum nor having any initial segment which is stationary at its supremum, so that $\beta \in S_p$ implies $\beta > \kappa$ and $\mathrm{cof}(\beta) = \kappa\}$, ordered by $q \geq p$ iff $q \supseteq p$ and $S_p = S_q \cap \sup(S_p)$, i.e., S_q is an end extension of S_p. It is well-known that for G V-generic over $\mathbb{P}$ (see [9] or [16]), in $V[G]$, if we assume GCH holds in V, a non-reflecting stationary set $S = S[G] = \cup\{S_p : p \in G\} \subseteq \lambda$ of ordinals of cofinality κ has been introduced, the bounded subsets of λ are the same as those in V, and cardinals, cofinalities, and GCH have been preserved. It is also virtually immediate that $\mathbb{P}$ is κ-directed closed, and it can be shown (see [9] or [16]) that $\mathbb{P}$ is $\prec \lambda$-strategically closed.

We mention that we are assuming familiarity with the large cardinal notions of measurability, strongness, superstrongness, strong compactness, and supercompactness. We will also at the end of this paper refer to the large cardinal notions of Woodinness and Shelahness. Interested readers may consult [15], [20], or [22] for further details. We mention only that unlike [15], we will say that the cardinal κ is λ strong for $\lambda > \kappa$ if there is $j : V \to M$ an elementary embedding having critical point κ so that $j(\kappa) > \lambda$ and $V_\lambda \subseteq M$. As always, κ is strong if κ is λ strong for every $\lambda > \kappa$. We will also say the cardinal κ is superstrong with target λ if there is $j : V \to M$ an elementary embedding having critical point κ so that $j(\kappa) = \lambda$ and $V_\lambda \subseteq M$. If $j_0 : V \to M$ witnesses that κ is superstrong with target λ and $j_1 : M \to N$ witnesses the measurability of λ in M, then it is easily verified that $j_1 \circ j_0 : V \to N$ witnesses that κ is λ strong.

We mention that we are also assuming some familiarity with the basics of extender technology and the transference of generic objects via elementary embeddings. The section on background material of [10] is extremely useful in this regard. We will freely, particularly in the proofs of Lemmas 2.4 and 2.5, use notation, definitions, and terminology found here. Readers may also consult [20] for additional details concerning extenders.

Finally, both authors wish to express their gratitude to Menachem Magidor for his explanations to them given at the January 7-13, 1996 meeting in Set Theory held at the Mathematics Research Institute, Oberwolfach, Germany on his method of forcing to make the first n measurable and strongly compact cardinals coincide, for any finite n.

2 The Proof of Theorem 1 for One Cardinal

In this section, we will construct, starting with a supercompact cardinal, a model in which the least strongly compact cardinal κ is the same as the least strong cardinal and κ isn't $2^\kappa = \kappa^+$ supercompact. We begin with the following lemma, which also appears as Lemma 3.1 of [5].

Lemma 2.1 *Let κ be at least 2^κ supercompact and strong. Assume $j : V \to M$ is an elementary embedding witnessing at least the 2^κ supercompactness of κ, and let μ be the normal measure over κ associated with j. Then $\{\delta < \kappa : \delta$ is a strong cardinal$\} \in \mu$.*

Proof: We first show, for j and μ as in the statement of Lemma 2.1, that $\{\delta < \kappa : \delta \text{ is } \kappa \text{ strong}\} \in \mu$. (See also the proof of Proposition 26.11 of [15].)

To see this, note that since $M^{2^\kappa} \subseteq M$, $j \restriction V_{\kappa+1} \in M$. Thus, as in [3], page 203, there is $\mathcal{E} \in M$ a $(\kappa, j(\kappa))$ extender and $k : M \to \mathrm{Ult}(M, \mathcal{E})$ so that κ is the critical point of k and M and $\mathrm{Ult}(M, \mathcal{E})$ agree through rank $j(\kappa)$. This means $M \models$ "κ is superstrong with target $j(\kappa)$", so by reflection, $\{\delta < \kappa : \delta$ is superstrong with target $\kappa\} \in \mu$. By our remarks in Section 1, $\{\delta < \kappa : \delta$ is κ strong$\} \in \mu$.

Fix now $\delta < \kappa$ so that $V \models$ "δ is κ strong". We show that if $\lambda > \kappa$ is arbitrary, $V \models$ "δ is λ strong". Let $\lambda' > \lambda$ be so that any extender $\mathcal{E}$ witnessing the λ strongness of δ is such that $\mathcal{E} \in V_{\lambda'}$. By the strongness of κ, let $j^* : V \to M^*$ be an embedding having critical point κ witnessing that κ is λ' strong. Since $V \models$ "δ is κ strong", $M^* \models$ "$j^*(\delta) = \delta$ is $j^*(\kappa) > \lambda' > \lambda$ strong". As $V_{\lambda'} \subseteq M^*$ and $M^* \models$ "δ is λ strong", $V \models$ "δ is λ strong". This proves Lemma 2.1.

$\square$

We observe that in the above proof, it will actually be the case that $M \models$ "κ is a strong limit of strong cardinals". This is since $M \models$ "κ is $j(\kappa)$ strong and $j(\kappa)$ is strong", so by the second paragraph of the above proof, $M \models$ "κ is strong". Further, if $\delta < \kappa$ is so that $V \models$ "δ is strong", then $M \models$ "$j(\delta) = \delta$ is strong". Thus, by reflection, we have the more powerful fact that $\{\delta < \kappa : \delta$ is a strong limit of strong cardinals$\} \in \mu$.

We turn now to the proof of Theorem 1.

Proof: Let $V \models$ "ZFC + κ is supercompact". Without loss of generality, by first doing a preliminary forcing if necessary, we may also assume that

$V \models \mathrm{GCH}$.

By Lemma 2.1, let $A = \langle \delta_\alpha : \alpha < \kappa \rangle$ be an enumeration of the strong cardinals below κ. The partial ordering $\mathbb{P}^\kappa$ we use in the proof of Theorem 1 given in this section is the Easton support iteration $\langle \langle \mathbb{P}^\kappa_\alpha, \dot{\mathbb{Q}}^\kappa_\alpha \rangle : \alpha < \kappa \rangle$, where $\mathbb{P}^\kappa_0$ is the partial ordering $\mathrm{Add}(\omega, 1)$ and $\Vdash_{\mathbb{P}^\kappa_\alpha}$ "$\dot{\mathbb{Q}}^\kappa_\alpha$ adds a non-reflecting stationary set of ordinals of cofinality ω to δ_α".

Lemma 2.2 $V^{\mathbb{P}^\kappa} \models$ *"No cardinal $\delta < \kappa$ is strong".*

Proof: Let $\delta < \kappa$ be so that $V \models$ "δ is strong". It must therefore be the case that $\delta = \delta_\alpha$ for some $\alpha < \kappa$. This allows us to write $\mathbb{P}^\kappa = \mathbb{P}^\kappa_\alpha * \dot{\mathbb{Q}}^\kappa_\alpha * \dot{\mathbb{R}} = \mathbb{P}^\kappa_{\alpha+1} * \dot{\mathbb{R}}$.

By the definition of $\mathbb{P}^\kappa$ and the fact that any stationary subset of a measurable (or weakly compact) cardinal must reflect, $V^{\mathbb{P}^\kappa_{\alpha+1}} \models$ "δ isn't measurable (and hence isn't strong) since there is $S \subseteq \delta$ which is a non-reflecting stationary set of ordinals of cofinality ω". Since by the definition of $\mathbb{P}^\kappa$, $\Vdash_{\mathbb{P}^\kappa_{\alpha+1}}$ "$\dot{\mathbb{R}}$ is δ'-strategically closed for δ' the least inaccessible above δ", $V^{\mathbb{P}^\kappa_{\alpha+1}} = V^{\mathbb{P}^\kappa} \models$ "$S \subseteq \delta$ is a non-reflecting stationary set of ordinals of cofinality ω, so δ isn't measurable". Thus, $V^{\mathbb{P}^\kappa} \models$ "No V-strong cardinal $\delta < \kappa$ is measurable". The proof of Lemma 2.2 will therefore be complete once we have shown there is no cardinal $\delta < \kappa$ so that $V^{\mathbb{P}^\kappa} \models$ "δ is strong".

Write $\mathbb{P}^\kappa$ as $\mathbb{P}^\kappa_0 * \dot{\mathbb{Q}}$. By the definition of $\mathbb{P}$, $|\mathbb{P}^\kappa_0| = \omega$ and $\Vdash_{\mathbb{P}^\kappa_0}$ "$\dot{\mathbb{Q}}$ is $\aleph_1$-strategically closed". Therefore, using Hamkins' terminology of [12], [13], and [14], $\mathbb{P}^\kappa$ is a "gap forcing admitting a very low gap", so by the results of [12], [13] and [14], $V^{\mathbb{P}^\kappa} \models$ "Any strong cardinal was already strong in V". This means $V^{\mathbb{P}^\kappa} \models$ "No cardinal $\delta < \kappa$ is strong". This proves Lemma 2.2.

$\square$

Lemma 2.3 $V^{\mathbb{P}^\kappa} \models$ "*No cardinal $\delta < \kappa$ is strongly compact*".

Proof: By Lemmas 2.1 and 2.2, $V^{\mathbb{P}^\kappa} \models$ "There are unboundedly in κ many cardinals $\delta < \kappa$ containing a non-reflecting stationary set of ordinals of cofinality ω". It is a theorem of [22] that if a cardinal γ contains a non-reflecting stationary set of ordinals of cofinality ρ, then there are no strongly compact cardinals in the interval $(\rho, \gamma]$. Thus, $V^{\mathbb{P}^\kappa} \models$ "No cardinal $\delta < \kappa$ is strongly compact". This proves Lemma 2.3.

$\square$

Lemma 2.4 $V^{\mathbb{P}^\kappa} \models$ "*κ is strongly compact*".

Proof: The proof of Lemma 2.4 uses the unpublished ideas of Magidor referred to at the beginning of this paper. (See also the proof of Lemma 4 of [6].) Let $\lambda > 2^\kappa = \kappa^+$ be an arbitrary successor of a regular cardinal, and let $k_1 : V \to M$ be an embedding witnessing the λ supercompactness of κ so that $M \models$ "κ is $< \lambda$ supercompact but κ isn't λ supercompact". λ has been chosen large enough so that we may assume by choosing a normal ultrafilter of Mitchell order 0 over κ that $k_2 : M \to N$ is an embedding witnessing the measurability of κ definable in M so that $N \models$ "κ isn't measurable".

It is the case that if $k : V \to N$ is an elementary embedding with critical point κ and for any $x \subseteq N$ with $|x| \le \lambda$, there is some $y \in N$ so that $x \subseteq y$ and $N \vDash$ "$|y| < j(\kappa)$", then k witnesses the λ strong compactness of κ. Using this fact, it is easily verifiable that $j = k_2 \circ k_1$ is an elementary embedding witnessing the λ strong compactness of κ. We show that j extends to $j : V^{\mathbb{P}^\kappa} \to N^{j(\mathbb{P}^\kappa)}$. Since this extended embedding witnesses the λ strong compactness of κ in $V^{\mathbb{P}^\kappa}$, this proves Lemma 2.4.

To do this, write $j(\mathbb{P}^\kappa)$ as $\mathbb{P}^\kappa * \dot{\mathbb{Q}}^\kappa * \dot{\mathbb{R}}^\kappa$, where $\dot{\mathbb{Q}}^\kappa$ is a term for the portion of $j(\mathbb{P}^\kappa)$ between κ and $k_2(\kappa)$ and $\dot{\mathbb{R}}^\kappa$ is a term for the rest of $j(\mathbb{P}^\kappa)$, i.e., the part above $k_2(\kappa)$. Note that since $N \vDash$ "κ isn't measurable", $\kappa \notin \text{field}(\dot{\mathbb{Q}}^\kappa)$. Also, since Lemma 2.1 and the succeeding paragraph imply that $M \vDash$ "κ is strong", by elementarity, $N \vDash$ "$k_2(\kappa)$ is strong". Thus, the field of $\dot{\mathbb{Q}}^\kappa$ is composed of all N-strong cardinals in the interval $(\kappa, k_2(\kappa)]$ (so $k_2(\kappa) \in \text{field}(\dot{\mathbb{Q}}^\kappa)$), and the field of $\dot{\mathbb{R}}^\kappa$ is composed of all N-strong cardinals in the interval $(k_2(\kappa), k_2(k_1(\kappa)))$.

Let G_0 be V-generic over $\mathbb{P}^\kappa$. We construct in $V[G_0]$ an $N[G_0]$-generic object G_1 over $\mathbb{Q}^\kappa$ and an $N[G_0][G_1]$-generic object G_2 over $\mathbb{R}^\kappa$. Since $\mathbb{P}^\kappa$ is an Easton support iteration of length κ, a direct limit is taken at stage κ, and no forcing is done at stage κ, the construction of G_1 and G_2 automatically guarantees that $j''G_0 \subseteq G_0 * G_1 * G_2$. This means that $j : V \to N$ extends to $j : V[G_0] \to N[G_0][G_1][G_2]$.

To build G_1, note that since k_2 can be assumed to be generated by an ultrafilter $\mathcal{U}$ over κ and since in both V and M, $2^\kappa = \kappa^+$, $|k_2(\kappa^+)| =$

$|k_2(2^\kappa)| = |\{f : f : \kappa \to \kappa^+ \text{ is a function}\}| = |[\kappa^+]^\kappa| = \kappa^+$. Thus, as $N[G_0] \models$ "$|\wp(\mathbb{Q}^\kappa)| = k_2(2^\kappa)$", we can let $\langle D_\alpha : \alpha < \kappa^+ \rangle$ enumerate in $V[G_0]$ the dense open subsets of $\mathbb{Q}^\kappa$ present in $N[G_0]$. Since the κ closure of N with respect to either M or V implies the least element of the field of $\mathbb{Q}^\kappa$ is $> \kappa^+$, the definition of $\mathbb{Q}^\kappa$ as the Easton support iteration which adds a non-reflecting stationary set of ordinals of cofinality ω to each $N[G_0]$-strong cardinal in the interval $(\kappa, k_2(\kappa)]$ implies that $N[G_0] \models$ "$\mathbb{Q}^\kappa$ is $\prec \kappa^+$-strategically closed". By the fact the standard arguments show that forcing with the κ-c.c. partial ordering $\mathbb{P}^\kappa$ preserves that $N[G_0]$ remains κ-closed with respect to either $M[G_0]$ or $V[G_0]$, $\mathbb{Q}^\kappa$ is $\prec \kappa^+$-strategically closed in both $M[G_0]$ and $V[G_0]$.

We can now construct G_1 in either $M[G_0]$ or $V[G_0]$ as follows. Player I picks $p_\alpha \in D_\alpha$ extending $\sup(\langle q_\beta : \beta < \alpha \rangle)$ (initially, q_{-1} is the empty condition) and player II responds by picking $q_\alpha \geq p_\alpha$ (so $q_\alpha \in D_\alpha$). By the $\prec \kappa^+$-strategic closure of $\mathbb{Q}^\kappa$ in both $M[G_0]$ and $V[G_0]$, player II has a winning strategy for this game, so $\langle q_\alpha : \alpha < \kappa^+ \rangle$ can be taken as an increasing sequence of conditions with $q_\alpha \in D_\alpha$ for $\alpha < \kappa^+$. Clearly, $G_1 = \{p \in \mathbb{Q}^\kappa : \exists \alpha < \kappa^+ [q_\alpha \geq p]\}$ is our $N[G_0]$-generic object over $\mathbb{Q}^\kappa$.

It remains to construct in $V[G_0]$ the desired $N[G_0][G_1]$-generic object G_2 over $\mathbb{R}^\kappa$. To do this, we first note that as $M \models$ "κ is strong", we can write $k_1(\mathbb{P}^\kappa)$ as $\mathbb{P}^\kappa * \dot{\mathbb{S}}^\kappa * \dot{\mathbb{T}}^\kappa$, where $\Vdash_{\mathbb{P}^\kappa}$ "$\dot{\mathbb{S}}^\kappa$ adds a non-reflecting stationary set of ordinals of cofinality ω to κ", and $\dot{\mathbb{T}}^\kappa$ is a term for the rest of $k_1(\mathbb{P}^\kappa)$.

Note now that $M \vDash$ "No cardinal $\delta \in (\kappa, \lambda]$ is strong". To see this, assume to the contrary $\delta \in (\kappa, \lambda]$ is so that $M \vDash$ "δ is strong". If $\ell : M \to M^*$ is an elementary embedding witnessing the λ' strongness of δ for some cardinal $\lambda' > \lambda \geq \delta > \kappa$, then as $M \vDash$ "κ is $< \lambda$ supercompact", $M^* \vDash$ "$\ell(\kappa) = \kappa$ is $< \ell(\lambda)$ supercompact". Since $\ell(\delta)$ can be made arbitrarily high in the universe by increasing the amount of strongness ℓ witnesses, $\ell(\lambda)$ can be made arbitrarily high in the universe also, so by choosing λ' large enough, the fact $M^* \vDash$ "κ is $< \ell(\lambda)$ supercompact" is sufficient to deduce that κ is λ supercompact in M. As this contradicts the choice of M, we must have that $M \vDash$ "δ isn't strong". Thus, the field of $\dot{\mathbb{T}}^\kappa$ is composed of all M-strong cardinals in the interval $(\lambda, k_1(\kappa))$, which implies that in M, $\Vdash_{\mathbb{P}^\kappa * \dot{\mathbb{S}}^\kappa}$ "$\dot{\mathbb{T}}^\kappa$ is $\prec \lambda^+$-strategically closed". Further, since $V \vDash$ GCH and λ is regular, $|[\lambda]^{<\kappa}| = \lambda$ and $2^\lambda = \lambda^+$. Therefore, as k_1 can be assumed to be generated by an ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda)$, $|k_1(\lambda^+)| = |k_1(2^\lambda)| = |2^{k_1(\lambda)}| = |\{f : f : P_\kappa(\lambda) \to \lambda^+ \text{ is a function}\}| = |[\lambda^+]^\lambda| = \lambda^+$.

Work until otherwise specified in M. Consider the "term forcing" partial ordering $\mathbb{T}^*$ (see [10], Section 1.2.5, page 8) associated with $\dot{\mathbb{T}}^\kappa$, i.e., $\tau \in \mathbb{T}^*$ iff τ is a term in the forcing language with respect to $\mathbb{P}^\kappa * \dot{\mathbb{S}}^\kappa$ and $\Vdash_{\mathbb{P}^\kappa * \dot{\mathbb{S}}^\kappa}$ "$\tau \in \dot{\mathbb{T}}^\kappa$", ordered by $\tau \geq \sigma$ iff $\Vdash_{\mathbb{P}^\kappa * \dot{\mathbb{S}}^\kappa}$ "$\tau \geq \sigma$". Clearly, $\mathbb{T}^* \in M$. Also, since $\Vdash_{\mathbb{P}^\kappa * \dot{\mathbb{S}}^\kappa}$ "$\dot{\mathbb{T}}^\kappa$ is $\prec \lambda^+$-strategically closed", it can easily be verified that $\mathbb{T}^*$ itself is $\prec \lambda^+$-strategically closed in M and, since $M^\lambda \subseteq M$, in V as well. Therefore, as $\Vdash_{\mathbb{P}^\kappa * \dot{\mathbb{S}}^\kappa}$ "$|\dot{\mathbb{T}}^\kappa| = k_1(\lambda)$ and $2^{k_1(\lambda)} = (k_1(\lambda))^+ = k_1(\lambda^+)$", we can assume without loss of generality that in M, $|\mathbb{T}^*| = k_1(\lambda)$. This means we can let

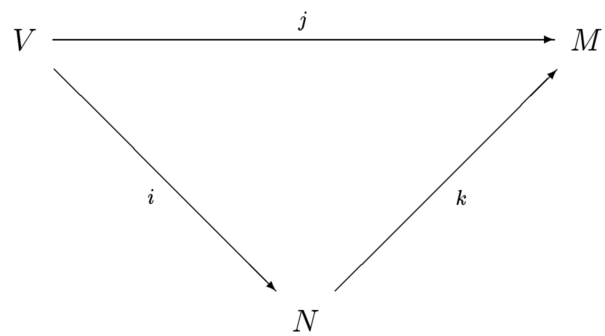
$\langle D_\alpha : \alpha < \lambda^+\rangle$ enumerate in V the dense open subsets of $\mathbb{T}^*$ present in M and argue as before to construct in V an M-generic object H_2 over $\mathbb{T}^*$.

Note now that since N can be assumed to be given by an ultrapower of M via a normal ultrafilter $\mathcal{U} \in M$ over κ, Fact 2 of Section 1.2.2 of [10] tells us that $k_2'' H_2$ generates an N-generic object G_2^* over $k_2(\mathbb{T}^*)$. By elementariness, $k_2(\mathbb{T}^*)$ is the term forcing in N defined with respect to $k_2(k_1(\mathbb{P}_\kappa)_{\kappa+1}) = \mathbb{P}^\kappa * \dot{\mathbb{Q}}^\kappa$. Therefore, since $j(\mathbb{P}^\kappa) = k_2(k_1(\mathbb{P}^\kappa)) = \mathbb{P}^\kappa * \dot{\mathbb{Q}}^\kappa * \dot{\mathbb{R}}^\kappa$, G_2^* is N-generic over $k_2(\mathbb{T}^*)$, and $G_0 * G_1$ is $k_2(\mathbb{P}^\kappa * \dot{\mathbb{S}}^\kappa)$-generic over N, Fact 1 of Section 1.2.5 of [10] tells us that for $G_2 = \{i_{G_0 * G_1}(\tau) : \tau \in G_2^*\}$, G_2 is $N[G_0][G_1]$-generic over $\mathbb{R}^\kappa$. Thus, in $V[G_0]$, $j : V \to N$ extends to $j : V[G_0] \to N[G_0][G_1][G_2]$. This proves Lemma 2.4.

$\square$

Lemma 2.5 $V^{\mathbb{P}^\kappa} \models$ "*κ is strong*".

Proof: We use for the proof of this lemma notation and terminology from the introductory section of [10]. Fix $\lambda > \kappa^+$, λ a cardinal so that $\lambda = \aleph_\lambda$. Let $j : V \to M$ be an elementary embedding witnessing the λ strongness of κ generated by a (κ, λ)-extender of width κ so that $M \models$ "κ isn't λ strong", and let $i : V \to N$ be the elementary embedding witnessing the measurability of κ generated by the normal ultrafilter $\mathcal{U} = \{x \subseteq \kappa : \kappa \in j(x)\}$. We then have the commutative diagram



where $j = k \circ i$ and the critical point of k is above κ.

Observe that $M \vDash$ "No cardinal $\rho \in (\kappa, \lambda]$ is strong", for if this were false, then since $V_\lambda \subseteq M$, $M \vDash$ "κ is $< \rho$ strong". By the argument in the second paragraph of the proof of Lemma 2.1, $M \vDash$ "κ is strong", contradicting the choice of M. This means that in M, the least strong cardinal $\delta > \kappa$ is so that $\delta > \lambda$.

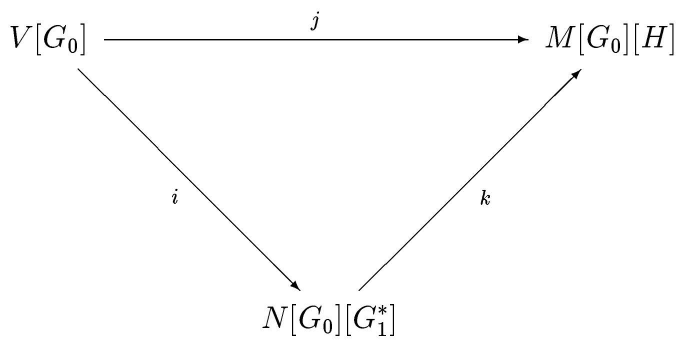
For any ordinal α, define σ_α as the least ordinal $> \alpha$ so that α isn't σ_α strong if such an ordinal exists, and $\sigma_\alpha = 0$ otherwise. Define $f : \kappa \to \kappa$ as $f(\alpha) =$ The least inaccessible cardinal $> \sigma_\alpha$. By our choice of λ and the preceding paragraph, $\kappa < \lambda < j(f)(\kappa) < \delta$, where δ is the least strong cardinal in $M \geq \kappa$, i.e., the least element of the field of $j(\mathbb{P}^\kappa) - \kappa$.

Note now that $M = \{j(g)(a) : a \in [\lambda]^{<\omega}, \text{dom}(g) = [\kappa]^{|a|}, g : [\kappa]^{|a|} \to V\} = \{k(i(g))(a) : a \in [\lambda]^{<\omega}, \text{dom}(g) = [\kappa]^{|a|}, g : [\kappa]^{|a|} \to V\}$. By defining $\gamma = i(f)(\kappa)$, we have $k(\gamma) = k(i(f)(\kappa)) = j(f)(\kappa) > \lambda$. This means $j(g)(a) = k(i(g))(a) = k(i(g) \restriction [\gamma]^{|a|})(a)$, i.e., $M = \{k(h)(a) : a \in [\lambda]^{<\omega}, h \in N, \text{dom}(h) = [\gamma]^{|a|}, h : [\gamma]^{|a|} \to N\}$. By elementariness, we must have $N \vDash$ "κ isn't strong and $\kappa < \gamma = i(f)(\kappa) < \delta_0 =$ The least strong cardinal in $N \geq \kappa =$

The least element of the field of $i(\mathbb{P}^\kappa) - \kappa$", since $M \vDash$ "$k(\kappa) = \kappa$ isn't strong and $k(\kappa) = \kappa < k(\gamma) = k(i(f)(\kappa)) = j(f)(\kappa) < k(\delta_0) = \delta$". Therefore, k can be assumed to be generated by an N-extender of width $\gamma \in (\kappa, \delta_0)$.

Write $i(\mathbb{P}^\kappa) = \mathbb{P}^\kappa * \dot{\mathbb{Q}}^0$, where $\dot{\mathbb{Q}}^0$ is a term for the portion of $i(\mathbb{P}^\kappa)$ whose field is composed of ordinals in the interval $[\kappa, i(\kappa))$. Since $N \vDash$ "κ isn't a strong cardinal", the field of $\dot{\mathbb{Q}}^0$ is actually composed of ordinals in the interval $(\kappa, i(\kappa))$, or more precisely, of ordinals in the interval $[\delta_0, i(\kappa))$. This means that if G_0 is once again V-generic over $\mathbb{P}^\kappa$, the argument from Lemma 2.4 for the construction of the generic object G_1 can be applied here as well to construct in $V[G_0]$ an $N[G_0]$-generic object G_1^* over $\mathbb{Q}^0$. Since $i''G_0 \subseteq G_0 * G_1^*$, i extends to $i : V[G_0] \to N[G_0][G_1^*]$, and since $k''G_0 = G_0$ and $k(\kappa) = \kappa$, k extends to $k : N[G_0] \to M[G_0]$. By Fact 3 of Section 1.2.2 of [10], $k : N[G_0] \to M[G_0]$ can also be assumed to be generated by an extender of width $\gamma \in (\kappa, \delta_0)$.

In analogy to the preceding paragraph, write $j(\mathbb{P}^\kappa) = \mathbb{P}^\kappa * \dot{\mathbb{Q}}^1$. By the last sentence of the preceding paragraph and the fact δ_0 is the least ordinal in the field of $\dot{\mathbb{Q}}^0$, we can use Fact 2 of Section 1.2.2 of [10] to infer that $H = \{p \in \mathbb{Q}^1 : \exists q \in k''G_1^*[q \geq p]\}$ is $M[G_0]$-generic over $k(\mathbb{Q}^1)$. Thus, k extends to $k : N[G_0][G_1^*] \to M[G_0][H]$, and we get the new commutative diagram



Since $M \models$ "No cardinal $\rho \in [\kappa, \lambda]$ is strong", the field of $\dot{\mathbb{Q}}^1$ is composed of ordinals in the interval $(\lambda, j(\kappa))$. Therefore, as $V_\lambda \subseteq M$, $V_\lambda[G_0] \subseteq M[G_0]$, and as the field of $\mathbb{Q}^1$ is composed of ordinals in the interval $(\lambda, j(\kappa))$, $V_\lambda[G_0]$ is the set of all sets of rank $< \lambda$ in $M[G_0][H]$. Hence, j is a λ strong embedding. Since λ was arbitrary, this proves Lemma 2.5.

$\square$

Lemma 2.6 $V^{\mathbb{P}^\kappa} \models$ *"κ isn't $2^\kappa = \kappa^+$ supercompact".*

Proof: By Lemmas 2.2 and 2.5, $V^{\mathbb{P}^\kappa} \models$ "κ is a strong cardinal so that no cardinal $\delta < \kappa$ is strong". Thus, by Lemma 2.1, $V^{\mathbb{P}^\kappa} \models$ "κ isn't 2^κ supercompact". Since $|\mathbb{P}^\kappa| = \kappa$ and $V \models$ "$2^\kappa = \kappa^+$", $V^{\mathbb{P}^\kappa} \models$ "$2^\kappa = \kappa^+$". This proves Lemma 2.6.

$\square$

Lemmas 2.1 - 2.6 complete the proof of Theorem 1 for one cardinal.

$\square$

We remark that the use of non-reflecting stationary subsets of ordinals of cofinality ω in the preceding proof was completely arbitrary. We could just as easily have added non-reflecting stationary subsets of ordinals of cofinality γ, where for $\delta_0 < \kappa$ the least strong cardinal, $\gamma \in (\omega, \delta_0)$ is an arbitrary regular cardinal.

We conclude this section by noting that the large cardinal structure above κ in V can be completely arbitrary by the proof just given. This is quite different from the situation in Magidor's original proof of the consistency of the first $n \in \omega$ strongly compact cardinals being the first n measurable cardinals and the situation in [6], in which severe limitations are of necessity placed on the large cardinal structure of the ground model. The reason for this is that strongness, unlike measurability, is not a local property, so in the proofs of Lemmas 2.4 and 2.5, we don't have to worry about unwanted cardinals having a non-reflecting stationary set of ordinals added to them. The fact that these limitations don't exist will allow us in the next section to prove Theorem 1 for a proper class of cardinals.

3 The Proof of Theorem 1 in the General Case

We turn now to the proof of Theorem 1 for a proper class of cardinals.

Proof: Let $V \vDash$ "ZFC + $\langle \kappa_\alpha : \alpha \in \text{Ord} \rangle$ is the proper class of supercompact cardinals". Without loss of generality, we assume in addition that $V \vDash$ GCH and that by "cutting off" the universe if necessary at the least inaccessible limit of supercompact cardinals, for $\gamma_0 = \omega$ and $\gamma_\alpha = \cup_{\beta<\alpha}\kappa_\beta$ for $\alpha > 0$, $\gamma_\alpha < \kappa_\alpha$ is singular if α is a limit ordinal. Further, by the methods of either [4] or [1] (both of which generalize Laver's result of [17]), we can also assume without loss of generality that for $\mathbb{R} = \text{Add}(\omega, 1) * \dot{\mathbb{R}}^*$, $V_1 = V^{\mathbb{R}} \models$ "GCH + The supercompactness of each κ_α is indestructible under forcing with κ_α-directed closed set or class partial orderings not destroying GCH". Since it will be the case that $\Vdash_{\text{Add}(\omega,1)}$ "$\dot{\mathbb{R}}^*$ is $\aleph_1$-strategically closed" and $|\text{Add}(\omega, 1)| = \omega$, $\mathbb{R}$ is a gap forcing admitting a very low gap. Thus, once again by Hamkins' results of [12], [13], and [14], $V_1 \models$ "Any cardinal which is supercompact or strong must have been supercompact or strong in V".

Work in V_1. For each ordinal α, let $\langle \delta^\alpha_\beta : \beta < \kappa_\alpha \rangle$ be an enumeration of the V-strong cardinals in the interval $(\gamma_\alpha, \kappa_\alpha)$, and let $\mathbb{P}^{\kappa_\alpha} = \langle \langle \mathbb{P}^{\kappa_\alpha}_\beta, \dot{\mathbb{Q}}^{\kappa_\alpha}_\beta \rangle : \beta < \kappa_\alpha \rangle$ be the Easton support iteration where $\mathbb{P}^{\kappa_\alpha}_0 = \{\emptyset\}$ and $\Vdash_{\mathbb{P}^{\kappa_\alpha}_\beta}$ "$\dot{\mathbb{Q}}^{\kappa_\alpha}_\beta$ adds a non-reflecting stationary set of ordinals of cofinality γ^+_α to δ^α_β". We define $\mathbb{P}$ as the Easton support product $\prod_{\alpha \in \text{Ord}} \mathbb{P}^{\kappa_\alpha}$. Since each $\mathbb{P}^{\kappa_\alpha}$ is γ^+_α-directed closed, the standard Easton arguments show $V_1^{\mathbb{P}} \models \text{ZFC}$.

For each ordinal α, write $\mathbb{P} = \mathbb{P}_{<\alpha} \times \mathbb{P}^{\kappa_\alpha} \times \mathbb{P}^{>\alpha}$, where $\mathbb{P}_{<\alpha} = \prod_{\beta<\alpha} \mathbb{P}^{\kappa_\beta}$ and $\mathbb{P}^{>\alpha}$ is the remainder of $\mathbb{P}$. By the definition of $\mathbb{P}$ and the fact the supercompactness of κ_α is indestructible under set or class forcing not destroying GCH, $V_1^{\mathbb{P}^{>\alpha}} \models$ "GCH + κ_α is supercompact". Further, since $\mathbb{R} * (\dot{\mathbb{P}}^{>\alpha} \times \dot{\mathbb{P}}^{\kappa_\alpha}) = \text{Add}(\omega, 1) * (\dot{\mathbb{R}}^* * (\dot{\mathbb{P}}^{>\alpha} \times \dot{\mathbb{P}}^{\kappa_\alpha}))$ is so that $\Vdash_{\text{Add}(\omega,1)}$ "$\dot{\mathbb{R}}^* * (\dot{\mathbb{P}}^{>\alpha} \times \dot{\mathbb{P}}^{\kappa_\alpha})$ is $\aleph_1$-strategically closed", the results of [12], [13], and [14] once more apply to show that any cardinal which is strong in $V_1^{\mathbb{P}^{>\alpha} \times \mathbb{P}^{\kappa_\alpha}}$ must have been strong in V. Thus, we can apply the results of Section 2 to show that $V^{\mathbb{P}^{>\alpha} \times \mathbb{P}^{\kappa_\alpha}} \models$ "κ_α is both strongly compact and strong, there are no strongly compact or strong cardinals in the interval $(\gamma_\alpha, \kappa_\alpha)$, and κ_α isn't $2^{\kappa_\alpha} = \kappa_\alpha^+$ supercompact". Since $V_1 \models$ "$|\mathbb{P}_{<\alpha}| < 2^{\gamma_\alpha^+}$", the Lévy-Solovay results [18] show that $V_1^{\mathbb{P}^{>\alpha} \times \mathbb{P}^{\kappa_\alpha} \times \mathbb{P}_{<\alpha}} = V_1^{\mathbb{P}} \models$ "κ_α is both strongly compact and strong, there are no strongly compact or strong cardinals in the interval $(\gamma_\alpha, \kappa_\alpha)$, and κ_α isn't $2^{\kappa_\alpha} = \kappa_\alpha^+$ supercompact". Therefore, since any cardinal δ which is strongly compact or strong and is not a κ_α must be so that $\delta \in (\gamma_\alpha, \kappa_\alpha)$, $V_1^{\mathbb{P}}$ is our desired model. This proves Theorem 1 for a proper class of cardinals.

$\square$

We conclude this section by noting that a result of Menas from [21] shows that any measurable limit of strongly compact cardinals is strongly compact. This has as a consequence that if we assume large enough cardinals in the universe, there can never be a precise coincidence between the notions of strongly compact and strong. This is shown by the following, whose proof is essentially the same as Menas' proof of [21] that the least measurable limit κ of strongly compact or supercompact cardinals isn't 2^κ supercompact.

Fact 3.1 *If κ is the least measurable limit of cardinals which are both strongly compact and strong, then κ isn't $\kappa + 2$ strong.*

Proof: Assume to the contrary that κ is $\kappa + 2$ strong, and let $j : V \to M$ be an elementary embedding witnessing this fact. Since $M \models$ "κ is measurable" and $j \upharpoonright \kappa = \text{id}$, $M \models$ "κ is a measurable limit of cardinals which are both strongly compact and strong". This contradicts that $M \models$ "$j(\kappa) > \kappa$ is the least measurable limit of cardinals which are both strongly compact and strong". This proves Fact 3.1.

$\square$

4 Possible Generalizations and Concluding Remarks

We observe that by combining the techniques of this paper with those of [2], it is possible to prove the following.

Theorem 2 *Let $V \models$ "ZFC + Ω is an inaccessible limit of measurable limits of supercompact cardinals + $f : \Omega \to 3$ is a function". There is then a partial ordering $\mathbb{P} \in V$ so that for $\overline{V} = V^{\mathbb{P}}_{\Omega}$, the universe of $V^{\mathbb{P}}$ truncated at Ω, $\overline{V} \models$ "ZFC + If $f(\alpha) = 0$, then the α^{th} compact cardinal γ_α isn't 2^{γ_α} supercompact or $\gamma_\alpha + 2$ strong + If $f(\alpha) = 1$, then the α^{th} compact cardinal γ_α is supercompact + If $f(\alpha) = 2$, then the α^{th} compact cardinal γ_α is strong but isn't 2^{γ_α} supercompact".*

For Theorem 2, we take a compact cardinal as being one which is either supercompact or non-supercompact strongly compact. Also, since we will be able to assume GCH in $\overline{V}$, when $f(\alpha) = 0$ or $f(\alpha) = 2$, γ_α won't be γ_α^+ supercompact.

We will not give a detailed proof here, but we will explicitly describe the forcing conditions $\mathbb{P}$ used in the construction of $\overline{V}$. Readers of this paper and [2] should then fairly easily be able to combine the methods of these two papers to prove Theorem 2.

We begin as in the proof of Theorem 1 given in Section 3 by assuming $V \models$ GCH and that by using a partial ordering of the form $\mathbb{R} = \text{Add}(\omega, 1) * \dot{\mathbb{R}}^*$ that $V^{\mathbb{R}} \models$ "GCH + The supercompact and strongly compact cardinals coincide except at measurable limit points + Every supercompact cardinal κ is indestructible under κ-directed closed forcing not destroying GCH". Since the work of [1] and [2] tells us $\mathbb{R}$ can be presumed to preserve all V-supercompact cardinals, their measurable limits, and the regularity of Ω, we can assume without loss of generality that Ω is in $V^{\mathbb{R}}$ the least regular limit of measurable limits of supercompact cardinals.

Working in $V^{\mathbb{R}}$, we let $\langle \delta_\alpha : \alpha < \Omega \rangle$ enumerate the measurable limits of supercompact cardinals below Ω. For an arbitrary $\alpha < \Omega$, let $\langle \kappa^\alpha_\beta : \beta < \delta_\alpha \rangle$ enumerate the $V = V^{\mathbb{R}}$-supercompact cardinals in the interval $(\cup_{\gamma < \alpha} \delta_\gamma, \delta_\alpha)$. Define $\rho_\alpha = \omega$ when $\alpha = 0$ and $\rho_\alpha = (\cup_{\gamma < \alpha} \delta_\gamma)^+$ when $\alpha \in (0, \Omega)$. If $f(\alpha) = 0$, take $\mathbb{P}^\alpha$ as the Easton support iteration of partial orderings which add a non-reflecting stationary set of ordinals of cofinality ρ_α to each κ^α_β. If $f(\alpha) = 1$, take $\mathbb{P}^\alpha$ as the partial ordering which adds a non-reflecting stationary set of ordinals of cofinality κ^α_0 to δ_α. If $f(\alpha) = 2$, take $\mathbb{P}^\alpha$ as $\mathbb{Q}^\alpha_1 \times \mathbb{Q}^\alpha_2$, where $\mathbb{Q}^\alpha_1$ is the partial ordering which adds a non-reflecting stationary set of ordinals of cofinality κ^α_0 to δ_α, and $\mathbb{Q}^\alpha_2$ is the Easton support iteration of partial orderings which add a non-reflecting stationary set of ordinals of cofinality ρ_α to each V-strong cardinal in the interval $(\rho_\alpha, \kappa^\alpha_0)$. Let $\mathbb{P}^*$ be the Easton support product $\prod_{\alpha<\Omega} \mathbb{P}^\alpha$. $\mathbb{P} = \mathbb{R} * \dot{\mathbb{P}}^*$ is our desired partial ordering.

We remark that another possible generalization of Theorem 1 that one might wish to obtain is the construction, relative to a proper class of supercompact cardinals, of a model in which not only do the strongly compact and strong cardinals precisely coincide, but each strongly compact cardinal κ is κ^+ supercompact. In such a model, GCH would of necessity have to fail, since by Lemma 2.1, no strongly compact cardinal κ could be 2^κ supercompact. The techniques used to build this sort of model would doubtlessly involve a melding of the ideas of [6] and [8] with the ideas of this paper, along with the construction of the appropriate kinds of supercompact and strong embeddings. Although we feel attaining this result is within reach, we have not yet been able to come up with a concrete proof.

In conclusion to this paper, we note that it is tempting to want to prove an analogue to Theorem 1 for superstrong cardinals, i.e., to want to construct a model in which the strongly compact and superstrong cardinals precisely coincide. That this can't be, however, is shown by the following.

Fact 4.1 *Suppose κ is both strongly compact and superstrong. Then κ has a normal measure concentrating on strongly compact cardinals.*

Proof: Let $j : V \to M$ be an elementary embedding witnessing that κ is superstrong. Since $V_{j(\kappa)} \subseteq M$, $V \models$ "$j(\kappa)$ is a strong limit cardinal". Thus, $V_{j(\kappa)} \models$ "κ is $< j(\kappa)$ strongly compact", i.e., $M \models$ "κ is $< j(\kappa)$ strongly compact". This means, by elementarity, that $M \models$ "κ is $< j(\kappa)$ strongly compact and $j(\kappa)$ is strongly compact", so by a theorem of DiPrisco [11], $M \models$ "κ is strongly compact". Fact 4.1 now follows by reflection.

$\square$

Thus, an analogue to Theorem 1 for superstrong cardinals is impossible. We finish by asking, however, if an analogue to Theorem 1 can be proven for Woodin or Shelah cardinals, i.e., if it is consistent, relative to some large cardinal hypothesis, for the classes of strongly compact and Woodin cardinals or strongly compact and Shelah cardinals to coincide precisely.

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