

MARTIN'S MAXIMUM AND WEAK SQUARE

JAMES CUMMINGS AND MENACHEM MAGIDOR

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ABSTRACT. We analyse the influence of the forcing axiom Martin's Maximum on the existence of square sequences, with a focus on the weak square principle $\square_{\lambda,\mu}$.

1. INTRODUCTION

It is well known that strong forcing axioms (for example PFA, MM) and strong large cardinal axioms (strongly compact, supercompact) exert rather similar kinds of influence on the combinatorics of infinite cardinals. This is perhaps not so surprising when we consider the widely held belief that essentially the only way to obtain a model of PFA or MM is to collapse a supercompact cardinal to become ω_2 . We quote a few results which bring out the similarities:

- 1a) (Solovay [14]) If κ is supercompact then \square_λ fails for all $\lambda \geq \kappa$.
- 1b) (Todorćević [15]) If PFA holds then \square_λ fails for all $\lambda \geq \omega_2$.
- 2a) (Shelah [11]) If κ is supercompact then \square_λ^* fails for all λ such that $\text{cf}(\lambda) < \kappa < \lambda$.
- 2b) (Magidor, see Theorem 1.2) If MM holds then \square_λ^* fails for all λ such that $\text{cf}(\lambda) = \omega$.
- 3a) (Solovay [13]) If κ is supercompact then SCH holds at all singular cardinals greater than κ .
- 3b) (Foreman, Magidor and Shelah [7]) If MM holds then SCH holds.
- 3c) (Viale [16]) If PFA holds then SCH holds. In fact both the p-ideal dichotomy and the mapping reflection principle (which are well known consequences of PFA) suffice to prove SCH.

We will consider a hierarchy of principles introduced by Schimmerling [10] which are intermediate between the full square principle \square_λ and the weak square principle \square_λ^* .

Definition 1.1. Let λ be an infinite cardinal and let μ be a cardinal with $1 \leq \mu \leq \lambda$. A $\square_{\lambda,\mu}$ -sequence is a sequence $\langle \mathcal{C}_\alpha : \alpha < \lambda^+ \rangle$ such that for each α

- (1) \mathcal{C}_α is a non-empty family of club subsets of α , each with order type at most λ .
- (2) $|\mathcal{C}_\alpha| \leq \mu$.
- (3) For every $C \in \mathcal{C}_\alpha$ and every $\beta \in \text{lim}(C)$, $C \cap \beta \in \mathcal{C}_\beta$.

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We note that the square principle \square_λ is $\square_{\lambda,1}$, and the weak square principle \square_λ^* is $\square_{\lambda,\lambda}$. It is known [2, 8] that in general these principles are strictly decreasing in strength as μ increases.

In some joint work with Matt Foreman [2] we studied these square principles and their relationship with large cardinals, stationary reflection principles and PCF-theoretic scales. As part of that work we obtained quite sharp results on the extent to which these square principles can hold above a supercompact cardinal. As we already mentioned, if $\text{cf}(\lambda) < \kappa < \lambda$ and κ is supercompact then \square_λ^* fails. Burke and Kanamori [10] showed that if κ is supercompact, $\kappa \leq \lambda$ and $\mu < \text{cf}(\lambda)$ then $\square_{\lambda,\mu}$ fails.

We proved a consistency result which is complementary to these results. If λ is singular then there is a $\text{cf}(\lambda)$ -directed-closed forcing poset which adds no λ -sequences of ordinals (in particular it preserves cardinals and cofinalities up to λ^+) and adds a $\square_{\lambda,\text{cf}(\lambda)}$ -sequence. In particular if $\kappa \leq \text{cf}(\lambda) < \lambda$ and κ is a Laver indestructible supercompact cardinal, then in the generic extension κ is still supercompact, and $\square_{\lambda,\text{cf}(\lambda)}$ holds.

Some prior work by Magidor [9] gives sharp results for PFA. Inspection of the argument by Todorćević that PFA is incompatible with \square_λ shows that in fact PFA denies $\square_{\lambda,\omega_1}$ for all λ ; Magidor showed that it is consistent with PFA that $\square_{\lambda,\omega_2}$ holds for all λ .

In this note we will prove similarly sharp results for the extent of squares in the presence of MM. In fact we will prove

Theorem 1.2. *Suppose that MM holds and λ is an uncountable cardinal. Then*

- (1) *If $\text{cf}(\lambda) = \omega$, then \square_λ^* fails.*
- (2) *If $\text{cf}(\lambda) = \omega_1$, then $\square_{\lambda,\mu}$ fails for every $\mu < \lambda$.*
- (3) *If $\text{cf}(\lambda) \geq \omega_2$, then $\square_{\lambda,\mu}$ fails for every $\mu < \text{cf}(\lambda)$.*

Theorem 1.3. *It is consistent that MM holds and*

- (1) *\square_λ^* holds for all λ of cofinality ω_1 .*
- (2) *$\square_{\lambda,\text{cf}(\lambda)}$ holds for all λ of cofinality at least ω_2 .*

2. LIMITS ON SQUARE IN A MODEL OF MM

In this section we prove Theorem 1.2. The arguments are in the same spirit as those of [2], in that we are investigating and using the web of relationships between squares, PCF-theoretic scales and stationary reflection.

2.1. Countable cofinality. We will give two proofs that if MM holds and $\text{cf}(\lambda) = \omega$ then \square_λ^* fails. The first one is shorter but the second one (due to Magidor) gives more information.

The first proof uses results from [2], together with a standard consequence of MM. MM implies [7] the following form of stationary reflection: if λ is uncountable and S is a stationary subset of $[\lambda^+]^{\aleph_0}$, then there is $X \subseteq \lambda^+$ such that $|X| = \text{cf}(X) = \aleph_1$ and $S \cap [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$. We proved [2] that this form of stationary reflection is incompatible with \square_λ^* when $\text{cf}(\lambda) = \omega$, concluding our first proof.

In the second proof we will actually be proving from MM a result stronger than the failure of \square_λ^* . To state this result, we recall from [2] the notion of a *good scale*. Given a singular cardinal μ and a sequence $\langle \mu_i : i < \text{cf}(\mu) \rangle$ of regular cardinals which is increasing and cofinal and μ , a *scale of length μ^+ in $\prod_i \mu_i$* is a

sequence $\langle g_\alpha : \alpha < \mu^+ \rangle$ which is increasing and cofinal in $\prod_i \mu_i$ under the eventual domination ordering. The scale $\langle g_\alpha : \alpha < \mu^+ \rangle$ is *good* if almost every $\alpha < \mu^+$ (modulo the club filter) of cofinality greater than $\text{cf}(\mu)$ is a *good point*, that is to say there are $A \subseteq \alpha$ unbounded and $m < \omega$ such that $\langle g_\alpha(n) : \alpha \in A \rangle$ is strictly increasing for $m < n < \omega$.

A basic theorem in PCF theory is the result by Shelah [12] that for every singular μ there is some sequence $\prod_i \mu_i$ with a scale of length μ^+ in $\prod_i \mu_i$. A result by Foreman and Magidor [6] implies that if μ is singular and \square_μ^* holds (actually they used a weaker hypothesis known as the *approximation* or *approachability* property) then every scale of length μ^+ is good. We will prove that if λ is singular and $\text{cf}(\lambda) = \omega$, then under MM there is no good scale of length λ^+ , so that in particular \square_λ^* must fail.

We assume towards a contradiction that $\langle \lambda_i : i < \omega \rangle$ is an increasing sequence of regular cardinals which is cofinal in λ , and that $\langle g_\alpha : \alpha < \lambda^+ \rangle$ is a good scale. For technical reasons we will assume that $\lambda_0 > \aleph_1$. Before giving the details of the argument we outline the strategy: we will use MM to produce stationarily many $\delta \in \lambda^+ \cap \text{cof}(\omega_1)$ with the following property: there exist countable sets $S_n \subseteq \lambda_n$ with the property that for every $\alpha < \delta$ there exist $\beta < \delta$ and $g \in \prod_n S_n$ such that $g_\alpha <^* g <^* g_\beta$.

Such a δ can not be a good point: for given $A \subseteq \delta$ and $m < \omega$ as in the definition of goodness, we may find $\alpha_i \in A$ and $G_i \in \prod_n S_n$ for $i < \omega_1$ such that $g_{\alpha_i} <^* G_i <^* g_{\alpha_{i+1}}$. We may fix $n > m$ and $B \subseteq \omega_1$ unbounded such that $g_{\alpha_i}(n) < G_i(n) < g_{\alpha_{i+1}}(n)$ for all $i \in B$. This is a contradiction since $G_i(n) \in S_n$, S_n is countable and the sequence $\langle g_{\alpha_i}(n) : i \in B \rangle$ is strictly increasing.

We will produce stationarily many δ of this type by applying MM to a certain stationary preserving forcing poset \mathbb{P} , which is a variation on Namba forcing. The conditions are trees T of finite sequences such that (writing $s \triangleleft t$ for the relation “ s extends t ”, and $s \frown t$ for the concatenation of s and t)

- (1) For every $t \in T$, $t(i) \in \lambda_i \cap \text{cof}(\omega)$ for every $i < \text{lh}(t)$.
- (2) The tree T has a “stem” s , that is an element $s \in T$ such that
 - (a) For every $u \in T$, $u \triangleleft s$ or $s \triangleleft u$.
 - (b) For every $u \in T$ such that $s \triangleleft u$, if $\text{lh}(u) = n$ then $\{\alpha \in \lambda_n \cap \text{cof}(\omega) : u \frown \langle \alpha \rangle \in S\}$ is stationary in λ_n .

The ordering is inclusion.

The generic object added by \mathbb{P} is a function $h \in \prod_n (\lambda_n \cap \text{cof}(\omega))$. We will prove a series of facts about the forcing poset \mathbb{P} ; most of them are quite standard, see for example [3] for the analysis of a similar “Namba style” forcing.

We write $n(S)$ for the length of the stem of S . If S, T are conditions and $n < \omega$ then we write $S \leq_n T$ when $S \leq T$, $\text{stem}(S) = \text{stem}(T)$, and $u \in S \iff u \in T$ for all u such that $\text{lh}(u) \leq n(S) + n$. As usual we have a form of *fusion lemma*. The fusion lemma states that if $\langle S_i : i < \omega \rangle$ is a *fusion sequence*, that is $S_{i+1} \leq_i S_i$ for all $i < \omega$, then there is a condition S with $S \leq_i S_i$ for all i . When $T \leq S$ we say that T is an “ n -step extension” of S if $n(T) = n(S) + n$; it is easy to see that in this case $T \leq_0 S_t$, where $t = \text{stem}(T)$ and as usual $S_t = \{u \in S : u \triangleleft t \text{ or } t \triangleleft u\}$.

Fact one: If $\dot{\beta}$ is a name for an element of an ordinal μ , and S is a condition such that $\mu < \lambda_{n(S)}$, then there is $T \leq_0 S$ which decides $\dot{\beta}$.

Proof: We will say that a condition U is *bad* if the conclusion fails, that is $\mu < \lambda_n(U)$ and no $T \leq_0 U$ decides $\dot{\beta}$. Suppose for a contradiction that S is bad. Let $n(S) = n$, and let the stem of S be s . There are stationarily many $\alpha \in \lambda_n$ such that $s \frown \langle \alpha \rangle \in S$. We claim that the set of such α such that $S_{s \frown \langle \alpha \rangle}$ is bad is non-stationary. Otherwise we may find a fixed β and conditions $T_\alpha \leq_0 S_{s \frown \langle \alpha \rangle}$ forcing $\dot{\beta} = \beta$ for a stationary set A of α , but then if $U = \bigcup_{\alpha \in A} T_\alpha$ we have that $U \leq_o S$, and U decides $\dot{\alpha}$, contradicting the badness of S .

So let B be the stationary set of α such that $S_{s \frown \langle \alpha \rangle}$ is bad, and let $S_1 = \bigcup_{\alpha \in B} S_{s \frown \langle \alpha \rangle}$. So $S_1 \leq_0 S$, and S_1 has the property that no one-step extension of S_1 decides $\dot{\beta}$.

Repeating this argument we build a fusion sequence such that for each i there no i -step extension of S_i deciding $\dot{\beta}$; then there is S^* a lower bound for this sequence, and no extension of S^* decides $\dot{\beta}$, an immediate contradiction.

Fact two: If $\dot{\beta}$ is a name for an ordinal, and S is a condition, then there are $T \leq_0 S$ and $n < \omega$ such that every n -step extension of T decides $\dot{\beta}$.

Proof: The argument is similar to that for fact one. Again we say that a condition U is bad if it fails to satisfy the conclusion of the theorem, that is there do not exist n and $T \leq_0 U$ such that every n -step extension of T decides $\dot{\beta}$. We suppose for a contradiction that S is bad. Let $n(S) = n$, and let the stem of S be s .

We claim that the set of α with $s \frown \langle \alpha \rangle \in S$ and $S_{s \frown \langle \alpha \rangle}$ not bad is non-stationary. Otherwise we get a fixed n and a stationary set A , such that for $\alpha \in A$ there is $T_\alpha \leq_0 S_{s \frown \langle \alpha \rangle}$ with every n -step extension of T_α deciding $\dot{\beta}$. If we set $U = \bigcup_{\alpha \in A} T_\alpha$, we have that $U \leq_o S$ and every $n + 1$ -step extension of U decides $\dot{\alpha}$, a contradiction which proves the claim.

Repeating this argument we build a fusion sequence such that for each i every i -step extension of S_i is bad; then there is S^* a lower bound for this sequence, and no extension of S^* decides $\dot{\beta}$, an immediate contradiction.

Fact three: \mathbb{P} does not change the cofinality of $(\lambda^+)^V$ to ω .

Proof: Let \dot{f} be a name for a function from ω to λ^+ and let S be a condition. By repeated applications of fact two we may build an extension $U \leq S$ and a set B of size λ such that U forces the range of \dot{F} to be a subset of B .

Fact four: the generic function h added by \mathbb{P} is an exact upper bound (eub) in $V[h]$ for $\langle g_\alpha : \alpha < \lambda^+ \rangle$. That is to say, in $V[h]$ we have that

$$\{f \in \prod_n \lambda_n : f <^* h\} = \{f \in \prod_n \lambda_n : \exists \alpha < \lambda^+ f <^* g_\alpha\}.$$

Proof: Since $\langle g_\alpha : \alpha < \lambda^+ \rangle$ is a scale in V , it is enough to show that h eventually dominates every function in $(\prod_n \lambda_n)^V$, and every function $g < h$ is dominated by some function in $(\prod_n \lambda_n)^V$. The first part is easy: given a function $f \in (\prod_n \lambda_n)^V$ and a condition S , we may build $T \leq_0 S$ such that

$$\{\alpha : u \frown \langle \alpha \rangle \in T\} \subseteq \lambda_{lh(u)} \setminus g(lh(u))$$

for all $u \in T$ extending the stem of T . Then T forces that $f <^* h$.

For the second part we fix a name \dot{g} for a function $g < h$, and a condition S . Let $n(S) = n$ and let the stem of S be s . Using fact one, for each α with $s \frown \langle \alpha \rangle \in S$ we may find $T_\alpha \leq S_{s \frown \langle \alpha \rangle}$ such that T_α determines the value of $g(n)$, and we note

that since $g < h$ the value which T_α determines for $g(n)$ is less than α . Applying Fodor's theorem we may find $S_1 \leq_0 S$ and an ordinal α_s such that $S_1 \Vdash g(n) = \alpha_s$.

Repeating this argument we may build a fusion sequence such that for all $u \in S_i$ with $n \leq lh(u) < n + i$, there is an ordinal α_u such that $(S_i)_u \Vdash g(lh(u)) = \alpha_u$. So by fusion we obtain $U \leq_0 S$ such that $U_u \Vdash g(lh(u)) = \alpha_u$ for all $u \in U$ extending s . For each $m \geq n$ there are at most λ_{m-1} many $u \in U$ with $lh(u) = m$, so we may compute the supremum of the corresponding ordinals α_u and obtain some ordinal $f(m) < \lambda_m$. Clearly U forces that $g <^* f$, and we have proved fact four.

Fact five: \mathbb{P} is stationary preserving.

Proof: Fix a stationary subset T of ω_1 , a condition S , and a name \dot{C} for a club subset of ω_1 . As usual let $n(S) = n$ and let s be the stem of S . For each i such that $s \smallfrown \langle i \rangle \in S$, we choose a countable ordinal $\alpha_{s \smallfrown i}$, in such a way that each countable ordinal occurs as $\alpha_{s \smallfrown i}$ for stationarily many i . We then appeal to fact one and choose $U_i \leq S_{s \smallfrown \langle i \rangle}$ such that U_i determines $\min(\dot{C} \setminus (\alpha_{s \smallfrown i} + 1))$ as some countable ordinal $\beta_{s \smallfrown i}$. Forming the union of the U_i we obtain a condition $S_1 \leq_1 S$.

Repeating this construction we obtain (by fusion as usual) a condition $U \leq S$ with stem s , together with an assignment of ordinals $\alpha_u < \beta_u < \omega_1$ to each $u \in U$ with $lh(u) > n$. This assignment has the properties

- (1) U_u forces that $\beta_u = \min(\dot{C} \setminus (\alpha_u + 1))$.
- (2) For each $t \in U$ extending s and each countable ordinal η , there are stationarily many $i \in \lambda_{lh(t)}$ such that $\alpha_{t \smallfrown \langle i \rangle} = \eta$.

We now describe, for each countable ordinal δ , a game G_δ played on the tree, in which players I and II will build an increasing sequence u_i of elements of U , where $lh(u_i) = n + i$. We start by setting $u_0 = s$, the stem of U .

At the start of round i , u_i has been determined. In round i , player I chooses a non-stationary set $B_i \subseteq \lambda_{n+i}$ such that $u_i \smallfrown \langle \alpha \rangle \in U$ for all $\alpha \in B_i$, and also an ordinal $\eta_i < \delta$. Player II chooses $\gamma_i \notin B_i$ with $u_i \smallfrown \langle \gamma_i \rangle \in U$. We set $u_{i+1} = u_i \smallfrown \langle \gamma_i \rangle$. Player II wins if and only if $\eta_i < \alpha_{u_{i+1}} < \beta_{u_{i+1}} < \delta$ for all $i < \omega$.

The game G_δ is open, so by the Gale-Stewart theorem one of the two players has a winning strategy. We claim that the set of δ for which player I wins is non-stationary. To see this, suppose for contradiction that W is a stationary set such that for every $\delta \in W$, I has a winning strategy σ_δ . We fix some large regular θ , and choose a countable elementary $N \prec H_\theta$ such that N contains everything relevant and $\delta =_{\text{def}} N \cap \omega_1 \in W$.

We will construct a run of the game G_δ in which $u_i \in N$ for all i , player I plays according to σ_δ , yet player II wins. At round i in the construction, suppose that σ_δ instructs I to play a set B and an ordinal η . Now $\eta < \delta$ so $\eta \in N$, but there is no reason to believe that $B \in N$; however $\langle \sigma_\nu : \nu \in U \rangle \in N$, and the current position in the game is in N , so if we let B^* be the union of all the sets $\sigma_\nu(u_0, \dots, u_i)$ then $B \subseteq B^*$ and $B^* \in N$. By elementarity and the properties of the ordinal labelling we may choose $\gamma \in N$ such that $\gamma \notin B^*$, $u_i \smallfrown \langle \gamma \rangle \in U$ and $\eta < \alpha_{u_i \smallfrown \langle \gamma \rangle}$. Since the ordinal labelling on U is in N , $\beta_{u_i \smallfrown \langle \gamma \rangle} \in N$ and so $\beta_{u_i \smallfrown \langle \gamma \rangle} < \delta$. We may therefore choose $u_{i+1} = u_i \smallfrown \langle \gamma \rangle$.

To finish we choose $\delta \in T$ such that player I wins G_δ with some strategy τ . We also fix an ω -sequence of ordinals δ_i which is increasing and cofinal in δ . We then form a subtree V of U with the same root, by considering all runs of the game G_δ in which II plays according to τ and I plays δ_i as the ordinal part of his move in round

i. This subtree has stationary branching at every point above the root, because otherwise we could find a run in which I wins by choosing the right non-stationary set.

So $V \leq U$ and V is a condition. By construction V forces that \dot{C} is unbounded in δ , so V forces that $\delta \in \dot{C}$. This shows that the stationarity of T is preserved, concluding the proof of fact five.

We can now proceed to argue, using MM, that the scale $\langle g_\alpha : \alpha < \lambda^+ \rangle$ is not good. We will apply MM to the forcing poset $\mathbb{Q} = \mathbb{P} * \text{Coll}(\omega_1, \lambda^+)$.¹ The poset \mathbb{Q} is semi-proper, and in the generic extension by \mathbb{Q} we have that

- (1) $\text{cf}(\lambda^+) = |\lambda^+| = \omega_1$.
- (2) There is a function $h \in \prod_n (\lambda_n \cap \text{cof}(\omega))^\omega$ which is an eub for $\langle g_\alpha : \alpha < \lambda^+ \rangle$.

Working in the extension by \mathbb{Q} we may choose for each n a countable set $S_n \in V$ which is cofinal in $h(n)$. In the extension we may also find a sequence $\langle \alpha_i : i < \omega_1 \rangle$ which is increasing and cofinal in λ^+ , and is such that for every i there is $H_i \in \prod_n S_n$ with $g_{\alpha_i} <^* H_i <^* g_{\alpha_{i+1}}$ for all i .

We now work in V . Let C be club in λ^+ . Using MM we may obtain countable sets $S_n^* \subseteq \lambda_n$, an increasing sequence $\langle \alpha_i^* : i < \omega_1 \rangle$ with $\delta =_{\text{def}} \sup_i \alpha_i^* \in C$, and functions $H_i^* \in \prod_n S_n^*$ such that $g_{\alpha_i} <^* H_i^* <^* g_{\alpha_{i+1}}$ for all i .

As we argued above, no such δ can be a good point for the scale. We have shown that there are stationarily many points of cofinality ω_1 which are both good. This concludes the proof of claim 1 in Theorem 1.2.

2.2. Uncountable cofinality. We recall another stationary reflection principle which follows from MM [7]. Namely if MM holds, then for every regular $\kappa > \omega_1$ and every sequence $\langle S_i : i < \omega_1 \rangle$ of stationary subsets of $\kappa \cap \text{cof}(\omega)$, there are stationarily many $\alpha \in \kappa \cap \text{cof}(\omega_1)$ such that $S_i \cap \alpha$ is stationary for every $i < \omega_1$.

Claims 2 and 3 in Theorem 1.2 will follow immediately from the reflection fact which we just quoted and the following Lemmas.

Lemma 2.1. *If λ is singular and $\square_{\lambda, \mu}$ holds for some $\mu < \lambda$, then every stationary subset of λ^+ has $\text{cf}(\lambda)$ many stationary subsets which do not reflect simultaneously at any point of uncountable cofinality.*

Lemma 2.2. *If λ is uncountable and $\square_{\lambda, \mu}$ holds for some $\mu < \text{cf}(\lambda)$, then every stationary subset of λ^+ has a stationary subset which does not reflect at any point of uncountable cofinality.*

Lemma 2.1 is a very mild strengthening of a result from [2], where the conclusion is that the sets do not reflect simultaneously at points of cofinality greater than $\text{cf}(\lambda)$. Lemma 2.2 strengthens results of Cummings and Schimmerling [4]. Consistency results from [2] show that both these lemmas are sharp.

The proofs of both lemmas use the same construction, in which we start with a $\square_{\lambda, \mu}$ -sequence for some $\mu < \lambda$, and build a λ^+ sequence of functions in $\lambda^{\text{cf}(\lambda)}$ with some special properties. If λ is singular this is just the standard construction of a “very good scale”, but we will also allow the possibility that λ is regular.

We will fix a $\square_{\lambda, \mu}$ -sequence $\langle \mathcal{C}_\alpha : \alpha < \lambda^+ \rangle$. By a standard argument we may assume that for α with $\text{cf}(\alpha) < \lambda$, all the clubs which appear in \mathcal{C}_α have order type

¹It seems likely to us that under MM the forcing poset \mathbb{P} collapses λ^+ to an ordinal of cardinality and cofinality ω_1 , so with more work we could probably dispense with the additional collapse.

less than λ . Let $m(\alpha) = \min\{ot(C) : C \in \mathcal{C}_\alpha\}$, and $s(\alpha) = \sup\{ot(C) : C \in \mathcal{C}_\alpha\}$; clearly $m(\alpha) \leq s(\alpha) \leq \lambda$, and $m(\alpha) < \lambda$ for α such that $cf(\alpha) < \lambda$. In the case that $\mu < cf(\lambda)$ we also have $s(\alpha) < \lambda$ for α such that $cf(\alpha) < \lambda$.

Before proceeding we record a technical fact about the functions s and m . If $\eta < \lambda^+$ and there exist ordinals σ and μ such that there are stationarily many $\alpha < \eta$ with $s(\alpha) = \sigma$ and $m(\alpha) = \mu$, then $\mu \leq m(\eta) \leq s(\eta) \leq \sigma$. To see this observe that for every $E \in \mathcal{C}_\eta$ and every $\alpha \in \lim(E)$ we have $E \cap \alpha \in \mathcal{C}_\alpha$; by hypothesis there are unboundedly many $\alpha \in \lim(E)$ such that $\mu \leq ot(E \cap \alpha) \leq \sigma$, so that $\mu \leq ot(E) \leq \sigma$.

We now fix a sequence $\langle \lambda_i : i < cf(\lambda) \rangle$ of regular cardinals. If λ is singular then we choose this to be a strictly increasing sequence of regular cardinals which is cofinal in λ , and has $\max\{\mu, cf(\lambda)\} < \lambda_0$. If λ is regular we set $\lambda_i = \lambda$ for all $i < \lambda$. In either case it is easy to see that $\prod_i \lambda_i$ is λ^+ -directed under the ordering of eventual domination.

We build a sequence $\langle g_\alpha : \alpha < \lambda^+ \rangle$ of functions in $\prod_i \lambda_i$ which is increasing in the eventual domination ordering, along with some auxiliary functions h_C . For $C \in \mathcal{C}_\alpha$ we set $h_C(i) = \sup_{\alpha \in C} g_\alpha(i)$ for i such that $\lambda_i > ot(C)$, and $h_C(i) = 0$ otherwise. We construct the functions g_α in such a way that $g_\alpha(i) > \sup_{C \in \mathcal{C}_\alpha} h_C(i)$ for all i . The construction can proceed because $\prod_i \lambda_i$ is λ^+ -directed, and $\mu < \lambda_i$ for all i .

The key point is now that for every $\delta < \lambda^+$ of uncountable cofinality, there is a club set $E \subseteq \delta$ such that for all i with $m(\delta) \leq \lambda_i$ the sequence $\langle g_\alpha(i) : \alpha \in E \rangle$ is strictly increasing. To see this we choose some $D \in \mathcal{C}_\delta$ with $ot(D) = m(\delta)$, and let $E = \lim(D)$. Let i be such that $m(\delta) \leq \lambda_i$, and let $\beta, \gamma \in E$ with $\beta < \gamma$. Then

- $D \cap \gamma \in \mathcal{C}_\gamma$.
- $ot(D \cap \gamma) < ot(D) = m(\delta) \leq \lambda_i$, from which it follows that $h_{D \cap \gamma}(i) = \sup_{\alpha \in D \cap \gamma} g_\alpha(i)$.
- $\beta \in D \cap \gamma$, so $g_\beta(i) \leq h_{D \cap \gamma}(i) < g_\gamma(i)$.

Notice that if λ is singular then $m(\delta) < \lambda$, so $m(\delta) < \lambda_i$ for all large i ; if λ is regular then $m(\delta) \leq \lambda$ so $m(\delta) \leq \lambda_i$ for all i .

We are now ready to prove Lemma 2.1. Let λ be singular, let $\mu < \lambda$, fix a $\square_{\lambda, \mu}$ -sequence and construct functions $\langle g_\alpha : \alpha < \lambda^+ \rangle$ as above. Let S be a stationary subset of λ^+ . For each $i < cf(\lambda)$, fix a stationary set $S_i \subseteq S$ and an ordinal $\gamma_i < \lambda_i$ such that $g_\beta(i) = \gamma_i$ for all $\beta \in S_i$. Suppose (for contradiction) that δ is an ordinal of uncountable cofinality such that $S_i \cap \delta$ is stationary for every i , and fix a club subset E of δ such that $\langle g_\beta(i) : \beta \in E \rangle$ is strictly increasing for all large i . For any such i we may find $\beta < \beta^*$ where $\beta, \beta^* \in E \cap S_i$, and then reach a contradiction because $g_\beta(i) = \gamma_i = g_{\beta^*}(i)$ while $\langle g_\beta(i) : \beta \in E \rangle$ is strictly increasing. This concludes the proof of Lemma 2.1.

We now prove Lemma 2.2. Assuming now that $\mu < cf(\lambda)$, we fix a $\square_{\lambda, \mu}$ -sequence, and again construct functions $\langle g_\alpha : \alpha < \lambda^+ \rangle$ as above. Let S be a stationary subset of λ^+ . The set of points with cofinality λ is non-reflecting, so we may assume that S consists of ordinals with cofinality less than λ . Let $T \subseteq S$ be a stationary set such that the function s is constant on T , say $s(\alpha) = \sigma$ for all $\alpha \in S$. Now $\sigma < \lambda$, so we may find i such that $\sigma < \lambda_i$, and then find $U \subseteq T$ such that the function $\alpha \mapsto g_\alpha(i)$ is constant on U , say $g_\alpha(i) = \rho$ for all $\alpha \in U$.

Suppose (for contradiction) that $U \cap \eta$ is stationary for some $\eta < \lambda^+$ of uncountable cofinality. As we argued when we defined the functions s and m , $m(\eta) \leq \sigma$.

We may find a club subset $E \subseteq \eta$ such that $\langle g_\alpha(j) : \alpha \in E \rangle$ is strictly increasing for all j such that $m(\eta) \leq j$. Now $m(\eta) \leq \sigma < \lambda_i$, so that in particular $\langle g_\alpha(i) : \alpha \in E \rangle$ is strictly increasing; but this is impossible because $E \cap U$ is unbounded in η and $g_\alpha(i) = \rho$ for all $\alpha \in U$. This concludes the proof of Lemma 2.2.

We note that when λ is regular, we are actually proving a stronger conclusion. Namely there is a function from λ^+ to λ which is increasing on a club in every ordinal of uncountable cofinality. This is an instance of the “strong non-reflection” phenomenon first studied by Džamonja and Shelah[5, 1].

3. MAXIMISING SQUARE IN A MODEL OF MM

In this section we prove Theorem 1.3. The strategy is straightforward: we construct a model with a supercompact cardinal in which the extent of square is maximised, then perform the standard consistency proof for MM.

We will use two forcing posets. Given a singular cardinal λ , let P_λ be the forcing poset to add a $\square_{\lambda, \text{cf}(\lambda)}$ -sequence. The key properties of this poset are that

- (1) Player II wins the standard strategic closure game of length $\mu + 1$ played on P_λ , for every cardinal μ less than λ .
- (2) P_λ is $\text{cf}(\lambda)$ -directed closed.
- (3) P_λ adds a $\square_{\lambda, \text{cf}(\lambda)}$ -sequence.

We also need a poset for adding a certain partial version of \square_λ , in which we have one club set C_η for each η in a certain kind of stationary subset of λ^+ . Given a regular cardinal κ and a cardinal λ with $\lambda \geq \kappa$, let $Q(\kappa, \lambda)$ be the following poset: conditions are sequences $\langle C_\eta : \eta \in \beta + 1 \rangle$ where

- (1) $\beta < \lambda^+$.
- (2) C_η is either empty or a club subset of η with order type at most λ .
- (3) C_η is non-empty for every $\eta \in \beta + 1$ with $\text{cf}(\eta) \geq \kappa$.
- (4) For every η and every $\zeta \in \lim(C_\eta)$, $C_\zeta = C_\eta \cap \zeta$.

The ordering is end-extension. This poset (due to Baumgartner) has the following properties:

- (1) Player II wins the standard strategic closure game of length $\mu + 1$ played on $Q(\kappa, \lambda)$, for every cardinal μ less than λ .
- (2) $Q(\kappa, \lambda)$ is κ -directed closed.
- (3) $Q(\kappa, \lambda)$ adds a sequence $\langle C_\eta : \eta \in S \rangle$ which is a “partial square on points of cofinality at least κ ”, that is $\lambda^+ \cap \text{cof}(\geq \kappa) \subseteq S \subseteq \lambda^+$, and for every $\eta \in S$
 - (a) C_η is a club subset of η with order type at most λ .
 - (b) For every $\zeta \in \lim(C_\eta)$, $\zeta \in S$ and $C_\zeta = C_\eta \cap \zeta$.

Now we describe the construction of the model for Theorem 1.3. We start with a model V_0 in which GCH holds and κ is supercompact. We then force in the standard way to make κ Laver indestructible, and obtain a model V_1 . In V_1 we have that GCH holds at and above κ , and the supercompactness of κ is indestructible under κ -directed closed forcing.

We now do an iteration of length ON with Easton supports. For each singular cardinal $\lambda > \kappa$ we will force with P_λ if $\text{cf}(\lambda) \geq \kappa$, and with $Q(\kappa, \lambda)$ if $\text{cf}(\lambda) < \kappa$. We obtain a model V_2 , in which by standard arguments

- (1) κ is supercompact.
- (2) Cardinals and cofinalities are preserved.
- (3) GCH holds at and above κ .

- (4) For every singular $\lambda > \kappa$
- (a) If $\text{cf}(\lambda) \geq \kappa$, then $\square_{\lambda, \text{cf}(\lambda)}$ holds.
 - (b) If $\text{cf}(\lambda) < \kappa$ there is a partial square on points of cofinality at least κ .

Now we force over V_2 with the standard forcing for the consistency of MM. The forcing poset is semi-proper and κ -c.c. with cardinality κ . After forcing we get a model V_3 in which MM holds. By standard arguments ω_1 is preserved, κ is the new ω_2 , $2^\omega = \omega_2$ and GCH holds at and above ω_2 . We now analyse the extent of square in the model V_3 , by determining for each uncountable λ the range of μ for which $\square_{\lambda, \mu}$ holds.

- $\lambda = \omega_1$. Since MM implies that ω_2 has the tree property, there are no special ω_2 -trees and hence $\square_{\omega_1}^*$ fails.
- λ regular and $\lambda \geq \omega_2$. $\lambda^{<\lambda} = \lambda$ and so \square_λ^* holds. On the other hand it follows from Theorem 1.2 that $\square_{\lambda, \mu}$ fails for every $\mu < \lambda$.
- λ singular and $\text{cf}(\lambda) = \omega$. As we mentioned in the introduction, it follows from Theorem 1.2 that \square_λ^* fails.
- λ singular and $\text{cf}(\lambda) = \omega_1$. By the properties of V_2 , there is a partial square on points of cofinality at least ω_2 in λ^+ . We now choose for each point δ of cofinality ω_1 a club C_δ in δ . Since $\lambda^{\aleph_0} = \lambda$, for every $\alpha < \lambda^+$ we have $|\{C_\delta \cap \alpha : \text{cf}(\delta) = \omega_1\}| \leq \lambda$. So we may “fill in” by adding appropriate sets of clubs at points of cofinality ω , and obtain a \square_λ^* sequence. It follows from Theorem 1.2 that $\square_{\lambda, \mu}$ fails for every $\mu < \lambda$.
- λ singular and $\text{cf}(\lambda) \geq \omega_2$. By the properties of V_2 , and the fact that cardinals and cofinalities above κ agree between V_2 and V_3 , $\square_{\lambda, \text{cf}(\lambda)}$ holds. It follows from Theorem 1.2 that $\square_{\lambda, \mu}$ fails for every $\mu < \text{cf}(\lambda)$.

This shows that the model V_3 is a model of MM which has, for each λ , the strongest possible form of the square principle $\square_{\lambda, \mu}$.

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MATHEMATICAL SCIENCES DEPARTMENT, CARNEGIE MELLON UNIVERSITY, PITTSBURGH PA 15215, USA

E-mail address: `jcumming@andrew.cmu.edu`

EINSTEIN INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM 91904, ISRAEL

E-mail address: `menachem@math.huji.ac.il`