ARONSZAJN AND KUREPA TREES

JAMES CUMMINGS

Abstract. Monroe Eskew [3] asked whether the tree property at ω_2 implies there is no Kurepa tree (as is the case in the Mitchell model, or under PFA). We prove that the tree property at ω_2 is consistent with the existence of ω_1-trees with as many branches as desired.

1. Introduction

Monroe Eskew [3] observed that in the standard models where ω_2 has the tree property there is no Kurepa tree. He then raised the natural question whether the tree property at ω_2 implies that there is no Kurepa tree. We answer this question in the negative.

We build our model in two steps. In the first step we perform Mitchell's construction [5] to obtain the tree property at ω_2. In the second step we add an ω_1-tree with many branches, and argue that the tree property is preserved. The main point is that in the second step we use a certain variation of the standard forcing to add such an ω_1-tree *computed in the ground model*; a similar idea appears in Uri Abraham's construction [1] of a model where both ω_2 and ω_3 have the tree property.

The paper is organised as follows:

- In section 2 we review the arguments that there are no Kurepa trees under PFA or in the Mitchell model, and discuss properties of Mitchell's construction which will be used in the proof of the main result.
- In section 3 we define a poset for adding an ω_1-tree with many branches and prove some technical results.
- In section 4 we show first that the tree property at ω_2 is consistent with the existence of a Kurepa tree (Theorem 1) and then parlay this result into a proof that the tree property at ω_2 puts no bound on the number of branches of an ω_1-tree (Theorem 2).

We will use some standard facts about forcing, whose proofs we give here for the sake of completeness. We write $\mathrm{lh}(s)$ for the length of a sequence s, and $i_G(\dot{a})$ for the interpretation of a forcing term $\dot{a}$ by a generic object G.

Lemma 1 (Silver [7]). *Let $\mathbb{P}$ be a countably closed forcing poset and suppose that either T is an ω_1-tree or $2^\omega > \omega_1$ and T is an ω_2-tree. Then forcing with $\mathbb{P}$ adds no new cofinal branch in T.*

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Proof. Let $\dot{b}$ name a new cofinal branch. We will build a strictly increasing sequence of ordinals $(\alpha_i)_{i \le \omega}$ less than $\mathrm{ht}(T)$, together with a family of conditions $(p_s)_{s \in {}^{\le \omega}2}$ in $\mathbb{P}$ and a family of points $(x_s)_{s \in {}^{\le \omega}2}$ in T such that:

- If t extends s then $p_t \le p_s$.
- x_s is a point on level $\alpha_{\mathrm{lh}(s)}$ of T, and p_s forces that x_s is the unique point of $\dot{b}$ on this level.
- For each $s \in {}^{<\omega}2$, $x_{s \frown 0} \neq x_{s \frown 1}$.

This is easy using the hypotheses that $\mathbb{P}$ is countably closed and $\dot{b}$ is forced to be a new branch. Clearly $\{x_s : s \in {}^{\omega}2\}$ is a set of 2^ω distinct points on level α_ω, but this is impossible by our hypotheses. □

Lemma 2 (Mitchell [4])**.** *Let $\mathbb{P}$ be a poset such that $\mathbb{P} \times \mathbb{P}$ is ccc, and let T be a tree of height ω_1. Then forcing with T adds no new cofinal branch in T.*

Proof. Let $\dot{b}$ be forced to be a new cofinal branch. We will build a sequence of conditions $(p_i, q_i)_{i < \omega_1}$ in $\mathbb{P} \times \mathbb{P}$ and a strictly increasing sequence of countable ordinals $(\gamma_i)_{i < \omega_1}$ such that:

- p_i and q_i decide the unique point of $\dot{b}$ on level γ_i in the same way, say as x_i.
- p_i and q_i decide the unique point of $\dot{b}$ on level $\gamma_i + 1$ in different ways, say p_i as y_i and q_i as z_i.

This is easy using the hypothesis that $\dot{b}$ is forced to be a new branch.

By the hypothesis on $\mathbb{P}$ we find $i < j$ such that there exist conditions $p \le p_i, p_j$ and $q \le q_i, q_j$. Since both p_j and q_j force that x_j is the unique point of $\dot{b}$ on level γ_j, while p_i (resp q_i) forces that y_i (resp z_i) is the unique point on level $\gamma_i + 1$, we see that $y_i, z_i \le_T x_j$. This is a contradiction since T is a tree. □

We will also use a standard fact (which we learned from Menachem Magidor) about the preservation of the Knaster property.

Lemma 3. *If κ is a regular uncountable cardinal, $\mathbb{A}$ is κ-Knaster and $\mathbb{B}$ is κ-cc then $\mathbb{A}$ remains κ-Knaster after forcing with $\mathbb{B}$.*

Proof. Let b force that $(\dot{a}_\alpha)_{\alpha < \kappa}$ is a counterexample, and choose $b_\alpha \le b$ and a_α such that $b_\alpha \vDash \dot{a}_\alpha = a_\alpha$. Use the Knaster property to find unbounded $A \subseteq \kappa$ such that $(a_\alpha)_{\alpha \in A}$ is a sequence of pairwise compatible conditions. Then use κ-cc to find $\alpha \in A$ such that $b_\alpha \Vdash \{\beta \in A : b_\beta \in G\}$ is unbounded in κ. This is a contradiction since $i_G(\dot{a}_\beta) = a_\beta$ for $b_\beta \in G$, while b forces that $(\dot{a}_\alpha)_{\alpha < \kappa}$ has no pairwise comparable subsequence of length κ. □

2. PFA and the Mitchell model

We begin by reviewing the standard arguments that PFA implies there are no ω_2-Aronszajn trees and there are no Kurepa trees.

Fact 1 (Baumgartner [2])**.** *PFA implies that there are no ω_2-Aronszajn trees.*

Proof. Assume PFA, and suppose for a contradiction that T is an ω_2-Aronszajn tree. Let $\mathbb{P} = Coll(\omega_1, \omega_2)$, then it follows from Lemma 1 that T has no cofinal branches in the extension by $\mathbb{P}$. Working in this extension let U be a cofinal subtree of T with height ω_1, so that U has no cofinal branches, and let $\mathbb{Q}$ be the standard ccc forcing to specialise U. Applying PFA to the proper poset $\mathbb{P} * \mathbb{Q}$, we obtain some $\beta \in \omega_2 \cap \mathrm{cof}(\omega_1)$ together with a special cofinal subtree of $T \restriction \beta$, an immediate contradiction since every point of height β in T induces a cofinal branch in $T \restriction \beta$. $\square$

Fact 2 (Baumgartner [2]). *PFA implies that there are no Kurepa trees.*

Proof. Assume PFA, and suppose for a contradiction that T is a Kurepa tree. Let $\mathbb{P} = Coll(\omega_1, 2^{\omega_1})$, then it follows from Lemma 1 that $\mathbb{P}$ adds no new cofinal branch to T, so that T has at most ω_1 cofinal branches in the extension by $\mathbb{P}$. Let $\mathbb{Q}$ be the standard ccc forcing in $V^{\mathbb{P}}$ (see eg [2]) to "freeze" the set of branches of T by adding a certain function f with domain T, so that every branch of T in any ω_1-preserving extension of $V^{\mathbb{P}*\mathbb{Q}}$ lies in $V^{\mathbb{P}}$ and is simply definable from the function f and some point of T.

Applying PFA to the proper poset $\mathbb{P} * \mathbb{Q}$ we obtain a function with domain T which witnesses that T has at most ω_1 branches, an immediate contradiction since T is a Kurepa tree. $\square$

We turn now to the Mitchell model [5], and take the opportunity to record some facts for later use. Our treatment of Mitchell's construction follows Abraham [1] and we refer the reader to that paper for proofs of the facts quoted here. Given an inaccessible cardinal β, Mitchell constructed a forcing poset which we denote $\mathbb{P}^{\mathrm{Mi}}_\beta$ and which has the following properties:

- $\mathbb{P}^{\mathrm{Mi}}_\beta$ is a β-Knaster poset of cardinality β with $\mathbb{P}^{\mathrm{Mi}}_\beta \subseteq V_\beta$.
- $\mathbb{P}^{\mathrm{Mi}}_\beta$ can be written as the projection of $Add(\omega, \beta) \times \mathbb{Q}^{\mathrm{Mi}}_\beta$ for some countably closed term forcing $\mathbb{Q}^{\mathrm{Mi}}_\beta$.
- $\mathbb{P}^{\mathrm{Mi}}_\beta$ preserves ω_1 and collapses all cardinals between ω_1 and β, so that β becomes ω_2.
- $\mathbb{P}^{\mathrm{Mi}}_\beta$ adds β Cohen subsets of ω, and forces that $2^\omega = \beta$.
- For every inaccessible $\alpha < \beta$:
  - $\mathbb{P}^{\mathrm{Mi}}_\alpha = \mathbb{P}^{\mathrm{Mi}}_\beta \cap V_\alpha$.
  - $\mathbb{P}^{\mathrm{Mi}}_\beta$ can be factorised as $\mathbb{P}^{\mathrm{Mi}}_\alpha * \mathbb{R}^{\mathrm{Mi}}_{\alpha,\beta}$ where (in the extension by $\mathbb{P}^{\mathrm{Mi}}_\alpha$) the poset $\mathbb{R}^{\mathrm{Mi}}_{\alpha,\beta}$ can be written as the projection of $Add(\omega, \beta - \alpha) \times \mathbb{Q}^{\mathrm{Mi}}_{\alpha,\beta}$ for some countably closed term forcing $\mathbb{Q}^{\mathrm{Mi}}_{\alpha,\beta}$

Fact 3 (Mitchell [5]). *If κ is weakly compact then $\mathbb{P}^{\mathrm{Mi}}_\kappa$ forces that there are no ω_2-Aronszajn trees.*

Proof. Let G be $\mathbb{P}^{\mathrm{Mi}}_\kappa$-generic over V, and suppose for contradiction that T is an κ-Aronszajn tree in $V[G]$. By standard reflection properties of the weakly compact cardinal κ, there is an inaccessible $\alpha < \kappa$ such that $T \restriction \alpha$ is an Aronszajn tree in $V[G \restriction \alpha]$ where $G \restriction \alpha$ is the $\mathbb{P}^{\mathrm{Mi}}_\alpha$-generic object induced by G. Clearly $T \restriction \alpha$ has a cofinal branch in $V[G]$, and it follows from the analysis of $\mathbb{P}^{\mathrm{Mi}}_\kappa$ given above that forcing over $V[G \restriction \alpha]$ with $Add(\omega, \kappa - \alpha) \times \mathbb{Q}^{\mathrm{Mi}}_{\alpha,\kappa}$ adds a branch to T.

We may view forcing with $Add(\omega, \kappa - \alpha) \times \mathbb{Q}^{\mathrm{Mi}}_{\alpha,\kappa}$ as a two-step iteration, where we first force with the countably closed poset $\mathbb{Q}^{\mathrm{Mi}}_{\alpha,\kappa}$ and then with the $\aleph_1$-Knaster poset $Add(\omega, \kappa - \alpha)$. Using Lemmas 1 and 2, neither step can add a cofinal branch to the tree $T \restriction \alpha$ and we have an immediate contradiction. $\square$

The following fact, which is an easy variation on an argument of Silver [7] involving the Levy collapse of an inaccessible cardinal, has been observed independently by several authors.

Fact 4. *If κ is weakly compact then $\mathbb{P}^{\mathrm{Mi}}_\kappa$ forces that there are no Kurepa trees.*

Proof. Let G be $\mathbb{P}^{\mathrm{Mi}}_\kappa$-generic over V, and suppose for contradiction that T is a Kurepa tree in $V[G]$. Let $\alpha < \kappa$ be an inaccessible cardinal chosen large enough that $T \in V[G \restriction \alpha]$. Arguing much as in the proof of Fact 3, the forcing poset $Add(\omega, \kappa - \alpha) \times \mathbb{Q}^{\mathrm{Mi}}_{\alpha,\kappa}$ adds no new cofinal branches of the tree T, and so in particular all branches of T in $V[G]$ already appear in the inner model $V[G \restriction \alpha]$. Since κ is inaccessible in $V[G \restriction \alpha]$ and $\kappa = \omega_2^{V[G]}$, we see that T has at most $\aleph_1$ branches in $V[G]$, contradicting the assumption that it is a Kurepa tree. □

3. Adding trees

Let λ be a cardinal with $\lambda > \omega_1$. We define a poset $\mathbb{P}^{\mathrm{Ku}}_\lambda$ to add an ω_1-tree with λ branches. A condition p is a pair (t_p, f_p) where t_p is a countable normal tree of successor height $\alpha_p + 1$, and f_p is a countable partial function from λ to $\mathrm{Lev}_{\alpha_p}(t_p)$. Conditions are ordered as follows: $q \le p$ if and only if $\alpha_q \ge \alpha_p$, $t_q \restriction (\alpha_p + 1) = t_p$, $\mathrm{dom}(f_p) \subseteq \mathrm{dom}(f_q)$ and $f_p(\eta) \le_{t_q} f_q(\eta)$ for all $\eta \in \mathrm{dom}(p)$.

Remark 1. *This is not quite the standard poset due to Stewart* [6] *for adding trees with many branches, since we do not demand that f_p be injective.*

Lemma 4. *The poset $\mathbb{P}^{\mathrm{Ku}}_\lambda$ is countably closed, and adds an ω_1-tree with λ distinct branches.*

Proof. Let $(p_i)_{i<\omega}$ be a decreasing sequence of conditions, let $p_i = (t_i, f_i)$ and let t_i have height $\alpha_i + 1$. Let t^* be the unique tree of height $\sup_i(\alpha_i + 1)$ such that $t^* \restriction (\alpha_i + 1) = t_i$ for all i, and let $d = \bigcup_i \mathrm{dom}(f_i)$. For each $\zeta \in d$, the sequence $b_\zeta = (f_i(\zeta))_{i<\omega, \zeta \in \mathrm{dom}(f_i)}$ is increasing in t^*. To define a lower bound (t_ω, f_ω) we distinguish two cases:

- If the sequence $(\alpha_i)_{i<\omega}$ is eventually constant, then the sequence $(t_i)_{i<\omega}$ is also eventually constant with value t^*, and for every $\zeta \in d$ the sequence b_ζ is eventually constant with some value on the top level of t^*. In this case we set $t_\omega = t^*$, $\mathrm{dom}(f_\omega) = d$, and $f_\omega(\zeta)$ equal to the eventual constant value of b_ζ.
- If the sequence $(\alpha_i)_{i<\omega}$ is not eventually constant, then t^* has limit height and for each $\zeta \in d$ the sequence b_ζ forms a cofinal branch in t^*. We choose t_ω to be some countable normal tree of height $\mathrm{ht}(t^*) + 1$ such that $t_\omega \restriction \mathrm{ht}(t^*) = t^*$, and every branch b_ζ is bounded by some point on the top level of t_ω. We then set $f_\omega(\zeta)$ equal to the unique point which bounds b_ζ.

To finish, it will suffice to show that for all $\alpha < \omega_1$ and distinct $\zeta, \eta < \lambda$ the set of conditions q such that $\alpha_q > \alpha$ and $f_q(\zeta) \neq f_q(\eta)$ is dense. Given a condition p, we first form $q_0 \le p$ such that $\alpha_{q_0} > \alpha$ as follows: t_{q_0} is some sufficiently tall countable normal tree with $t_{q_0} \restriction (\alpha_p + 1) = t_p$, $\mathrm{dom}(f_p) = \mathrm{dom}(f_{q_0})$, and for $\theta \in \mathrm{dom}(f_p)$ we define $f_{q_0}(\theta)$ as some point on the top level of t_{q_0} which lies above $f_p(\theta)$. Then we find $q_1 \le q_0$ such that $\alpha_{q_1} = \alpha_{q_0}$ and $\zeta, \eta \in \mathrm{dom}(f_{q_1})$. Finally we find $q \le q_1$ such that $\alpha_q = \alpha_{q_1} + 1$, $\mathrm{dom}(f_q) = \mathrm{dom}(f_{q_1})$ and $f_q(\zeta) \neq f_q(\eta)$. □

Lemma 5. *Let μ be a regular uncountable cardinal such that $\gamma^{\aleph_0} < \mu$ for all $\gamma < \mu$. Then the poset $\mathbb{P}^{\mathrm{Ku}}_\mu$ has the μ-Knaster property.*

Proof. Consider a sequence of conditions $p_i = (t_i, f_i)$ for $i < \mu$. Using the Δ-system lemma we may find an unbounded set $A \subseteq \mu$ such that:

- For all $i \in A$, $t_i = t$ for some fixed countable tree t.
- $(\mathrm{dom}(f_i))_{i \in A}$ forms a Δ-system with some root r.
- For all $i \in A$, $f_i \restriction r = g$ for some fixed function g.

It is now easy to verify that $(p_i)_{i \in A}$ is a sequence of pairwise compatible conditions. $\square$

As an immediate corollary, if CH holds then $\mathbb{P}^{\mathrm{Ku}}_\mu$ has the $\aleph_2$-cc and hence preserves all cardinals.

Lemma 6. *If $\lambda < \lambda'$ let π from $\mathbb{P}^{\mathrm{Ku}}_{\lambda'}$ to $\mathbb{P}^{\mathrm{Ku}}_{\lambda}$ be given by $\pi : (t, f) \mapsto (t, f \restriction \lambda)$. Then π is a projection.*

Proof. It is routine to check that π is order-preserving and maps the trivial condition to the trivial condition. Let $q \in \mathbb{P}^{\mathrm{Ku}}_{\lambda'}$ and let $p \in \mathbb{P}^{\mathrm{Ku}}_{\lambda}$ with $p \leq \pi(q)$. Let $\alpha_q = \alpha$ and $\alpha_p = \beta$, and define $q' \leq q$ as follows: $\alpha_{q'} = \beta$, $f_{q'}(\eta) = f_p(\eta)$ for $\eta \in \mathrm{dom}(p)$ and $f_{q'}(\eta)$ is some point in $\mathrm{Lev}_\beta(t_p)$ lying above $f_q(\eta)$ for $\eta \in \mathrm{dom}(q) \setminus \mathrm{dom}(p) = \mathrm{dom}(q) \setminus \lambda$. Clearly $\pi(q') = p$. $\square$

Let H be a $\mathbb{P}^{\mathrm{Ku}}_{\lambda}$ generic filter, and let $T = \bigcup\{t : (t, f) \in G\}$. As usual let $\mathbb{P}^{\mathrm{Ku}}_{\lambda'}/H = \{p \in \mathbb{P}^{\mathrm{Ku}}_{\lambda'} : \pi(p) \in H\}$, so that forcing over $V[H]$ with $\mathbb{P}^{\mathrm{Ku}}_{\lambda}/H$ will give a $\mathbb{P}^{\mathrm{Ku}}_{\lambda'}$-generic object H' whose projection via π is H. It is easy to see that $\mathbb{P}^{\mathrm{Ku}}_{\lambda'}/H$ is equivalent to the poset $\mathbb{R}^{\mathrm{Ku}}_{\lambda,\lambda'} \in V[H]$ defined as follows. A condition in $\mathbb{R}^{\mathrm{Ku}}_{\lambda,\lambda'}$ is a pair $r = (\alpha_r, f_r)$ where f_r is a countable partial function from $\lambda' \setminus \lambda$ to $\mathrm{Lev}_{\alpha_r}(T)$. $s \leq r$ if and only $\alpha_s \geq \alpha_r$, $\mathrm{dom}(f_r) \subseteq \mathrm{dom}(f_s)$ and $f_r(\eta) \leq f_s(\eta)$ for all $\eta \in \mathrm{dom}(r)$.

Lemma 7. *Let CH hold in V, let H be $\mathbb{P}^{\mathrm{Ku}}_{\lambda}$-generic and let $\mathbb{R}^{\mathrm{Ku}}_{\lambda,\lambda'} \in V[H]$ be defined as above. Then:*

- *$\mathbb{R}^{\mathrm{Ku}}_{\lambda,\lambda'}$ is countably distributive in $V[H]$.*
- *The class of regular cardinals μ such that $\gamma^{\aleph_0} < \mu$ for all $\gamma < \mu$ is absolute between V and $V[H]$. Moreover for every cardinal μ in this class, $\mathbb{R}^{\mathrm{Ku}}_{\lambda,\lambda'}$ is μ-Knaster in $V[H]$.*

Proof. The first assertion is immediate because the forcing poset $\mathbb{P}^{\mathrm{Ku}}_{\lambda'}$ is countably closed, and $\mathbb{R}^{\mathrm{Ku}}_{\lambda,\lambda'}$ is equivalent to $\mathbb{P}^{\mathrm{Ku}}_{\lambda'}/H$. By Lemma 5 with $\mu = \aleph_2$, $\mathbb{P}^{\mathrm{Ku}}_{\lambda}$ is $\aleph_2$-Knaster and so easily $V[H]$ has the same cardinals and cofinalities as V; since V and $V[H]$ also have the same countable sequences of ordinals, the given class of cardinals is absolute between V and $V[H]$. Finally if μ lies in this class then a very similar argument to that for Lemma 5 shows that $\mathbb{R}^{\mathrm{Ku}}_{\lambda,\lambda'}$ is μ-Knaster in $V[H]$. $\square$

4. The main result

We are now ready to prove that the tree property at ω_2 is consistent with the existence of a Kurepa tree.

Theorem 1. *Let κ be weakly compact and let CH hold. Then in the generic extension by $\mathbb{P}^{\mathrm{Mi}}_\kappa \times \mathbb{P}^{\mathrm{Ku}}_\kappa$:*

- *ω_1 is preserved and κ is ω_2.*
- *There is an ω_1-Kurepa tree.*
- *ω_2 has the tree property.*

Proof. Let $G \times H$ be $\mathbb{P}^{\mathrm{Mi}}_\kappa \times \mathbb{P}^{\mathrm{Ku}}_\kappa$-generic over V. Since $\mathbb{P}^{\mathrm{Mi}}_\kappa$ and $\mathbb{P}^{\mathrm{Ku}}_\kappa$ are κ-Knaster in V, so is $\mathbb{P}^{\mathrm{Mi}}_\kappa \times \mathbb{P}^{\mathrm{Ku}}_\kappa$. We recall from Section 2 that we can write $\mathbb{P}^{\mathrm{Mi}}_\kappa$ as the projection of $Add(\omega, \kappa) \times \mathbb{Q}^{\mathrm{Mi}}_\kappa$ for some countably closed forcing poset $\mathbb{Q}^{\mathrm{Mi}}_\kappa$. It follows that $\mathbb{P}^{\mathrm{Mi}}_\kappa \times \mathbb{P}^{\mathrm{Ku}}_\kappa$ is the projection of $Add(\omega, \kappa) \times (\mathbb{Q}^{\mathrm{Mi}}_\kappa \times \mathbb{P}^{\mathrm{Ku}}_\kappa)$, where $\mathbb{Q}^{\mathrm{Mi}}_\kappa \times \mathbb{P}^{\mathrm{Ku}}_\kappa$ is countably closed, hence $\mathbb{P}^{\mathrm{Mi}}_\kappa \times \mathbb{P}^{\mathrm{Ku}}_\kappa$ preserves ω_1. It follows that ω_1 is preserved and κ is ω_2 in $V[G \times H]$.

By the construction of $\mathbb{P}^{\mathrm{Ku}}_\kappa$, in $V[H]$ there is an ω_1-tree with κ cofinal branches. In $V[G \times H]$ the same tree will be an ω_1-tree with ω_2 cofinal branches, that is to say a Kurepa tree.

It remains to see that ω_2 has the tree property in $V[G \times H]$, so assume for a contradiction that T is an κ-Aronszajn tree in $V[G \times H]$. By the usual reflection arguments we may may find an inaccessible $\alpha < \kappa$ such that $T \restriction \alpha$ is an α-Aronszajn tree in $V[G \restriction \alpha \times H \restriction \alpha]$, where $G \restriction \alpha \times H \restriction \alpha$ is the $\mathbb{P}^{\mathrm{Mi}}_\alpha \times \mathbb{P}^{\mathrm{Ku}}_\alpha$-generic object induced by $G \times H$.

Since $T \restriction \alpha$ has a branch in $V[G \times H]$, we will be done if we can show that forcing over $V[G \restriction \alpha \times H \restriction \alpha]$ with $\mathbb{R}^{\mathrm{Mi}}_{\alpha,\kappa} \times \mathbb{R}^{\mathrm{Ku}}_{\alpha,\kappa}$ adds no cofinal branch in $T \restriction \alpha$. We will do this by arguments which parallel those for Fact 3. We start by embedding $V[G]$ into $V[G_{Add} \times G_{term}]$ where G_{Add} is the $Add(\omega, \kappa)$-generic added by G, G_{term} is $\mathbb{Q}^{\mathrm{Mi}}_\kappa$-generic and G is the $\mathbb{P}^{\mathrm{Mi}}_\kappa$-generic object induced by projection from $G_{Add} \times G_{term}$. Similarly we embed $V[G \restriction \alpha]$ into $V[G_{Add} \restriction \alpha \times G_{term} \restriction \alpha]$.

Claim 1. *The models $V[G_{Add} \restriction \alpha]$, $V[G \restriction \alpha]$ and $V[G \restriction \alpha \times H \restriction \alpha]$ have the same ω-sequences of ordinals.*

Proof. $\mathbb{P}^{\mathrm{Mi}}_\alpha$ is a projection of $Add(\omega, \alpha) \times \mathbb{Q}^{\mathrm{Mi}}_\alpha$, so $\mathbb{P}^{\mathrm{Mi}}_\alpha \times \mathbb{P}^{\mathrm{Ku}}_\alpha$ is a projection of $Add(\omega, \alpha) \times \mathbb{Q}^{\mathrm{Mi}}_\alpha \times \mathbb{P}^{\mathrm{Ku}}_\alpha$. Since $\mathbb{Q}^{\mathrm{Mi}}_\alpha \times \mathbb{P}^{\mathrm{Ku}}_\alpha$ is countably closed, the claim follows immediately by Easton's Lemma. □

Claim 2. *The forcing poset $\mathbb{R}^{\mathrm{Ku}}_{\alpha,\kappa}$ is countably distributive and α-Knaster in $V[G \restriction \alpha \times H \restriction \alpha]$.*

Proof. By Lemma 7, $\mathbb{R}^{\mathrm{Ku}}_{\alpha,\kappa}$ is α-Knaster in $V[H \restriction \alpha]$. By Lemma 5, $\mathbb{P}^{\mathrm{Ku}}_\alpha$ is α-Knaster in V and so (by the Product Lemma) $\mathbb{R}^{\mathrm{Mi}}_\alpha$ is α-cc in $V[H \restriction \alpha]$. It follows from Lemma 3 that $\mathbb{R}^{\mathrm{Ku}}_{\alpha,\kappa}$ is α-Knaster in $V[G \restriction \alpha \times H \restriction \alpha]$.

Arguing as in the preceding claim we may embed $V[G \restriction \alpha][H]$ into $V[G_{Add} \restriction \alpha \times G_{term} \restriction \alpha \times H]$ and use Easton's lemma to show that every ω-sequence of ordinals in this model is in $V[G_{Add} \restriction \alpha]$. It follows that $\mathbb{R}^{\mathrm{Ku}}_{\alpha,\kappa}$ is countably distributive in $V[G \restriction \alpha \times H \restriction \alpha]$. □

By Claim 2 $\mathbb{R}^{\mathrm{Ku}}_{\alpha,\kappa}$ is α-Knaster in $V[G \restriction \alpha \times H \restriction \alpha]$, so we have that $\alpha = \omega_2$ and $T \restriction \alpha$ has no cofinal branch in $V[G \restriction \alpha \times H]$.

Recall from Section 2 that in $V[G \restriction \alpha]$ we may write $\mathbb{R}^{\mathrm{Mi}}_{\alpha,\kappa}$ as a projection of $Add(\omega, \kappa - \alpha) \times \mathbb{Q}^{\mathrm{Mi}}_{\alpha,\kappa}$, where $\mathbb{Q}^{\mathrm{Mi}}_{\alpha,\kappa}$ is a countably closed term forcing. The same analysis obtains in $V[G \restriction \alpha \times H]$, and by Claims 1 and 2 the poset $\mathbb{Q}^{\mathrm{Mi}}_{\alpha,\kappa}$ is still countably closed in this model. We may now argue exactly as in Fact 3 that $T \restriction \alpha$ has no branch in $V[G \times H]$, an immediate contradiction.

□

It is now easy to arrange a model of the tree property at ω_2 where an ω_1-tree has an arbitrary number of branches.

Theorem 2. *Let κ be weakly compact and let CH hold. Let $\lambda \geq \kappa$. Then in the generic extension by $\mathbb{P}^{\mathrm{Mi}}_\kappa \times \mathbb{P}^{\mathrm{Ku}}_\lambda$:*

- *ω_1 is preserved, κ is ω_2 and cardinals greater than κ are preserved.*

- *There is an ω_1-tree with at least λ branches.*
- *ω_2 has the tree property.*

Proof. Let $G \times H$ be $\mathbb{P}^{\mathrm{Mi}}_\kappa \times \mathbb{P}^{\mathrm{Ku}}_\lambda$-generic. Arguments as in the proof of Theorem 1 show that $\mathbb{P}^{\mathrm{Ku}}_\lambda$ is κ-cc in $V[G]$, so that cardinals above κ are preserved. For the tree property suppose that $T \in V[G \times H]$ is a κ-tree. Using the chain condition and evident homogeneity of $\mathbb{P}^{\mathrm{Ku}}_\lambda$, we may find H' such that $T \in V[G \times H'] \subseteq V[G \times H]$ and $G \times H'$ is $\mathbb{P}^{\mathrm{Mi}}_\kappa \times \mathbb{P}^{\mathrm{Ku}}_\kappa$-generic. Appealing to Theorem 1 T has a branch in $V[G \times H']$. $\square$

References

[1] U. Abraham, *Aronszajn trees on $\aleph_2$ and $\aleph_3$*, Annals of Pure and Applied Logic **24** (1983), 213-230.

[2] J. Baumgartner, *Applications of the Proper Forcing Axiom*, in the Handbook of Set-Theoretic Topology, edited by K. Kunen and J. Vaughan, North-Holland, 1984, 913–959.

[3] M. Eskew, *Tree properties on ω_1 and ω_2*, `https://mathoverflow.net/questions/217951/tree-properties-on-omega-1-and-omega-2`, accessed on 23.5.2016.

[4] W. Mitchell, *Aronszajn trees and the independence of the transfer property*, PhD thesis, University of California, Berkeley, 1970.

[5] W. Mitchell, *Aronszajn trees and the independence of the transfer property*, Annals of Mathematical Logic **5** (1972), 21–46

[6] D. Stewart, *The consistency of the Kurepa hypothesis with the axioms of set theory*, Master's thesis, University of Bristol, 1966.

[7] J. Silver, *The independence of Kurepa's conjecture and two-cardinal conjectures in model theory*, in Axiomatic Set Theory (Proceedings of Symposia in Pure Mathematics, Volume XIII, Part I), edited by D. Scott, American Mathematical Society, Providence RI, 1971, 383–390.