NOTE ON A PROBLEM OF MONROE ESKEW

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Abstract. Monroe Eskew asked whether the tree property at ω_2 implies there is no Kurepa tree (as is the case in the Mitchell model, or under PFA). We prove that the tree property at ω_2 is consistent with the existence of ω_1-trees with as many branches as desired.

1. Introduction

Monroe Eskew observed that in the standard models where ω_2 has the tree property, there is no Kurepa tree. He then raised the natural question whether the tree property at ω_2 implies that there is no Kurepa tree. In this note we answer this question in the negative.

We build our model in two steps. In the first step we perform Mitchell's construction to obtain the tree property at ω_2. In the second step we add an ω_1-tree with many branches, and argue that the tree property is preserved. The main point is that in the second step we use a certain variation of the standard forcing to add such an ω_1-tree *computed in the ground model*; a similar idea appears in Uri Abraham's construction [1] of a model where both ω_2 and ω_3 have the tree property.

We presume that the reader is quite familiar with Abraham's analysis of Mitchell forcing.

2. Adding trees

Let λ be a cardinal with $\lambda > \omega_1$. We define a poset $\mathbb{K}_\lambda$ to add an ω_1-tree with λ branches. A condition p is a pair (t_p, f_p) where t_p is a countable normal tree of successor height $\alpha_p + 1$, and f_p is a countable partial function from λ to $\mathrm{Lev}_{\alpha_p}(t_p)$. Conditions are ordered as follows: $q \leq p$ if and only if $\alpha_q \geq \alpha_p$, $t_q \restriction (\alpha_p + 1) = t_p$, $\mathrm{dom}(f_p) \subseteq \mathrm{dom}(f_q)$ and $f_p(\eta) \leq_{t_q} f_q(\eta)$ for all $\eta \in \mathrm{dom}(p)$.

Remark 1. *This is not quite the standard poset for adding trees with many branches, since we do not demand that f_p be injective.*

Lemma 1. *The poset $\mathbb{K}_\lambda$ is countably closed, and adds an ω_1-tree with λ distinct branches.*

Proof. Let $\langle p_i : i < \omega \rangle$ be a decreasing sequence of conditions, let $p_i = (t_i, f_i)$ and let t_i have height $\alpha_i + 1$. Let t^* be the unique tree of height $\sup_i(\alpha_i + 1)$ such that $t^* \restriction (\alpha_i + 1) = t_i$, and let $d = \bigcup_i \mathrm{dom}(f_i)$. For each $\zeta \in d$, the sequence $b_\zeta = \langle f_i(\zeta) : i < \omega, \zeta \in \mathrm{dom}(f_i) \rangle$ is increasing in t^*. To define a lower bound (t_ω, f_ω) we distinguish two cases:

- If α_i is eventually constant, then t_i is also eventually constant with value t^*, and for every $\zeta \in d$ the sequence b_ζ is eventually constant with some value on the top level of t^*. In this case we set $t_\omega = t^*$, $\mathrm{dom}(f_\omega) = d$, and $f_\omega(\zeta)$ equal to the eventual constant value of b_ζ.

- If α_i is not eventually constant, then t^* has limit height and for each $\zeta \in d$ the sequence b_ζ forms a cofinal branch in t^*. We choose t_ω to be some countable normal tree of height $\mathrm{ht}(t^*) + 1$ such that $t_\omega \restriction \mathrm{ht}(t^*) = t^*$, and every branch b_ζ is bounded by some point on the top level of t_ω. We then set $f_\omega(\zeta)$ equal to the unique point which bounds b_ζ.

To finish it will suffice to show that for all $\alpha < \omega_1$ and distinct $\zeta, \eta < \lambda$ the set of conditions q such that $\alpha_q > \alpha$ and $f_q(\zeta) \neq f_q(\eta)$ is dense. Given a condition p, we first form $q_0 \leq q$ such that $\alpha_{q_0} > \alpha$ as follows: t_{q_0} is some sufficiently tall countable normal tree with $t_{q_0} \restriction (\alpha_p + 1) = p$, $\mathrm{dom}(f_p) = \mathrm{dom}(f_{q_0})$, and $f_{q_0}(\theta)$ is some point on the top level of t_{q_0} which lies above $f_p(\theta)$ for all $\theta \in \mathrm{dom}(f_p)$. Then we find $q_1 \leq q_0$ such that $\alpha_{q_1} = \alpha_{q_0}$ and $\zeta, \eta \in \mathrm{dom}(f_{q_1})$. Finally we find $q \leq q_1$ such that $\alpha_q = \alpha_{q_1} + 1$, $\mathrm{dom}(f_q) = \mathrm{dom}(f_{q_1})$ and $f_q(\zeta) \neq f_q(\eta)$.

$\square$

Lemma 2. *Let CH hold, and let μ be a regular uncountable cardinal such that $\gamma^{\aleph_0} < \mu$ for all $\gamma < \mu$. Then the poset $\mathbb{K}_\lambda$ has the μ-Knaster property.*

Proof. Consider a sequence of conditions $p_i = (t_i, f_i)$ for $i < \mu$. Since $\aleph_1 < \mu$ and CH holds we may assume that $t_i = t$ for some fixed tree t of height $\alpha + 1$, (using the Delta system lemma) that the domains of the f_i form a Delta system with root r say, and finally (CH again) that $f_i \restriction r$ is constant. $\square$

If $\lambda < \lambda'$ then define a map π from $\mathbb{K}_{\lambda'}$ to K_λ by $\pi : (t, f) \mapsto (t, f \restriction \lambda)$. We claim this is a projection. Clearly it is order-preserving, so let $q \in \mathbb{K}_{\lambda'}$ and let $p \leq \pi(q)$. Let $\alpha_q = \alpha$ and $\alpha_p = \beta$, and define $q' \leq q$ as follows: $\alpha_{q'} = \beta$, $f_{q'}(\eta) = f_p(\eta)$ for $\eta \in \mathrm{dom}(p)$ and $f_{q'}(\eta)$ some point in $\mathrm{Lev}_\beta(t_p)$ lying above $f_q(\eta)$ for $\eta \in \mathrm{dom}(q) \setminus \mathrm{dom}(p) = \mathrm{dom}(q) \setminus \lambda$. Clearly $\pi(q') = p$.

Let H be a $\mathbb{K}_\lambda$ generic filter, and let $T = \bigcup\{t : (t, f) \in G\}$. As usual let $K_{\lambda'}/H = \{p \in K_{\lambda'} : \pi(p) \in H\}$. It is easy to see that this poset is equivalent to the poset $\mathbb{R} \in V[H]$ defined as follows. A condition is a pair $r = (\alpha_r, f_r)$ where f_r is a countable partial function from $\lambda' \setminus \lambda$ to $\mathrm{Lev}_{\alpha_r}(T)$. $s \leq r$ if and only $\alpha_s \geq \alpha_r$, $\mathrm{dom}(f_r) \subseteq \mathrm{dom}(f_s)$ and $f_r(\eta) \leq f_s(\eta)$ for all $\eta \in \mathrm{dom}(r)$.

It is easy to see that if CH holds in V then $\mathbb{R}$ is μ-Knaster in $V[H]$ for μ as above (the class of such μ is abolute between V and $V[H]$ in this situation). The countable closure of $\mathbb{K}_\lambda$ and $\mathbb{K}_{\lambda'}$ implies that $\mathbb{R}$ is countably distributive (adds no ω-sequences of ordinals) in $V[H]$.

2.1. **Preservation of the Knaster property.** The following is probably well-known (I think I learned it from Magidor).

Lemma 3. *If κ is regular uncountable, $\mathbb{A}$ is κ-Knaster and $\mathbb{B}$ is κ-cc then $\mathbb{A}$ remains κ-Knaster after forcing with $\mathbb{B}$.*

Proof. Let b force that $\langle \dot{a}_\alpha : \alpha < \kappa \rangle$ is a counterexample, and choose $b_\alpha \leq b$ and a_α such that $b_\alpha \vDash \dot{a}_\alpha = a_\alpha$. Use the Knaster property to find unbounded $A \subseteq \kappa$ such that $\langle a_\alpha : \alpha \in A \rangle$ are pairwise compatible. Then use κ-cc to find $\alpha \in A$ such that $b_\alpha \Vdash \{\beta \in A : b_\beta \in G\}$ is unbounded in κ. Forcing below b_α we see that in $V[G]$ the sequence $\langle i_G(\dot{a}_\beta) = a_\beta : \beta \in A, b_\beta \in G\}$ is pairwise compatible. $\square$

2.2. **The tree property.** Let κ be measurable (probably an overkill, weak compact should be enough) and let CH hold. Let $\mathbb{P}$ be Mitchell forcing and let $\mathbb{Q}$ be $\mathbb{K}_\kappa$ (as defined in V). We force over V with $\mathbb{P} \times \mathbb{Q}$, and claim that $\kappa = \aleph_2$ and has the tree property in the final model. Let $G \times H$ be the generic object.

Since $\mathbb{P}$ and $\mathbb{Q}$ are κ-Knaster in V, so is their product. It follows that $\kappa = \aleph_2$ in $V[G \times H]$. We may as usual force with $j(\mathbb{P})/G \times j(\mathbb{Q})/H$ and extend the embedding j to get $j : V[G \times H] \to M[j(G) \times j(H)]$. Certainly the generic ω_1-tree added by H is a Kurepa tree in $V[G \times H]$. Let $T \in V[G \times H]$ be a κ-tree and suppose for contradiction that T has no branch. As usual $T \in M[G \times H]$ and T has a branch in $M[j(G) \times j(H)]$, so we will be done if we can argue that $j(\mathbb{P})/G \times j(\mathbb{Q})/H$ adds no branch to T.

As usual, it is helpful to write $\mathbb{P}$ as the projection of a product $Add(\omega, \kappa) \times \mathbb{S}$ where $\mathbb{S}$ is a certain countably closed term forcing. This means that we may embed $V[G]$ into $V[G_0 \times G_1]$ where G_0 is the $Add(\omega, \kappa)$-generic added by G, G_1 is $\mathbb{S}$-generic and G is the $\mathbb{P}$-generic induced by $G_0 \times G_1$. Similarly we may embed $M[G]$ into $M[G_0 \times G_1]$.

Lemma 4. *$\mathbb{Q}$ is countably distributive in $M[G]$.*

Proof. Let H be $\mathbb{Q}$-generic over $M[G]$. Forcing over $M[G][H]$ with $Add \times \mathbb{S}/G$ we embed $M[G][H]$ into $M[H \times G_0 \times G_1]$ where $H \times G_0 \times G_1$ is $\mathbb{Q} \times Add \times \mathbb{S}$-generic. By Easton's lemma every ω-sequence of ordinals in $M[H \times G_0 \times G_1]$ lies in $M[G_0]$. $\square$

Let $\mathbb{R} = j(\mathbb{Q})/H$.

Lemma 5. *$\mathbb{R}$ is countably distributive and κ-Knaster in $M[G \times H]$.*

Proof. Since $\mathbb{R}$ is κ-Knaster in $M[H]$ and $\mathbb{P}$ is κ-cc (in fact κ-Knaster) in $M[H]$, it follows from Lemma 3 that $\mathbb{R}$ is κ-Knaster in $M[G \times H]$.

Arguing as in the last lemma we may embed $M[G][j(H)]$ into $M[G_0 \times G_1 \times j(H)]$ and use Easton's lemma to show that every ω-sequence of ordinals in this model is in $M[G_0]$. It follows that $\mathbb{R}$ is countably distributive in $M[G \times H]$. $\square$

Working in $M[G]$ we may write $j(\mathbb{P})/G$ as the projection of a product $Add(\omega, j(\kappa) - \kappa) \times \mathbb{S}^*$ where $\mathbb{S}^*$ is a countably closed term forcing. Since $\mathbb{Q}$ is countably distributive in $M[G]$, the same analysis obtains in the model $M[G \times H]$. Since $\mathbb{R}$ is κ-Knaster in $M[G \times H]$, we have that $\kappa = \omega_2$ in $M[G][j(H)]$ and T has no branch in $M[G][j(H)]$. Since $\mathbb{R}$ is countably distributive in $M[G \times H]$, the product analysis for $j(\mathbb{P})/G$ still obtains in $M[G][j(H)]$. As usual we may now argue that T has no branch in $M[j(G)][j(H)]$.

References

[1] Uri Abraham, *Aronszajn trees on $\aleph_2$ and $\aleph_3$*, Annals of Pure and Applied Logic **24** (1983), 213-230.