SQUARES, SCALES AND STATIONARY REFLECTION

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Abstract. Since the work of Gödel and Cohen, which showed that Hilbert's First Problem (the Continuum Hypothesis) was independent of the usual assumptions of mathematics (axiomatized by Zermelo-Fraenkel Set Theory with the Axiom of Choice, ZFC), there have been a myriad of independence results in many areas of mathematics.

These results have led to the systematic study of several combinatorial principles that have proven effective at settling many of the important independent statements. Among the most prominent of these are the principles *diamond*($\diamondsuit$) and *square*($\square$) discovered by Jensen. Simultaneously, attempts have been made to find suitable natural strengthenings of ZFC, primarily by Large Cardinal or Reflection Axioms. These two directions have tension between them in that Jensen's principles, which tend to suggest a rather rigid mathematical universe, are at odds with reflection properties. A third development was the discovery by Shelah of "PCF Theory", a generalization of cardinal arithmetic that is largely determined inside ZFC.

In this paper we consider interactions between these three theories in the context of singular cardinals, focusing on the various implications between square and *scales* (a fundamental notion in PCF theory), and on consistency results between relatively strong forms of square and stationary set reflection.

1. Introduction

Since the work of Gödel and Cohen which showed that Hilbert's First Problem (the Continuum Hypothesis) was independent of the usual assumptions of mathematics (Zermelo-Fraenkel Set Theory with

1991 *Mathematics Subject Classification.* Primary 03E35, 03E55; Secondary 03E05.

Key words and phrases. Forcing, large cardinals, combinatorial principles.

The first author was partially supported by grant number 517/94 of the Israel Science Foundation and DMS 9703945.

The second author was partially supported by DMS 9500484.

The third author was partially supported by grant number 517/94 of the Israel Science Foundation.

the Axiom of Choice, ZFC), there have been a myriad of independence results in many areas of mathematics.

These results have led to the systematic study of several combinatorial principles that have proven effective at settling many of the important independent statements. Among the most prominent of these are the principles diamond($\diamondsuit$) and square($\Box$) discovered by Jensen. These two principles are far ranging in their application. Among the first applications were the results of Jensen that the principle, $\diamondsuit_{\omega_1}$ implies the existence of a Souslin tree on ω_1 and hence the failure of Souslin's Hypothesis. Gregory [17] showed that the principle $\Box_{\omega_1}$, together with the Continuum Hypothesis implies the existence of a Souslin tree on ω_2.

Simultaneously, attempts have been made to find suitable natural strengthenings of ZFC, primarily by Large Cardinal (or Reflection) Axioms. These two directions have tension between them, in that Jensen's principles, which tend to suggest a rather rigid mathematical universe, are at odds with reflection properties. One of the basic and most useful types of reflection property that follows from large cardinals is stationary set reflection, which contradicts square.

These issues are particularly piquant at successors of singular cardinals, where Shelah's "PCF Theory" has become an important tool. This theory provides a generalization of the usual cardinal arithmetic, with the advantage that many of the properties of the PCF theory are determined in ZFC alone. Fundamental to the PCF theory is the existence of various types of "Scales". A typical application of the PCF theory is the ZFC theorem [46] that $\aleph_{\omega+1}$ is not a Jonsson cardinal.

In this paper we consider interactions of these three theories in the context of singular cardinals, focusing on the various implications between square and scales, and on consistency results between relatively strong forms of square and stationary set reflection.

The differences in behavior of singular and regular cardinals have been recognized for quite some time. For example [9] showed that the behavior of cardinal exponentiation at regular cardinals was essentially arbitrary, subject to the relatively weak requirements of König's Theorem [26]. The behavior of cardinal exponentiation at singular cardinals is still not fully understood, but it is known to be significantly more complicated. Work of Silver [47] and Galvin and Hajnal [15] put severe restrictions on the size of $\mathrm{P}(\lambda)$ in terms of the behavior of the power function for smaller cardinals, for λ a singular cardinal of uncountable cofinality. In particular, if the Generalized Continuum Hypothesis (GCH) holds below λ then it holds at λ.

Magidor [32] showed that it is consistent to have the GCH hold below $\aleph_\omega$, but fail at $\aleph_\omega$. Thus singular cardinals of countable cofinality behave differently. Shelah [46] in a particularly dramatic development, showed the inequality:

$$\aleph_\omega^\omega \leq \aleph_{\omega_4} \times 2^{\aleph_0}$$

For this he developed the theory known as "PCF theory", a theory primarily concerned with the behavior of reduced products of cofinal sets of regular cardinals $A \subseteq \lambda$ for a singular cardinal λ.

The differences between regular and singular cardinals are apparent in the behavior of various square principles as well. For example, collapsing a Mahlo cardinal to be the successor of a regular cardinal μ makes square fail at μ^+, whereas it is known that the failure of square at the successor of singular cardinal is a much stronger property, though how strong is still not completely settled [40]. It is a theorem of Shelah [13, 41] that even quite weak forms of square (the approachability property) fail at the successor of singular cardinals above a supercompact cardinal. By suitable collapsing it is possible to make these forms of square fail at $\aleph_\omega$ as well [13].

The main tools for determining lower bounds for the consistency strength of the failure of square are "Core Models" and "Covering Lemmas". A "core model" is a particular kind of canonical model of ZFC and a large cardinal axiom [38, 48]. A particularly striking feature of these core models is that the failure of "covering" at a singular cardinal for one of the core models yields an inner model with a large cardinal of the strength of that core model. The core models tend to be models of square, or square-like properties [39]. In particular, the failure of square or of the singular cardinals hypothesis implies the existence of inner models of fairly large cardinals.

In his study of core models for Woodin cardinals Schimmerling was able to establish versions of square ($\Box_{\kappa,\lambda}$ and $\Box_{\kappa,<\lambda}$) that are slightly weaker than the usual square, but stronger than Jensen's principle "Weak Square". The usual square provides an organized and coherent way of building bijections between a cardinal κ and ordinals $\alpha < \kappa^+$, by coherently cofinalizing each α with a closed unbounded set. Roughly speaking, Schimmerling's alternative square principles provide more than one closed unbounded set for each $\alpha < \kappa^+$. These principles have much the same utility as the original square. (For a study of weak square principles and their efficacy, see [13].)

By specifying the number of cofinalizing sets involved one gets a hierarchy of square principles. Jensen showed that this hierarchy is strict (in the sense of implication) at successors of regular cardinals, and this paper uses his techniques in a slightly different way to prove the same result for successors of singular cardinals.

These square principles are all stronger then Jensen's principle "Weak Square". Weak square, in Schimmerling's language, is $\square_{\kappa,\kappa}$. This principle was shown by Jensen [25] to be equivalent to the existence of a special Aronszajn tree.

It follows from an easy variation on the classical proof that Schimmerling's $\square_{\kappa,<\omega}$ implies the existence of a non-reflecting stationary set. In this paper we show that variations of square that allow infinitely many (but fewer than κ) cofinalizing sets have strong consequences for the existence of various types of scales and their continuity properties. In particular the existence of "Good Scales" and "Very Good Scales" follow from these square principles. These scale principles, in turn, limit what kind of reflection is possible.

On the other hand, we show it is consistent that $\square_{\aleph_\omega,\omega}$ holds and still every stationary subset of $\aleph_{\omega+1}$ reflects. Further it is consistent that weak square holds and every small collection of stationary subsets of $\aleph_{\omega+1}$ consisting of points of a bounded cofinality simultaneously reflects.

The first five sections of the paper give "ZFC-results", that various combinatorial principles imply various others. The rest of the paper is devoted to relative consistency results, showing limitations on what can be proved in ZFC, as well as providing universes with desirable combinations of combinatorial properties. A rough outline of the paper is as follows:

- In section 2, we give the basic definitions explicitly, and describe known results.
- In section 3, we define the principles of reflection and simultaneous reflection, scales, good and very good scales. We show that $\square_{\kappa,\lambda}$ (for $\lambda < \kappa$) implies the existence of very good scales, and that these in turn imply that simultaneous reflection fails.
- In section 4, we discuss reflection for stationary collections of countable sets and its relation to various square principles. As an intermediary device of independent interest, we consider a principle due to Shelah, called ADS_κ. We show that either weak square or the existence of a very good scale implies ADS_κ, and that ADS_κ implies the negation of stationary reflection for stationary collections of countable sets.
- In section 5, we discuss "Improved Square" and its consequences, and limitations on the consistency of "Improved Square".

- In section 6, we begin the development of techniques for forcing various squares. Key to our results is the method of "threading" a square. The concept of an "indestructibly generically supercompact cardinal" is reviewed, as well as standard results for constructing models where they exist.
- In section 7, we show that Jensen's result distinguishing between different $\square_{\kappa,\lambda}$'s for κ regular extends to the case of κ singular.
- In section 8, we show that it is sometimes possible to add $\square_\kappa$ using forcing of cardinality less than κ.
- In section 9, we show the consistency of various forms of square (e.g. $\square_{\lambda,cof(\lambda)}$ for λ singular) above a supercompact. The notion of "Indexed Square" is introduced, and it is shown that it implies a square-like principle S_λ which Shelah has shown is equivalent to the upwards two cardinal transfer property $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$.
- In section 10, it is shown that it is consistent to have indexed $\square_{\aleph_\omega,\omega}$ and still have simultaneous reflection for any finite collection of stationary subsets of $\aleph_{\omega+1}$. The proof goes by adding a generic indexed square sequence, and then generically killing "fragile" stationary sets. Those stationary sets that remain are preserved by appropriate thread forcings that recreate generic supercompactness.

  A comment on priority is in order here. After the authors had proved the main result of section 10, it was pointed out that a slightly weaker result, namely the consistency of the two cardinal transfer property with every stationary subset of $\aleph_{\omega+1}$ reflecting was claimed in [4]. However the proof in that paper contains an error; Lemma 16 (page 2384) is false, since with the stated definition of $Q^{\lambda_i}_S$ the sequence $\langle q_j : j < \rho \rangle$ does not supply the "threading" information needed to apply Lemma 6. Moreover a repair to that proof cannot be done easily: results of [13] show that if V is a model of $\square^*_\lambda$ where λ is a cardinal of cofinality ω, then there is no $\mu < \lambda$ which is generically supercompact via countably closed forcing in any generic extension by a countably closed forcing (e.g. Shelah and Ben-David's $\mathbb{Q}^{\lambda_i}_S$). Nor do these authors know of a suitable threading forcing for S_λ or indexed $\square_{\lambda,\omega}$ that provably preserves stationary subsets of λ^+; at this point in time we do not know how to finish the proof in [4] along the outlines of their argument.
- In section 11, we investigate square and reflection in the model obtained by adding a Prikry sequence to a measurable cardinal κ. We show that there is a non-reflecting stationary subset of

κ^+, such that any stationary subset of its complement reflects. Further, there is a very good scale in this model.
- In section 12, we show that it is consistent to have weak square at $\aleph_\omega$ and simultaneous reflection of less than $\aleph_\omega$ stationary sets consisting of ordinals of a bounded cofinality. The technique here is to fortify a weak square sequence by a "Diamond-Plus Sequence", and then thread it. The threading forcing is shown to be stationary set preserving (for this special weak square sequence) and to resurrect generically supercompact cardinals.
- In section 13, it is shown that reflection for stationary collections of countable sets does not imply simultaneous reflection, even for two stationary sets. Similar results have been obtained independently by Larson [29].

Our notation is fairly standard, here we give a brief review. $|X|$ is the cardinality of X, $\mathrm{ot}(X)$ is the order type of X, and $\mathrm{P}(X)$ is the power set of X. ${}^Y X$ is the set of all functions from Y to X, and if α is an ordinal then ${}^{<\alpha}X$ is the set of all sequences of length less than α from X. If κ is a cardinal then $[X]^\kappa$ is the set of subsets of X with cardinality κ, and $[X]^{<\kappa}$ and $[X]^{\leq\kappa}$ are defined in the natural way. For κ, λ cardinals κ^λ is the cardinal $|{}^\lambda\kappa|$. $\mathrm{cf}(\alpha)$ is the cofinality of α.

Definition 1.1. Let κ be a cardinal. Then
1. $\mathrm{cof}(\kappa) = \{ \alpha : \mathrm{cf}(\alpha) = \kappa \}$.
2. $\mathrm{cof}(< \kappa) = \{ \alpha : \mathrm{cf}(\alpha) < \kappa \}$.
3. $\mathrm{cof}(\leq \kappa) = \{ \alpha : \mathrm{cf}(\alpha) \leq \kappa \}$.

For forcing we mostly follow the conventions used in Kunen's book [28]. $p \leq q$ means that p is stronger than q, a *κ-closed* poset is one in which any descending sequence of length less than κ has a lower bound, and a *(λ, ∞)-distributive* poset is one in which any sequence of dense open sets of length less than λ has nonempty intersection. A *κ-directed closed* poset is one in which any directed set of size less than κ has a lower bound. We write $p \parallel \phi$ for "p decides ϕ", which is to say that either $p \Vdash \phi$ or $p \Vdash \neg\phi$.

$\mathrm{Add}(\kappa, \lambda)$ is the poset for adding λ Cohen subsets of κ, $\mathrm{Coll}(\kappa, \lambda)$ is the Lévy collapse to make λ have cardinality κ, and $\mathrm{Coll}(\kappa, < \lambda)$ is the Lévy collapse to make all cardinals in $[\kappa, \lambda)$ have cardinality κ.

2. Square sequences and non-reflecting stationary sets

The combinatorial principle $\square_\kappa$ was introduced by Jensen [25] who showed that if $V = L$ then $\square_\kappa$ holds for every uncountable cardinal κ. $\square_\kappa$ is typically used in building objects of size κ^+, and was originally motivated by Jensen's work on constructing κ^+-Souslin trees in L.

Definition 2.1. Let κ be an uncountable cardinal. A $\Box_\kappa$*-sequence* is a sequence $\langle C_\alpha : \alpha < \kappa^+, \lim(\alpha)\rangle$ such that for all $\alpha < \kappa^+$

1. C_α is closed and unbounded in α.
2. If $\mathrm{cf}(\alpha) < \kappa$, then $\mathrm{ot}(C_\alpha) < \kappa$.
3. For all $\beta \in \lim(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$.

We say that $\Box_\kappa$ *holds* iff there exists a $\Box_\kappa$-sequence.

Notice that if $\vec{C}$ is a $\Box_\kappa$-sequence then $\mathrm{ot}(C_\alpha) \leq \kappa$ for all α, and if in addition κ is singular then $\mathrm{ot}(C_\alpha) < \kappa$ for all κ. Some writers adopt a slightly different definition, in which property 2 is replaced by the weaker demand that $\mathrm{ot}(C_\alpha) \leq \kappa$ for all α; the existence of a sequence with this weaker property implies the existence of a $\Box_\kappa$-sequence.

A classical application of $\Box_\kappa$ is to show that there are many *non-reflecting stationary subsets* of κ^+.

Definition 2.2. Let κ be an uncountable regular cardinal. Let S be a stationary subset of κ.

1. S *reflects at* α iff $\alpha < \kappa$, $\mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α.
2. $\mathrm{Refl}(S)$ *holds* iff every stationary subset of S reflects at some α.
3. S is *non-reflecting* iff S does not reflect at any α.

Stationary reflection is a phenomenon of considerable interest in combinatorial set theory. For example Gregory [17] showed that if GCH holds and there is a non-reflecting stationary subset of $\aleph_2 \cap \mathrm{cof}(\omega)$ then there is an $\aleph_2$-Souslin tree.

The next result is well known but we give the proof for the sake of completeness. It will be a paradigm for many of the results that we prove later.

Theorem 1. *Let* $\Box_\kappa$ *hold and let* S *be a stationary subset of* κ^+. *Then* $\mathrm{Refl}(S)$ *fails.*

Proof. Let $\vec{C}$ be a $\Box_\kappa$-sequence, and let $F(\alpha) = \mathrm{ot}(C_\alpha)$ for each limit $\alpha < \kappa^+$. By Fodor's lemma there is a stationary set $T \subseteq S$ on which F is constant. We claim that T does not reflect. To see this fix $\alpha < \kappa^+$ with $\mathrm{cf}(\alpha) > \omega$, and observe that if $\beta \in \lim(C_\alpha)$ then $F(\beta) = \mathrm{ot}(C_\beta) = \mathrm{ot}(C_\alpha \cap \beta)$. It follows that F is 1-1 on $\lim(C_\alpha)$, hence $\lim(C_\alpha)$ meets T in at most one point and so $T \cap \alpha$ is not stationary in α. $\Box$

It is natural to ask whether the hypothesis of Theorem 1 can be weakened. Jensen [25] defined a principle $\Box^*_\kappa$ ("weak square") as follows.

Definition 2.3. Let κ be an infinite cardinal. A $\Box^*_\kappa$*-sequence* is a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+, \lim(\alpha)\rangle$ such that

1. $\mathcal{C}_\alpha \subseteq \mathrm{P}(\alpha)$, $1 \leq |\mathcal{C}_\alpha| \leq \kappa$, and $\mathcal{C}_\alpha$ is a set of closed and unbounded subsets of α.
2. If $\mathrm{cf}(\alpha) < \kappa$ then $\forall C \in \mathcal{C}_\alpha$ $\mathrm{ot}(C) < \kappa$.
3. $\forall C \in \mathcal{C}_\alpha$ $\forall \beta \in \lim(C)$ $C \cap \beta \in \mathcal{C}_\beta$.

We say that $\square^*_\kappa$ *holds* iff there exists a $\square^*_\kappa$-sequence.

It is easy to see that if $\kappa^{<\kappa} = \kappa$ then there is a $\square^*_\kappa$-sequence, so that $\square^*_\kappa$ is of greatest interest when κ is singular. It can be shown [13] that if $\square^*_\kappa$ holds then we may choose a $\square^*_\kappa$-sequence with the additional property that each $\mathcal{C}_\alpha$ contains a set of order type $\mathrm{cf}(\alpha)$. We would get an equivalent definition of $\square^*_\kappa$ if we weakened clause 2 in the definition to demand only that $\forall \alpha$ $\forall C \in \mathcal{C}_\alpha$ $\mathrm{ot}(C) \leq \kappa$.

$\square^*_\kappa$ is a powerful construction principle, for example Jensen [25] showed that the existence of a $\square^*_\kappa$-sequence is equivalent to the existence of a special κ^+-Aronszajn tree. However it turns out that $\square^*_\kappa$ is not strong enough to manufacture non-reflecting stationary sets, in fact we see in Theorem 21 that from a suitable large cardinal assumption we can show the consistency of $\square^*_{\aleph_\omega}$ plus a strong form of reflection for stationary subsets of $\aleph_{\omega+1}$.

Schimmerling [39] introduced a natural hierarchy of principles intermediate in strength between $\square_\kappa$ and $\square^*_\kappa$, in connection with his work on the core model for one Woodin cardinal.

Definition 2.4. Let κ be a cardinal. A $\square_{\kappa,\lambda}$-*sequence* is a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+, \lim(\alpha) \rangle$ such that

1. $\mathcal{C}_\alpha \subseteq \mathrm{P}(\alpha)$, $1 \leq |\mathcal{C}_\alpha| \leq \lambda$, and $\mathcal{C}_\alpha$ is a set of closed and unbounded subsets of α.
2. If $\mathrm{cf}(\alpha) < \kappa$ then $\forall C \in \mathcal{C}_\alpha$ $\mathrm{ot}(C) < \kappa$.
3. $\forall C \in \mathcal{C}_\alpha$ $\forall \beta \in \lim(C)$ $C \cap \beta \in \mathcal{C}_\beta$.

A $\square_{\kappa,<\lambda}$-*sequence* is defined similarly, only we demand that $1 \leq |\mathcal{C}_\alpha| < \lambda$.

It is clear that $\square_\kappa = \square_{\kappa,1}$, $\square^*_\kappa = \square_{\kappa,\kappa}$, and the strength of $\square_{\kappa,\lambda}$ decreases with increasing λ. Jensen [24] showed that for κ regular the strength of $\square_{\kappa,\lambda}$ is strictly decreasing, and in Theorem 13 we prove the corresponding result for singular κ. Schimmerling (and independently Foreman and Magidor) observed that for sufficiently small values of λ we can manufacture non-reflecting stationary sets using $\square_{\kappa,<\lambda}$.

Theorem 2. *Let* $\kappa^{<\lambda} = \kappa$ *and let* $\square_{\kappa,<\lambda}$ *hold. If* $S \subseteq \kappa^+$ *is stationary then there exists a stationary set* $T \subseteq S$ *such that* T *does not reflect at any* α *with* $\mathrm{cf}(\alpha) \geq \lambda$.

Proof. Let $\vec{\mathcal{C}}$ be a $\square_{\kappa,<\lambda}$-sequence. Let $F(\alpha) = \{ \operatorname{ot}(C) \ : \ C \in \mathcal{C}_\alpha \}$. By Fodor's lemma there is $T \subseteq S$ a stationary set on which F is constant, say with value X. Now if $\operatorname{cf}(\alpha) \geq \lambda$ and $C \in \mathcal{C}_\alpha$ then for every $\beta \in \lim(C) \cap T$ we have $\operatorname{ot}(C \cap \beta) \in X$. So $\lim(C) \cap T$ is a set of size less than λ, and hence T is not stationary in α. $\square$

In particular, $\square_{\kappa,<\omega}$ implies the failure of $\operatorname{Refl}(S)$ for every stationary $S \subseteq \kappa^+$.

If $\kappa^{<\lambda} = \kappa$ then necessarily $\lambda \leq \operatorname{cf}(\kappa)$. It is worth noticing that we can still get some information in this situation without any cardinal arithmetic assumptions.

Theorem 3. *Let κ be singular. Let $\square_{\kappa,<\operatorname{cf}(\kappa)}$ hold, and let $S \subseteq \kappa^+$ be stationary. Then there exist a stationary $T \subseteq S$ and $\beta < \kappa$ such that T fails to reflect at points of cofinality greater than β.*

Proof. Let $\vec{\mathcal{C}}$ be a $\square_{\kappa,<\operatorname{cf}(\kappa)}$-sequence. Define $F : \kappa^+ \longrightarrow \kappa$ by setting

$$F : \alpha \longmapsto \sup(\{ \operatorname{ot}(C) \ : \ C \in \mathcal{C}_\alpha \}).$$

Let $T \subseteq S$ be stationary such that $F \restriction T$ is constant, with constant value $\beta < \kappa$. Then if $\operatorname{cf}(\gamma) > \beta$ and $C \in \mathcal{C}_\gamma$ we know $\operatorname{ot}(C) \geq \operatorname{cf}(\gamma) > \beta$, so that for all sufficiently large $\delta \in \lim(C)$ we have $F(\delta) > \beta$ and hence $\delta \notin T$. $\square$

Remark 2.5. We will see in Theorem 18 that $\square_{\aleph_\omega,\omega}$ is consistent with "every stationary subset of $\aleph_{\omega+1}$ reflects at all sufficiently large cofinalities".

An important fact about square principles is that they will necessarily hold unless there are inner models of large cardinals. For example, Solovay showed that $\square_{\aleph_1}$ holds unless $\aleph_2$ is Mahlo in L, and a complementary consistency result by Mitchell [37] shows that if κ is Mahlo it can be collapsed to $\aleph_2$ in such a way that $\square^*_{\aleph_1}$ fails in the generic extension.

For κ singular, it can be shown that $\square_\kappa$ or something close to it will hold unless there are inner models for sizeable large cardinals. Here we just quote some of the more important results, and refer the reader to Schimmerling's survey [40] for more details and a comprehensive list of references.

Fact 2.6 (Jensen)**.** *If κ is singular and $\square_\kappa$ fails then there is an inner model with a strong cardinal.*

Fact 2.7 (Mitchell, Schimmerling,Steel)**.** *If κ is a measurable cardinal and $\alpha < \kappa$ is singular and $\square_{\alpha,<\omega}$ fails then there is an inner model with a Woodin cardinal.*

Fact 2.8 (Mitchell, Schimmerling, Steel, Woodin). *If α is a singular strong limit cardinal and $\Box_{\alpha,<\omega}$ fails then PD holds (so in particular for every $n < \omega$ there is a model with n Woodin cardinals).*

However, sufficiently large cardinals will serve to make even weak forms of square fail at a singular cardinal. For example

Fact 2.9 (Solovay). *If κ is supercompact and λ is a cardinal with $\kappa < \lambda$ then $\Box_\lambda$ fails.*

Fact 2.10 (Shelah). *If κ is supercompact and $\mathrm{cf}(\lambda) < \kappa < \lambda$ then $\Box^*_\lambda$ fails.*

In fact Shelah proved that under the hypotheses of Fact 2.10 even more is true, the weaker principle AP_λ (see Definition 6.13) fails. For more information about these matters see [13].

There is also a connection between weak squares and forcing axioms, as explored by Todorcevic [49] and Magidor [31].

Fact 2.11 (Magidor). *PFA implies that $\Box_{\kappa,\aleph_1}$ fails for $\kappa \geq \aleph_1$, while "PFA $+\forall\kappa \geq \aleph_2\ \Box_{\kappa,\aleph_2}$" is consistent.*

It is interesting to notice that Martin's Maximum has much more impact than PFA on the existence of weak squares. Foreman and Magidor [13] introduce a principle VWS_κ ("Very Weak Square") which is substantially weaker than $\Box^*_\kappa$ and the principle AP_κ, but still strong enough for many applications in algebra and topology.

Fact 2.12 (Magidor). *Martin's Maximum implies that VWS_κ fails for κ of cofinality ω.*

3. Simultaneous reflection and very good scales

We will say that a family of stationary sets "reflects simultaneously" if there is a point at which each set in the family reflects.

Definition 3.1. Let κ be a regular uncountable cardinal, let $S \subseteq \kappa$ be stationary. Then $\mathrm{Refl}(\mu, S)$ *holds* iff for every sequence $\langle T_i : i < \mu\rangle$ of stationary subsets of S there exists α such that T_i reflects at α for every $i < \mu$.

One reason for taking an interest in simultaneous reflection is that this phenomenon often occurs in the natural models for reflection properties. For example

Fact 3.2 (Baumgartner [2]). *If κ is a weakly compact cardinal and $\mathbb{P} = \mathrm{Coll}(\aleph_1, <\kappa)$ then the reflection property $\mathrm{Refl}(\aleph_1, \aleph_2 \cap \mathrm{cof}(\omega))$ holds in $V^{\mathbb{P}}$.*

In general simultaneous reflection is stronger than ordinary reflection. For example the principle $\mathrm{Refl}(\aleph_2 \cap \mathrm{cof}(\omega))$ is equiconsistent [18] with the existence of a Mahlo cardinal, while the principle $\mathrm{Refl}(2, \aleph_2 \cap \mathrm{cof}(\omega))$ is equiconsistent [33] with the existence of a weakly compact cardinal.

We will also be interested in the cofinality of the point at which reflection occurs.

Definition 3.3. Let $\lambda = \mathrm{cf}(\lambda) > \omega$, and let $S \subseteq \lambda$ be a stationary subset of λ.

1. $\mathrm{Refl}(S, \nu)$ *holds* iff every stationary $T \subseteq S$ reflects at some $\alpha < \lambda$ with $\mathrm{cf}(\alpha) = \nu$.
2. $\mathrm{Refl}(S, \geq \nu)$ *holds* iff every stationary $T \subseteq S$ reflects at some $\alpha < \lambda$ with $\mathrm{cf}(\alpha) \geq \nu$.
3. $\mathrm{Refl}(\mu, S, \nu)$ *holds* iff for every μ-sequence $\langle T_i : i < \mu \rangle$ of stationary subsets of S there exists $\alpha < \lambda$ such that $\mathrm{cf}(\alpha) = \nu$ and all the T_i reflect at α.
4. $\mathrm{Refl}(\mu, S, \geq \nu)$ *holds* iff for every μ-sequence $\langle T_i : i < \mu \rangle$ of stationary subsets of S there exists $\alpha < \lambda$ such that $\mathrm{cf}(\alpha) \geq \nu$ and all the T_i reflect at α.

We will prove that $\square_{\kappa,\lambda}$ for $\lambda < \kappa$ puts a bound on the extent of simultaneous reflection in κ^+. For this we will use some ideas from Shelah's PCF theory.

PCF stands for "Possible CoFinalities". PCF theory is a sophisticated combinatorial theory in which a singular cardinal κ is analysed by looking at reduced products of sets of regular cardinals which are unbounded in κ. No detailed knowledge of PCF theory is required to understand our results, as long as the reader is prepared to take on trust the basic Fact 3.6. For more information about PCF theory see [5] or [46].

We begin with the key concept of a *scale*.

Definition 3.4. Let κ be a singular cardinal. *A scale of length β for κ* is a triple $(\vec{\kappa}, \vec{f}, I)$ where

1. $\vec{\kappa} = \langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ is an increasing sequence of regular cardinals such that $\sup_i \kappa_i = \kappa$.
2. I is an ideal on $\mathrm{cf}(\kappa)$, from which we define a relation $<_I$ on ${}^I ON$ by
$$f <_I g \iff \{ i : f(i) \geq g(i) \} \in I.$$
3. $\vec{f} = \langle f_\alpha : \alpha < \beta \rangle$ is a sequence of functions such that
   (a) $f_\alpha \in \prod_{i<\mathrm{cf}(\kappa)} \kappa_i$.

(b) If $\gamma < \delta < \beta$ then $f_\gamma <_I f_\delta$.
(c) $\forall f \in \prod_{i<\mathrm{cf}(\kappa)} \kappa_i \, \exists \alpha < \beta \; f <_I f_\alpha$.

This is more generality than we really need, we will only be concerned with scales of length κ^+ and the ideal of bounded sets.

Definition 3.5. Let κ be singular.

1. *A scale for* κ is a pair $(\vec{\kappa}, \vec{f})$ such that $(\vec{\kappa}, \vec{f}, I_{\mathrm{bdd}})$ is a scale of length κ^+ for κ, where $I_{\mathrm{bdd}} = \{ A \subseteq \mathrm{cf}(\kappa) \; : \; \sup(A) < \mathrm{cf}(\kappa) \}$
2. Let $f, g \in \prod_{i<\mathrm{cf}(\kappa)} \kappa_i$. Then
   (a) $f <^* g \iff f <_{I_{\mathrm{bdd}}} g$.
   (b) $f < g \iff \forall i \; f(i) < g(i)$.
   (c) $f \leq g \iff \forall i \; f(i) \leq g(i)$.

The following fact is one of the early triumphs of PCF theory.

Fact 3.6 (Shelah)**.** *If κ is singular then there exists a scale for κ.*

The idea of a "good point" or "flat point" is a key one in analysing the properties of scales.

Definition 3.7. Let κ be singular and let $(\vec{\kappa}, \vec{f})$ be a scale for κ. A point $\alpha < \kappa^+$ is *good for* $(\vec{\kappa}, \vec{f})$ iff there exist $A \subseteq \alpha$ unbounded in α and $i < \mathrm{cf}(\kappa)$ such that $\forall \beta, \gamma \in A \; \forall j > i \; (\beta < \gamma \implies f_\beta(j) < f_\gamma(j))$.

It is easy to see that all points of cofinality less than $\mathrm{cf}(\kappa)$ are good.

Definition 3.8. Let $\vec{g} = \langle g_\alpha : \alpha < \beta \rangle$ be increasing in $(\prod_{i<\mathrm{cf}(\kappa)} \kappa_i, <_I)$. $g \in \prod_{i<\mathrm{cf}(\kappa)} \kappa_i$ is an *exact upper bound (eub) for* $\vec{g}$ iff

1. $\forall \alpha < \beta \; g_\alpha <_I g$.
2. $\forall h \in \prod_{i<\mathrm{cf}(\kappa)} \kappa_i \; (h <_I g \implies \exists \alpha \; h <_I g_\alpha)$.

If $\mathrm{cf}(\alpha) > \mathrm{cf}(\kappa)$ then α is good if and only if there is an eub g for $\langle f_\beta : \beta < \alpha \rangle$ with the property that $\mathrm{cf}(g(i)) = \mathrm{cf}(\alpha)$ for all i. We will be interested in the question of how many points are good, where it is worth noticing that the set of good points is essentially a function of $\vec{\kappa}$.

Fact 3.9 (Shelah)**.** *The set of good points in a scale $(\vec{\kappa}, \vec{f})$ is a stationary set. It is determined modulo the club filter on κ^+ by the sequence $\vec{\kappa}$.*

Definition 3.10. $(\vec{\kappa}, \vec{f})$ is a *good scale for* κ iff it is a scale for κ, and modulo the club filter on κ^+ almost every point of cofinality greater than $\mathrm{cf}(\kappa)$ is good for $(\vec{\kappa}, \vec{f})$. GS_κ *holds* iff there exists a good scale for κ.

It turns out that quite weak versions of square at κ (for example the principle AP_κ) imply that all scales for κ are good. See [13] for more on this subject and on situations in which there are scales which fail to be good. For the purposes of this paper we are also interested in a stronger notion, that of a "very good scale".

Definition 3.11. $(\vec{\kappa}, \vec{f})$ is a *very good scale for* κ iff

1. $(\vec{\kappa}, \vec{f})$ is a scale for κ.
2. For every point $\alpha < \kappa^+$ such that $\mathrm{cf}(\alpha) > \mathrm{cf}(\kappa)$ there exists a closed and unbounded set $C \subseteq \alpha$ and $i < \mathrm{cf}(\kappa)$ such that $\forall \beta, \gamma \in C\ \forall j > i\ (\beta < \gamma \implies f_\beta(j) < f_\gamma(j))$.

VGS_κ *holds* iff there exists a very good scale for κ.

Theorem 4. *Let κ be singular, let $\lambda < \kappa$. Then $\square_{\kappa,\lambda}$ implies VGS_κ.*

Proof. Let $\langle \mathcal{C}_\alpha : \alpha < \kappa^+, \lim(\alpha) \rangle$ witness $\square_{\kappa,\lambda}$. Let $(\vec{\kappa}, \vec{f})$ be any scale for κ, where the existence of such a scale is guaranteed by Fact 3.6. Without loss of generality we may assume that $\lambda < \kappa_0$. We will inductively build $\vec{g}$ such that $f_\alpha < g_\alpha$ and $(\vec{\kappa}, \vec{g})$ is a very good scale.

The successor stage is easy, we just choose $g_{\alpha+1}$ such that $g_{\alpha+1} > f_{\alpha+1}$ and $g_{\alpha+1} > g_\alpha$.

For γ a limit, we choose g_γ such that

1. $g_\gamma > f_\gamma$.
2. $\forall \alpha < \gamma\ g_\alpha <^* g_\gamma$.
3. For all i, $g_\gamma(i) > \sup \{ \sup_{\beta \in C} g_\beta(i) \ : \ C \in \mathcal{C}_\gamma, |C| < \kappa_i \}$.

We need to show that this gives a very good scale. Fix $\gamma < \kappa^+$ a limit ordinal with $\mathrm{cf}(\gamma) > \mathrm{cf}(\kappa)$, and choose $C \in \mathcal{C}_\gamma$. Fix i such that $|C| < \kappa_i$, and let $j \geq i$. If $\alpha, \beta \in \lim(C)$ then $C \cap \beta \in \mathcal{C}_\beta$, $\alpha \in C \cap \beta$ and $|C \cap \beta| < \kappa_j$, so that $g_\alpha(j) < g_\beta(j)$. So the club set $\lim(C)$ witnesses that $(\vec{\kappa}, \vec{g})$ has the defining property of a very good scale at the point γ. □

We now prove that VGS_κ puts some limit on the possibilities for stationary reflection in κ^+.

Theorem 5. *Let κ be singular, and let VGS_κ hold. Then for every stationary $T \subseteq \kappa^+$ the principle $\mathrm{Refl}(\mathrm{cf}(\kappa), T, \geq \mathrm{cf}(\kappa)^+)$ fails.*

Proof. Let $(\vec{\kappa}, \vec{f})$ be a very good scale for κ. Assume for a contradiction that $\mathrm{Refl}(\mathrm{cf}(\kappa), T, \geq \mathrm{cf}(\kappa)^+)$ holds. For each $i < \mathrm{cf}(\kappa)$, consider the function g_i from T to κ_i given by $g_i(\alpha) = f_\alpha(i)$, and let $S_i \subseteq T$ be stationary such that $\forall \alpha \in S_i\ g_i(\alpha) = g(i)$.

Now apply the hypothesis $\mathrm{Refl}(\mathrm{cf}(\kappa), T, \geq \mathrm{cf}(\kappa)^+)$ to the sequence of sets $\langle S_i : i < \mathrm{cf}(\kappa) \rangle$, and find some $\alpha < \kappa^+$ such that $\mathrm{cf}(\alpha) > \mathrm{cf}(\kappa)$ and $S_i \cap \alpha$ is stationary in α for all $i < \mathrm{cf}(\kappa)$.

Since $(\vec{\kappa}, \vec{f})$ is a very good scale, we may find $C \subseteq \alpha$ club and $i < \kappa$ such that if $\gamma, \delta \in C$, $\gamma < \delta$ and $j > i$ then $f_\gamma(j) < f_\delta(j)$.

Let $j = i + 1$. Since C is club in α and $S_j \cap \alpha$ is stationary in α we may find two points $\beta < \gamma < \alpha$ with $\beta, \gamma \in C \cap S_j$. But then $f_\beta(j) = g(j) = f_\gamma(j)$, contradicting the choice of C and i. $\square$

Remark 3.12. Actually all we needed here was that an unbounded subsequence of the S_i should reflect simultaneously.

Remark 3.13. Schimmerling [7] proved that $\square_{\kappa,<\kappa}$ has some influence on simultaneous reflection.

Remark 3.14. In the model of Theorem 18 we will have $\square_{\aleph_\omega,\omega}$ (and so $\mathrm{VGS}_{\aleph_\omega}$) along with simultaneous reflection for any finite family of stationary subsets of $\aleph_{\omega+1}$. It follows that Theorem 5 is fairly sharp.

Remark 3.15. By Theorem 21, $\square^*_{\aleph_\omega}$ is consistent with a strong form of simultaneous reflection and so by Theorem 5 $\square^*_{\aleph_\omega}$ does not imply $\mathrm{VGS}_{\aleph_\omega}$.

It is sometimes convenient to consider scales with an additional "continuity" property.

Definition 3.16. Let $(\vec{\kappa}, \vec{f})$ be a scale for κ, let $\alpha < \kappa^+$. $(\vec{\kappa}, \vec{f})$ is *continuous at* α iff

$$(\exists g \; g \text{ is an eub for } \vec{f} \restriction \alpha) \implies f_\alpha \text{ is an eub for } \vec{f} \restriction \alpha.$$

Theorem 6. *If there is a very good scale for* κ*, there is a very good scale which is continuous at every point of cofinality greater than* $\mathrm{cf}(\kappa)$*.*

Proof. Let $(\vec{\kappa}, \vec{f})$ be a very good scale for κ. We define a new scale $(\vec{\kappa}, \vec{f}^*)$.

$f^*_\gamma = f_\gamma$ unless

- $\mathrm{cf}(\gamma) > \omega$.
- There exists D club in γ and $i < \mathrm{cf}(\kappa)$ such that $\langle f_\alpha(j) : \alpha \in D \rangle$ is strictly increasing for all $j \geq i$.

In this case choose i_γ to be the least i such that a club D as above exists and $\kappa_i > \mathrm{cf}(\gamma)$, choose such D with order type $\mathrm{cf}(\gamma)$, and define

$$f^*_\gamma(j) = \begin{cases} 0 & \text{if } j < i_\gamma \\ \sup\{f_\alpha(j) : \alpha \in D\} & \text{if } j \geq i_\gamma \end{cases}$$

This definition is independent of the club D which is chosen.

We claim that the scale $\vec{f}^*$ is very good and is continuous at every point of cofinality greater than $\mathrm{cf}(\kappa)$. To see this let $\gamma < \kappa^+$ be such that $\mathrm{cf}(\gamma) > \mathrm{cf}(\kappa)$. Choose C club in γ such that $\langle f_\alpha(j) : \alpha \in C\rangle$ is strictly increasing for $j \geq i_\gamma$, and $\mathrm{ot}(C) = \mathrm{cf}(\gamma)$.

We claim that for $j \geq i_\gamma$

- $f^*_\alpha(j) \leq f_\alpha(j)$ for all $\alpha \in \lim(C)$.
- $\langle f^*_\alpha(j) : \alpha \in \lim(C)\rangle$ is strictly increasing.
- $f^*_\gamma(j) = \sup\{f^*_\alpha(j) : \alpha \in \lim(C)\}$.

We prove by induction on $\alpha \in \lim(C)$ that $f^*_\alpha(j) \leq f_\alpha(j)$ and that $\langle f^*_\beta(j) : \beta \in \lim(C) \cap (\alpha + 1)\rangle$ is strictly increasing for every $j \geq i_\gamma$. Fix such a j.

Successor case: If α is a successor point in $\lim(C)$ then $\mathrm{cf}(\alpha) = \omega$, so $f^*_\alpha = f_\alpha$ and in particular $f_\alpha(j) = f^*_\alpha(j)$. Let α_0 be the preceding point, then $f^*_{\alpha_0}(j) \leq f_{\alpha_0}(j) < f_\alpha(j) = f^*_\alpha(j)$, and so the sequence $\langle f^*_\beta(j) : \beta \in \lim(C) \cap (\alpha + 1)\rangle$ is strictly increasing.

Limit case: Let α be a limit point in $\lim(C)$. By induction the sequence $\langle f^*_\beta(j) : \beta \in \lim(C) \cap \alpha\rangle$ is strictly increasing. On an unbounded set of $\beta \in \lim(C) \cap \alpha$ (namely the successors) $f^*_\beta = f_\beta$, and so $f^*_\beta(j) = f_\beta(j) < f_\alpha(j)$. This implies that for **all** $\beta \in \lim(C) \cap \alpha$, $f^*_\beta(j) < f_\alpha(j)$.

There are now two subcases:

Subcase 1: $\mathrm{cf}(\alpha) = \omega$, and so $f^*_\alpha = f_\alpha$. In this case we are done because $f^*_\alpha(j) = f_\alpha(j)$ and $\langle f^*_\beta(j) : \beta \in \lim(C) \cap (\alpha + 1)\rangle$ is strictly increasing.

Subcase 2: $\mathrm{cf}(\alpha) > \omega$. In this case $\langle f_\delta(k) : \delta \in \lim(C) \cap \alpha\rangle$ is increasing for all $k \geq i_\gamma$, and also $\mathrm{cf}(\alpha) < \mathrm{cf}(\gamma) < \kappa_{i_\gamma}$, so we defined $i_\alpha \leq i_\gamma$ and $f^*_\alpha(k) = \sup\{f_\delta(k) : \delta \in \lim(C) \cap \alpha\}$ for all $k \geq i_\alpha$.

In particular we defined $f^*_\alpha(j) = \sup\{f_\beta(j) : \beta \in \lim(C) \cap \alpha\}$. So $f^*_\alpha(j) \leq f_\alpha(j)$, and $\langle f^*_\beta(j) : \beta \in \lim(C) \cap (\alpha + 1)\rangle$ is strictly increasing because $f^*_\beta = f_\beta$ unboundedly often in $\lim(C) \cap \alpha$.

When we reach γ the argument of Subcase 2 shows that $f^*_\gamma(j) = \sup\{f^*_\alpha(j) : \alpha \in \lim(C)\}$. This implies continuity because $\mathrm{cf}(\gamma) > \mathrm{cf}(\kappa)$. □

4. Reflection in $[\lambda]^{\aleph_0}$ and almost disjoint sets

The following generalised form of stationarity is a key idea in modern set theory. This notion is due to Jech [19].

Definition 4.1. Let X be uncountable. $S \subseteq [X]^{\aleph_0}$ is *stationary* iff for all $F : {}^{<\omega}X \longrightarrow X$ there exists $A \in S$ such that $A \neq \emptyset$ and $F``{}^{<\omega}A \subseteq A$.

Remark 4.2. This form of the definition was given by Kueker [27], and is equivalent to Jech's. There is an extensive treatment of generalised stationarity in [14].

An analogue of Fodor's lemma is true for this version of stationarity.

Fact 4.3. *If $S \subseteq [X]^{\aleph_0}$ is stationary and $F : S \longrightarrow X$ is a function such that $F(A) \in A$ for all $A \in S$, then F is constant on a stationary subset of S.*

There is a notion of reflection appropriate for this notion of stationarity.

Definition 4.4. Let X be an uncountable transitive set.

1. A stationary set $S \subseteq [X]^{\aleph_0}$ *reflects to* $Y \subseteq X$ iff $|Y| \subseteq Y$ and $S \cap [Y]^{\aleph_0}$ is stationary in $[Y]^{\aleph_0}$.
2. Let $S \subseteq [X]^{\aleph_0}$ be stationary, then $\mathrm{Refl}^*(S)$ *holds* iff every stationary $T \subseteq S$ reflects to some $Y \in [X]^{\aleph_1}$ with $\mathrm{cf}(\mathrm{ot}(Y \cap ON)) = \aleph_1$.

Typically X is either a cardinal or a set of the form H_θ. The following facts are particularly significant for the purposes of this paper.

Fact 4.5 (Foreman, Magidor and Shelah [14])**.** *If κ is a supercompact cardinal and G is generic over V for $\mathrm{Coll}(\aleph_1, < \kappa)$, then in $V[G]$ the reflection principle $\mathrm{Refl}^*([\lambda]^{\aleph_0})$ holds for every λ with $\lambda = \mathrm{cf}(\lambda) \geq \aleph_2$.*

Fact 4.6 (Foreman, Magidor and Shelah [14])**.** *MM (Martin's Maximum) implies that $\mathrm{Refl}^*([\lambda]^{\aleph_0})$ holds for every λ with $\lambda = \mathrm{cf}(\lambda) \geq \aleph_2$.*

We will prove that $\Box^*_\kappa$ and VGS_κ each imply the failure of the reflection principle $\mathrm{Refl}^*([\kappa^+]^{\aleph_0})$. As in the last section the proof will involve some intermediate combinatorial principles. The following principle was studied by Shelah [42]. We have dubbed it ADS_κ where "ADS" stands for "Almost Disjoint Sets".

Definition 4.7. ADS_κ *holds* iff there exists $\langle A_\alpha : \alpha < \kappa^+ \rangle$ such that

1. A_α is unbounded in κ, $\mathrm{ot}(A_\alpha) = \mathrm{cf}(\kappa)$.
2. For all $\beta < \kappa^+$ there exists $g : \beta \longrightarrow \kappa$ such that the sequence $\langle A_\alpha \setminus g(\alpha) : \alpha < \beta \rangle$ consists of pairwise disjoint sets.

The existence of such a sequence $\vec{A}$ is a kind of "incompactness" phenomenon in κ^+, of the sort studied in [35]. We digress briefly into the relationship between ADS_κ and the principles studied in [35]; recall that a *transversal* for a set of non-empty sets is a one-one choice function for that set.

Definition 4.8. $\mathrm{NPT}(\kappa, \lambda)$ *holds* iff there exists a family of sets $\mathcal{B}$ such that

- $|\mathcal{B}| = \kappa$.
- $\forall B \in \mathcal{B}\ |B| < \lambda$.
- For all $\mathcal{C} \subseteq \mathcal{B}$, if $|\mathcal{C}| < \kappa$ then $\mathcal{C}$ has a *transversal.*
- $\mathcal{B}$ does not have a transversal.

It is clear that any sequence witnessing that ADS_κ is true will also witness that $\mathrm{NPT}(\kappa^+, \mathrm{cf}(\kappa)^+)$ is true. In general the implication will not be reversible; Magidor and Shelah proved [35] that $\mathrm{NPT}(\aleph_{\omega+1}, \aleph_1)$ always holds, while we will see in Theorem 8 that $\mathrm{ADS}_{\aleph_\omega}$ can fail.

Shelah [42, page 440] proved that

Fact 4.9. *If κ is regular, or is a singular cardinal such that $\Box_\kappa$ holds, then* ADS_κ *holds.*

Shelah showed that ADS_κ has some influence on the cardinal structure of extensions of the universe.

Fact 4.10 (Shelah [42])**.** *If $V \subseteq W$ are two models of set theory such that*

$$V \models \text{“}\kappa \textit{ is a cardinal and } \mathrm{ADS}_\kappa \textit{ holds”}$$

and κ^+_V is a cardinal in W, then $W \models \mathrm{cf}(\kappa) = \mathrm{cf}(|\kappa|)$.

See [6] for a generalisation of this result.

Theorem 7. *If* VGS_κ *or* $\Box^*_\kappa$ *holds then* ADS_κ *holds.*

Proof. It will be convenient to introduce a concept intermediate between “very good scale” and “good scale”. Let us say that a scale $(\vec{\kappa}, \vec{f})$ is *better* iff it has the following property: for all $\alpha < \kappa^+$ with $\mathrm{cf}(\alpha) > \mathrm{cf}(\kappa)$ there is D club in α such that

1. $\mathrm{ot}(D) = \mathrm{cf}(\alpha)$.
2. For all $\beta \in D$ there is $i < \mathrm{cf}(\kappa)$ such that $f_\gamma(j) < f_\beta(j)$ for all $j > i$ and all $\gamma \in D \cap \beta$.

Clearly a very good scale is better and a better scale is good. We start with the easy remark that points of cofinality less than or equal to $\mathrm{cf}(\kappa)$ automatically enjoy the property in question.

Claim. *If $(\vec{\kappa}, \vec{f})$ is any scale for κ, $\alpha < \kappa^+$ with $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\kappa)$, and D is any unbounded subset of α with order type $\mathrm{cf}(\alpha)$, then for every $\beta \in D$ there is $i < \mathrm{cf}(\kappa)$ such that $f_\gamma(j) < f_\beta(j)$ for all $j > i$ and all $\gamma \in D \cap \beta$.*

Proof. For all $\gamma \in D \cap \beta$ we may choose $i(\gamma)$ such that $\forall j \geq i(\gamma)\ f_\gamma(j) < f_\beta(j)$. Let $i = \sup_\gamma i(\gamma)$. Since $|D \cap \beta| < \mathrm{cf}(\alpha) \leq \mathrm{cf}(\kappa)$, $i < \mathrm{cf}(\kappa)$. ☐

Claim. *If $\Box^*_\kappa$ holds then there is a better scale.*

Proof. Let $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ witness $\square^*_\kappa$. Let $(\vec{\kappa}, \vec{g})$ be a scale for κ.

We define a new scale $(\vec{\kappa}, \vec{f})$ by induction.

Successor case: Let $f_{\alpha+1} = g_\beta$ for β so large that $f_\alpha <^* g_\beta$.

Limit case: For each $C \in \mathcal{C}_\alpha$ define f_C by setting $f_C(i) = \sup_{\gamma \in C} f_\gamma(i)$ if $\mathrm{ot}(C) < \kappa_i$, and $f_C(i) = 0$ otherwise. Let $f_\alpha = g_\beta$ for β so large that

1. $\forall C \in \mathcal{C}_\alpha \; f_C <^* g_\beta$.
2. $\forall \gamma < \alpha \; f_\gamma <^* g_\beta$.

It is clear that $(\vec{\kappa}, \vec{f})$ is a scale. Given α with $\mathrm{cf}(\alpha) > \mathrm{cf}(\kappa)$ fix some $C \in \mathcal{C}_\alpha$. Let $C_1 = \lim(C)$. Now if $\beta \in C_1$ then $C \cap \beta \in \mathcal{C}_\beta$, so $f_{C\cap\beta} <^* f_\beta$. For sufficiently large j we have $\mathrm{ot}(C \cap \beta) < \kappa_j$ and so $f_{C\cap\beta}(j) = \sup_{\gamma \in C\cap\beta} f_\gamma(j)$. If we now choose D to be a club subset of C_1 with order type $\mathrm{cf}(\alpha)$, then D will witness that $(\vec{\kappa}, \vec{f})$ is better. $\square$

For the rest of the proof we fix a better scale $(\vec{\kappa}, \vec{f})$. We may assume without loss of generality that $\mathrm{cf}(\kappa) < \kappa_0$ and $\forall \alpha \, \forall i \; f_\alpha(i) \geq \sup_{j<i} \kappa_j$. We claim that $\langle \mathrm{range}(f_\alpha) : \alpha < \kappa^+ \rangle$ exemplifies ADS_κ.

We will establish the following fact for all $\beta < \kappa^+$.

$(*)_\beta$: There exists $G : \beta \longrightarrow \mathrm{cf}(\kappa)$ such that if $\gamma < \delta < \beta$ and $i > \max\{G(\gamma), G(\delta)\}$ then $f_\gamma(i) < f_\delta(i)$.

To see that this gives the desired property, fix β and let G witness $(*)_\beta$. For $\alpha < \beta$ let B_α be the final segment of $\mathrm{range}(f_\alpha)$ given by $\{ f_\alpha(i) \; : \; i \geq G(\alpha) \}$, then we claim that the sets B_α are mutually disjoint.

Suppose that $\alpha_0 < \alpha_1 < \beta$ and $x \in B_{\alpha_0} \cap B_{\alpha_1}$, so that $x = f_{\alpha_0}(i) = f_{\alpha_1}(j)$ for $i \geq G(\alpha_0)$, $j \geq G(\alpha_1)$. i must equal j because otherwise $f_{\alpha_0}(i)$ and $f_{\alpha_1}(j)$ would lie in disjoint intervals. Therefore $i \geq \max\{G(\alpha_0), G(\alpha_1)\}$, but then $f_{\alpha_0}(i) < f_{\alpha_1}(i)$. So the B_α are disjoint, and we have shown that $\langle \mathrm{range}(f_\alpha) : \alpha < \kappa^+ \rangle$ witnesses ADS_κ.

We will prove $(*)_\beta$ by induction on β, where the case $\beta = 0$ is trivial.

Case 1: $\beta = \gamma + 1$. Let $G : \gamma \longrightarrow \mathrm{cf}(\kappa)$ witness $(*)_\gamma$. Choose $H : \beta \longrightarrow \mathrm{cf}(\kappa)$ such that $H(\gamma) = 0$, $\forall \delta < \gamma \; G(\delta) \leq H(\delta)$, and $\forall \delta < \gamma \; \forall i > H(\delta) \; f_\delta(i) < f_\gamma(i)$. It is easy to check that H witnesses $(*)_\beta$.

Case 2: $\lim(\beta)$.

Since the scale $(\vec{\kappa}, \vec{f})$ is better, we may fix C club in β such that

1. $\mathrm{ot}(C) = \mathrm{cf}(\beta)$.
2. For all $\gamma \in C$ there is $i < \mathrm{cf}(\kappa)$ such that $f_\delta(j) < f_\gamma(j)$ for all $\delta \in C \cap \gamma$ and all $j > i$.

We enumerate C in increasing order as $\langle \beta_\zeta : \zeta < \mathrm{cf}(\beta) \rangle$, and choose $\langle G_\zeta : \zeta < \mathrm{cf}(\beta) \rangle$ a sequence of functions such that G_ζ witnesses $(*)_{\beta_\zeta}$. We define $H(\gamma)$ for $\beta_\zeta \leq \gamma < \beta_{\zeta+1}$ so that

1. $H(\gamma) \geq G_{\zeta+1}(\gamma)$.
2. $\forall i > H(\gamma)\ \forall \eta < \zeta\ f_{\beta_\eta}(i) < f_{\beta_\zeta}(i)$.
3. $\forall i > H(\gamma)\ f_{\beta_\zeta}(i) \leq f_\gamma(i) < f_{\beta_{\zeta+1}}(i)$.

Claim. *H witnesses $(*)_\beta$.*

Proof. Let $\gamma < \delta < \beta$. There are 2 subcases.

Subcase a: $\exists \eta\ \beta_\eta \leq \gamma < \delta < \beta_{\eta+1}$. In this case we know by clause 1 in the definition of H that $H(\gamma) \geq G_{\eta+1}(\gamma)$ and $H(\delta) \geq G_{\eta+1}(\delta)$. $G_{\eta+1}$ witnesses $(*)_{\beta_{\eta+1}}$, so $\forall i > \max\{H(\gamma), H(\delta)\}\ f_\gamma(i) < f_\delta(i)$ and we are done.

Subcase b: $\exists \eta\ \exists \theta\ \beta_\eta \leq \gamma < \beta_{\eta+1} \leq \beta_\theta \leq \delta < \beta_{\theta+1}$. In this case let $i > \max\{H(\gamma), H(\delta)\}$, then we see that

$$f_\gamma(i) < f_{\beta_{\eta+1}}(i) \leq f_{\beta_\theta}(i) \leq f_\delta(i).$$

Here the inequalities follow from clause 3 in the definition of $H(\gamma)$, clause 2 in the definition of $H(\delta)$ and clause 3 in the definition of $H(\delta)$ respectively. □

This concludes the proof of Theorem 7. □

Finally we return to the problem of stationary reflection.

Theorem 8. *Let κ be singular of cofinality ω. If ADS_κ holds then $\mathrm{Refl}^*([\kappa^+]^{\aleph_0})$ fails.*

Proof. Let $\langle A_\alpha : \alpha < \kappa^+ \rangle$ be a sequence witnessing ADS_κ and define $S = \{\ x \in [\kappa^+]^{\aleph_0}\ :\ A_{\sup(x)} \subseteq x\ \}$.

Claim. *S is stationary in $[\kappa^+]^{\aleph_0}$.*

Proof. Let $F : [\kappa^+]^{<\aleph_0} \longrightarrow \kappa^+$, and let $\alpha > \kappa$ be such that $\mathrm{cf}(\alpha) = \omega$ and $F``[\alpha]^{<\aleph_0} \subseteq \alpha$. Now choose x to be a countable cofinal subset of α such that $A_\alpha \subseteq x$ and x is closed under F, then $\sup(x) = \alpha$ so that $x \in S$. □

Applying the hypothesis that $\mathrm{Refl}^*([\kappa^+]^{\aleph_0})$ holds, let $Y \subseteq \kappa^+$ be such that $\aleph_1 \subseteq Y$, $\mathrm{cf}(\mathrm{ot}(Y)) = |Y| = \aleph_1$, and $S \cap [Y]^{\aleph_0}$ is stationary in $[Y]^{\aleph_0}$. Notice that for each $X \in [Y]^{\aleph_0}$ we have $\sup(X) < \sup(Y)$.

By the hypothesis that $\vec{A}$ witnesses ADS_κ, we fix $g : \sup(Y) \longrightarrow \kappa$ such that $\langle A_\beta \setminus g(\beta) : \beta < \sup(Y) \rangle$ is a sequence of mutually disjoint sets. For each $X \in S \cap [Y]^{\aleph_0}$ let $h(X) = \min(A_{\sup(X)} \setminus g(\sup(X)))$, and observe that $\forall X \in S \cap [Y]^{\aleph_0}\ A_{\sup(X)} \subseteq X$ and so $\forall X \in S\ h(X) \in X$.

By Fodor's theorem there exists $T \subseteq S$ with T stationary in $[Y]^{\aleph_0}$ such that h is constant on T. It is easy to see that if $X, X' \in T$ then $\sup(X) = \sup(X')$, because otherwise we will have $h(X) \neq h(X')$. Suppose $\sup(X) = \beta$ for all $X \in T$.

We know $\mathrm{cf}(\beta) = \omega$ and $\mathrm{cf}(\mathrm{ot}(Y)) = \aleph_1$, so $\beta < \sup(Y)$. It is easy to see that $\{ X \in [Y]^{\aleph_0} : \sup(X) = \beta \}$ is non-stationary, contradicting the stationarity of T. □

Remark 4.11. We have not used the full power of ADS_κ, since we only needed to be able to make any subsequence of size $\aleph_1$ disjoint in order to make the proof work.

Remark 4.12. The last result uses ideas of Shelah [43]. See the remarks after Theorem 2.12 in Foreman and Magidor's paper [12].

5. Improved squares

As we remarked after Definition 2.3 the principle $\square^*_\kappa$ implies there is a $\square^*_\kappa$-sequence $\vec{\mathcal{C}}$ with the additional property that each set $\mathcal{C}_\alpha$ contains a set of order type $\mathrm{cf}(\alpha)$. Here we consider the principle obtained by adding this property to the list of properties demanded of a $\square_{\kappa,\lambda}$ sequence.

Definition 5.1. A $\square^{\mathrm{imp}}_{\kappa,\lambda}$*-sequence (improved* $\square_{\kappa,\lambda}$*-sequence)* is a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+, \lim(\alpha) \rangle$ such that

1. $\mathcal{C}_\alpha \subseteq \mathrm{P}(\alpha)$, $1 \leq |\mathcal{C}_\alpha| \leq \lambda$, and $\mathcal{C}_\alpha$ is a set of closed and unbounded subsets of α with order types less than or equal to κ.
2. If $\mathrm{cf}(\alpha) < \kappa$ then $\forall C \in \mathcal{C}_\alpha$ $\mathrm{ot}(C) < \kappa$.
3. $\exists C \in \mathcal{C}_\alpha$ $\mathrm{ot}(C) = \mathrm{cf}(\alpha)$.
4. $\forall C \in \mathcal{C}_\alpha$ $\forall \beta \in \lim(C)$ $C \cap \beta \in \mathcal{C}_\beta$.

Remark 5.2. If $\square^{\mathrm{imp}}_{\kappa,\lambda}$ holds then $\square^{\mathrm{imp}}_{\mu,\lambda}$ holds for every μ with $\lambda \leq \mu < \kappa$.

The proof is immediate. This shows that in general $\square_\kappa$ does not imply $\square^{\mathrm{imp}}_{\kappa,\lambda}$ for any $\lambda < \kappa$. For example Mitchell [37] starts with a Mahlo cardinal and builds a model where there is no special $\aleph_2$-Aronszajn tree, which is to say that $\square_{\aleph_1,\aleph_1}$ fails; it is possible (see Section 6) to force $\square_{\aleph_2}$ without adding any $\aleph_2$-sequence of ordinals, and doing this forcing construction over Mitchell's model will give a model of $\square_{\aleph_2} + \neg\square^{\mathrm{imp}}_{\aleph_2,\aleph_1}$.

Improved squares have slightly more influence on stationary reflection than their unimproved cousins. The next result should be contrasted with Theorem 18.

Theorem 9. *Let $\square^{\text{imp}}_{\kappa,\lambda}$ hold. Let $\lambda < \text{cf}(\mu) \leq \mu \leq \kappa$ and let $S \subseteq \kappa^+$ be stationary. Then there exists a stationary $T \subseteq S$ and a $\beta < \mu$ such that T does not reflect at any point α with $\beta < \text{cf}(\alpha) \leq \mu$.*

Proof. Let $\langle \mathcal{C}_\alpha : \alpha < \kappa^+, \lim(\alpha) \rangle$ witness $\square^{\text{imp}}_{\kappa,\lambda}$. Define $F : \kappa^+ \longrightarrow \mu$ by setting

$$F : \alpha \longmapsto \sup(\{ \text{ot}(C) \ : \ C \in \mathcal{C}_\alpha, \text{ot}(C) < \mu \}).$$

Find $T \subseteq S$ a stationary set such that $F \restriction T$ is constant with value $\beta < \mu$. Now if $\beta < \text{cf}(\alpha) \leq \mu$, there is $C \in \mathcal{C}_\alpha$ such that $\text{ot}(C) = \text{cf}(\alpha)$. For each $\gamma \in \lim(C)$ we have $C \cap \gamma \in \mathcal{C}_\gamma$ and $\text{ot}(C \cap \gamma) < \mu$, so that $F(\gamma) \geq \text{ot}(C \cap \gamma)$ for each such γ. It follows that $F(\gamma) > \beta$ for all sufficiently large $\gamma \in \lim(C)$, so that T cannot be stationary in α. □

Next we ask how small λ can be in an improved square.

Lemma 5.3. *Let $\langle C_\alpha : \alpha < \mu \rangle$ be a sequence such that C_α is club in α and $\text{ot}(C_\alpha) = \text{cf}(\alpha)$. If $\alpha < \mu$ is such that $\text{cf}(\alpha) = \aleph_n$ then for every $i < n$ there is $\beta < \alpha$ such that $\text{cf}(\beta) = \aleph_i$ and*

$$|\{ C_\gamma \cap \beta \ : \ \gamma \geq \beta, \beta \in \lim(C_\gamma) \}| \geq n - i.$$

Proof. Let $\gamma_0 = \alpha$. Choose inductively $\gamma_1, \dots, \gamma_n$ such that

1. $\gamma_j \in \lim(C_{\gamma_i})$ for all $i < j$.
2. $\text{cf}(\gamma_j) = \aleph_{n-j}$.
3. $\text{ot}(C_{\gamma_i} \cap \gamma_j) > \aleph_{n-i-1}$ for all $i < j$.

For each j and $i < j$ we have arranged that

$$\aleph_{n-i-1} < \text{ot}(C_{\gamma_i} \cap \gamma_j) < \text{ot}(C_{\gamma_i}) = \text{cf}(\gamma_i) = \aleph_{n-i},$$

so that the sets in the sequence $\langle C_{\gamma_i} \cap \gamma_j : i < j \rangle$ are all distinct. So γ_{n-i} is the desired point of cofinality $\aleph_i$. □

Corollary 5.4. *If $n < \omega$ then $\square^{\text{imp}}_{\aleph_n,j}$ cannot hold if $j \leq n$. For $\kappa \geq \aleph_\omega$, $\square^{\text{imp}}_{\kappa,j}$ is false for every $j < \omega$.*

This result is fairly sharp, because $\square^{\text{imp}}_{\rho,<\omega}$ is consistent for every ρ. To see this we recall Jensen's Global □ principle.

Definition 5.5. A *Global □-sequence* is a sequence $\langle C_\alpha : \text{cf}(\alpha) < \alpha \rangle$ such that

1. C_α is club in α, with $\text{ot}(C_\alpha) < \alpha$.
2. If $\beta \in \lim(C_\alpha)$ then $\text{cf}(\beta) < \beta$ and $C_\beta = C_\alpha \cap \beta$.

Jensen [25] showed that if $V = L$ then there is a Global □-sequence.

Theorem 10. *If there is a Global □-sequence then $\square^{\text{imp}}_{\rho,<\omega}$ holds for every cardinal ρ.*

Proof. Let $\vec{C}$ be a Global $\Box$-sequence. First we derive a form of "Global improved square", then we derive $\Box^{\rm imp}_{\rho,<\omega}$-sequences from it.

Claim. *There is a sequence $\langle \mathcal{D}_\alpha : \mathrm{cf}(\alpha) < \alpha \rangle$ such that for all α*

1. $1 \le |\mathcal{D}_\alpha| < \omega$.
2. $\forall D \in \mathcal{D}_\alpha$ $\mathrm{ot}(D) < \alpha$.
3. *For all $D \in \mathcal{D}_\alpha$ and all $\beta \in \lim(D)$, $\mathrm{cf}(\beta) < \beta$ and $D \cap \beta \in \mathcal{D}_\beta$.*
4. *There exists $D \in \mathcal{D}_\alpha$ with $\mathrm{ot}(D) = \mathrm{cf}(\alpha)$.*

Proof. For each α with $\mathrm{cf}(\alpha) < \alpha$ we will define by induction on α a sequence of sets $\langle D^i_\alpha : i \le i(\alpha) \rangle$ where $i(\alpha) < \omega$.

We start by defining $D^0_\alpha = C_\alpha$. Let $\beta =_{\rm def} \mathrm{ot}(C_\alpha)$. If $\beta = \mathrm{cf}(\alpha)$ then we set $i(\alpha) = 0$ and stop. Otherwise we know that $\mathrm{cf}(\beta) = \mathrm{cf}(\alpha) < \beta$, so that we have already defined $\langle D^j_\beta : j \le i(\beta) \rangle$; in this case we define $i(\alpha) = i(\beta) + 1$ and $D^{j+1}_\alpha = \{ \gamma \in C_\alpha : \mathrm{ot}(C_\alpha \cap \gamma) \in D^j_\beta \}$ for each $j \le i(\beta)$; in other words we copy the sequence at β from β up to α, using C_α.

It is easy to see by induction that $\langle D^i_\alpha : i \le i(\alpha) \rangle$ is a decreasing sequence of club subsets of C_α, and that $\mathrm{ot}(D^{i(\alpha)}_\alpha) = \mathrm{cf}(\alpha)$.

Subclaim. *If $j \le i(\alpha)$ and $\gamma \in \lim(D^j_\alpha)$ then $j \le i(\gamma)$ and $D^j_\alpha \cap \gamma = D^j_\gamma$.*

Proof. We prove this by induction on j, for all α simultaneously. The claim is immediate for $j = 0$ by the coherence properties of the $\Box$-sequence $\vec{C}$.

Now let $j = i + 1$. Let $\beta = \mathrm{ot}(C_\alpha)$, then we know since $i(\alpha) > 0$ that

- $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) < \beta$.
- $i \le i(\beta)$.
- $D^j_\alpha = \{ \delta \in C_\alpha : \mathrm{ot}(C_\alpha \cap \delta) \in D^i_\beta \}$.

Since $D^j_\alpha \subseteq C_\alpha$ we know that $\gamma \in \lim(C_\alpha)$, so that $\mathrm{cf}(\gamma) < \gamma$ and $C_\gamma = C_\alpha \cap \gamma$. Let $\rho =_{\rm def} \mathrm{ot}(C_\gamma)$.

Since γ is a limit point of D^j_α, $\rho = \mathrm{ot}(C_\alpha \cap \gamma) \in \lim(D^i_\beta)$. By induction

- $i \le i(\rho)$.
- $D^i_\rho = D^i_\beta \cap \rho$.

Therefore

$$\begin{aligned} D^j_\alpha \cap \gamma &= \{ \delta \in C_\alpha \cap \gamma : \mathrm{ot}(C_\alpha \cap \delta) \in D^i_\beta \cap \rho \} \\ &= \{ \delta \in C_\gamma : \mathrm{ot}(C_\gamma \cap \delta) \in D^i_\rho \} \end{aligned}$$

Now consider the construction of the D^i_γ's. $\mathrm{cf}(\gamma) = \mathrm{cf}(\rho) < \rho = \mathrm{ot}(C_\gamma)$, and $i \le i(\rho)$, so that $j \le i(\gamma)$ and $D^j_\gamma = \{ \delta \in C_\gamma : \mathrm{ot}(C_\gamma \cap \delta) \in D^i_\rho \}$.

We have shown that $D^j_\gamma = D^j_\alpha \cap \gamma$, concluding the proof of the subclaim. □

We can now set $\mathcal{D}_\alpha = \{ D^i_\alpha \ : \ i \leq i(\alpha) \}$ to conclude the proof of the claim. □

Claim. *For every cardinal ρ there exists a $\square^{\text{imp}}_{\rho,<\omega}$-sequence.*

Proof. Observe that for every $\alpha \in (\rho, \rho^+)$ we have $\text{cf}(\alpha) \leq \rho < \alpha$, so that the family $\mathcal{D}_\alpha$ defined in the previous claim exists. For each $D \subseteq ON$ define $D^* = \{ \gamma \ : \ \rho + \gamma \in D \}$, and let

$$\mathcal{E}_\alpha = \{ D^* \ : \ D \in \mathcal{D}_{\rho+\alpha}, \text{ot}(D) \leq \rho \}.$$

It is routine to check that $\langle \mathcal{E}_\alpha : \alpha < \rho^+ \rangle$ is a $\square^{\text{imp}}_{\rho,<\omega}$-sequence. □

This concludes the proof of Theorem 10. □

We can also prove results with a more local flavour.

Theorem 11. *Suppose that $\square_{\aleph_\omega,\lambda}$ holds, and that $\square_{\aleph_n}$ holds for all n with $1 \leq n < \omega$. Then $\square^{\text{imp}}_{\aleph_\omega,\lambda+\aleph_0}$ holds.*

Proof. Let $\langle \mathcal{C}_\alpha : \alpha < \aleph_{\omega+1} \rangle$ be a $\square_{\aleph_\omega,\lambda}$-sequence, and for each $n < \omega$ let $\langle C^n_\beta : \beta < \aleph_{n+1} \rangle$ be a $\square_{\aleph_n}$-sequence. We now define a sequence of approximations to an improved square sequence.

Let $\mathcal{C}^0_\alpha = \mathcal{C}_\alpha$. Then for each $m < \omega$ let

$$\mathcal{C}^{m+1}_\alpha = \mathcal{C}^m_\alpha \cup \mathcal{D}^m_\alpha,$$

where $\mathcal{D}^m_\alpha$ is

$$\{ \{ \gamma \in C \ : \ \text{ot}(C \cap \gamma) \in C^n_\delta \} \ : \ C \in \mathcal{C}^m_\alpha, n < \omega, \text{ot}(C) = \delta < \aleph_{n+1} \}.$$

Finally let $\mathcal{C}^\omega_\alpha = \bigcup_{n<\omega} \mathcal{C}^n_\alpha$.

Clearly $|\mathcal{C}^\omega_\alpha| \leq \lambda + \aleph_0$. We claim that $\langle \mathcal{C}^\omega_\alpha : \alpha < \aleph_{\omega+1} \rangle$ is a $\square^{\text{imp}}_{\aleph_\omega,\lambda+\aleph_0}$-sequence.

First we check that $\forall \alpha \ \exists C \in \mathcal{C}^\omega_\alpha \ \text{ot}(C) = \text{cf}(\alpha)$. To see this observe that if $C \in \mathcal{C}^m_\alpha$ and $\text{ot}(C) \neq \text{cf}(\alpha)$ then $\text{ot}(C)$ is not a regular cardinal, so for some n we have $\text{ot}(C) = \delta \in (\aleph_n, \aleph_{n+1})$; then if $C^* = \{ \gamma \in C \ : \ \text{ot}(C \cap \gamma) \in C^n_\delta \}$ we have $\text{ot}(C^*) = \text{ot}(C^n_\delta) \leq \aleph_n < \text{ot}(C)$ and $C^* \in \mathcal{C}^{m+1}_\alpha$. It follows that any element of $\mathcal{C}^\omega_\alpha$ whose order type is minimal has order type $\text{cf}(\alpha)$.

To finish we prove $\forall C \in \mathcal{C}^m_\alpha \ \forall \gamma \in \lim(C) \ C \cap \gamma \in \mathcal{C}^m_\gamma$, by induction on m. If $m = 0$ this is immediate. Suppose we have it for m. Let $C \in \mathcal{D}^m_\alpha$. Then $C = \{ \gamma \in D \ : \ \text{ot}(D \cap \gamma) \in C^n_\delta \}$, where $D \in \mathcal{C}^m_\alpha$ and $\text{ot}(D) = \delta < \aleph_{n+1}$. Fix $\gamma \in \lim(C)$ and let $\rho = \text{ot}(D \cap \gamma)$, then $\rho \in \lim(C^n_\delta)$ and therefore $C^n_\rho = C^n_\delta \cap \rho$.

Now by induction $D \cap \gamma \in \mathcal{C}^m_\gamma$, and we also see that $\mathrm{ot}(D \cap \gamma) < \delta < \aleph_{n+1}$.

$$\begin{aligned} C \cap \gamma &= \{ \sigma \in D : \mathrm{ot}(D \cap \sigma) \in C^n_\delta \cap \rho \} \\ &= \{ \sigma \in D \cap \gamma : \mathrm{ot}(D \cap \sigma) \in C^n_\rho \}, \end{aligned}$$

so $C \cap \gamma \in \mathcal{C}^{m+1}_\gamma$ because it is defined from $D \cap \gamma \in \mathcal{C}^m_\gamma$. □

Remark 5.6. Similar methods show that if $n < \omega$ and $\Box_{\aleph_m}$ holds for $m \leq n$, then $\Box^{\mathrm{imp}}_{\aleph_n, n+1}$ holds.

Burke and Kanamori observed [39] that if κ is λ^+-strongly compact then $\Box_{\lambda, <\mathrm{cf}(\lambda)}$ fails. A similar result is true for the "improved square" principles $\Box^{\mathrm{imp}}_{\lambda,\mu}$. Both proofs are based on the following well-known fact, which is due to Solovay and is the key to the proof of Fact 2.9.

Fact 5.7. *Let γ be regular and uncountable. Let $j : V \longrightarrow M$ be an embedding such that $\mathrm{crit}(j) = \kappa \leq \gamma$ and let $\rho =_{\mathrm{def}} \sup(j``\gamma)$. If C is club in ρ, then $\{ \alpha < \gamma : j(\alpha) \in C \}$ is $< \kappa$-club in γ.*

Theorem 12. *Let κ be λ^+-strongly compact, where $\lambda > \kappa$. Then $\Box^{\mathrm{imp}}_{\lambda,<\lambda}$ fails.*

Proof. Suppose that $\vec{\mathcal{C}}$ witnesses $\Box^{\mathrm{imp}}_{\lambda,<\lambda}$, and choose $C_\alpha \in \mathcal{C}_\alpha$ such that $\mathrm{ot}(C_\alpha) = \mathrm{cf}(\alpha)$ for each limit $\alpha < \lambda^+$. Let $j : V \longrightarrow M$ be such that $\mathrm{crit}(j) = \kappa$, and if $\rho =_{\mathrm{def}} \sup(j``\lambda^+)$ then $\rho < j(\lambda^+)$ and $M \models \mathrm{cf}(\rho) < j(\kappa)$. Let $\vec{D} = j(\vec{C})$, $D = D_\rho$, and $C = \{ \alpha < \lambda^+ : j(\alpha) \in D_\rho \}$.

Then C is $< \kappa$-club. Let β be the $\lambda + \omega^{\mathrm{th}}$ element of C, so that $\beta \in \lim(C)$ and $|C \cap \beta| = \lambda$. Since $\mathrm{cf}(\beta) = \omega$, j is continuous at β and so $j(\beta) \in \lim(D)$.

For each $\alpha \in C \cap \beta$ we have $j(\alpha) \in D \cap j(\beta)$, so by elementarity there is $\sigma < \lambda^+$ such that $\beta \in \lim(C_\sigma)$, $\mathrm{cf}(\sigma) < \kappa$, and $\alpha \in C_\sigma \cap \beta$. There are $|\mathcal{C}_\beta|$ possibilities for $C_\sigma \cap \beta$ and each has cardinality less than κ, so that $|C \cap \beta| \leq |\mathcal{C}_\beta| . \kappa < \lambda$. This is a contradiction. □

6. Preliminaries on forcing

We make several uses of the following theorem.

Fact 6.1 (Laver [30])**.** *Let κ be supercompact, let $\alpha < \kappa$. Then there is an α-directed closed and κ-c.c. poset $\mathbb{P}$ such that $|\mathbb{P}| = \kappa$, κ is supercompact in $V^{\mathbb{P}}$, and*

$V^{\mathbb{P}} \models$ *"κ-directed closed forcing preserves the supercompactness of κ".*

A supercompact cardinal κ whose supercompactness is preserved by any κ-directed closed forcing is said to be "Laver indestructible".

We collect some information related to forcing square sequences and reflection properties, which we will use repeatedly in the sections that follow.

6.1. **Forcing square sequences.** Let κ be any cardinal, and let $1 \leq \lambda \leq \kappa$. We shall describe a forcing poset $\mathbb{P}(\kappa, \lambda)$ which adds a $\square_{\kappa,\lambda}$-sequence and preserves all cardinals up to and including κ^+. The conditions in $\mathbb{P}(\kappa, \lambda)$ are just proper initial segments of the desired sequence. More precisely

Definition 6.2. $p \in \mathbb{P}(\kappa, \lambda)$ iff

1. p is a function with $\mathrm{dom}(p) = \{ \beta \leq \alpha \ : \ \lim(\beta) \}$ for some limit ordinal $\alpha < \kappa^+$.
2. For all $\beta \in \mathrm{dom}(p)$, $1 \leq |p(\beta)| \leq \lambda$.
3. For all $\beta \in \mathrm{dom}(p)$ and $C \in p(\beta)$, C is club in β and $\mathrm{ot}(C) \leq \kappa$.
4. If $\mathrm{cf}(\beta) < \kappa$ then $\mathrm{ot}(C) < \kappa$ for all $C \in p(\beta)$.
5. For all $\beta \in \mathrm{dom}(p)$, $C \in p(\beta)$ and $\gamma \in \lim(C)$, $C \cap \gamma \in p(\gamma)$.

If $p, q \in \mathbb{P}(\kappa, \lambda)$ then $q \leq p$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $q \restriction \mathrm{dom}(p) = p$.

Remark 6.3. The idea of forcing square sequences by initial segments is due to Jensen.

To show that $\mathbb{P}(\kappa, \lambda)$ is a reasonable forcing poset, we use the idea of *strategic closure.*

Remark 6.4. Strategic closure properties of posets and Boolean algebras were introduced by Jech [20, 21] and have also been studied by Foreman [10] and Gray [16].

Definition 6.5. Let $\mathbb{P}$ be a partial ordering, let α be an ordinal. Then the game $G^{II}_\alpha(\mathbb{P})$ is played as follows: players I and II take turns to write down the entries of a decreasing sequence $\langle p_\beta : \beta < \alpha \rangle$ of conditions in $\mathbb{P}$, with I playing at odd stages and II playing at even stages (including all limit stages). If play reaches a stage where player II cannot move then player I wins, otherwise player II wins.

The poset $\mathbb{P}$ is *α-strategically closed* iff player II has a winning strategy for the game $G^{II}_\alpha(\mathbb{P})$. $\mathbb{P}$ is *$< \lambda$-strategically closed* iff player II wins $G^{II}_\alpha(\mathbb{P})$ for all $\alpha < \lambda$.

The following fact is well-known.

Fact 6.6. *Let κ be a cardinal. If $\mathbb{P}$ is $(\kappa + 1)$-strategically closed then forcing with $\mathbb{P}$ adds no κ-sequences of ordinals.*

Lemma 6.7. *The poset $\mathbb{P}(\kappa, \lambda)$ is $(\kappa + 1)$-strategically closed.*

Proof. We describe a winning strategy for player II in the appropriate game. The only subtle point is that in case κ is singular we cannot let player II play a club of order type κ at move κ; we describe a strategy that works for κ regular or singular.

Let $\lambda = \mathrm{cf}(\kappa)$ and let $C \subseteq \kappa$ be a club set of order type λ. We assume that at stage α the players have produced a sequence of conditions $\langle p_\beta : \beta < \alpha \rangle$, where $\max \mathrm{dom}(p_\beta) = \gamma_\beta$.

Case 1: α is an even successor ordinal, say $\alpha = 2\beta + 2$. In this case player II plays a condition $p_{2\beta+2} \leq p_{2\beta+1}$ such that $\gamma_{2\beta+2} = \gamma_{2\beta+1} + \omega$ and $p(\gamma_{2\beta+2}) = \{ \{ \gamma_{2\beta+1} + n \ : \ n < \omega \} \}$

Case 2: α is a limit ordinal. We define $\gamma_\alpha = \sup_{\beta < \alpha} \gamma_\beta$.

Subcase 2a: $\sup(C \cap \alpha) = \alpha$. In this case we define p_α by setting $p_\alpha \restriction \gamma_\alpha = \bigcup_{\beta < \alpha} p_\beta$ and $p_\alpha(\gamma_\alpha) = \{ \gamma_\beta \ : \ \beta \in C \cap \alpha \}$.

Subcase 2b: $\sup(C \cap \alpha) < \alpha$. In this case we define p_α by setting $p_\alpha \restriction \gamma_\alpha = \bigcup_{\beta < \alpha} p_\beta$ and $p_\alpha(\gamma_\alpha) = \{ \gamma_\beta \ : \ \sup(C \cap \alpha) < \beta < \alpha \}$.

It is routine to verify that this is a winning strategy for player II. □

It follows that forcing with $\mathbb{P}(\kappa, \lambda)$ adds no κ-sequences of ordinals, and so preserves all cardinals and cofinalities up to and including κ^+. It is routine to check that if G is $\mathbb{P}(\kappa, \lambda)$-generic then $\bigcup G$ is a $\square_{\kappa,\lambda}$-sequence in $V[G]$; the proof that $\{ p \ : \ \mathrm{lh}(p) > \beta \}$ is dense for each $\beta < \kappa^+$ is similar to the proof of Lemma 6.7.

6.2. **Threading a generic square sequence.** Let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ be a $\square_{\kappa,\lambda}$-sequence in some model of set theory V. Let $W \supseteq V$ be a larger model of set theory. We say that $C \in W$ *threads* the sequence $\vec{\mathcal{C}}$ if C is club in κ^+ and $\forall \alpha \in \lim(C)\ C \cap \alpha \in \mathcal{C}_\alpha$. It is clear that no such C can exist in V (or any extension of V in which κ^+ is regular) because every initial segment of C can have order type at most κ. However it is sometimes possible to add such a club C by reasonably distributive forcing; in particular this will be possible if $\vec{\mathcal{C}}$ was added by the forcing $\mathbb{P}(\kappa, \lambda)$ given in Definition 6.2.

Definition 6.8. Let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ be a $\square_{\kappa,\lambda}$-sequence. Let $\gamma = \mathrm{cf}(\gamma) \leq \kappa$. Then $c \in \mathbb{T}_\gamma(\vec{\mathcal{C}})$ iff

1. c is a closed bounded subset of κ^+.
2. $\mathrm{ot}(c) < \gamma$.
3. $\forall \beta \in \lim(c)\ c \cap \beta \in \mathcal{C}_\beta$.

If $c, d \in \mathbb{T}_\gamma(\vec{\mathcal{C}})$ then $c \leq d$ iff $c \cap (\max(d) + 1) = d$.

It is clear that $\mathbb{T}_\gamma(\vec{\mathcal{C}})$ will add C which is club and threads $\vec{\mathcal{C}}$. In general it is not clear what the order type of C will be, and how much damage the forcing will do.

Lemma 6.9. *Let κ be a cardinal, let $1 \leq \lambda \leq \kappa$, $\gamma = \mathrm{cf}(\gamma) \leq \kappa$. Let $\mathbb{P} = \mathbb{P}(\kappa, \lambda)$, and $\mathbb{T} = \mathbb{T}_\gamma(\vec{\mathcal{C}})_{V^{\mathbb{P}}}$ where $\vec{\mathcal{C}}$ is the generic $\square_{\kappa,\lambda}$-sequence added by $\mathbb{P}$. Then*

1. *$\mathbb{P} * \mathbb{T}$ has a γ-closed dense subset.*
2. *$\mathbb{T}$ adds a generic club which threads $\vec{\mathcal{C}}$ and has order type γ. κ^+_V has cofinality γ in $V^{\mathbb{P}*\mathbb{T}}$.*

Proof. Observe that $\mathbb{T} \subseteq V$, since forcing with $\mathbb{P}$ adds no κ-sequences of ordinals and conditions in $\mathbb{T}$ are bounded subsets of κ^+. We take each claim in turn.

1. We define D to be the set of conditions $(p, \dot{c}) \in \mathbb{P} * \mathbb{T}$ such that p decides $\dot{c}$ (say $p \Vdash \dot{c} = \check{c}$ for some $c \in V$) and $\max \mathrm{dom}(p) = \max(c)$. We claim that D is dense in $\mathbb{P} * \mathbb{T}$ and that it is γ-closed.
   For density, let $(q, \dot{d})$ be any condition in $\mathbb{P} * \mathbb{T}$. We refine q to p such that p decides $\dot{d}$ (say $p \Vdash \dot{d} = \check{d}$) and $\gamma =_{\mathrm{def}} \max \mathrm{dom}(p) > \max(d)$. Now the condition $(p, d \cup \{\gamma\})$ refines $(q, \dot{d})$ and lies in D.
   For closure, fix a decreasing sequence $\langle (p_\delta, c_\delta) : \delta < \beta \rangle$ of conditions in D for some $\beta < \gamma$. Let $\alpha_\delta = \max \mathrm{dom}(p_\delta) = \max(c_\delta)$ for each δ, and define $\alpha = \sup_{\delta<\beta} \alpha_\delta$. Now define $c = \bigcup_{\delta<\beta} c_\delta \cup \{\alpha\}$, and let p be such that $\max \mathrm{dom}(p) = \alpha$, $p \restriction \alpha = \bigcup_{\delta<\beta} p_\delta$, $p(\alpha) = \{c \cap \alpha\}$. It is routine to check that (p, c) is a condition and that $(p, c) \in D$.
2. An easy density argument shows that $\mathbb{T}$ adds a club C which threads $\vec{\mathcal{C}}$. Since each initial segment of C is determined by some condition, $\mathrm{ot}(C) \leq \gamma$. On the other hand $\mathbb{P} * \mathbb{T}$ has a γ-closed dense subset, so that
   $$V^{\mathbb{P}*\mathbb{T}} \models \text{“}\gamma \text{ is regular and } \kappa^+_V \text{ has cofinality at least } \gamma\text{”}$$
   and thus $V^{\mathbb{P}*\mathbb{T}} \models \mathrm{cf}(\kappa^+_V) = \gamma = \mathrm{ot}(C)$.

$\square$

It follows from this that $\mathbb{T}$ is (γ, ∞)-distributive in $V^{\mathbb{P}(\kappa,\lambda)}$.

6.3. **Stationary reflection in $\aleph_{\omega+1}$.** Recall that “Refl(μ, S, ν)” is the principle stating that any μ stationary subsets of S reflect simultaneously to a point of cofinality ν. Magidor [33] showed that it is possible to start with ω supercompact cardinals and to make a model in which $\mathrm{Refl}(\aleph_{n+1}, \aleph_{\omega+1} \cap \mathrm{cof}(\leq \aleph_n), \aleph_{n+1})$ holds for each $n < \omega$. In subsequent sections we will be building some related models, so we summarise the key ideas of Magidor's proof.

We start by fixing $\langle \kappa_n : n < \omega \rangle$ an increasing ω-sequence of supercompact cardinals, and define an iteration $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n : n < \omega \rangle$ where $\mathbb{Q}_0 = \mathrm{Coll}(\omega, < \kappa_0)$ and $\mathbb{Q}_n = \mathrm{Coll}(\kappa_{n-1}, < \kappa_n)_{V^{\mathbb{P}_n}}$ for $n > 0$. We let $\mathbb{P}_\omega$ be the inverse limit of this iteration. The final model is $V^{\mathbb{P}_\omega}$.

The following fact captures some of the key features of $V^{\mathbb{P}_\omega}$.

Fact 6.10. *Let $\kappa =_{\mathrm{def}} \sup_n \kappa_n$, and $V_1 =_{\mathrm{def}} V^{\mathbb{P}_\omega}$.*

1. *$V_1 \models$ "$\kappa_n = \aleph_{n+1}$, $\kappa = \aleph_\omega$, and $\kappa^+_V = \aleph_{\omega+1}$".*
2. *Let $\mathbb{R} \in V_1$ be a forcing where $V_1 \models$ "$\mathbb{R}$ is $\aleph_{n+1}$-directed closed". Let G be $\mathbb{R}$-generic over V_1 and let $\lambda = \mathrm{cf}(\lambda) > \kappa_n$. Then there is $\bar{\mathbb{R}} \in V_1[G]$ such that*
   (a) *$V_1[G] \models$ "$\bar{\mathbb{R}}$ is $\aleph_n$-closed".*
   (b) *If H is $\bar{\mathbb{R}}$-generic over $V_1[G]$ then in $V_1[G * H]$ there is an elementary embedding $j : V_1[G] \longrightarrow M \subseteq V_1[G * H]$ such that*
      - $\mathrm{crit}(j) = \kappa_n = \aleph^{V_1}_{n+1}$.
      - $j \restriction \lambda \in M$.
      - $j(\kappa_n) > \lambda$.
      - $\sup(j``\lambda) < j(\lambda)$.
      - $M \models |\kappa_n| = |\lambda| = \mathrm{cf}(\lambda) = \aleph_n = \aleph^{V_1}_n$.

In the more picturesque language of [11], $\aleph_{n+1}$ is "indestructibly generically supercompact".

One of the main points in the proof of Fact 6.10 is

Fact 6.11. *Let ρ be regular, let $\rho < \lambda < \mu$, and let*

$$V^{\mathrm{Coll}(\rho, <\lambda)} \models \text{“}\mathbb{P} \text{ is a } \rho\text{-closed poset with } |\mathbb{P}| < \mu\text{”}.$$

Let i be the natural complete embedding of $\mathrm{Coll}(\rho, < \lambda)$ into $\mathrm{Coll}(\rho, < \mu)$. Then i can be extended to a complete embedding j of $\mathrm{Coll}(\rho, < \lambda) * \mathbb{P}$ into $\mathrm{Coll}(\rho, < \mu)$ in such a way that the quotient forcing $\mathrm{Coll}(\rho, < \mu)/j``[\mathrm{Coll}(\rho, < \lambda) * \mathbb{P}]$ is ρ-closed.

Let us now try to check that V_1 has the desired stationary reflection property. For simplicity we begin by considering only one stationary set. Let

$$V_1 \models \text{“}S \text{ is a stationary subset of } \aleph_{\omega+1} \cap \mathrm{cof}(\leq \aleph_n)\text{”}$$

for some $n < \omega$. Let $\lambda = \kappa^+$ and apply Lemma 6.10 with $\mathbb{R} = 0$. This gives a map $j : V_1 \longrightarrow M \subseteq V_1^{\bar{\mathbb{R}}}$ where

- $j \restriction \lambda \in M$.
- $\mathrm{crit}(j) = \kappa_{n+1} = \aleph_{n+2}$.
- $M \models \mathrm{cf}(\lambda) = \aleph_{n+1} = \aleph^{V_1}_{n+1}$.

- $V_1 \models$ "$\bar{\mathbb{R}}$ is $\aleph_{n+1}$-closed".
- $\sup(j``\lambda) < j(\lambda)$.

Let $\mu = \sup j``\lambda$. It is enough to prove that $V_1^{\bar{\mathbb{R}}} \models$ "S is stationary"; for since j is continuous at points of cofinality less than κ_{n+1} and $M \subseteq V_1^{\mathbb{R}}$, it will follow that $M \models$ "$j(S) \cap \mu$ is stationary in μ". We can then conclude by the elementarity of j that in V_1 the stationary set S reflects to some point of cofinality $\aleph_{n+1}$.

Unfortunately it is not true in general [41] that $\aleph_{n+1}$-closed forcing preserves the stationarity of stationary subsets of $\aleph_{\omega+1} \cap \mathrm{cof}(\leq \aleph_n)$. We collect some information about preservation of stationarity by sufficiently closed forcing, and then indicate how to finish the argument that $\mathrm{Refl}(\aleph_{n+1}, \aleph_{\omega+1} \cap \mathrm{cof}(\leq \aleph_n), \aleph_{n+1})$ holds.

Lemma 6.12. *Let $\gamma = \mathrm{cf}(\gamma) > \omega$ and let $S \subseteq \gamma$ be stationary. Then the stationarity of S is preserved by γ-closed forcing.*

Proof. Let $\mathbb{P}$ be γ-closed and let $p \Vdash$ "$\dot{C}$ is club in γ". Define sequences $\langle p_\alpha : \alpha < \gamma \rangle$ and $\langle \beta_\alpha : \alpha < \gamma \rangle$ such that

1. $p_0 \leq p$, and $\langle p_\alpha : \alpha < \gamma \rangle$ is a decreasing sequence of elements of $\mathbb{P}$.
2. $\langle \beta_\alpha : \alpha < \gamma \rangle$ is an increasing and continuous sequence of elements of γ.
3. $p_\alpha \Vdash \beta_\alpha \in \dot{C}$.

Since $\langle \beta_\alpha : \alpha < \gamma \rangle$ enumerates a club subset of γ which lies in V, there exists α such that $\beta_\alpha \in S$. So p_α is a refinement of p which forces $\dot{C} \cap \check{S} \neq \emptyset$. □

To get an appropriate generalisation of the well-known fact that countably closed forcing preserves the stationarity of a stationary set of cofinality ω ordinals, we introduce yet another variant of $\square_\kappa$. "AP" stands for "APproachability".

Definition 6.13 (Shelah)**.** $\langle C_\alpha : \alpha < \kappa^+ \rangle$ is an AP_κ*-sequence* iff

1. If $\lim(\alpha)$ then C_α is club in α, $\mathrm{ot}(C_\alpha) = \mathrm{cf}(\alpha)$.
2. For a club set of $\alpha < \kappa^+$, $\forall \beta < \alpha\ \exists \gamma < \alpha\ C_\alpha \cap \beta = C_\gamma$.

As usual we say that "AP_κ holds" iff there is an AP_κ-sequence. It is not hard to see that $\square^*_\kappa \implies \mathrm{AP}_\kappa$. AP_κ is known [13] to be substantially weaker than $\square^*_\kappa$, and by Remark 6.15 AP_κ says essentially nothing about the existence of non-reflecting stationary sets. On the other hand AP_κ does have a strong influence on scales, since for κ singular AP_κ implies that every scale for κ is good [13].

Lemma 6.14 (Shelah)**.** *Let $V \models \mathrm{AP}_\kappa$. Let $S \subseteq \kappa^+ \cap \mathrm{cof}(< \mu)$ be stationary, and let $\mathbb{Q}$ be μ-closed. Then S is stationary in $V^{\mathbb{Q}}$.*

Proof. Fix $\vec{C}$ witnessing AP_κ. Suppose for a contradiction that $\dot{D}$ names a club and $q \Vdash \dot{D} \cap S = \emptyset$. Fix some large regular H_θ and a wellordering $<_\theta$ of H_θ, and then let $N \prec (H_\theta, \in, <_\theta)$ be such that

- $\kappa \subseteq N$, $\delta =_{\mathrm{def}} N \cap \kappa^+ \in S$, $|N| = \kappa$.
- $q, \mathbb{Q}, S, \dot{D}, \vec{C} \in N$.
- $\forall \beta < \delta\ \exists \gamma < \delta\ C_\delta \cap \beta = C_\gamma$.

Notice that all initial segments of C_δ are in N. Enumerate C_δ in increasing order as $\langle \delta_i : i < \mathrm{cf}(\delta) \rangle$. Now build $\langle q_i : i < \mathrm{cf}(\delta) \rangle$ as follows.

- q_{i+1} is the $<_\theta$-least condition such that $q_{i+1} \leq q_i$ and there is β greater than δ_i such that $q_{i+1} \Vdash \beta \in \dot{D}$.
- For j limit, q_j is the $<_\theta$-least lower bound for $\langle q_i : i < j \rangle$.

It is clear that $\langle q_i : i < j \rangle$ is definable from the parameters $\{C_\delta \cap \delta_j, q, \dot{D}\}$ which are all in N. So $\langle q_i : i < j \rangle \in N$, as is each condition q_i. If we now let q be a lower bound for $\langle q_i : i < \mathrm{cf}(\delta) \rangle$ then q forces $\delta \in \dot{D} \cap S$, contradiction! □

The last key point in the argument from [33] is to show that AP_κ holds in $V^{\mathbb{P}_\omega}$. We refer the reader to [33] for this proof; the key points are that for each $n \geq 0$

- The cardinal $\aleph_{n+1}$ started off as a supercompact cardinal κ_n in V.
- The forcing $\mathbb{P}_\omega$ can be factored as a κ_n-c.c. forcing followed by a κ_n-closed forcing.

Granted that $\mathrm{AP}_{\aleph_\omega}$ holds, we can argue as outlined above that S reflects to a point of cofinality $\aleph_{n+1}$.

We still owe a proof that $\aleph_{n+1}$ sets reflect simultaneously; fix an $\aleph_{n+1}$-sequence $\langle S_i : i < \aleph_{n+1} \rangle$ of stationary subsets of $\aleph_{\omega+1} \cap \mathrm{cof}(\leq \aleph_n)$, and construct $j : V_1 \longrightarrow M \subseteq V_1^{\bar{\mathbb{R}}}$ as before. Since $\mathrm{crit}(j) = \aleph_{n+2}^{V_1}$ the same argument as before shows that for every $i < \aleph_{n+1} = j(\aleph_{n+1})$ the set $j(S_i) \cap \mu$ is stationary, so by elementarity there is a point of cofinality $\aleph_{n+1}$ at which every set S_i reflects.

Remark 6.15. It is also possible to show that in $V^{\mathbb{P}_\omega}$ the principle $\mathrm{Refl}^*([\aleph_{\omega+1}]^{\aleph_0})$ holds. So $\mathrm{AP}_{\aleph_\omega}$ does not have much power to construct non-reflecting stationary sets either in $\aleph_{\omega+1}$ or in $[\aleph_{\omega+1}]^{\aleph_0}$.

7. Distinguishing squares

Jensen [24] proved that for λ regular the principles $\square_{\lambda,\mu}$ for $\mu \leq \lambda$ have strictly decreasing strength as μ increases; in this section we prove a similar result for $\aleph_\omega$. The case of $\aleph_\omega$ is considered for simplicity, similar methods will work at other singular cardinals.

Remark 7.1. In the model of Theorem 21 $\square^*_{\aleph_\omega}$ holds but $\square_{\aleph_\omega,\lambda}$ fails for all $\lambda < \aleph_\omega$. Accordingly we only need to distinguish principles of the form $\square_{\aleph_\omega,\lambda}$ where $\lambda < \aleph_\omega$ in Theorem 13.

Theorem 13. *Let κ be supercompact, and suppose $2^{\kappa^{+\omega}} = \kappa^{+\omega+1}$. Let μ, ν be two cardinals (one or both can be finite) such that $1 \leq \mu < \nu < \aleph_\omega$. Then there is a generic extension in which*

1. *All cardinals less than or equal to ν are preserved.*
2. $\aleph_\omega = \kappa_V^{+\omega}$.
3. $\square_{\aleph_\omega,\nu}$ *holds.*
4. $\square_{\aleph_\omega,\mu}$ *fails.*

Proof. Let $\rho = \max\{\nu, \aleph_1\}$ and let $\lambda = \kappa^{+\omega}$. Let $\mathbb{P} = \mathrm{Coll}(\rho, <\kappa)$, so that $\mathbb{P}$ is ρ-closed and $\lambda = \aleph_\omega^{V^{\mathbb{P}}}$.

Working in $V^{\mathbb{P}}$ we define a two-step forcing iteration $\mathbb{Q} * \mathbb{R}$. The model in which the conclusion of the theorem holds will be $V^{\mathbb{P}*\mathbb{Q}}$, while $\mathbb{R}$ is an auxiliary forcing whose use is explained below.

1. $\mathbb{Q} = \mathbb{P}(\lambda,\nu)_{V^{\mathbb{P}}}$, where $\mathbb{P}(\lambda,\nu)$ is the forcing to add a $\square_{\lambda,\nu}$-sequence defined in Definition 6.2.
2. $\mathbb{R} = \mathbb{T}_\rho(\vec{\mathcal{C}})_{V^{\mathbb{P}*\mathbb{Q}}}$ where $\vec{\mathcal{C}}$ is the $\square_{\lambda,\nu}$-sequence added by $\mathbb{Q}$ and $\mathbb{T}_\rho(\vec{\mathcal{C}})$ is the "threading" forcing defined in Definition 6.8.

By Lemma 6.9, $\mathbb{Q} * \mathbb{R}$ has a dense ρ-closed subset. Our cardinal arithmetic assumptions imply $V^{\mathbb{P}} \models |\mathbb{Q} * \mathbb{R}| = \lambda^+$. We fix G, H such that G is $\mathbb{Q}$-generic over V and H is $\mathbb{R}$-generic over $V[G]$.

Now we fix an elementary embedding $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda^+$, ${}^{\lambda^+}M \subseteq M$. $j \restriction \mathbb{P}$ is the identity map and gives a complete embedding of $\mathbb{P}$ into $j(\mathbb{P})$, with the quotient forcing being $\mathrm{Coll}(\rho, [\kappa, j(\kappa)))$. By Lemma 6.11 we can extend this to give an embedding of $\mathbb{P}*\mathbb{Q}*\mathbb{R}$ into $j(\mathbb{P})$ in such a way that the quotient forcing is ρ-closed.

Let I be $\mathbb{R}$-generic over $V[G*H]$ and let J be $j(\mathbb{P})/G*H*I$-generic over $V[G*H*I]$, then we can build $j : V[G] \longrightarrow M[G*H*I*J]$ extending $j : V \longrightarrow M$. We want to extend this embedding to one with domain $V[G*H]$.

Let $\gamma = \bigcup j``\lambda^+$, then $\gamma < j(\lambda^+)$ and $M \models \mathrm{cf}(\gamma) = \lambda^+$. Let C be the club set of order type ρ in λ^+ which is added by I; then $C \notin V[G*H]$ but every initial segment of C is in V, and so every limit point of C has cofinality less than ρ in V. Now let $E = j``C$. Clearly E is unbounded in γ, and E is closed because j is continuous at points of cofinality $< \rho$.

Define $\vec{\mathcal{D}} = \langle \mathcal{D}_\alpha : \alpha < \gamma \rangle = \bigcup_{s \in H} j(s)$. $\vec{\mathcal{D}}$ is not a condition in $j(\mathbb{Q})$ because it lacks a top element. However if $\alpha \in \lim(E)$ then $\alpha = j(\beta)$ for some $\beta \in \lim(C)$, so that $E \cap \alpha = j``(C \cap \beta) = j(C \cap \beta) \in j(\mathcal{C}_\beta) = \mathcal{D}_\alpha$.

It follows that if $Q = \vec{\mathcal{D}} \cup \{(\gamma, \{E\})\}$ then Q is a condition in $j(\mathbb{Q})$ and what is more $Q \leq j(q)$ for every $q \in H$.

Now force below the "master condition" Q to get K which is $j(\mathbb{Q})$-generic. Since $j``H \subseteq K$ we may extend j to get an elementary embedding $j : V[G * H] \longrightarrow M[G * H * I * J * K]$, which we will use to show that there is no $\Box_{\lambda,\mu}$-sequence in $V[G * H]$. Towards a contradiction, suppose that $\vec{\mathcal{E}} = \langle \mathcal{E}_\alpha : \alpha < \lambda^+ \rangle$ is such a sequence.

Notice that all the models which we discuss agree in their computation of the set of $< \rho$-sequences of ordinals, in particular cofinalities and cardinal arithmetic below ρ will look the same in all these models. λ^+_V is the $\aleph_{\omega+1}$ of $V[G * H]$, and is an ordinal of cardinality and cofinality ρ in $V[G * H * I]$.

Claim. *In $V[G * H * I * J]$ there exists F club in λ^+ such that $\forall \alpha \in \lim(F)$ $F \cap \alpha \in \mathcal{E}_\alpha$ (that is to say F threads $\vec{\mathcal{E}}$ in the sense of Section 6.2).*

Proof. Let $\vec{\mathcal{F}} = j(\vec{\mathcal{E}})$, and observe that because $\nu \leq \rho < \kappa = \mathrm{crit}(j)$ we have $\mathcal{F}_{j(\alpha)} = j(\mathcal{E}_\alpha) = j``\mathcal{E}_\alpha$. Choose some $A \in \mathcal{F}_\gamma$, and observe that $A \in M[G * H * I * J]$ because $K$ is generic over $M[G * H * I * J]$ for $(j(\lambda^+), \infty)$-distributive forcing; from now on we will work in the model $V[G * H * I * J]$. $j``\lambda^+$ is an ω-club subset of γ, so that $\lim(A) \cap j``\lambda^+$ is unbounded in γ. Define $F_0 = \{ \alpha < \lambda^+ : j(\alpha) \in \lim(A) \}$, so that F_0 is unbounded in λ^+.

If $\alpha \in F_0$ then $j(\alpha) \in \lim(A)$, so that $A \cap j(\alpha) \in j``\mathcal{E}_\alpha$. For each $\alpha \in F_0$, let $X_\alpha \in \mathcal{E}_\alpha$ be such that $j(X_\alpha) = A \cap j(\alpha)$. By elementarity, if $\alpha < \beta$ are two points in F_0 then $\alpha \in \lim(X_\beta)$ and $X_\alpha = X_\beta \cap \alpha$. If $F = \bigcup_{\alpha \in F_0} X_\alpha$ then it is routine to check that F is club and $\forall \alpha \in \lim(F)$ $F \cap \alpha \in \mathcal{E}_\alpha$. $\square$

We denote $j(\mathbb{P})/G * H * I$ by $\mathbb{S}$, so that $I * J$ is $\mathbb{R} * \mathbb{S}$-generic over $V[G * H]$ and $\mathbb{S}$ is ρ-closed.

Claim. $F \in V[G * H * I]$.

Proof. Suppose not. Then $F = \dot{F}^J$ for some $\mathbb{S}$-name $\dot{F}$ such that

$$\Vdash^{V[G*H*I]}_{\mathbb{S}} \text{“}\dot{F} \text{ is not in the ground model”}.$$

We work in $V[G * H * I]$.

If μ is infinite, let σ be the least cardinal such that $2^\sigma > \mu$. If μ is finite let $\sigma = \omega$. In either case:

1. σ is an infinite cardinal, with $\sigma < \rho$.
2. $\forall \tau < \sigma$ $2^\tau < \rho$.
3. $2^\sigma > \mu$.

We will build $\langle p_t : t \in {}^{\leq\sigma}2\rangle$ a tree of conditions in $\mathbb{S}$ and a continuous increasing sequence $\langle \alpha_i : i \leq \sigma\rangle$ of ordinals less than λ^+ such that

1. If u extends t then $p_u \leq p_t$.
2. If $i < \sigma$ then for every $t \in {}^i2$ the conditions $p_{t\frown 0}$ and $p_{t\frown 1}$ force incompatible information about $\dot{F}\cap[\alpha_i, \alpha_{i+1})$, and they both force that $\dot{F} \cap [\alpha_i, \alpha_{i+1}) \neq \emptyset$.

The key points are that $\mathbb{S}$ is ρ-closed and $V[G * H * I] \models \mathrm{cf}(\lambda^+) = \rho$. If $\tau \leq \sigma$ is a limit ordinal then we set $\alpha_\tau = \sup_{i<\tau} \alpha_i$, and for each $t \in {}^\tau 2$ we choose p_t to be a lower bound for the decreasing τ-sequence $\langle p_{t\restriction i} : i < \tau\rangle$. Suppose now that $\tau < \sigma$ and we have defined $\{ p_t \ : \ t \in {}^\tau 2 \}$ and α_τ; we choose $p_{t\frown i}$, γ_t, δ^i_t so that

1. $\alpha_\tau < \gamma_t < \delta^i_t < \lambda^+$.
2. $p_{t\frown i} \leq p_t$.
3. $p_{t\frown 0}$ and $p_{t\frown 1}$ decide the statement "$\gamma_t \in \dot{F}$" in different ways.
4. $p_{t\frown i} \Vdash \delta^i_t \in \dot{F}$.

Now we let $\alpha_{i+1} = \sup_{t\in{}^\tau 2, i<2} \delta^i_t + 1$.

We have constructed a family $\{ p_t \ : \ t \in {}^\sigma 2 \}$ of 2^σ conditions in $\mathbb{S}$. For each $t \in {}^\sigma 2$, p_t forces that $\alpha_\sigma \in \lim(\dot{F})$, so p_t forces that $\dot{F} \cap \alpha_\sigma \in \mathcal{E}_{\alpha_\sigma}$; we extend p_t to p^*_t which decides the value of $\dot{F} \cap \alpha_\sigma$, say $p^*_t \Vdash \dot{F} \cap \alpha_\sigma = C_t$. By construction the function which takes t to C_t is injective, but this is impossible because $2^\sigma > \mu \geq |\mathcal{E}_{\alpha_\sigma}|$. □

Claim. $F \in V[G * H]$.

Proof. The proof is quite similar to that of the last claim; the main difference is that $\mathbb{R}$ is not ρ-closed, so we work with $\mathbb{Q} * \mathbb{R}$ instead. Since $F \in V[G * H * I]$, we may fix some $\mathbb{Q} * \mathbb{R}$-name $\dot{f} \in V[G]$ such that $\dot{f}^{H*I} = F$ and

$$\Vdash^{V[G]}_{\mathbb{Q}*\mathbb{R}} \dot{f}^{G_{\mathbb{Q}}*G_{\mathbb{R}}} \notin V[G][G_{\mathbb{R}}].$$

Subclaim. *For all $(q, r) \in \mathbb{Q} * \mathbb{R}$ there exist $\bar{q} \leq q$, r_0, r_1 and γ such that*

1. $(\bar{q}, r_0), (\bar{q}, r_1) \leq (q, r)$
2. *The conditions $(\bar{q}, r_0)$, and $(\bar{q}, r_1)$ decide the statement $\check{\gamma} \in \dot{f}$ in different ways.*

Proof. Suppose that the subclaim fails. Working in $V[G]$, define a $\mathbb{Q}$-name $\dot{f}_0$ such that for all $\bar{q} \leq q$ and all γ,

$$\bar{q} \Vdash^{V[G]}_{\mathbb{Q}} \check{\gamma} \in \dot{f}_0 \iff \exists \bar{r}\ (\bar{q}, \bar{r}) \leq (\bar{q}, r), (\bar{q}, \bar{r}) \Vdash^{V[G]}_{\mathbb{Q}*\mathbb{R}} \check{\gamma} \in \dot{f}.$$

It is now routine to check that $(q, r) \Vdash \dot{f} = \dot{f}_0$, contradicting the choice of $\dot{f}$. □

We now distinguish between the cases where μ is finite and μ is infinite.

Case 1. μ is infinite: Let σ be as in the proof of the last claim. Working in $V[G]$ we will define $\langle r_t : t \in {}^{<\sigma}2 \rangle$, $\langle q_i : i < \sigma \rangle$ and $\langle \alpha_i : i < \sigma \rangle$ such that

- $(q_{\langle\rangle}, r_{\langle\rangle}) \Vdash \forall \alpha \in \lim(\dot{f})\ \dot{f} \cap \alpha \in \mathcal{E}_\alpha$.
- For each t, $(q_{\mathrm{lh}(t)}, r_t) \in \mathbb{Q}*\mathbb{R}$ and $\max \mathrm{dom}(q_{\mathrm{lh}(t)}) = \max r_t = \alpha_{\mathrm{lh}(t)}$.
- If u extends t then $(q_{\mathrm{lh}(u)}, r_u) \leq (q_{\mathrm{lh}(t)}, r_t)$.
- If $i < \sigma$ then for every $t \in {}^i 2$ the conditions $(q_{\mathrm{lh}(t)+1}, r_{t \frown 0})$ and $(q_{\mathrm{lh}(t)+1}, r_{t \frown 1})$ force incompatible information about $\dot{f} \cap [\alpha_i, \alpha_{i+1})$, and both force that $\dot{f} \cap [\alpha_i, \alpha_{i+1}) \neq \emptyset$.

Successor stages: Suppose $\tau < \sigma$ and we have defined the sequences $\langle r_t : t \in {}^{\leq\tau}2 \rangle$, $\langle q_i : i \leq \tau \rangle$ and $\langle \alpha_i : i \leq \tau \rangle$. We observe that $2^\tau < \lambda$. Enumerate ${}^\tau 2$ as $\langle s_j : j < 2^\tau \rangle$. We will build conditions $q^\tau_j \in \mathbb{Q}$ and $r^*_{s_j \frown 0}, r^*_{s_j \frown 1} \in \mathbb{R}$ such that

1. $\langle q^\tau_j : j < 2^\tau \rangle$ is decreasing and has a lower bound (this is possible by the strategic closure of $\mathbb{Q}$).
2. $(q^\tau_j, r^*_{s_j \frown 0})$ and $(q^\tau_j, r^*_{s_j \frown 1})$ force incompatible information about $\dot{f} \cap [\alpha_\tau, \max \mathrm{dom}(q_j))$, and also force that $\dot{f} \cap [\alpha_\tau, \max \mathrm{dom}(q_j)) \neq \emptyset$.

We then let $q_{\tau+1}$ be a lower bound for $\langle q^\tau_j : j < 2^\tau \rangle$, let $\alpha_{\tau+1} = \max \mathrm{dom}(q_{\tau+1})$, and $r_{s_j \frown k} = r^*_{s_j \frown k} \cup \{\alpha_{\tau+1}\}$.

Limit stages: Suppose that $\tau < \sigma$ is limit and we have defined the sequences $\langle r_t : t \in {}^{<\tau}2 \rangle$, $\langle q_i : i < \tau \rangle$ and $\langle \alpha_i : i < \tau \rangle$. Let $\alpha_\tau = \sup_i \alpha_i$. For each sequence $t \in {}^\tau 2$ define $r_t = \bigcup_{i<\tau} r_{t \restriction i} \cup \{\alpha_\tau\}$, and let $F_\tau = \{\ r_t \cap \alpha_\tau\ :\ t \in {}^\tau 2\ \}$. $|F_\tau| = 2^\tau < \mu$ by the minimal choice of σ, so that we may set $q_\tau = \bigcup_{i<\tau} q_i \cup \{(\alpha_\tau, F_\tau)\}$.

After σ steps we choose $B \subseteq {}^\sigma 2$ such that $|B| = \mu^+$, and let $\alpha_\sigma = \sup_{i<\sigma} \alpha_i$. For each $t \in B$ let $r_t = \bigcup_{i<\sigma} r_{t \restriction i} \cup \{\alpha_\sigma\}$, and define $q_\sigma = \bigcup_{i<\sigma} q_i \cup \{(\alpha_\sigma, \{\ r_t \cap \alpha_\sigma\ :\ t \in B\ \})\}$. q_σ is a legitimate condition because $|B| = \mu^+ \leq \nu$, and what is more $(q_\sigma, r_t) \in \mathbb{Q} * \mathbb{R}$ for all $t \in B$.

We may now extend q_σ to a condition q^* which decides the set $\mathcal{E}_{\alpha_\sigma}$, say as some family E of club subsets of α_σ where $|E| \leq \mu$. This is a contradiction, because for every $t \in B$ we may extend (q^*, r_t) to decide $\dot{f} \cap \alpha_\sigma$, and distinct values of t must give distinct elements of E.

Case 2. μ is finite: In this case we fix (q, r) such that $(q, r) \Vdash \forall \alpha \in \lim(\dot{f})\ \dot{f} \cap \alpha \in \mathcal{E}_\alpha$. We then fix $\bar{q} \leq q$ and $r_0, \dots r_\mu$ and an ordinal α such that

1. $(\bar{q}, r_i) \leq (q, r)$.
2. The conditions $(\bar{q}, r_i)$ force incompatible information about $\dot{f} \cap \alpha$.

We now construct $\bar{q}^i_j$, r^i_j, α^i_j such that

1. $\bar{q} \geq \bar{q}^0_0 \geq \bar{q}^0_1 \geq \ldots \geq \bar{q}^0_\mu \geq \bar{q}^1_0 \geq \ldots \bar{q}^1_\mu \geq \bar{q}^2_0 \geq \ldots$
2. $\alpha < \alpha^0_0 < \alpha^0_1 < \ldots \alpha^0_\mu < \alpha^1_0 < \ldots < \alpha^1_\mu < \alpha^2_0 < \ldots$
3. For each $i \leq \mu$, $(\bar{q}, r_i) \geq (\bar{q}^0_i, r^0_i) \geq (\bar{q}^1_i, r^1_i) \geq (\bar{q}^2_i, r^2_i) \geq \ldots$
4. For each $i \leq \mu$ and $j < \omega$, $(\bar{q}^i_j, r^i_j) \Vdash \alpha^i_j \in \dot{f}$.

We then define $\alpha^* = \sup_{i,j} \alpha_{i,j}$, $r^*_j = \bigcup_i r^i_j \cup \{\alpha^*\}$, and $q^* = \bigcup_{i,j} q^i_j \cup \{r^*_j \cap \alpha^* : j \leq \mu\}$. $q^* \in \mathbb{Q}$ and $(q^*, r_j) \in \mathbb{Q} * \mathbb{R}$ for all $j \leq \mu$, and so we may derive a contradiction exactly as in the case where μ is infinite.

This concludes the proof that $F \in V[G * H]$. □

We have now arrived at a contradiction. Since λ^+ is regular in $V[G * H]$, $\mathrm{ot}(F) = \lambda^+$. Therefore we may find $\alpha \in \lim(F)$ such that $\mathrm{ot}(F \cap \alpha) > \lambda$. But $F \cap \alpha \in \mathcal{E}_\alpha$ and $\mathcal{E}_\alpha$ is a set of club sets which all have order type less than or equal to λ.

It follows that the principle $\square_{\lambda,\mu}$ fails in $V[G * H]$. This concludes the proof of Theorem 13. □

8. Adding square by small forcing

In this section we prove that it is sometimes possible to do small forcing over a model in which $\square_\kappa$ (or even in some cases $\square^*_\kappa$) fails and obtain a model in which $\square_\kappa$ holds.

Theorem 14. *Let κ be supercompact. Then there is a generic extension W such that*

1. $\kappa = \aleph_2^W$.
2. $\square^*_{\aleph_\omega}$ *fails in* W.
3. *If H is* $\mathrm{Coll}(\omega, \omega_1)$*-generic over W then $\square_{\aleph_\omega}$ holds in $W[H]$.*

Proof. By unpublished work of Baumgartner [50], we may force to add a sequence $\langle C_\alpha : \alpha \in S \rangle$ such that

1. $\kappa^{+\omega+1} \cap \mathrm{cof}(\geq \kappa) \subseteq S \subseteq \kappa^{+\omega+1}$.
2. For all $\alpha \in S$
   (a) $\mathrm{ot}(C_\alpha) \leq \kappa^{+\omega}$.
   (b) For all $\beta \in \lim(C_\alpha)$, $\beta \in S$ and $C_\beta = C_\alpha \cap \beta$.

preserving all cardinals up to $\kappa^{+\omega+1}$ and the supercompactness of κ. See [50] or [1] for details.

If we then force with $\mathrm{Coll}(\omega_1, < \kappa)$, we obtain a model W in which $\square^*_{\aleph_\omega}$ fails. Let H be $\mathrm{Coll}(\omega, \omega_1)$-generic over W and observe that if $\alpha \in \aleph_{\omega+1} \setminus S$ then $W[H] \models \mathrm{cf}(\alpha) = \omega$. We may therefore define a $\square_{\aleph_\omega}$-sequence in $W[H]$ by setting $D_\alpha = C_\alpha$ if $\alpha \in S$, and letting D_α be any set of order type ω cofinal in α when $\alpha \notin S$. □

Remark 8.1. A similar issue arises in [36]. There the problem is that forcing over a model which has no $\aleph_{\omega+\omega+1}$-Aronszajn tree with the poset $\mathrm{Coll}(\aleph_0, \aleph_\omega)$ may *a priori* give a model which has an $\aleph_{\omega+1}$-Aronszajn tree.

A similar argument shows that for λ regular we can create a situation in which $\square_\lambda$ fails, and forcing with $\mathrm{Coll}(\omega, \omega_1)$ makes $\square_\lambda$ hold. However, in general it may not be possible to force $\square_\lambda$ with mild forcing if we demand that $\square^*_\lambda$ should fail in the ground model.

Theorem 15. *Let $1 \leq n < \omega$ and let $\lambda = \aleph_n$. Let $\mathbb{P}$ be λ-c.c. and suppose that $\Vdash_{\mathbb{P}}$ "$\square_\lambda$ holds". Then $\square^*_\lambda$ holds in V.*

Proof. By results of Shelah [45] if $T = \{ \alpha < \lambda^+ \ : \ \mathrm{cf}(\alpha) < \lambda \}$ then there exists $\langle \mathcal{D}_\alpha : \alpha \in T \rangle$ such that

1. $\mathcal{D}_\alpha$ is a nonempty family of size at most λ of club subsets of α with order type less than λ.
2. If $D \in \mathcal{D}_\alpha$ and $\beta \in \lim(D)$ then $D \cap \beta \in \mathcal{D}_\beta$.

Let $\langle \dot{C}_\alpha : \alpha < \lambda^+ \rangle$ name a $\square_\lambda$-sequence. If $\alpha < \lambda^+$ has cofinality λ then $\dot{C}_\alpha$ names a club subset of α with order type λ, and so since $\mathbb{P}$ is λ-c.c. it is easy to see that

$$C^*_\alpha = \{ \gamma < \alpha \ : \Vdash_{\mathbb{P}} \gamma \in \dot{C}_\alpha \}$$

is a club subset of α with order type λ.

If $\gamma \in \lim(C^*_\alpha)$ then $\Vdash_{\mathbb{P}} \gamma \in \lim(\dot{C}_\alpha)$, and so $\Vdash_{\mathbb{P}} \dot{C}_\gamma = \dot{C}_\alpha \cap \gamma$ and

$$C^*_\alpha \cap \gamma = \{ \delta < \gamma \ : \Vdash_{\mathbb{P}} \delta \in \dot{C}_\gamma \}.$$

Let

$$S = \{ \gamma < \lambda^+ \ : \ \exists \alpha \, \gamma \in \lim(C^*_\alpha) \},$$

and for $\gamma \in S$ let

$$C^*_\gamma = \{ \delta < \gamma \ : \Vdash_{\mathbb{P}} \delta \in \dot{C}_\gamma \}.$$

Now let

$$\mathcal{E}_\alpha = \begin{cases} \{C^*_\alpha\} & \text{if } \mathrm{cf}(\alpha) = \lambda \\ \mathcal{D}_\alpha & \text{if } \mathrm{cf}(\alpha) < \lambda, \, \alpha \notin S \\ \mathcal{D}_\alpha \cup \{C^*_\alpha\} & \text{if } \mathrm{cf}(\alpha) < \lambda, \, \alpha \in S \end{cases}$$

It is routine to verify that $\vec{\mathcal{E}}$ is a $\square^*_\lambda$-sequence. $\square$

9. Square above a supercompact cardinal

As we mentioned in Section 5, the existence of a supercompact cardinal puts some constraints on the possibilities for the existence of square sequences. To be more precise if κ is supercompact and $\lambda \geq \kappa$ then

- $\square_{\lambda,<\mathrm{cf}(\lambda)}$ fails.
- If $\mathrm{cf}(\lambda) < \kappa$ then $\square^*_\lambda$ fails.

Here we will show that if κ is supercompact and $\kappa \leq \mathrm{cf}(\lambda) < \lambda$ then $\square_{\lambda,\mathrm{cf}(\lambda)}$ can hold.

Theorem 16. *Let λ be a singular cardinal. Then there exists a forcing poset $\mathbb{P}$ such that*

1. $\mathbb{P}$ *is* $\mathrm{cf}(\lambda)$*-directed closed.*
2. $\mathbb{P}$ *is* $<\lambda$*-strategically closed.*
3. $\Vdash_{\mathbb{P}}$ *"*$\square_{\lambda,\mathrm{cf}(\lambda)}$ *holds"*

Proof. Let $\mathrm{cf}(\lambda) = \mu$ and fix $\langle \lambda_i : i < \mu \rangle$ an increasing sequence of regular cardinals such that $\mu < \lambda_0$ and $\sup_i \lambda_i = \lambda$.

Conditions in $\mathbb{P}$ have the form

$$p = \langle C_{\alpha,i} : \lim(\alpha), \alpha \leq \gamma, i(\alpha) \leq i < \mu \rangle,$$

where

1. γ is a limit ordinal less than λ^+. We refer to γ as the *length* of p and write $\gamma = \mathrm{lh}(p)$.
2. $i(\alpha) < \mu$ for each limit $\alpha \leq \gamma$.
3. If $i(\alpha) \leq i < \mu$ then $C_{\alpha,i}$ is club in α and $\mathrm{ot}(C_{\alpha,i}) < \lambda_i$.
4. If $i(\alpha) \leq i < j < \mu$ then $C_{\alpha,i} \subseteq C_{\alpha,j}$.
5. If $i(\beta) \leq i < \mu$ and $\alpha \in \lim(C_{\beta,i})$ then $i(\alpha) \leq i$ and $C_{\alpha,i} = C_{\beta,i} \cap \alpha$.
6. If α and β are limit ordinals with $\alpha < \beta \leq \gamma$ then $\alpha \in \lim(C_{\beta,i})$ for all sufficiently large $i < \mu$.

If

$$\begin{aligned} p &= \langle C^p_{\alpha,i} : \alpha \leq \gamma_p, i_p(\alpha) \leq i < \mu \rangle \\ q &= \langle C^q_{\alpha,i} : \alpha \leq \gamma_q, i_q(\alpha) \leq i < \mu \rangle \end{aligned}$$

then $q \leq p$ iff

1. $\gamma_p \leq \gamma_q$.
2. For all limit $\alpha \leq \gamma_p$, $i_p(\alpha) = i_q(\alpha)$ and $C^p_{\alpha,i} = C^q_{\alpha,i}$ for all i with $i_p(\alpha) \leq i < \mu$.

Claim. $\mathbb{P}$ *is* μ*-directed closed.*

Proof. The ordering on $\mathbb{P}$ is *treelike*, that is to say that if $p \leq q, r$ then q is comparable with r. So the directed subsets of $\mathbb{P}$ are exactly the linearly ordered ones and it suffices to show that $\mathbb{P}$ is μ-closed.

Let $\rho < \mu$ and let $\langle p_\sigma : \sigma < \rho \rangle$ be a strictly decreasing sequence of conditions with $\gamma(\sigma) =_{\text{def}} \text{lh}(p_\sigma)$. Let $\gamma =_{\text{def}} \sup_\sigma \gamma(\sigma)$.

By the definition of the ordering on $\mathbb{P}$, the union of the p_σ for $\sigma < \rho$ is a matrix of sets $\langle C_{\alpha,i} : \alpha < \gamma, i(\alpha) \leq i < \mu \rangle$. We need to define $i(\gamma)$ and $\langle C_{\gamma,i} : i(\gamma) \leq i < \mu \rangle$ in such a way that $\langle C_{\alpha,i} : \alpha \leq \gamma, i(\alpha) \leq i < \mu \rangle$ is a condition in $\mathbb{P}$.

For each pair of ordinals (σ, τ) with $\sigma < \tau < \rho$, let $f(\sigma, \tau)$ be the least $i < \mu$ such that $\gamma(\sigma) \in \lim(C_{\gamma(\tau),i})$. Such an i must exist by clause 6 in the definition of $\mathbb{P}$.

Observe that if $f(\sigma, \tau) \leq i < \mu$ then $\gamma(\sigma) \in \lim(C_{\gamma(\tau),i})$ and so $C_{\gamma(\sigma),i} = C_{\gamma(\tau),i} \cap \gamma(\sigma)$. We define $i(\gamma) =_{\text{def}} \sup_{\sigma,\tau} f(\sigma,\tau)$, where $i(\gamma) < \mu$ because $\rho < \mu = \text{cf}(\mu)$. We let $C_{\gamma,i} =_{\text{def}} \bigcup_{\sigma<\rho} C_{\gamma(\sigma),i}$ for all $i \geq i(\gamma)$.

It is now routine to check that $\langle C_{\alpha,i} : \alpha \leq \gamma, i(\alpha) \leq i < \mu \rangle$ is a condition. The key points are that

- $\rho < \mu < \lambda_0$ so that $\text{ot}(C_{\gamma,i}) < \lambda_i$.
- The choice of $i(\gamma)$ guarantees that $C_{\gamma,i} \cap \gamma(\sigma) = C_{\gamma(\sigma),i}$ for all $\sigma < \rho$ and $i \geq i(\gamma)$.

□

Claim. $\mathbb{P}$ *is* $< \lambda$*-strategically closed.*

Proof. For a fixed $\theta < \lambda$, we describe a winning strategy for player II in $G^{II}_{\theta+1}(\mathbb{P})$. We begin by fixing $j < \mu$ such that $\theta < \lambda_j$. We will denote the condition played at move ρ in the game by

$$p_\rho = \langle C_{\alpha,i} : \alpha \leq \gamma(\rho), i(\alpha) \leq i < \mu \rangle.$$

Player II will play in such a way that

- $i(\gamma(2\alpha)) = j$ for all α.
- If $\alpha < \beta$ and $i \geq j$ then $C_{\gamma(2\alpha),i} = C_{\gamma(2\beta),i} \cap \gamma(2\alpha)$.

Successor stages: Suppose that player I plays $p_{2\alpha+1}$. Player II responds with $p_{2\alpha+2}$ where

1. $\gamma(2\alpha + 2) = \gamma(2\alpha + 1) + \omega$, $i(\gamma(2\alpha + 2)) = j$.
2. If $i \geq j$ and $\gamma(2\alpha) \in \lim(C_{\gamma(2\alpha+1),i})$ then $C_{\gamma(2\alpha+2),i} = C_{\gamma(2\alpha+1),i} \cup \{ \gamma(2\alpha + 1) + n : n < \omega \}$. Notice that this case will apply for all sufficiently large $i < \mu$.
3. If $\gamma(2\alpha) \notin \lim(C_{\gamma(2\alpha+1),i})$ then $C_{\gamma(2\alpha+2),i} = C_{\gamma(2\alpha),i} \cup \{\gamma(2\alpha)\} \cup \{ \gamma(2\alpha + 1) + n : n < \omega \}$.

We have arranged matters so that $C_{\gamma(2\alpha+2),i}$ increases with i, always end-extends $C_{\gamma(2\alpha),i}$ and end-extends $C_{\gamma(2\alpha+1),i}$ for all large i. Using this it is routine to check that $p(2\alpha+2)$ is a condition.

Limit stages: Let $\delta \leq \theta$ be a limit stage in the game. Let $\gamma(\delta) = \sup_{\alpha<\delta} \gamma(\alpha)$, $i(\gamma(\delta)) = j$, and $C_{\gamma(\delta),i} = \bigcup_{2\alpha<\delta} C_{\gamma(2\alpha),i}$ for $i \geq j$. Player II played at the even stages 2α preceding δ in such a way that $C_{\gamma(\delta),i} \cap \gamma(2\alpha) = C_{\gamma(2\alpha),i}$. Using this it is routine to check that p_δ is a condition. □

Since $\mathbb{P}$ is $< \lambda$-strategically closed, forcing with $\mathbb{P}$ adds no $< \lambda$-sequence of ordinals. Since λ is singular, forcing with $\mathbb{P}$ adds no λ-sequences of ordinals, and so preserves all cardinals and cofinalities up to and including λ^+.

Essentially the same argument as for strategic closure shows that for all $\gamma < \lambda^+$ the set $\{ p \in \mathbb{P} : \mathrm{lh}(p) > \gamma \}$ is dense in $\mathbb{P}$. So the generic object is a $\square_{\lambda,\mathrm{cf}(\lambda)}$-sequence in $V^{\mathbb{P}}$. □

The following is an immediate consequence of Theorem 16.

Theorem 17. *Let κ be a Laver indestructible supercompact cardinal and let $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. Then there is a forcing extension in which κ is still supercompact, cardinals and cofinalities up to λ^+ are preserved, and $\square_{\lambda,\mathrm{cf}(\lambda)}$ holds.*

Remark 9.1. In the model of Theorem 17

- $\square^{\mathrm{imp}}_{\lambda,<\lambda}$ fails as a result of Theorem 12.
- If $S = \lambda^+ \cap \mathrm{cof}(< \kappa)$ then the supercompactness of κ implies that any family of size less than κ of stationary subsets of S reflects simultaneously. Theorem 5 implies that there are $\mathrm{cf}(\lambda)$ subsets of S which do not reflect simultaneously.

We observe that in Theorem 16 we have actually forced a principle which is (formally) stronger than $\square_{\lambda,\mathrm{cf}(\lambda)}$.

Definition 9.2. A $\square^{\mathrm{ind}}_{\lambda,\mathrm{cf}(\lambda)}$*-sequence (indexed $\square_{\lambda,\mathrm{cf}(\lambda)}$ sequence)* is a matrix of sets $\langle C_{\alpha,i} : \alpha < \lambda^+, i(\alpha) \leq i < \mathrm{cf}(\lambda)\rangle$ such that for some increasing sequence $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ of regular cardinals with limit λ

1. $i(\alpha) < \mathrm{cf}(\lambda)$ for all $\alpha < \lambda^+$.
2. $\mathrm{ot}(C_{\alpha,i}) < \lambda_i$ for all α.
3. $C_{\alpha,i}$ is club in α.
4. If $i(\alpha) \leq i < j < \mathrm{cf}(\lambda)$ then $C_{\alpha,i} \subseteq C_{\alpha,j}$.
5. If $i(\beta) \leq i < \mathrm{cf}(\lambda)$ and $\alpha \in \lim(C_{\beta,i})$ then $i(\alpha) \leq i$ and $C_{\alpha,i} = C_{\beta,i} \cap \alpha$.
6. If α and β are limit ordinals with $\alpha < \beta < \lambda^+$ then $\alpha \in \lim(C_{\beta,i})$ for all sufficiently large $i < \mu$.

Remark 9.3. Shelah [44] introduces a combinatorial principle S_λ and shows that S_λ is equivalent to the model-theoretic transfer principle $(\aleph_1, \aleph_0) \longrightarrow (\lambda^+, \lambda)$. If $\langle C_{\alpha,i} : \alpha < \lambda^+, i(\alpha) \leq i < \mathrm{cf}(\lambda) \rangle$ is a $\Box^{\mathrm{ind}}_{\lambda,\mathrm{cf}(\lambda)}$-sequence, and we set $S_{\alpha,i} = \{ \gamma < \alpha \, : \, \omega\gamma \in \lim(C_{\omega\alpha,i}) \}$ for $i \geq i(\alpha)$ and $S_{\alpha,i} = \emptyset$ otherwise, then $\langle S_{\alpha,i} : \alpha < \lambda^+, i < \mathrm{cf}(\lambda) \rangle$ is a S_λ-sequence.

Remark 9.4. Cummings and Schimmerling [7] have studied a number of variations on the concept of an indexed square sequence.

For use in the next section we develop a theory of "threading" appropriate for generic $\Box^{\mathrm{ind}}_{\lambda,\mathrm{cf}(\lambda)}$-sequences.

We fix a singular cardinal λ, and an increasing sequence $\langle \lambda_i : i < \mu \rangle$ of regular cardinals such that $\mu = \mathrm{cf}(\lambda)$ and $\sup_i \lambda_i = \lambda$. Let $\mathbb{P}$ be the forcing to add a generic $\Box^{\mathrm{ind}}_{\lambda,\mu}$-sequence which was defined in the proof of Theorem 16. Let $\vec{C} = \langle C_{\alpha,i} : \alpha < \lambda^+, i(\alpha) \leq i < \mathrm{cf}(\lambda) \rangle$ be the generic $\Box^{\mathrm{ind}}_{\lambda,\mu}$-sequence added by $\mathbb{P}$. Working in $V^{\mathbb{P}}$ we define a family of forcing posets.

Definition 9.5. Let $j < \mu$. $c \in \mathbb{Q}_j(\vec{C})$ iff

1. c is a closed bounded subset of λ^+.
2. If $\gamma = \max(c)$ then $i(\gamma) \leq j$ and $c = C_{\gamma,j} \cup \{\gamma\}$.

$\mathbb{Q}_j(\vec{C})$ is ordered by end-extension.

It is worthwhile to notice that each condition in $\mathbb{Q}_j(\vec{C})$ lies in V. The following result is analogous to Lemma 6.9.

Lemma 9.6. *$\mathbb{P} * \mathbb{Q}_j(\vec{C})$ has a λ_j-directed closed dense subset.*

Proof. It will suffice to produce a dense λ_j-closed subset on which the ordering is treelike. Let D be the set of conditions of the form $(p, \check{c})$ where $\mathrm{lh}(p) = \max(c)$.

If $p = \langle C_{\alpha,i} : \alpha \leq \gamma, i(\alpha) \leq i < \mu \rangle$ and $(p, \check{c}) \in D$ then $\gamma = \max(c)$, $i(\gamma) \leq j$ and $c = C_{\gamma,j} \cup \{\gamma\}$.

Claim. *D is dense in $\mathbb{P} * \mathbb{Q}_j(\vec{C})$.*

Proof. Let $(p, \dot{c}) \in \mathbb{P} * \mathbb{Q}_j(\vec{C})$. Extending p if necessary, we may assume that p decides $\dot{c}$ (say $p \Vdash \dot{c} = \check{c}$) and also that $\mathrm{lh}(p) > \max(c)$. Let $p = \langle C_{\alpha,i} : \alpha \leq \gamma, i(\alpha) \leq i < \mu \rangle$. Let $\delta = \max(c)$, so that $c = C_{\delta,j} \cup \{\delta\}$.

We will define a condition $q = \langle C_{\alpha,i} : \alpha \leq \gamma + \omega, i(\alpha) \leq i < \mu \rangle$ of length $\gamma + \omega$ extending p, with $i(\gamma + \omega) = j$.

Let k be minimal such that $k > j$ and $\delta \in \lim(C_{\gamma,k})$. We define

$$C_{\gamma+\omega,i} = C_{\delta,i} \cup \{\delta\} \cup \{\gamma + n : n < \omega\}$$

for $j \leq i < k$, and

$$C_{\gamma+\omega,i} = C_{\gamma,i} \cup \{\gamma + n : n < \omega\}$$

for $k \leq i < \mu$.

Now let $d = C_{\gamma+\omega,j} \cup \{\gamma + \omega\}$. $(q, \check{d}) \in D$ and $(q, \check{d}) \leq (p, \dot{c})$ so that D is dense. □

Claim. *D is λ_j-closed.*

Proof. Let $\rho < \lambda_j$ and let $\langle (p_\sigma, c_\sigma) : \sigma < \rho \rangle$ be a strictly decreasing sequence of conditions in D, where $\mathrm{lh}(p_\sigma) = \gamma(\sigma)$. Let $\gamma = \sup_\sigma \gamma(\sigma)$, and let the union of the sequence $\langle p_\sigma : \sigma < \rho \rangle$ be the matrix of sets $\langle C_{\alpha,i} : \alpha < \gamma, i(\alpha) \leq i < \mu \rangle$. Let $c = \bigcup_\sigma c_\sigma$.

For each $\sigma < \rho$, $\gamma(\sigma) \in \lim(c)$ and $c \cap \gamma(\sigma) = C_{\gamma(\sigma),j}$. It follows that if $\sigma < \tau < \rho$ and $i \geq j$ then $\gamma(\sigma) \in \lim(C_{\gamma(\tau),i})$, and so $C_{\gamma(\tau),i} \cap \gamma(\sigma) = C_{\gamma(\sigma),i}$. We now define $p = \langle C_{\alpha,i} : \alpha \leq \gamma, i(\alpha) \leq i < \mu \rangle$ by setting $i(\gamma) = j$ and $C_{\gamma,i} = \bigcup_{\sigma<\rho} C_{\gamma(\sigma),i}$ for all $i \geq j$; in particular, $C_{\gamma,j} = c$.

Finally we let $d = c \cup \{\gamma\}$. $(p, d) \in D$ and $(p, d) \leq (p_\sigma, c_\sigma)$ for all $\sigma < \rho$, so that D is λ_j-closed. □

This concludes the proof that $\mathbb{P} * \mathbb{Q}_j(\vec{C})$ has a λ_j-closed subset. □

We conclude this section by proving that if $i < j$ then in $V^{\mathbb{P}}$ there is natural projection from $\mathbb{Q}_i$ to $\mathbb{Q}_j$.

Definition 9.7. In $V^{\mathbb{P}}$ define a map $\pi_{ij} : \mathbb{Q}_i(\vec{C}) \longrightarrow \mathbb{Q}_j(\vec{C})$ by $\pi_{ij}(c) = C_{\max(c),j} \cup \{\max(c)\}$.

Lemma 9.8. *In $V^{\mathbb{P}}$*

1. *π_{ij} is a projection from $\mathbb{Q}_i$ to $\mathbb{Q}_j$.*
2. *If $i < j < k$ then $\pi_{ik} = \pi_{jk} \circ \pi_{ij}$.*
3. *If $(p, c) \in D$ then p determines $\pi_{ij}(c)$ for all $j \geq i$.*

Proof. We prove that π_{ij} is a projection, the remaining claims are immediate. Let p be a condition which forces that $d \in \mathbb{Q}_j$, $c \in \mathbb{Q}_i$ and $d \leq \pi_{ij}(c)$. Let $\max(c) = \delta$, $\max(d) = \gamma$. We may as well assume that $\mathrm{lh}(p) = \gamma$. If $p = \langle C_{\alpha,k} : \alpha \leq \gamma, i(\alpha) \leq k < \mu \rangle$ then we have $c = C_{\delta,i} \cup \{\delta\}$, $d = C_{\gamma,j} \cup \{\gamma\}$, and $\delta \in \lim(C_{\gamma,j})$.

Arguing exactly as in the proof of Lemma 9.6 we may build $q \leq p$ of height $\gamma + \omega$ such that $i(\gamma + \omega) = i$, $C_{\gamma+\omega,i} \cap \delta = c$ and $C_{\gamma+\omega,j} \cap \gamma = d$. If $c^* = C_{\gamma+\omega,i} \cup \{\gamma + \omega\}$ then q forces that $c^* \in \mathbb{Q}_i$, $c^* \leq c$ and $\pi_{ij}(c^*) \leq d$. □

10. Square and reflection

Theorem 2 shows that if $\square_{\kappa,<\omega}$ holds then every stationary subset of κ^+ has a non-reflecting stationary subset. Here we show it is consistent with $\square^{\mathrm{ind}}_{\aleph_\omega,\omega}$ that every finite family of stationary subsets of $\aleph_{\omega+1}$ should reflect simultaneously. This result is close to optimal, because it follows from Theorem 5 that if $\square_{\aleph_\omega,\omega}$ holds then any stationary subset of $\aleph_{\omega+1}$ has ω stationary subsets which do not reflect simultaneously.

Remark 10.1. We have presented the case of $\aleph_\omega$ here but the idea of the proof works in more general situations.

Theorem 18. *Let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of supercompact cardinals. Let $\kappa = \sup_n \kappa_n$. Then there is a generic extension in which*

1. *$\kappa = \aleph_\omega$, and κ^+ is preserved.*
2. *$\square^{\mathrm{ind}}_{\aleph_\omega,\omega}$ holds.*
3. *For every finite set f of stationary subsets of $\aleph_\omega$ there exists $N < \omega$ such that if $N \leq n < \omega$ then there exists α of cofinality $\aleph_n$ such that all sets in f reflect at α.*

Proof. The idea of the proof is to collapse the supercompact cardinals as in section 6.3, force $\square^{\mathrm{ind}}_{\aleph_\omega,\omega}$ as in Theorem 16, and finally to do an iteration destroying certain "bad" stationary sets. Those stationary subsets of $\aleph_{\omega+1}$ which survive the final stage of the construction will be shown to reflect by an argument in the same spirit as that of 6.3.

Let

$$V_0 \models \text{“}\langle \kappa_n : n < \omega \rangle \text{ is an increasing sequence of supercompact cardinals”}.$$

Let $V = V_0^{\mathbb{R}}$ where $\mathbb{R}$ is the iterated Lévy collapse described in section 6.3. In V we have $\kappa_n = \aleph_{n+1}$, $\kappa = \aleph_\omega$, $\kappa^+_{V_0} = \aleph_{\omega+1}$.

Until further notice we will consider V as the ground model. The fact that $\aleph^V_{n+1}$ was supercompact in V_0 will not be important until we come to prove the reflection properties of the final model.

Working in V we will define an iteration $\mathbb{S} * \mathbb{P}$, and our final model will be $V^{\mathbb{S}*\mathbb{P}}$. Let $\mathbb{S}$ be the forcing to add a $\square^{\mathrm{ind}}_{\aleph_\omega,\omega}$-sequence from Theorem 16, where we set $\lambda_n = \aleph_n$ for each $n < \omega$. Let $\mathbb{T}_n = \mathbb{Q}_n(\vec{C})_{V^{\mathbb{P}}}$, where $\vec{C}$ is the generic $\square^{\mathrm{ind}}_{\aleph_\omega,\omega}$-sequence added by $\mathbb{P}$ and $\mathbb{Q}_n(\vec{C})_{V^{\mathbb{P}}}$ is the threading forcing from Definition 9.5.

Working in $V^{\mathbb{S}}$ we will define posets $\langle \mathbb{P}_\alpha : \alpha \leq \aleph_{\omega+2} \rangle$ by induction on α, maintaining a certain inductive hypothesis which will imply that $\mathbb{S} * \mathbb{P}_\alpha$ adds no $\aleph_\omega$-sequences of ordinals. We will also be inductively defining $\dot{S}_\alpha \in V^{\mathbb{S}*\mathbb{P}_\alpha}$ such that $\Vdash$ "$\dot{S}_\alpha$ is a subset of $\aleph_{\omega+1}$".

Conditions in $\mathbb{P}_\alpha$ are functions p such that

1. $\mathrm{dom}(p) \subseteq \alpha$, $|\mathrm{dom}(p)| \leq \aleph_\omega$.
2. If $\beta \in \mathrm{dom}(p)$ then
   (a) $p(\beta)$ is a closed bounded subset of $\aleph_{\omega+1}$.
   (b) $p \upharpoonright \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \cap \dot{S}_\beta = \emptyset$.

If $p, q \in \mathbb{P}_\alpha$ then $p \leq q$ iff

1. $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$.
2. For all $\beta \in \mathrm{dom}(q)$, $p(\beta) \cap (\max q(\beta) + 1) = q(\beta)$.

Remark 10.2. Since $\mathbb{P}_\alpha$ adds no $\aleph_\omega$-sequences, $\mathbb{P}_{\alpha+1}$ is equivalent to the iteration $\mathbb{P}_\alpha * CU(\aleph_{\omega+1} \setminus \dot{S}_\alpha)$ where $CU(\aleph_{\omega+1} \setminus \dot{S}_\alpha)$ is the standard forcing to shoot a closed unbounded subset through $\aleph_{\omega+1} \setminus S_\alpha$.

Remark 10.3. Similar iterations appear in [33, Theorem 1] and [23, Theorem 3].

Remark 10.4. It is easy to see that $\mathbb{P}_{\aleph_{\omega+2}}$ will be $\aleph_{\omega+2}$-c.c. in $V^{\mathbb{S}}$.

Definition 10.5. Let $\alpha \leq \aleph_{\omega+2}$ and suppose that

$$V[\mathbb{S} * \mathbb{P}_\alpha] \models S \subseteq \aleph_{\omega+1}.$$

S is *fragile* iff

$$V[\mathbb{S} * \mathbb{P}_\alpha * \mathbb{T}_n] \models \text{“}S \text{ is non-stationary”}$$

for all $n < \omega$.

We will choose the sequence of names $\dot{S}_\alpha$ so that each one names a fragile set, in such a way that

$$V^{\mathbb{S} * \mathbb{P}_{\aleph_{\omega+2}}} \models \text{“every fragile set is non-stationary”}.$$

This is possible because

$$V^{\mathbb{S}} \models \text{“}2^{\aleph_\omega} = \aleph_{\omega+1} \text{ and } \mathbb{P}_{\aleph_{\omega+2}} \text{ is } \aleph_{\omega+2}\text{-c.c.”}$$

As the construction proceeds we will fix names $\dot{C}^n_\alpha$ such that

$$\Vdash_{\mathbb{S} * \mathbb{P}_\alpha * \mathbb{T}_n} \dot{C}^n_\alpha \cap \dot{S}_\alpha = \emptyset.$$

By an argument like that for Lemma 9.6 It is easy to see that the set of conditions in $\mathbb{S} * \mathbb{P}_\alpha * \mathbb{T}_n$ which are of the form $(s, \check{p}, \check{f})$ and have $\mathrm{lh}(s) = \max(f)$ is dense. We abuse notation by using “$\mathbb{S} * \mathbb{P}_\alpha * \mathbb{T}_n$” to refer to this dense set of conditions.

Definition 10.6. Let A be a set with $|A| \leq \aleph_\omega$. $\vec{P}$ is a *good partition* of A iff $\vec{P} = \langle P_n : n < \omega \rangle$ where the P_n form a partition of A and $|P_n| < \aleph_n$ for all n.

If $\vec{P}$ is a good partition then “$\vec{P} \upharpoonright k = 0$” means that $P_i = \emptyset$ for all $i < k$.

If $\vec{P}$ is a good partition of $A \subseteq \alpha$ and $\beta < \alpha$, then $\vec{P} \upharpoonright\upharpoonright \beta$ will denote the good partition $\vec{Q}$ of $A \cap \beta$ given by $Q_j = P_j \cap \beta$.

Definition 10.7. Let $q = (s, p, f) \in \mathbb{S} * \mathbb{P}_\alpha * \mathbb{T}_n$. q is *correct for* $\vec{P}$ iff

1. $\vec{P}$ is a good partition of $\mathrm{supp}(p)$ and $\vec{P} \upharpoonright n = 0$.
2. For all $k \geq n$ and all $\beta \in P_k$, $(s, p \upharpoonright \beta, \pi_{nk}(f)) \Vdash \max p(\beta) \in \dot{C}^k_\beta$.

Definition 10.8. Let $q = (s, p, f)$ and $q' = (s', p', f')$ be two conditions in $\mathbb{S} * \mathbb{P}_\alpha * \mathbb{T}_n$ Let $\vec{P}$, $\vec{P'}$ be good partitions of $\mathrm{supp}(p)$ and $\mathrm{supp}(p')$ respectively such that $\vec{P} \upharpoonright n = 0$ and $\vec{P'} \upharpoonright n = 0$.

Then $(q', P') \leq (q, P)$ iff

1. $q' \leq q$ in $\mathbb{S} * \mathbb{P}_\alpha * \mathbb{T}_n$.
2. $P'_k \cap \mathrm{supp}(p) = P_k$ for all $k \geq n$.

Lemma 10.9. *Let $\langle (q_i, \vec{P}^i) : i < \mu \rangle$ be such that*

1. $q_i = (s_i, p_i, f_i) \in \mathbb{S} * \mathbb{P}_\alpha * \mathbb{T}_n$.
2. q_i *is correct for* $\vec{P}^i$.
3. $\mu < \aleph_n$.
4. *If* $i < j < \mu$ *then* $(q_j, \vec{P}^j) \leq (q_i, \vec{P}^i)$.

Then there exists $(q, \vec{P})$ such that q is correct for $\vec{P}$ and $(q, \vec{P}) \leq (q_i, \vec{P}^i)$ for all $i < \mu$.

Proof. We are assuming that $\mathrm{lh}(s_i) = \max(f_i)$ for all $i < \mu$, so by Lemma 9.6 there is $(s, f) \in \mathbb{S} * \mathbb{T}_n$ such that $\mathrm{lh}(s) = \max(f)$ and $(s, f) \leq (s_i, f_i)$ for all $i < \mu$.

We will define a condition $p \in \mathbb{P}_\alpha$ with $\mathrm{supp}(p) = \bigcup_i \mathrm{supp}(p_i)$. We define $\vec{P}$ by setting $P_j = \bigcup_{i<\mu} P^i_j$ for all $j < \omega$, where $\vec{P}$ is good because $\mu < \aleph_n$ and $\vec{P} \upharpoonright n = 0$. To define p we will determine $p \upharpoonright \beta$ by induction on β. Limit stages in the construction of p are not problematic because the intended support of p is of cardinality $\aleph_\omega$.

Suppose that $\beta \in \bigcup_i \mathrm{supp}(p_i)$ and we have defined $p \upharpoonright \beta$ so that $(s, p \upharpoonright \beta, f) \leq (s_i, p_i \upharpoonright \beta, f_i)$ for all $i < \mu$. There is a fixed $N < \omega$ such that $\beta \in P^i_N$ for all large $i < \mu$. Since q_i is correct for $\vec{P}^i$,

$$(s_i, p_i \upharpoonright \beta, \pi_{nN}(f_i)) \Vdash \max p_i(\beta) \in C^N_\beta$$

and so

$$(s, p \upharpoonright \beta, \pi_{nN}(f)) \Vdash \max p_i(\beta) \in C^N_\beta.$$

Let $\delta_\beta = \sup_{i<\mu} p_i(\beta)$ and $p(\beta) = \bigcup_{i<\mu} p_i(\beta) \cup \{\delta_\beta\}$. Since $\dot{C}^N_\beta$ is forced to be club,

$$(s, p \upharpoonright \beta, \pi_{nN}(f)) \Vdash \delta_\beta \in C^N_\beta.$$

Claim. $(s, p \upharpoonright \beta + 1) \in \mathbb{S} * \mathbb{P}_{\beta+1}$.

Proof. It suffices to check that $(s, p \upharpoonright \beta) \Vdash p(\beta) \cap \dot{S}_\beta = \emptyset$. Since $(s_i, p_i \upharpoonright \beta) \Vdash p_i(\beta) \cap \dot{S}_\beta = \emptyset$ and $(s, p \upharpoonright \beta) \leq (s_i, p_i \upharpoonright \beta)$, we need only to check $(s, p \upharpoonright \beta) \Vdash \delta_\beta \notin \dot{S}_\beta$. Suppose for a contradiction that this is not the case and choose $(t, q) \leq (s, p \upharpoonright \beta)$ such that $(t, q) \Vdash \delta_\beta \in \dot{S}_\beta$. Then $(t, q, \pi_{nN}(f)) \leq (s, p \upharpoonright \beta, \pi_{nN}(f))$, so $(t, q, \pi_{nN}(f)) \Vdash \delta_\beta \in C^N_\beta$ contradicting the choice of $\dot{C}^N_\beta$ as a name for a club disjoint from $\dot{S}_\beta$. □

This construction produces a condition $q = (s, p, f)$ which is correct for $\vec{P}$ and has the property that $(q, \vec{P}) \leq (q_i, \vec{P^i})$ for all $i < \mu$. □

The next lemma is the key to the analysis of $\mathbb{S} * \mathbb{P}_\alpha$. It will follow from this lemma that $\mathbb{S} * \mathbb{P}_\alpha * \mathbb{T}_n$ adds no $< \aleph_n$-sequences of ordinals, which in turn implies that $\mathbb{S} * \mathbb{P}_\alpha$ adds no $\aleph_\omega$-sequences of ordinals.

Lemma 10.10. *Let $q = (s, p, f) \in \mathbb{S} * \mathbb{P}_\alpha * \mathbb{T}_n$. Let $k \geq n$. Let $\vec{P}$ be a good partition of* $\mathrm{supp}(p)$ *such that $\vec{P} \upharpoonright k = 0$. Then there exists $q^* = (s^*, p^*, f^*) \in \mathbb{S} * \mathbb{P}_\alpha * \mathbb{T}_n$ and $\vec{P}^*$ such that*

1. $(q^*, \vec{P}^*) \leq (q, \vec{P})$.
2. *q^* is correct for $\vec{P}^*$.*
3. $\vec{P}^* \upharpoonright k = 0$.

Proof. We prove the Lemma by induction on α for all n and k simultaneously. There are various cases.

Case 1: $\mathrm{cf}(\alpha) \geq \aleph_{\omega+1}$. $|\mathrm{supp}(p)| \leq \aleph_\omega$ so there exists $\beta < \alpha$ such that $\mathrm{supp}(p) \subseteq \beta$. Apply the induction hypothesis to $(s, p \upharpoonright \beta, f)$.

Case 2: α is a successor, say $\alpha = \beta + 1$. If $\beta \notin \mathrm{supp}(p)$ then apply the induction hypothesis to $(s, p \upharpoonright \beta, f)$. If $\beta \in \mathrm{supp}(p)$ then $\beta \in P_m$ for some $m \geq k$. $\dot{C}^m_\beta$ is a $\mathbb{S} * \mathbb{P}_\beta * \mathbb{T}_m$-name for a club subset of $\aleph_{\omega+1}$ so we may find $(s', p', f') \in \mathbb{S} * \mathbb{P}_\beta * \mathbb{T}_n$ and $\gamma > \max p(\beta)$ such that $(s', p', f') \leq (s, p \upharpoonright \beta, f)$ and $(s', p', \pi_{nm}(f')) \Vdash \check{\gamma} \in \dot{C}^m_\beta$. Arguing as in the proof of Lemma 10.9, $(s', p') \Vdash \check{\gamma} \notin \dot{S}_\beta$.

Now choose $\vec{Q}$ a good partition of $\mathrm{supp}(p')$ such that

- $\vec{Q} \upharpoonright k = 0$.
- $Q_j \cap \mathrm{supp}(p) = P_j$.
- $Q_m \cap \mathrm{supp}(p) = P_m \setminus \{\beta\}$.

By the induction hypothesis we may find $(s'', p'', f'') \in \mathbb{S} * \mathbb{P}_\beta * \mathbb{T}_n$ and $\vec{R}$ such that

1. (s'', p'', f'') is correct for $\vec{R}$.
2. $\vec{R} \upharpoonright k = 0$.

3. $((s'', p'', f''), \vec{R}) \leq ((s', p', f'), \vec{Q})$.

Now let $s^* = s''$, $f^* = f''$ and let $p^* \in \mathbb{P}_{\beta+1}$ be such that $p^* \restriction \beta = p''$, $p^*(\beta) = p(\beta) \cup \{\gamma\}$. Define a good partition $\vec{P}^*$ of $\mathrm{supp}(p^*)$ by setting $P^*_j = R_j$ for $j \neq m$, $P^*_m = R_m \cup \{\beta\}$. Then $q^* = (s^*, p^*, f^*)$ and $\vec{R}^*$ are as required.

Case 3: $\mathrm{cf}(\alpha) = \aleph_N$ where $N < \omega$.

Subcase 3a: $N < k$. Fix a sequence $\langle \alpha_i : i < \aleph_N \rangle$ which is continuous, increasing and cofinal in α. Let $\alpha_{\aleph_N} = \alpha$. We will define $q_i = (s_i, p_i, f_i)$ and $\vec{P^i}$ for $i \leq \aleph_N$ so that

1. $q_i \in \mathbb{S} * \mathbb{P}_{\alpha_i} * \mathbb{T}_k$.
2. q_i is correct for $\vec{P^i}$.
3. If $i < j$ then

$$((s_j, p_j \restriction \alpha_i, f_j), \vec{P^j} \restriction\restriction \alpha_i) \leq ((s_i, p_i, f_i), \vec{P^i}).$$

4. $((s_i, p_i, f_i), \vec{P^i}) \leq ((s, p \restriction \alpha_i, \pi_{nk}(f)), \vec{P} \restriction\restriction \alpha_i)$.

For $i = 0$ we choose q_0 and $\vec{P}^0$ by applying the induction hypothesis to the condition $(s, p \restriction \alpha_0, \pi_{nk}(f))$ and the partition $\vec{P} \restriction\restriction \alpha_0$.

Suppose that we have defined q_i and $\vec{P^i}$. Consider the condition $(s_i, p_i {}^\frown p \restriction [\alpha_i, \alpha_{i+1}), f_i)$ and the partition $\vec{Q}$ of $\mathrm{supp}(p \restriction [\alpha_i, \alpha_{i+1}))$ given by $Q_m = P^i_m \cup (P_m \cap [\alpha_i, \alpha_{i+1}))$. Applying the induction hypothesis gives q_{i+1} and $\vec{P}^{i+1}$ as desired.

Suppose now that i is limit. Choose (s_i, f_i) to be a lower bound for $\langle (s_j, f_j) : j < i \rangle$ as in Lemma 9.6. We now define p_i with support $\bigcup_{j<i} \mathrm{supp}(p_j)$. If $\beta \in \bigcup_{j<i} \mathrm{supp}(p_j)$ then let $\gamma_j(\beta) = \max p_j(\beta)$ for all j such that $\beta \in \mathrm{supp}(p_j)$, and let $\gamma(\beta) = \sup_{\beta \in supp(p_j)} \gamma_j(\beta)$. Define $p_i(\beta) = \bigcup_{j<i} p_j(\beta) \cup \{\gamma(\beta)\}$. An inductive proof similar to that of 10.9 shows that $q_i = (s_i, p_i, f_i)$ is a condition in $\mathbb{S} * \mathbb{P}_{\alpha_i} * \mathbb{T}_k$, and that if we define $\vec{P^i}$ by $P^i_m = \bigcup_{j<i} P^j_m$ then q_i is correct for $\vec{P^i}$.

Now let $s^* = s_{\aleph_n}$, $p^* = p_{\aleph_n}$, $\vec{P}^* = \vec{P}^{\aleph_n}$, and choose f^* such that $\pi_{nk}(f^*) \leq f_{\aleph_n}$. Then $q^* = (s^*, p^*, f^*)$ and $\vec{P}^*$ are as required.

Subcase 3b: $k \leq N$. Let $\bar{k} = N + 1$ and define a new partition $\vec{Q}$ of $\mathrm{supp}(p)$ by $\vec{Q} \restriction \bar{k} = 0$, $Q_{\bar{k}} = \bigcup_{m \leq \bar{k}} P_{\bar{k}}$, $Q_m = P_m$ for $m > \bar{k}$.

By the construction of Subcase 3a we may find $(q', \vec{R}) \leq (q, Q)$ where $q' = (s', p', f')$ is correct for $\vec{R}$. Observe that since $\vec{P}$ is a good partition, $|\bigcup_{m<\bar{k}} P_m| < \aleph_N$ and so $\bigcup_{m<\bar{k}} P_m$ is bounded in α, say $\bigcup_{m<\bar{k}} P_m \subseteq \beta$.

Define a good partition $\vec{S}$ of $\operatorname{supp}(p^* \restriction \beta)$ by setting

$$S_i = \begin{cases} \emptyset & i < k \\ P_i \cap \beta & k \leq i < \bar{k} \\ (R_{\bar{k}} \setminus \bigcup_{m<\bar{k}} P_m) \cap \beta & i = \bar{k} \\ R_i \cap \beta & i > \bar{k} \end{cases}$$

By induction we find $q'' = (s'', p'', f'')$ and $\vec{T}$ such that $(q'', \vec{T}) \leq ((s', p' \restriction \beta, f'), \vec{S})$ and q'' is correct for $\vec{T}$. Finally we let $s^* = s''$, $f^* = f''$, $p^* = p'' \frown p' \restriction [\beta, \alpha)$ and define $\vec{P}^*$ by setting $P^*_i = T_i \cup (S_i \cap [\beta, \alpha))$. □

Now we use results of Foreman [10] about games played on posets and Boolean algebras.

Definition 10.11. Let $\mathbb{P}$ be a partial ordering, let μ be regular. The game $G^I_\mu(\mathbb{P})$ is played as follows: player I and player II take turns to write down $p_\alpha \in \mathbb{P}$ for $\alpha < \mu$, where $\forall \alpha < \beta \; p_\beta \leq p_\alpha$. Player I plays at all even α, including limit α. Player II loses if there exists a limit $\lambda < \mu$ such that $\langle p_\alpha : \alpha < \lambda \rangle$ has no lower bound, and wins otherwise. If $\mathbb{B}$ is a Boolean algebra then G^I_μ is $G^I_\mu(\mathbb{B} - \{0\})$.

Lemma 10.12. *Player II has a winning strategy for the game $G^I_{\aleph_n}(\mathbb{S} * \mathbb{P}_{\omega+2} * \mathbb{T}_n)$.*

Proof. Whenever II plays a condition $q_i = (s_i, p_i, f_i)$, they will also record a partition $\vec{P}^i$ of $\operatorname{supp}(p_i)$ such that q_i is correct for $\vec{P}^i$. Lemmas 10.10 and 10.9 ensure that this is possible, and gives a winning strategy in the game. □

We are finally ready to prove that $\operatorname{Refl}(\aleph_{\omega+1})$ holds in $V^{\mathbb{S}*\mathbb{P}}$. Recall that V_0 was the original model in which the cardinals κ_i were supercompact, and that κ_i became $\aleph_{i+1}$. Let $V_1 = V$, $V_2 = V_1^{\mathbb{S}}$, $V_3 = V_2^{\mathbb{P}_{\omega+2}}$.

Suppose that $V_3 \models$ "S is a stationary subset of $\aleph_{\omega+1}$". Since S is not fragile we may find n so large that

- $S \cap \operatorname{cof}(< \aleph_n)$ is stationary.
- S is stationary in $V_3^{\mathbb{T}_{n+1}}$.

Without loss of generality we replace S by $S \cap \operatorname{cof}(< \aleph_n)$.

Now let $\mathbb{Q} = \operatorname{Coll}(\aleph_{n+1}, \aleph_{n+2})_{V_1}$. It is easy to see that II still wins $G^I_{\aleph_{n+1}}(\mathbb{S} * \mathbb{P}_{\omega+2} * \mathbb{T}_{n+1})$ in $V^1_{\mathbb{Q}}$, and that $V^1_{\mathbb{Q}} \models |\mathbb{S} * \mathbb{P}_{\omega+2} * \mathbb{T}_{n+1}| = \aleph_{n+1}$.

We now appeal to a theorem of Foreman [10].

Fact 10.13. *If $\mu = \mu^{<\mu}$, $|\mathbb{P}| = \mu$, and II has a winning strategy for $G^I_\mu(\mathbb{P})$ then the Boolean algebra $\operatorname{ro}(\mathbb{P})$ has a μ-directed closed dense subset.*

So if $V_4 = V_1^{\mathbb{Q}\times\mathbb{S}*\mathbb{P}_{\omega+2}*\mathbb{T}_{n+1}}$ then we can view V_4 as an extension of V_1 by an $\aleph_{n+1}$-directed closed poset $\mathbb{R}$.

At this point we will use the "indestructible generic supercompactness" property described in Lemma 6.10. This gives us a generic supercompact embedding $j : V_4 \longrightarrow M \subseteq V_4^{\bar{\mathbb{R}}}$, where

- $\mathrm{crit}(j) = \kappa_n$.
- $j(\kappa_n) > \kappa^+$.
- $\sup(j``\kappa^+) < j(\kappa^+)$.
- $j \restriction \kappa^+ \in M$.
- $M \models \mathrm{cf}(\kappa^+) = \aleph_n$.
- $V_4 \models$ "$\bar{\mathbb{R}}$ is $\aleph_n$-closed".

Now $V_4^{\bar{\mathbb{R}}}$ is an extension of V_1 by $\aleph_n$-closed forcing, and V_1 is a model of $\Box_{\aleph_\omega,\omega}$ and so *a fortiori* is a model of $\mathrm{AP}_{\aleph_\omega}$. By Lemma 6.14, it follows that S is stationary in $V_4^{\bar{\mathbb{R}}}$. If $\mu = \sup j``\kappa^+$ then

$$M \models \text{"}j(S) \cap \mu \text{ is stationary in } \mu,\ \mathrm{cf}(\mu) = \aleph_n\text{"}.$$

By elementarity,

$$V_4 \models \text{"there is } \alpha < \kappa^+ \text{ such that } S \cap \alpha \text{ is stationary, } \mathrm{cf}(\alpha) = \aleph_n\text{"}.$$

$V_4 = V_3^{\mathbb{T}_n*\mathbb{Q}} \supseteq V_3$, so in V_3 we see that S reflects to some ordinal of cofinality $\aleph_n$.

This concludes the proof of the theorem for one stationary set. Essentially the same proof will show that any finite family of stationary sets reflects simultaneously; however, as we would expect in the light of Theorems 4 and 5, the proof breaks down for an ω-sequence of stationary sets because we may not be able to choose a single value of n which works for every set in the sequence. $\square$

11. Prikry forcing

In this section we investigate the model obtained by doing Prikry forcing at κ where κ is κ^+-supercompact. This continues the work of Magidor and Ben-David [34].

Throughout this section κ is a measurable cardinal, U is a normal measure on κ, and $\mathbb{P}$ is the Prikry forcing defined from U. Magidor and Ben-David [34] showed that if U is the normal measure generated by a κ^+-supercompact embedding then $V^{\mathbb{P}} \models \Box^*_\kappa + \neg\Box_\kappa$.

Theorem 19. *In V let $S_0 =_{\mathrm{def}} \{ \alpha < \kappa^+ \ :\ \mathrm{cf}(\alpha) < \kappa \}$ and let $S_1 =_{\mathrm{def}} \{ \alpha < \kappa^+ \ :\ \mathrm{cf}(\alpha) = \kappa \}$. Then in $V^{\mathbb{P}}$*

1. *S_1 is a non-reflecting stationary set of cofinality ω ordinals.*
2. *If κ is κ^+-supercompact,* $\mathrm{Refl}(<\omega, S_0)$ *holds.*
3. $\mathrm{Refl}(\omega, S_0)$ *fails.*

Proof. We take each claim in turn.

1. S_0 and S_1 are stationary in V and $\mathbb{P}$ is κ^+-c.c. so that S_0 and S_1 are stationary in $V^{\mathbb{P}}$. $\mathrm{cf}(\kappa) = \omega$ in $V^{\mathbb{P}}$ so that S_1 is a set of ordinals with cofinality ω in $V^{\mathbb{P}}$. Finally if $\mathrm{cf}(\alpha) > \omega$ in $V^{\mathbb{P}}$ then $\omega < \mathrm{cf}(\alpha) < \kappa$ in V, so that if $C \in V$ is a club set in α with $\mathrm{ot}(C) = \mathrm{cf}(\alpha)$ then $C \cap S_1 = \emptyset$.
2. Suppose that $\mathrm{Refl}(n, S_0)$ fails for some $n < \omega$, and let

   $$\Vdash_{\mathbb{P}} \text{“}\langle \dot{T}_i : i < n \rangle \text{ are a counterexample to } \mathrm{Refl}(n, S_0)\text{”}.$$

   Let G be a $\mathbb{P}$-generic filter and for each i let $T_i \in V[G]$ be the interpretation of $\dot{T}_i$. If $j < \omega$ let G_j be the set of conditions $(s, A) \in G$ such that $\mathrm{lh}(s) = j$. Working in $V[G]$ we define

   $$T_i^j = \{ \alpha \in S_0 \ : \ \exists p \in G_j \ p \Vdash \check{\alpha} \in T_i \}.$$

   It is easy to see that $\langle T_i^j : j < \omega \rangle$ is an increasing sequence of sets, whose union is T_i.

   We choose j so large that T_i^j is stationary in $V[G]$ for each $i < n$. Let s be the sequence consisting of the first j elements of the generic ω sequence added by G; every element of G_j has form (s, A) for some $A \in U$. Working in V, define

   $$U_i = \{ \alpha \in S_0 \ : \ \exists A \in U \ (s, A) \Vdash \check{\alpha} \in \dot{T}_i \}.$$

   Clearly $T_i^j \subseteq U_i$, and so U_i must be stationary in V.

   κ is κ^+-supercompact so there exists $\alpha < \kappa^+$ such that $\omega < \mathrm{cf}(\alpha) < \kappa$ and $U_i \cap \alpha$ is stationary in V for all $i < n$. Let D be club in α with $\mathrm{ot}(D) = \mathrm{cf}(\alpha)$, and for each $\beta \in U_i \cap D$ choose $A_{\beta,i} \in U$ such that $(s, A_{\beta,i}) \Vdash \beta \in \dot{T}_i$. Let $A =_{\mathrm{def}} \bigcap_{\beta,i} A_\beta$, where $A \in U$ because $|D| < \kappa$.

   Now $(s, A) \Vdash U_i \cap D \subseteq \dot{T}_i \cap \alpha$. $U_i \cap D$ is still a stationary set in $V^{\mathbb{P}}$ because $\mathrm{cf}(\alpha) < \kappa$ and $\mathbb{P}$ does not affect the universe below κ. So

   $$(s, A) \Vdash \text{“}\dot{T}_i \text{ reflects at } \alpha \text{ for all } i < n\text{”},$$

   which is a contradiction. So $\mathrm{Refl}(< \omega, S_0)$ holds.
3. By a well known result of Solovay, in V we may write $S_0 = \bigcup_{i<\kappa} T_i$ where the T_i are disjoint and stationary. In $V^{\mathbb{P}}$ consider the sequence of stationary sets $\langle T_{\kappa_n} : n < \omega \rangle$ where $\langle \kappa_n : n < \omega \rangle$ is the cofinal ω-sequence in κ added by $\mathbb{P}$.

   Suppose that in $V^{\mathbb{P}}$ there is γ with $\mathrm{cf}(\gamma) > \omega$ such that $T_{\kappa_n} \cap \gamma$ is stationary for each n. In V we have $\omega < \mathrm{cf}(\gamma) < \kappa$. Working in V define $B =_{\mathrm{def}} \{ i < \kappa \ : \ T_i \cap \gamma \text{ is stationary} \}$.

Now on the one hand $\kappa_n \in B$ for all n, so that B is cofinal in κ. On the other hand $|B| \leq \mathrm{cf}(\gamma)$ because the T_i are disjoint, and since $\mathrm{cf}(\gamma) < \kappa$ this is a contradiction.

□

Theorem 20. *There is a very good scale at κ in $V^{\mathbb{P}}$.*

Proof. Let $\langle \kappa_n : n < \omega \rangle$ be the generic ω-sequence added by $\mathbb{P}$. Jech proved [22] that if we choose f_ζ with $[f_\zeta]_U = \zeta$ for each $\zeta < \kappa^+$, and then define $f^*_\zeta \in \prod_n \kappa_n^+$ by $f^*_\zeta : n \longmapsto f_\zeta(\kappa_n)$, then $V^{\mathbb{P}} \models$ "$\langle f^*_\zeta : \zeta < \kappa^+ \rangle$ is a scale".

We will show that this scale is very good. Fix $\zeta < \kappa^+$ such that $V^{\mathbb{P}} \models \mathrm{cf}(\zeta) = \lambda > \omega$. Then $\lambda < \kappa$ and $V \models \mathrm{cf}(\zeta) = \lambda$, so we choose $D \in V$ such that D is club in ζ and $\mathrm{ot}(D) = \lambda$. We will use D to witness that the scale is very good.

Let $A =_{\mathrm{def}} \{\, \alpha < \kappa \,:\, \langle f_\gamma(\alpha) : \gamma \in D \rangle$ is strictly increasing $\}$. $\kappa \in j(A)$, because $j(D) = j``D$ and so

$$\langle j(f)_\gamma(\kappa) : \gamma \in j(D) \rangle = \langle j(f_\gamma)(\kappa) : \gamma \in D \rangle = \langle \gamma : \gamma \in D \rangle.$$

Therefore $A \in U$, and hence for all sufficiently large n we have $\kappa_n \in A$. This is precisely what is demanded in the definition of very good scale. □

Remark 11.1. Cummings and Schimmerling show in [7] that Prikry forcing at κ always adds a $\square_{\kappa,\omega}$-sequence.

12. Square and reflection revisited

In this section we prove the following theorem.

Theorem 21. *Let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of supercompact cardinals. Let $\kappa = \sup_n \kappa_n$, and assume that GCH holds above κ. Then there is a forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$*

1. *$\kappa = \aleph_\omega$ and GCH holds.*
2. *$\square^*_{\aleph_\omega}$ holds.*
3. *$\mathrm{Refl}(\aleph_n, \aleph_{\omega+1} \cap \mathrm{cof}(< \aleph_n), \aleph_n)$ holds for $1 \leq n < \omega$.*

We start by giving a rough outline of the proof.

1. We iterate Laver's indestructibility forcing [30] to make each κ_n be indestructible under κ_n-directed closed forcing, and then force a certain combinatorial principle $\diamondsuit^+_{\kappa^+}$ using κ^+-directed closed forcing. Let the resulting model be V'.
2. Working in V' we iterate $\mathrm{Coll}(\kappa_n, < \kappa_{n+1})$ as in Section 6, so that κ_n becomes $\aleph_{n+1}$ and κ becomes $\aleph_\omega$. We argue that $\diamondsuit^+_{\aleph_{\omega+1}}$ now holds, and we know that each κ_n is "generically indestructibly

supercompact" in the sense of Lemma 6.10. Let the resulting model be V_1.

3. Working in V_1 we force $\square^*_{\aleph_\omega}$ using some forcing $\mathbb{S}$ whose definition involves the $\diamondsuit^+_{\aleph_{\omega+1}}$-sequence; the aim is to add a $\square^*_{\aleph_\omega}$-sequence with some strong closure properties. Let $V_2 = V_1^{\mathbb{S}}$.
4. Now we need to prove that stationary reflection holds in V_2. Given a sequence $\langle S_k : k < \aleph_n \rangle \in V_2$ as in the statement of the theorem, we force over V_2 with $\mathbb{T}_{n+1}$ which is the forcing to add a "threading" of the generic $\square^*_{\aleph_\omega}$-sequence in order type $\aleph_{n+1}$.

   The strong closure properties of the generic $\square^*_{\aleph_\omega}$-sequence guarantee that $\mathbb{T}_{n+1}$ preserves the stationarity of the sets S_k. $\mathbb{S} * \mathbb{T}_{n+1}$ has an $\aleph_{n+1}$-directed closed dense subset, the S_k are still stationary in $V_2^{\mathbb{T}_{n+1}} = V_1^{\mathbb{S}*\mathbb{T}_{n+1}}$, and this enables us to appeal to the indestructible generic supercompactness of κ_n and show that in $V_2^{\mathbb{T}_{n+1}}$ there is a single β of cofinality $\aleph_n$ at which every set S_k reflects. Stationarity is downwards absolute, so the S_k reflect simultaneously in V_2.

12.1. **Strong diamond.** The principle $\diamondsuit^+_{\lambda^+}$ was introduced by Jensen [8]. He showed that it holds in L and gave a forcing construction to add a $\diamondsuit^+_{\lambda^+}$-sequence; we give a different forcing construction here, which is better adapted to our purposes.

Definition 12.1. A $\diamondsuit^+_{\lambda^+}$-sequence is a sequence $\langle \mathcal{S}_\alpha : \alpha < \lambda^+ \rangle$ such that

1. $\mathcal{S}_\alpha \subseteq \mathrm{P}(\alpha)$, $|\mathcal{S}_\alpha| \leq \lambda$.
2. For every $X \subseteq \lambda^+$ there is $C \subseteq \lambda^+$ closed and unbounded, such that if $\alpha \in \lim(C)$ then $X \cap \alpha \in \mathcal{S}_\alpha$ and $C \cap \alpha \in \mathcal{S}_\alpha$.

We describe a forcing construction which will add such a sequence.

Theorem 22. *Let $2^\lambda = \lambda^+$ and $2^{\lambda^+} = \lambda^{++}$. Then there exists a λ^+-directed-closed and λ^{++}-c.c. forcing $\mathbb{P}$ such that $\diamondsuit^+_{\lambda^+}$ holds in $V^{\mathbb{P}}$.*

Proof. The idea of the proof will be to add a sequence $\vec{\mathcal{S}}$ of the right general form, and then to add club sets which will eventually witness that $\vec{\mathcal{S}}$ is a $\diamondsuit^+_{\lambda^+}$-sequence.

Definition 12.2. $p \in \mathbb{Q}_0$ if

1. p is a function with $\mathrm{dom}(p) \in \lambda^+$, $dom(p)$ a successor ordinal.
2. For each $\alpha \in \mathrm{dom}(p)$, $p(\alpha) \subseteq \mathrm{P}(\alpha)$.
3. $|p(\alpha)| \leq \lambda$ for all $\alpha \in \mathrm{dom}(p)$.

If $p, q \in \mathbb{Q}_0$ then $p \leq q$ iff $\mathrm{dom}(q) \leq \mathrm{dom}(p)$ and $p \restriction \mathrm{dom}(q) = q$.

It is easy to see that $\mathbb{Q}_0$ is λ^+-directed-closed and λ^{++}-c.c. In fact $\mathbb{Q}_0$ is just a version of the forcing to add one Cohen subset of λ^+. Forcing with $\mathbb{Q}_0$ adds a sequence of sets which has the right form to be a $\diamondsuit^+_{\lambda^+}$-sequence.

We will now build a forcing iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda^{++} \rangle$ of length λ^{++}, with supports of size $\leq \lambda$. Let $\mathbb{Q}_0$ be the forcing we just defined, and let $\vec{\mathcal{S}}$ be the λ^+-sequence of sets added by $\mathbb{Q}_0$. At stages $\alpha \geq 1$ we will choose a $\mathbb{P}_\alpha$-name $\dot{X}_\alpha$ for a subset of λ^+; we will see that $\mathbb{P}_{\lambda^{++}}$ is a λ^{++}-c.c. forcing poset of size λ^{++}, and so will be able to choose $\langle \dot{X}_\alpha : \alpha < \lambda^{++} \rangle$ in such a way that every subset of λ^+ in $V^{\mathbb{P}_{\lambda^{++}}}$ is denoted by some $\dot{X}_\alpha$. $\mathbb{Q}_\alpha$ will then be the forcing defined in $V^{\mathbb{P}_\alpha}$ whose conditions are closed bounded sets $c \subseteq \lambda^+$ such that

$$\forall \beta \; (\beta \in \lim(c) \implies c \cap \beta, X_\alpha \cap \beta \in \mathcal{S}_\beta),$$

and whose ordering is end-extension.

A priori it is not clear that $\mathbb{Q}_\alpha$ will be well-behaved, for example it seems that it may add a generic club set of order type ω. We will see that this does not happen; the key point in the proof will be that $\mathbb{P}_\alpha$ adds no λ-sequences of ordinals to V, so that each condition in $\mathbb{Q}_\alpha$ and each initial segment of X_α actually lies in V.

To analyse the properties of the iteration $\mathbb{P}_\alpha$, we will study a certain subset of the conditions.

Definition 12.3. Let $1 \leq \alpha < \lambda^{++}$, and let $\mathbb{P}_\alpha$ be stage α of the forcing iteration described above. Let $p \in \mathbb{P}_\alpha$. Then p is *flat* iff there exists $\beta < \lambda^+$ such that

1. $\mathrm{dom}(p(0)) = \beta + 1$.
2. For all $\alpha \in \mathrm{dom}(p)$ with $\alpha > 0$,
   (a) $p \restriction \alpha$ decides $\dot{X}_\alpha \cap \beta$.
   (b) $p \restriction \alpha \Vdash p(\alpha) = \check{C}_\alpha$, where $C_\alpha \in V$ is a closed set with maximum element β.

$\mathbb{P}^*_\alpha$ is the subset of $\mathbb{P}_\alpha$ consisting of flat conditions. If $p \in \mathbb{P}^*_\alpha$, the *height* of p is the unique β with the properties above (that is $\mathrm{dom}(p(0)) - 1$)

If p is a flat condition, we will not distinguish between $p(\alpha)$ (which is technically a $\mathbb{P}_\alpha$-name) and the set in V to which $p \restriction \alpha$ forces it to be equal. Notice that if $p \in \mathbb{P}^*_\alpha$ then $p \restriction \beta \in \mathbb{P}^*_\beta$ for all $\beta < \alpha$.

Lemma 12.4. *Let* $1 \leq \alpha < \lambda^{++}$. *Then*

1. $\mathbb{P}^*_\alpha$ *is* λ^+*-directed-closed. Moreover, if* $\langle p_\gamma : \gamma < \mu \rangle$ *is a directed set of flat conditions for some* $\mu < \lambda^+$*, and the height of* p_γ *is* σ_γ*, then there is a greatest lower bound (in* $\mathbb{P}_\alpha$*)* p *which is given by*

(a) $\mathrm{supp}(p) = \bigcup \mathrm{supp}(p_\gamma)$.
(b) $\mathrm{dom}(p(0)) = \sigma + 1$, *where* $\sigma = \sup_\gamma \sigma_\gamma$.
(c) $p(0) \restriction \sigma = \bigcup p_\gamma(0)$.
(d) *For* $\beta > 0$, $p(\beta) = \bigcup \{ p_\gamma(\beta) : \beta \in \mathrm{dom}(p_\gamma) \} \cup \{\sigma\}$.
(e) $p(0)(\sigma) = \{ p(\beta) \cap \sigma : \beta \in \mathrm{dom}(p) \} \cup \{ x_\beta : \beta \in \mathrm{dom}(p) \}$, *where* x_β *is the subset of* σ *such that* $p \restriction \beta \Vdash \dot{X}_\beta \cap \sigma = x_\beta$.

$p \in \mathbb{P}^*_\alpha$ *and* p *has height* σ.

2. $\mathbb{P}^*_\alpha$ *is dense in* $\mathbb{P}_\alpha$.

Proof. The first claim is easy to verify. We prove the second claim by induction on α. Suppose $\mathbb{P}^*_{\bar{\alpha}}$ is dense in $\mathbb{P}_{\bar{\alpha}}$ for all $\bar{\alpha} < \alpha$. We show $\mathbb{P}^*_\alpha$ is dense in $\mathbb{P}_\alpha$.

Case 1: $\alpha = 1$, $\mathbb{P}^*_1 = \mathbb{P}_1 \simeq \mathbb{Q}_0$ and there is nothing to prove.

Case 2: $\alpha = \beta + 1$. Fix a condition $p \in \mathbb{P}^*_{\beta+1}$.

By induction, we know that $\mathbb{P}^*_\beta$ is λ^+-closed and dense in $\mathbb{P}_\beta$. In particular $\mathbb{P}_\beta$ adds no bounded subsets of λ^+, and so we may choose $p_0 \le p \restriction \beta$ such that p_0 decides the value of $p(\beta)$, say that p_0 forces $p(\beta)$ to equal c where $\max(c) = \gamma$. As $\mathbb{P}^*_\beta$ is dense, we may choose $p_0 \in \mathbb{P}^*_\beta$ and by extending if necessary may also assume that the height of p_0 is greater than γ.

Now we argue in a similar vein to build a decreasing ω-sequence of conditions $p_0 > p_1 > p_2 > \dots$ and an increasing ω-sequence of ordinals $\rho_0 < \rho_1 < \dots$, where p_n is a condition in $\mathbb{P}^*_\beta$ of height ρ_n, and p_{n+1} decides $\dot{X}_\beta \cap \rho_n$; say $p_{n+1} \Vdash \dot{X}_\beta \cap \rho_n = x_n$. By the first claim of the Lemma, we may form a greatest lower bound for $\vec{p}$ which will be a condition $q \in \mathbb{P}^*_\beta$ of height $\rho = \sup \rho_n$.

Now we define q^+ as follows. $\mathrm{dom}(q^+) = \beta + 1$, and $q^+(\gamma) = q(\gamma)$ for $0 < \gamma < \beta$. $q^+(\beta) = c \cup \{\rho_0, \rho_1, \dots, \rho\}$. $q^+(0) \restriction \rho = q(0) \restriction \rho$, and $q^+(0)(\rho) = q(0)(\rho) \cup \{q^+(\beta), \bigcup_n x_n\}$.

It is now routine to check that $q^+ \in \mathbb{P}_{\beta+1}$, q^+ is flat of height ρ, and q^+ refines p.

Case 3: α is limit with $\mathrm{cf}(\alpha) \ge \lambda^+$. Fix $p \in \mathbb{P}_\alpha$, then the support of p is bounded by some $\beta < \alpha$. By induction we may find $q \le p \restriction \beta$ with $q \in \mathbb{P}^*_\beta$; if $q^+ \in \mathbb{P}_\alpha$ is defined by $q^+ \restriction \beta = q$ and $q^+ \restriction [\beta, \alpha) = 0$ then $q^+ \in \mathbb{P}^*_\alpha$ and $q^+ \le p$, as required.

Case 4: α is limit and $\mathrm{cf}(\alpha) \le \lambda$. Choose a sequence $\langle \alpha_i : i < \mathrm{cf}(\alpha) \rangle$ which is increasing, continuous and cofinal in α. Fix $p \in \mathbb{P}_\alpha$.

We will define a decreasing sequence of conditions $\langle p_i : i \le \mathrm{cf}(\alpha) \rangle$ such that $p_0 \le p$, $p_i \restriction \alpha_i \in \mathbb{P}^*_{\alpha_i}$ for each i, and $p_{\mathrm{cf}(\alpha)} \in \mathbb{P}^*_\alpha$. We let σ_i denote the height of $p_i \restriction \alpha_i$.

$i = 0$. Let $q_0 \leq p \upharpoonright \alpha_0$, $q_0 \in \mathbb{P}^*_{\alpha_0}$. Let $p_0 \upharpoonright \alpha_0 = q_0$, $p_0 \upharpoonright [\alpha_0, \alpha) = p \upharpoonright [\alpha_0, \alpha)$

$i = j + 1$. Let $q_i \leq p_j \upharpoonright \alpha_i$, $q_i \in \mathbb{P}^*_{\alpha_i}$. Now let $p_i \upharpoonright \alpha_i = q_i$, and $p_i \upharpoonright [\alpha_i, \alpha) = p \upharpoonright [\alpha_i, \alpha)$.

i is limit. For each $j < i$ consider the sequence $\langle p_k \upharpoonright \alpha_j : j \leq k < i \rangle$. This is a decreasing sequence from $\mathbb{P}^*_{\alpha_j}$ so by the first claim of the Lemma we can form a greatest lower bound r_j, where $r_j \in \mathbb{P}^*_{\alpha_j}$ and r_j has height $\sigma = \sup_{k<i} \sigma_k$.

It is easy to see that if $m < n$ then $r_m \upharpoonright (0, \alpha_m) = r_n \upharpoonright (0, \alpha_m)$, $r_m(0) \upharpoonright \sigma = r_n(0) \upharpoonright \sigma$, and $r_m(0)(\sigma) \subseteq r_n(0)(\sigma)$. We define q_i such that $\mathrm{dom}(q_i) = \alpha_i$, $q_i \upharpoonright (0, \alpha_j) = r_j \upharpoonright (0, \alpha_j)$ for all j, $q_i(0) \upharpoonright \sigma$ is the common value of $r_j(0) \upharpoonright \sigma$, and $q_i(0)(\sigma) = \bigcup_{j<i} r_j(0)(\sigma)$.

It is routine to check that $q_i \in \mathbb{P}^*_{\alpha_i}$ and q_i has height σ, also that $q_i \leq p_j \upharpoonright \alpha_i$ for each $j < i$. Now let $p_i \upharpoonright \alpha_i = q_i$, $p_i \upharpoonright [\alpha_i, \alpha) = p \upharpoonright [\alpha_i, \alpha)$.

The construction for the limit step also works for $i = \mathrm{cf}(\alpha)$, and produces $p_{\mathrm{cf}(\alpha)}$ which is in $\mathbb{P}^*_\alpha$ and refines p. □

Using Lemma 12.4, it is easy to see that $\mathbb{P} =_{\mathrm{def}} \mathbb{P}^*_{\lambda^{++}}$ is λ^+-directed-closed however the sets $\dot{X}_\alpha$ are chosen. $\mathbb{P}$ is λ^{++}-c.c. by a standard Δ-system argument. Therefore we may choose the names $\dot{X}_\alpha$ in such a way that every subset of λ^+ in $V^{\mathbb{P}}$ is denoted by some $\dot{X}_\alpha$.

Now if $X \in V^{\mathbb{P}}$ is any subset of λ^+, and $X = X_\alpha$, then the club set C_α added at stage α in the iteration will have the property that $\forall \beta \in \lim(C)\ X \cap \beta, C \cap \beta \in \mathcal{S}_\beta$. This is to say that $\vec{\mathcal{S}}$ is a $\diamondsuit^+_{\lambda^+}$-sequence in $V^{\mathbb{P}}$. □

Remark 12.5. Jensen also defined $\diamondsuit^+_\kappa$ for κ inaccessible, but that definition demands $|\mathcal{S}_\alpha| \leq \alpha$. In general (say for κ a Laver indestructible supercompact) it will not be possible to force $\diamondsuit^+_\kappa$ with κ-directed closed forcing, because $\diamondsuit^+_\kappa$ implies the existence of a κ-Kurepa tree.

We now prove that forcing of size λ^+ which preserves the regularity of λ^+ will also preserve $\diamondsuit^+_{\lambda^+}$. For technical reasons we need some precise information about the definability of a $\diamondsuit^+_{\lambda^+}$-sequence in the extension from data in the ground model and the generic filter.

Lemma 12.6. *Let $\mathbb{P}$ be such that $|\mathbb{P}| = \lambda^+$ and $\Vdash_{\mathbb{P}}$ "λ^+ is regular". Let $\langle q_\alpha : \alpha < \lambda^+ \rangle$ be such that $\mathbb{P} \subseteq \{\, q_\alpha \ :\ \alpha < \lambda^+ \,\}$.*

Let f be a bijection between λ^+ and $\lambda^+ \times \lambda^+ \times 3$, and let g be a bijection between λ^+ and $\lambda^+ \times \lambda^+ \times \lambda^+ \times 3$. Let $\langle \mathcal{S}_\alpha : \alpha < \lambda^+ \rangle$ be a $\diamondsuit^+_{\lambda^+}$-sequence.

Then

1. *There exists* $\langle \dot{\mathcal{T}}_\alpha : \alpha < \lambda^+ \rangle$ *such that if* G *is* $\mathbb{P}$*-generic and* $\mathcal{T}_\alpha = \dot{\mathcal{T}}_\alpha^G$ *then*
   - $\langle \mathcal{T}_\alpha : \alpha < \lambda^+ \rangle$ *is a* $\diamondsuit^+_{\lambda^+}$ *sequence in* $V[G]$.
   - $\mathcal{T}_\alpha$ *is uniformly* Δ_0*-definable from* $\{ \gamma < \alpha \ : \ q_\gamma \in G \}$, $\mathcal{S}_\alpha$, *and* f.
2. *There exists* $\langle \dot{\mathcal{U}}_\alpha : \alpha < \lambda^+ \rangle$ *such that if* G *is* $\mathbb{P}$*-generic and* $\mathcal{U}_\alpha = \dot{\mathcal{U}}_\alpha^G$ *then*
   - *For all* $\alpha < \lambda^+$, $|\mathcal{U}_\alpha| \leq \lambda$.
   - *For all* $A \in V[G]$, *if* $A \subseteq \lambda^+ \times \lambda^+$ *there is* $C \in V[G]$ *such that* C *is club in* λ^+ *and* $\forall \alpha \in \lim(C)\ A \cap \alpha \times \alpha, C \cap \alpha \in \mathcal{U}_\alpha$.
   - $\mathcal{U}_\alpha$ *is uniformly* Δ_0 *in* $\{ \gamma < \alpha \ : \ q_\gamma \in G \}, \mathcal{S}_\alpha, g$.

Proof. We describe the construction of $\dot{\mathcal{T}}_\alpha$, the construction for $\dot{\mathcal{U}}_\alpha$ will be exactly parallel (and so we omit it). Let E be the club set of $\alpha < \lambda^+$ such that $f \restriction \alpha$ is a bijection between α and $\alpha \times \alpha \times 3$.

For $\alpha \notin E$ we let $\dot{\mathcal{T}}_\alpha$ name $\emptyset$. For $\alpha \in E$ and $x, y \in \mathcal{S}_\alpha$ we define

$$B_0(x) = \{ \beta < \alpha \ : \ \exists \gamma < \alpha\ q_\gamma \in G, (\gamma, \beta, 1) \in f``x \},$$

and

$$B_1(x,y) = \{ \delta \in y \cap E \ : \ \forall \beta < \delta\ \exists \gamma < \delta\ \exists i < 2\ q_\gamma \in G, (\gamma, \beta, i) \in f``x \}.$$

We now define $\dot{\mathcal{T}}_\alpha$ to be a name for

$$\{ B_0(x) \ : \ x \in \mathcal{S}_\alpha \} \cup \{ B_1(x,y) \ : \ x, y \in \mathcal{S}_\alpha \}.$$

Notice that this is uniformly Δ_0 in the parameters $\{ \gamma < \alpha \ : \ q_\gamma \in G \}$, $\mathcal{S}_\alpha$, f.

We now check that we have defined a name for a $\diamondsuit^+_{\lambda^+}$-sequence. Let G be $\mathbb{P}$-generic, and for each α let $\mathcal{T}_\alpha = \dot{\mathcal{T}}_\alpha^G$. Let $A \in V[G]$ be a subset of λ^+, and let $A = \dot{\tau}^G$ for some $\mathbb{P}$-name $\dot{\tau}$.

Define $Y \subseteq \lambda^+ \times \lambda^+ \times 3$ as follows; $(\gamma, \beta, i) \in Y$ iff $q_\gamma \in \mathbb{P}$ and one of the following holds.

- $i = 0$ and $q_\gamma \Vdash \check{\beta} \notin \dot{\tau}$.
- $i = 1$ and $q_\gamma \Vdash \check{\beta} \in \dot{\tau}$.
- $i = 2$ and $q_\gamma \nparallel \check{\beta} \in \dot{\tau}$.

Let $X = \{ \alpha \ : \ f(\alpha) \in Y \}$.

Since $\vec{\mathcal{S}}$ is a $\diamondsuit^+_{\lambda^+}$-sequence we may find $C \subseteq \lambda^+$ club in λ^+ such that $\forall \alpha \in \lim(C)\ X \cap \alpha, C \cap \alpha \in \mathcal{S}_\alpha$. Working in $V[G]$, we now define

$$D = \{ \alpha \ : \ \forall \beta < \alpha\ \exists \gamma < \alpha\ q_\gamma \in G, q_\gamma \parallel \check{\beta} \in \dot{\tau} \}.$$

Since $V[G] \models$ "λ^+ is regular", D is a club subset of λ^+. Let $F = C \cap D \cap E$, then F is a club subset of λ^+.

Claim. $\forall \alpha \in \lim(F)\ A \cap \alpha, F \cap \alpha \in \mathcal{T}_\alpha$.

Proof. Fix $\alpha \in \lim(F)$. As $\alpha \in \lim(C)$, $X \cap \alpha \in \mathcal{S}_\alpha$ and $C \cap \alpha \in \mathcal{S}_\alpha$.

As $\alpha \in E$, $f``X \cap \alpha = Y \cap \alpha \times \alpha \times 3$. That is to say, for all $\gamma, \beta < \alpha$ if $q_\gamma \in \mathbb{P}$ then

- $q_\gamma \Vdash \check{\beta} \in \dot{\tau} \iff (\gamma, \beta, 1) \in f``X \cap \alpha$.
- $q_\gamma \parallel \check{\beta} \in \dot{\tau} \iff \exists i < 2\ (\gamma, \beta, i) \in f``X \cap \alpha$.

Now let $\beta < \alpha$.

$$\begin{aligned} \beta \in A &\iff \exists p \in G\ p \Vdash \beta \in \dot{\tau} \\ &\iff \exists \gamma < \alpha\ q_\gamma \in G, q_\gamma \Vdash \check{\beta} \in \dot{\tau} \text{ (as } \alpha \in D) \\ &\iff \exists \gamma < \alpha\ q_\gamma \in G, (\gamma, \beta, 1) \in f``X \cap \alpha \\ &\iff \beta \in B_0(X \cap \alpha) \end{aligned}$$

Hence $A \cap \alpha = B_0(X \cap \alpha) \in \mathcal{T}_\alpha$.

Similarly if $\delta < \alpha$ then

$$\begin{aligned} \delta \in D &\iff \forall \beta < \delta\ \exists \gamma < \delta\ q_\gamma \in G, q_\gamma \parallel \check{\beta} \in \dot{\tau} \\ &\iff \forall \beta < \delta\ \exists \gamma < \delta\ q_\gamma \in G, \exists i < 2\ (\gamma, \beta, i) \in f``X \cap \alpha \end{aligned}$$

Hence $F \cap \alpha = B_1(X \cap \alpha, C \cap \alpha) \in \mathcal{T}_\alpha$. The claim is proved. □

This concludes the proof of Lemma 12.6. □

12.2. **The models V' and V_1.** Recall that we are beginning with a model V in which GCH holds above κ, and $\langle \kappa_n : n < \omega \rangle$ is an increasing sequence of supercompact cardinals. We start by iterating the forcing described in Fact 6.1 to make each κ_n be Laver indestructible, and then force $\diamondsuit^+_{\kappa^+}$ using the forcing iteration of the last section. We obtain a model V' in which $\diamondsuit^+_{\kappa^+}$ holds, and in which it is still true that each κ_n is supercompact and that GCH holds above κ.

Now we force over V' with an iteration of the sort described in Fact 6.10. The iteration has the form $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n : n < \omega \rangle$ where $\mathbb{Q}_0 = \mathrm{Coll}(\omega, < \kappa_0)$ and $\mathbb{Q}_n = \mathrm{Coll}(\kappa_{n-1}, < \kappa_n)_{(V')^{\mathbb{P}_n}}$ for $n > 0$. We let $\mathbb{P}_\omega$ be the inverse limit of this iteration, and then let $V_1 = (V')^{\mathbb{P}_\omega}$.

We refer the reader back to Fact 6.10 for a detailed account of the properties of V_1. Recall in particular that in V_1 we have $\kappa_n = \aleph_{n+1}$, $\kappa = \aleph_\omega$, $\kappa^+ = \aleph_{\omega+1}$. Since $|\mathbb{P}_\omega| = \aleph_{\omega+1}$ and $\mathbb{P}_\omega$ preserves the regularity of κ^+, Lemma 12.6 tells us that $\diamondsuit^+_{\aleph_{\omega+1}}$ holds in V_1.

12.3. **The model V_2.** Working in V_1, we define a forcing $\mathbb{S}$ to add a $\square^*_{\aleph_\omega}$-sequence. For technical reasons (see Lemma 12.15) we will need to use a forcing more complex than the one defined in Definition 6.2. Conditions will be functions where $\mathrm{dom}(p) = \alpha + 1$ for some limit ordinal $\alpha < \aleph_{\omega+1}$ (we call α the *height* of p and write $\alpha = \mathrm{ht}(p)$) and we define the set of conditions of height α by induction on α. We will fix some large regular cardinal λ and a well ordering $<_\lambda$ of H_λ; given $X \subseteq H_\lambda$ we denote by $\mathrm{Sk}_\lambda(X)$ the Skolem hull of X in the structure $(H_\lambda, \in, <_\lambda)$. Let $\vec{\mathcal{S}}$ be the $<_\lambda$-least $\diamondsuit^+_{\aleph_{\omega+1}}$-sequence.

Remark 12.7. Since $\aleph_\omega$ is strong limit, if $\aleph_\omega \subseteq X \subseteq H_\lambda$ then every bounded subset of $\aleph_\omega$ is in $\mathrm{Sk}_\lambda(X)$.

Remark 12.8. In what follows we will consider $\mathrm{Sk}_\lambda(A)$ for various sets A defined in a generic extension $V_1^{\mathbb{S}}$ of V_1. Every A for which we do this will actually lie in V_1 and will be a subset of $H_\lambda^{V_1}$.

Definition 12.9. $p \in \mathbb{S}$ iff

1. $p = \langle \mathcal{C}_\beta : \beta \leq \alpha, \lim(\beta) \rangle$ where $\alpha < \aleph_{\omega+1}$, $|\mathcal{C}_\beta| \leq \aleph_\omega$, and $\mathcal{C}_\beta$ is a nonempty set of club subsets of β each with order type less than $\aleph_\omega$.
2. For $\beta \leq \alpha$, if $C \in \mathcal{C}_\beta$ and $\gamma \in \lim(C)$ then $C \cap \gamma \in \mathcal{C}_\gamma$.
3. For all $\beta < \alpha$, $p \restriction \beta + 1 \in \mathbb{S}$.
4. There is a set $A^p_\alpha \subseteq H_\lambda$ such that $\aleph_\omega \cup \alpha \subseteq A^p_\alpha$, $|A^p_\alpha| = \aleph_\omega$ and $\mathcal{C}_\alpha$ is equal to the set of C such that
   (a) C is club in α, $\mathrm{ot}(C) < \aleph_\omega$.
   (b) $C \in \mathrm{Sk}_\lambda(A^p_\alpha \cup \{p \restriction \alpha\} \cup \{\vec{\mathcal{S}} \restriction \alpha + 1\})$.
   (c) $\forall \gamma \in \lim(C)\ C \cap \gamma \in \mathcal{C}_\gamma$.

If $p, q \in \mathbb{S}$ then $p \leq q$ iff $ht(q) \leq \mathrm{ht}(p)$ and $p \restriction \mathrm{ht}(q) + 1 = q$.

Lemma 12.10. *Let $p = \langle \mathcal{C}_\beta : \beta \leq \alpha \rangle \in \mathbb{S}$. If $C \in \mathcal{C}_\beta$ then every closed and unbounded subset of C is also in $\mathcal{C}_\beta$. In particular, $\mathcal{C}_\beta$ contains a set of order type* $\mathrm{cf}(\beta)$.

Proof. Fix $\langle A^p_\beta : \beta \leq \alpha \rangle$ witnessing clause 4 above for $\beta \leq \alpha$. Let D be a closed and unbounded subset of C. Since $\mathrm{ot}(C) < \aleph_\omega$ and $\aleph_\omega \subseteq A^p_\beta$ we see by Remark 12.7 that $\{ \mathrm{ot}(C \cap \gamma) : \gamma \in D \} \in \mathrm{Sk}_\lambda(A^p_\beta)$, so that $D \in \mathrm{Sk}_\lambda(A^p_\beta \cup \{p \restriction \beta\} \cup \{\vec{\mathcal{S}} \restriction \beta + 1\})$. We need to check that $\forall \gamma \in \lim(D)\ D \cap \gamma \in \mathcal{C}_\gamma$, which we will do by induction.

If $\gamma \in \lim(D)$ then $\gamma \in \lim(C)$, so $C \cap \gamma \in \mathcal{C}_\gamma$. $D \cap \gamma$ is a closed unbounded subset of $C \cap \gamma$, so by induction $D \cap \gamma \in \mathcal{C}_\gamma$ and we are done. □

It will be important to know that $\mathbb{S}$ adds no new $\aleph_\omega$-sequences of ordinals. Since $\aleph_\omega$ is singular the next lemma will suffice for this.

Lemma 12.11. *If $n < \omega$ then $\mathbb{S}$ is $(\aleph_n + 1)$-strategically closed.*

Proof. We describe a strategy for player II in the game $G^{II}_{\aleph_n+1}(\mathbb{S})$. The strategy is similar to that for the forcing $\mathbb{P}(\kappa, \lambda)$ from Definition 6.2. Let the move made at stage i be $p_i = \langle \mathcal{C}_\beta : \beta \leq \alpha_i \rangle$.

At limit stages i: let $\alpha_i = \sup_{j<i} \alpha_j$, and let $C = \{ \alpha_j : j < i \}$. Let $A = \aleph_\omega \cup \alpha_i \cup \{C\}$ and define $\mathcal{C}_{\alpha_i}$ to be the set of all those D such that

1. D is club in α_i with order type less than $\aleph_\omega$.
2. $\forall \beta \in \lim(D)\ D \cap \beta \in \mathcal{C}_\beta$.
3. $D \in \mathrm{Sk}_\lambda(A \cup \{\langle \mathcal{C}_\beta : \beta < \alpha_i \rangle\} \cup \{\mathcal{S} \restriction \alpha_i + 1\})$.

II then plays $p_{i+2} = \langle \mathcal{C}_\beta : \beta \leq \alpha_i \rangle$.

The key point here is that by induction $\{ \alpha_j : j < i \} \in \mathcal{C}_{\alpha_i}$ for every limit $i \leq \aleph_n$, so that at each limit stage the strategy gives a nonempty set $\mathcal{C}_{\alpha_i}$ and player II makes a legal move.

At successor stages $i+2$, i even: let $\alpha_{i+2} = \alpha_{i+1} + \omega$, $C = \alpha_{i+2} \setminus \alpha_{i+1}$. Let $A = \aleph_\omega \cup \alpha_{i+2} \cup \{C\}$ and define $\mathcal{C}_{\alpha_i}$ to be the set of all those D such that

1. D is club in α_i with order type less than $\aleph_\omega$.
2. $\forall \beta \in \lim(D)\ D \cap \beta \in \mathcal{C}_\beta$.
3. $D \in \mathrm{Sk}_\lambda(A \cup \{p_{i+1}\} \cup \{\mathcal{S} \restriction \alpha_{i+2} + 1\})$.

II then plays $p_i = \langle \mathcal{C}_\beta : \beta \leq \alpha_{i+2} \rangle$. In this case it is clear that $C \in \mathcal{C}_{\alpha_i}$ since C has no limit points. □

Now let $\vec{q} = \langle q_\alpha : \alpha < \aleph_{\omega+1} \rangle$ be the $<_\lambda$-least sequence such that

1. $\vec{q}$ enumerates the set of all functions q such that $\mathrm{dom}(q)$ is a successor ordinal less than $\aleph_{\omega+1}$ and $\mathrm{range}(q) \subseteq H_{\aleph_{\omega+1}}$.
2. For all α, $\mathrm{dom}(q_\alpha) \leq \alpha$.

Notice that $\mathbb{S} \subseteq \{ q_\alpha : \alpha < \aleph_{\omega+1} \}$.

Let g be the $<_\lambda$-minimal bijection between $\aleph_{\omega+1}$ and $\aleph_{\omega+1} \times \aleph_{\omega+1} \times \aleph_{\omega+1} \times 3$. Applying Lemma 12.6 we fix $\langle \dot{\mathcal{U}}_\alpha : \alpha < \aleph_{\omega+1} \rangle$ such that if G is $\mathbb{S}$-generic over V_1 and $\mathcal{U}_\alpha = \dot{\mathcal{U}}_\alpha^G$ then

- For all $\alpha < \aleph_{\omega+1}$, $|\mathcal{U}_\alpha| \leq \aleph_\omega$.
- For all $A \in V_1[G]$, if $A \subseteq \aleph_{\omega+1} \times \aleph_{\omega+1}$ there is $C \in V_1[G]$ such that C is club in $\aleph_{\omega+1}$ and $\forall \alpha \in \lim(C)\ A \cap \alpha \times \alpha, C \cap \alpha \in \mathcal{U}_\alpha$.
- $\mathcal{U}_\alpha$ is uniformly Δ_0 in $\{ \gamma < \alpha : q_\gamma \in G \}, \mathcal{S}_\alpha, g$.

Lemma 12.12. *Let $\alpha < \aleph_{\omega+1}$.*

1. *If G is generic and $\mathcal{U}_\beta = \dot{\mathcal{U}}_\beta^G$ for all β, then $\mathcal{U}_\alpha \in V_1$ and $\vec{\mathcal{U}} \restriction \alpha \in V_1$.*
2. *If G is $\mathbb{S}$-generic over V_1, then $\{ \gamma < \alpha : q_\gamma \in G \} \in V_1$.*
3. *If $p \in \mathbb{S}$ is a condition of height α and G is some $\mathbb{S}$-generic filter with $p \in G$ then*

$$\{ \gamma < \alpha : q_\gamma \in G \} = \{ \gamma < \alpha : q_\gamma \textit{ is an initial segment of } p \restriction \alpha \}$$

and so $\{ \gamma < \alpha : q_\gamma \in G \} \in \mathrm{Sk}_\lambda(\{p \restriction \alpha\})$.

4. *If $p \in \mathbb{S}$ is a condition of height α and G is some $\mathbb{S}$-generic filter with $p \in G$ then $\mathcal{U}_\alpha \in \mathrm{Sk}_\lambda(\{p \upharpoonright \alpha, \mathcal{S}_\alpha\})$.*

Proof. The first two claims are immediate from the fact that $\mathbb{S}$ adds no $\aleph_\omega$-sequences of ordinals. For the third claim, observe that if $\gamma < \alpha$ then $\mathrm{dom}(q_\gamma) \leq \gamma < \alpha$, so that $q_\gamma \in G$ exactly when q_γ is an initial segment of $p \upharpoonright \alpha$. $\vec{q} \in \mathrm{Sk}_\lambda(\emptyset)$ and $\{ \gamma < \alpha : q_\gamma \in G \}$ is definable in H_λ from $p \upharpoonright \alpha$ and $\vec{q}$, so $\{ \gamma < \alpha : q_\gamma \in G \} \in \mathrm{Sk}_\lambda(\{p \upharpoonright \alpha\})$.

For the last claim, observe that $\mathcal{U}_\alpha$ is definable (in fact Δ_0-definable) from the parameters $\{ \gamma < \alpha : q_\gamma \in G \}$, $\mathcal{S}_\alpha$, and g. By the choice of g, $g \in \mathrm{Sk}_\lambda(\emptyset)$. We saw in the third claim that $\{ \gamma < \alpha : q_\gamma \in G \} \in \mathrm{Sk}_\lambda(\{p \upharpoonright \alpha\})$, so $\mathcal{U}_\alpha \in \mathrm{Sk}_\lambda(\{p \upharpoonright \alpha, \mathcal{S}_\alpha\})$. □

Lemma 12.13. *Let G be $\mathbb{S}$-generic and let $\langle \mathcal{C}_\alpha : \alpha < \aleph_{\omega+1} \rangle$ be the $\square^*_{\aleph_\omega}$-sequence added by G. Let $\alpha < \aleph_{\omega+1}$. Let $C \in V_1$ be such that*

1. *C is club in α, with order type less than $\aleph_\omega$.*
2. *$\forall \beta \in \lim(C)\ C \cap \beta \in \mathcal{C}_\beta$.*
3. *$C \in \mathrm{Sk}_\lambda(\aleph_\omega \cup \mathcal{U}_\alpha \cup \mathcal{C}_\alpha)$.*

Then $C \in \mathcal{C}_\alpha$.

Proof. Let $p = \langle \mathcal{C}_\beta : \beta \leq \alpha \rangle$. From the definition of $\mathbb{S}$ we may fix a set $A^p_\alpha \supseteq \aleph_\omega$ such that $\mathcal{C}_\alpha$ is equal to the set of D such that

1. D is club in α, $\mathrm{ot}(D) < \aleph_\omega$.
2. $D \in \mathrm{Sk}_\lambda(A^p_\alpha \cup \{p \upharpoonright \alpha\} \cup \{\vec{\mathcal{S}} \upharpoonright \alpha + 1\})$.
3. $\forall \gamma \in \lim(D)\ D \cap \gamma \in \mathcal{C}_\gamma$.

Since $|\mathcal{U}_\alpha| \leq \aleph_\omega$, we see by Remark 12.12 that $\mathcal{U}_\alpha \subseteq \mathrm{Sk}_\lambda(\aleph_\omega \cup \{p \upharpoonright \alpha\} \cup \{\mathcal{S}_\alpha\})$. Thus $C \in \mathrm{Sk}_\lambda(A^p_\alpha \cup \{p \upharpoonright \alpha\} \cup \{\vec{\mathcal{S}} \upharpoonright \alpha + 1\})$, and it follows that $C \in \mathcal{C}_\alpha$. □

Let $V_2 = V_1^{\mathbb{S}}$. Working in V_2, we define a forcing $\mathbb{T}_n$ for each n by setting $\mathbb{T}_n =_{\mathrm{def}} \mathbb{T}_{\aleph_n}(\vec{\mathcal{C}})_{V_2}$ where $\mathbb{T}_{\aleph_n}(\vec{\mathcal{C}})$ is the thread forcing defined in Definition 6.8. Of course $\mathbb{T}_n \notin V_1$, but the strategic closure of $\mathbb{S}$ implies that $\mathbb{T}_n \subseteq V_1$.

In V_1 let $\vec{x} = \langle x_\alpha : \alpha < \aleph_{\omega+1} \rangle$ be the $<_\lambda$-minimal sequence such that

- $\vec{x}$ enumerates all bounded subsets of $\aleph_{\omega+1}$.
- For all $\alpha < \aleph_{\omega+1}$, $x_\alpha \subseteq \alpha$.

Notice that $\mathbb{T}_n \subseteq \{ x_\alpha : \alpha < \aleph_{\omega+1} \}$.

The following lemma is analogous to Lemmas 6.9 and 9.6.

Lemma 12.14. *$\mathbb{S} * \mathbb{T}_n$ has an $\aleph_n$-directed closed subset.*

Proof. We let D be the set of conditions $(p, \check{c})$ such that $\mathrm{ht}(p) = \max(c)$. Arguing as in the proof of Lemma 6.9 we may show that D is dense and the ordering on D is treelike, so to finish the proof it suffices to show that D is $\aleph_n$-directed closed.

Let $\gamma < \aleph_n$ and let $\langle (p_i, c_i) : i < \gamma \rangle$ be a decreasing sequence of conditions in D, where $\mathrm{ht}(p_i) = \alpha_i$ and $p_i = \langle \mathcal{C}_\delta : \delta \leq \alpha_i \rangle$. We define $\alpha = \sup_{i<\gamma} \alpha_i$ and $C = \bigcup_{i<\gamma} c_i$, from which it follows easily that C is club in α and $\forall \beta \in \lim(C)\ C \cap \beta \in \mathcal{C}_\beta$.

Let $A = \aleph_\omega \cup \alpha \cup \{C\}$ and define $\mathcal{C}_\alpha$ to be the set of all those D such that

1. D is club in α_i with order type less than $\aleph_\omega$.
2. $\forall \beta \in \lim(D)\ D \cap \beta \in \mathcal{C}_\beta$.
3. $D \in \mathrm{Sk}_\lambda(A \cup \{\langle \mathcal{C}_\beta : \beta < \alpha_i \rangle\} \cup \{\mathcal{S} \restriction \alpha_i + 1\})$.

Notice that $C \in \mathcal{C}_\alpha$.

Then $(\langle \mathcal{C}_i : i \leq \alpha \rangle, C \cup \{\alpha\})$ is a condition in D which is a lower bound for $\langle (p_i, c_i) : i < \gamma \rangle$. □

The following lemma is the key to the proof of Theorem 21.

Lemma 12.15. *Let* $1 \leq n < \omega$. *If*

$$V_2 \models \text{“}S \textit{ is a stationary subset of } \aleph_{\omega+1} \cap \mathrm{cof}(< \aleph_n)\text{”}$$

then S *is stationary in* $V_2^{\mathbb{T}_n}$.

Proof. Let us fix G which is $\mathbb{S}$-generic over V_1, and let $\vec{\mathcal{C}}$ be the weak square sequence added by G. Let $V_2 = V_1[G]$, and work in V_2. Suppose that S is a counterexample to the statement of the lemma, and fix $p_0, \dot{D}$ such that

$$p_0 \Vdash^{V_2}_{\mathbb{T}_n} \dot{D} \text{ is club in } \aleph_{\omega+1} \text{ and } \dot{D} \cap \check{S} = \emptyset.$$

For each $\alpha < \aleph_{\omega+1}$ let

$$D_\alpha = \{\, q \in \mathbb{T}_n \ :\ \max(q) \geq \alpha, \exists \beta > \alpha\ q \Vdash \check{\beta} \in \dot{D} \,\}.$$

D_α is dense below p_0 in $\mathbb{T}_n$. Now we define $Z \subseteq \aleph_{\omega+1} \times \aleph_{\omega+1}$ by $Z = \{\, (\beta, \gamma) \ :\ x_\gamma \in D_\beta \,\}$.

By the construction of $\vec{\mathcal{U}}$, we may fix C club in $\aleph_{\omega+1}$ such that for all $\delta \in \lim(C)$

- $C \cap \delta \in \mathcal{U}_\delta$.
- $Z \cap \delta \times \delta \in \mathcal{U}_\delta$.

It follows that if $\delta \in \lim(C)$ then $\langle D_\beta \cap \{\, x_\gamma \ :\ \gamma < \delta \,\} : \beta < \delta \rangle \in \mathrm{Sk}_\lambda(\mathcal{U}_\delta)$.

For each $\delta < \aleph_{\omega+1}$ let

$$N_\delta = \mathrm{Sk}_\lambda(\aleph_\omega \cup \delta \cup \{\, \mathcal{U}_\alpha \ :\ \alpha < \delta \,\} \cup \{\, \mathcal{C}_\alpha \ :\ \alpha < \delta \,\}),$$

and observe that $N_\delta \cap \aleph_{\omega+1} \in \aleph_{\omega+1}$ and that $\langle N_\delta \cap \aleph_{\omega+1} : \delta < \aleph_{\omega+1} \rangle$ is continuous. Also $\langle N_\delta \cap \aleph_{\omega+1} : \delta < \aleph_{\omega+1} \rangle \in V_2$.

Since S is stationary in V_2 we may find ζ such that

1. $p_0 \in N_\zeta$.
2. $\zeta \in \lim(C) \cap S$.
3. $N_\zeta \cap \aleph_{\omega+1} = \zeta$.
4. $N_\zeta \cap \mathbb{T}_n = \{ x_\alpha \ : \ \alpha < \zeta, x_\alpha \in \mathbb{T}_n \}$.
5. For all $\beta < \zeta$, $D_\beta \cap N_\zeta$ is dense below p_0 in $\mathbb{T}_n \cap N_\zeta$.

As $\zeta \in S$ we know that $\mathrm{cf}(\zeta) < \aleph_n$, say $\mathrm{cf}(\zeta) = \aleph_m$ for some $m < n$. Using Lemma 12.13, we choose $D \in \mathcal{C}_\zeta$ such that $\mathrm{ot}(D) = \aleph_m$. Let D be enumerated in increasing order as $\langle \alpha_i : i < \aleph_m \rangle$. We will build $\langle p_i : i \leq \aleph_m \rangle$ in such a way that

1. For $i \leq \aleph_m$, $p_i \in \mathbb{T}_n$.
2. For $i < \aleph_m$, $p_i \in N_\zeta$ and $\vec{p} \restriction i \in N_\zeta$.

Let $\gamma_j = \max(p_j)$. We describe the construction of $\vec{p}$.

Case 1: Suppose that $i < \aleph_m$ and we have defined $\langle p_j : j \leq i \rangle$. Choose β_i to be the least ordinal greater than γ_i such that both C and D have nonempty intersection with the interval (γ_i, β_i).

$D_{\beta_i} \cap N_\zeta$ is dense in $\mathbb{T}_n \cap N_\zeta$ and $N_\zeta \cap \mathbb{T}_n = \{ x_\alpha \ : \ \alpha < \zeta, x_\alpha \in \mathbb{T}_n \}$. We choose p_{i+1} to be x_γ, where $\gamma < \zeta$ is chosen minimal such that $x_\gamma \leq p_i$ and $x_\gamma \in D_{\beta_i}$.

Case 2: $i < \aleph_m$ is limit and we have defined $\langle p_j : j < i \rangle$. Let $\gamma_i = \sup_{j<i} \gamma_j$, $c_i = \bigcup_j p_j$, $p_i = c_i \cup \{\gamma_i\}$. We need to show that $p_i \in \mathbb{T}_n \cap N_\zeta$ and that $\langle p_j : j < i \rangle \in N_\zeta$.

Notice that by construction $\gamma_i \in \lim(C) \cap \lim(D)$. Since $\gamma_i \in \lim(D)$, $D \cap \gamma_i \in \mathcal{C}_{\gamma_i}$. Since $\gamma_i \in \lim(C)$, $C \cap \gamma_i \in \mathcal{U}_{\gamma_i}$ and

$$\langle D_\beta \cap \langle x_\delta : \delta < \gamma_i \rangle : \beta < \gamma_i \rangle \in \mathrm{Sk}_\lambda(\aleph_\omega \cup \mathcal{U}_{\gamma_i}).$$

Now $\langle p_j : j < i \rangle$ is definable in H_λ from the parameters $D \cap \gamma_i$, $C \cap \gamma_i$, $\langle D_\beta \cap \langle x_\delta : \delta < \gamma_i \rangle : \beta < \gamma_i \rangle$. Therefore $\langle p_j : j < i \rangle \in \mathrm{Sk}_\lambda(\aleph_\omega \cup \mathcal{U}_{\gamma_i} \cup \mathcal{C}_{\gamma_i})$. Since c_i and p_i are definable from $\langle p_j : j < i \rangle$, we see that

$$c_i, p_i \in \mathrm{Sk}_\lambda(\aleph_\omega \cup \mathcal{U}_{\gamma_i} \cup \mathcal{C}_{\gamma_i})$$

By the definition of N_ζ, we see that p_i and $\vec{p} \restriction i$ are in N_ζ.

By construction c_i is a club subset of γ_i such that $\forall \delta \in \lim(c_i)\ c_i \cap \delta \in \mathcal{C}_\delta$, so by Lemma 12.13 $c_i \in \mathcal{C}_{\gamma_i}$. Since $i < \aleph_m < \aleph_n$ and each p_j is in $\mathbb{T}_n$, $\mathrm{ot}(c_i) < \aleph_n$ and so $p_i \in \mathbb{T}_n$.

Case 3: We have defined $\langle p_j : j < \aleph_m \rangle$. By the construction, $\zeta = \sup_{j<\aleph_m} \gamma_j$. Let $c_{\aleph_m} = \bigcup_{j<\aleph_m} p_j$, $p_{\aleph_m} = c_{\aleph_m} \cup \{\zeta\}$.

We chose D to be in $\mathcal{C}_\zeta$. Since $\zeta \in \lim(C)$, $C \cap \zeta \in \mathcal{U}_\zeta$ and

$$\langle D_\beta \cap \langle x_\delta : \delta < \zeta \rangle : \beta < \zeta \rangle \in \mathrm{Sk}_\lambda(\aleph_\omega \cup \mathcal{U}_\zeta).$$

Now $\langle p_j : j < \aleph_m \rangle$ is definable in H_λ from the parameters D, $C \cap \zeta$, $\langle D_\beta \cap \langle x_\delta : \delta < \zeta \rangle : \beta < \zeta \rangle$. Therefore $\langle p_j : j < \aleph_m \rangle \in \mathrm{Sk}_\lambda(\aleph_\omega \cup \mathcal{U}_\zeta \cup \mathcal{C}_\zeta)$. Since $c_{\aleph_m}$ and $p_{\aleph_m}$ are definable from $\langle p_j : j < \aleph_m \rangle$,

$$c_{\aleph_m}, p_{\aleph_m} \in \mathrm{Sk}_\lambda(\aleph_\omega \cup \mathcal{U}_\zeta \cup \mathcal{C}_\zeta).$$

By construction $c_{\aleph_m}$ is a club subset of ζ such that

$$\forall \delta \in \lim(c_{\aleph_m})\ c_{\aleph_m} \cap \delta \in \mathcal{C}_\delta,$$

so by Lemma 12.13 $c_{\aleph_m} \in \mathcal{C}_\zeta$. Since $\aleph_m < \aleph_n$ and each p_j is in $\mathbb{T}_n$, $\mathrm{ot}(c_{\aleph_m}) < \aleph_n$ and so $p_{\aleph_m} \in \mathbb{T}_n$.

The construction of p gives us that $p \Vdash \zeta \in \lim(\dot{D})$, and since p_0 forces $\dot{D}$ to be club $p \Vdash \zeta \in \dot{D}$. This contradicts the choice of $\dot{D}$ as a name for a club set disjoint from S.

Therefore forcing with $\mathbb{T}_n$ over V_2 preserves the stationarity of S. □

12.4. **Reflection holds in V_2.** In V_2, let $\langle S_k : k < \aleph_n \rangle$ be a sequence of stationary subsets of the stationary set $\aleph_{\omega+1} \cap \mathrm{cof}(< \aleph_n)$. We finish the proof of Theorem 21 by proving that there exists β such that $\mathrm{cf}(\beta) = \aleph_n$ and all of the S_k reflect at β.

We saw in the last section that $\mathbb{S} * \mathbb{T}_{n+1}$ has an $\aleph_{n+1}$-directed closed subset, and that forcing over V_2 with $\mathbb{T}_{n+1}$ preserves the stationarity of each S_k. Notice that $\mathbb{T}_{n+1}$ has added a club set of order type $\aleph_{n+1}$ to $\aleph_{\omega+1}$, and that $\aleph_{\omega+1}$ has become an ordinal of cofinality $\aleph_{n+1}$ in $(V_2)^{\mathbb{T}_n}$.

Let $\vec{C}$ be $\mathbb{S}$-generic over V_1 and let c be $\mathbb{T}_{n+1}$-generic over $V_2 = V_1[\vec{C}]$. By 6.10 there is a generic extension of $V_2[c]$ by $\aleph_n$-closed forcing, V_3 say, in which there exists $j : V_2[c] \longrightarrow M \subseteq V_3$ such that

- $\mathrm{crit}(j) = \aleph_{n+1}^{V_1}$.
- $j \restriction \aleph_{\omega+1}^{V_1} \in M$.
- $j(\aleph_{n+1}^{V_1}) > \aleph_{\omega+1}^{V_1}$.
- $\sup(j``\aleph_{\omega+1}^{V_1}) < j(\aleph_{\omega+1}^{V_1})$.
- $M \models \mathrm{cf}(\aleph_{\omega+1}^{V_1}) = \aleph_n = \aleph_n^{V_1}$.

We claim that each S_k is stationary in V_3. To see this observe that since $\mathrm{cf}(\aleph_{\omega+1}^{V_1}) = \aleph_{n+1}$ in $V_2[c]$, we can fix $f : \aleph_{n+1} \longrightarrow \aleph_{\omega+1}^{V_1}$ continuous increasing and cofinal. Let $T_k = \{ \alpha : f(\alpha) \in S_k \}$, then T_k is a stationary subset of $\aleph_{n+1} \cap \mathrm{cof}(< \aleph_n)$ in $V_2[c]$. It will be enough to prove that T_k is stationary in V_3.

Now $V_2[c]$ is a model of GCH and so $\square^*_{\aleph_n}$ holds in $V_2[c]$. A fortiori $AP_{\aleph_n}$ holds in $V_2[c]$ and so we can appeal to Lemma 6.14.

Since V_3 is an extension of $V_2[c]$ by $\aleph_n$-closed forcing, each of the sets T_k (and hence each of the sets S_k) is stationary in V_3. Now we can use a familiar line of argument to finish the proof. Let $\mu =$

$\sup j''\aleph_{\omega+1}$, then in V_3 each of the sets $j``S_k$ is stationary in μ (j has critical point $\aleph_{n+1}$ so is continuous at points of smaller cofinality). μ has cofinality $\aleph_n$ in V_3. Let $\langle U_k : k < \aleph_n \rangle = j(\langle S_k : k < \aleph_n \rangle)$, then $M \models$ "for all $k < \aleph_n$, $U_k \cap \mu$ is stationary". By the elementarity of j, there is β such that in V_2 the cofinality of β is $\aleph_n$ and each $S_k \cap \beta$ is stationary.

This concludes the proof of Theorem 21.

Corollary 12.16. *By Theorem 5,* $\mathrm{VGS}_{\aleph_\omega}$ *fails in the model of Theorem 21. So* $\Box^*_{\aleph_\omega}$ *does not imply* $\mathrm{VGS}_{\aleph_\omega}$.

Remark 12.17. It is natural to ask whether Theorem 21 can be improved to give the consistency of $\mathrm{Refl}^*([\kappa^+]^{\aleph_0})$ with $\Box^*$ (say by making the $\mathbb{T}_n$ forcings proper). This is impossible because by Theorem 7 $\Box^*_\kappa \implies \mathrm{ADS}_\kappa$, while by Theorem 8 $\mathrm{Refl}^*([\kappa^+]^{\aleph_0}) \implies \neg\mathrm{ADS}_\kappa$.

13. More on reflection

The model of Theorem 21 has a very strong form of simultaneous reflection for stationary subsets of $\aleph_{\omega+1}$, but by Remark 12.17 the principle $\mathrm{Refl}^*([\aleph_{\omega+1}]^{\aleph_0})$ fails there. It is easy to see that $\mathrm{Refl}^*([\kappa^+]^{\aleph_0})$ implies $\mathrm{Refl}(T)$, where $T = \{ \alpha < \kappa^+ : \mathrm{cf}(\kappa) = \omega \}$. In this section we show that Refl^* does not imply any form of simultaneous reflection. We use the forcing axiom MM^+ from [14].

Theorem 23. *Let* MM^+ *hold. Let* κ *be regular and uncountable. Then there is a forcing extension in which*

- $\mathrm{Refl}(2, \kappa)$ *fails.*
- $\mathrm{Refl}^*([\lambda]^{\aleph_0})$ *holds for every* λ.

Proof. Let $V \models MM^+$. Define a forcing $\mathbb{P}$ to add a pair of stationary subsets of κ which do not reflect simultaneously. A condition in $\mathbb{P}$ is a pair $p = (p_0, p_1)$ where

- $\mathrm{dom}(p_0) = \mathrm{dom}(p_1) < \kappa$ (we abuse notation by referring to the common domain as "*dom*(p)").
- $\mathrm{range}(p_i) \subseteq 2$ for $i = 0, 1$.
- $\forall \alpha\, p_i(\alpha) = 1 \implies \mathrm{cf}(\alpha) = \omega$ for $i = 0, 1$.
- If $\mathrm{cf}(\beta) > \omega$ and $\beta \leq \mathrm{dom}(p)$ then there exist c club in β and $i < 2$ such that $\forall \alpha \in c\; p_i(\alpha) = 0$.

The ordering on $\mathbb{P}$ is extension.

It is easy to check that $\mathbb{P}$ is countably closed, and is also $<\kappa$-strategically closed. $\mathbb{P}$ adds a pair of sets X_0, X_1 which do not reflect simultaneously; a straightforward density argument shows that each of the sets X_0, X_1 and $Y =_{\mathrm{def}} \kappa - (X_1 \cup X_2)$ is stationary in $V^{\mathbb{P}}$.

We now claim that $\mathrm{Refl}^*([\lambda]^{\aleph_0})$ holds in $V^{\mathbb{P}}$ for every uncountable λ. Let λ be uncountable and suppose

$$\Vdash_{\mathbb{P}} \text{“}S \text{ is a stationary subset of } [\lambda]^{\aleph_0}\text{”}.$$

Let $\mathbb{Q} = \mathrm{Coll}(\aleph_1, \max\{\kappa, \lambda\})$. If $H : \aleph_1 \longmapsto \max\{\kappa, \lambda\}$ is the generic surjection added by $\mathbb{Q}$ then we can define “collapsed” versions of the sets X_i, Y and S in $V^{\mathbb{P}*\mathbb{Q}}$. To be more precise let

$$s = \{\ \alpha < \aleph_1\ :\ H“\alpha \cap \lambda \in S\ \},$$

$$x_i = \{\ \alpha < \aleph_1\ :\ \sup(H“\alpha \cap \kappa) \in X_i\ \},$$

and

$$y = \{\ \alpha < \aleph_1\ :\ \sup(H“\alpha \cap \kappa) \in Y\ \}.$$

Since $\mathbb{Q}$ is countably closed each of the sets X_i, Y, S is stationary in $V^{\mathbb{P}*\mathbb{Q}}$, and so easily each of the sets x_i, y, s is also stationary.

Let z be the first set in the list x_0, x_1, y such that $s \cap z$ is stationary, and define $\mathbb{R}$ to be the standard forcing [3] to add a closed unbounded subset of z; conditions in $\mathbb{R}$ are closed bounded subsets of z, ordered by extension. We make the remark that (by the closure of $\mathbb{P}$ and of $\mathbb{Q}$) conditions in the forcings $\mathbb{Q}$ and $\mathbb{R}$ are actually sets in V; we will identify $\mathbb{P}*\mathbb{Q}*\mathbb{R}$ with the dense subset of conditions (p, q, r) such that q and r are in V, and in which $\mathrm{dom}(p) = \sup(\mathrm{range}(q) \cap \kappa)$ (the latter assumption will be technically convenient later).

Claim. *Forcing with $\mathbb{P} * \mathbb{Q} * \mathbb{R}$ preserves stationary subsets of $\aleph_1$.*

Proof. Let $T \subseteq \aleph_1$ be stationary. Let (p, q, r) be a condition in $\mathbb{P}*\mathbb{Q}*\mathbb{R}$ and let $\dot{C}$ be a name for a club set in $\aleph_1$. We may assume that p is strong enough to determine the choice of z, and for the sake of definiteness we assume that $p \Vdash z = x_0$ (the other cases will be similar). As usual we fix some large regular θ and some well ordering $<_\theta$ of H_θ, and then choose $N \prec (H_\theta, \in, <_\theta)$ such that $(p, q, r), \dot{C}, \mathbb{P} * \mathbb{Q} * \mathbb{R} \in N$, $|N| = \omega$ and $\delta =_{\mathrm{def}} N \cap \aleph_1 \in T$.

Now we build a decreasing chain of conditions $\langle (p_n, q_n, r_n) : n < \omega \rangle$ from $\mathbb{P} * \mathbb{Q} * \mathbb{R} \cap N$ so as to meet every dense subset of $\mathbb{P} * \mathbb{Q} * \mathbb{R}$ which lies in N. Let $p_\omega =_{\mathrm{def}} \bigcup_n p_n$ and $q_\omega =_{\mathrm{def}} \bigcup_n q_n$, then $(p_\omega, q_\omega) \in \mathbb{P} * \mathbb{Q}$; p_ω is a pair of functions with common domain $\alpha =_{\mathrm{def}} \sup(N \cap \kappa)$, and q_ω is a function with $\mathrm{dom}(q_\omega) = \delta$ and $\mathrm{range}(q_\omega) = N \cap \max\{\kappa, \lambda\}$. $r_\omega = \bigcup_n r_n$ is a closed and unbounded subset of δ.

In particular q_ω forces that $\sup(H“\delta \cap \kappa) = \alpha$. This means that if we extend p_ω to a condition p^* to force $\alpha \in X_0$, then $(p^*, q_\omega) \Vdash \delta \in x_0$; since $p \Vdash x_0 = z$ this implies that $(p^*, q_\omega, r_\omega \cup \{\delta\})$ is a condition in

$\mathbb{P}*\mathbb{Q}*\mathbb{R}$. As usual the condition $(p^*, q_\omega, r_\omega \cup \{\delta\})$ forces that $\delta \in \dot{C} \cap T$, so we have shown that the stationarity of T is preserved. ☐

Claim. *s is a stationary subset of $\aleph_1$ in $V^{\mathbb{P}*\mathbb{Q}*\mathbb{R}}$.*

Proof. It is a standard fact that shooting a club set through a stationary subset $A \subset \aleph_1$ preserves the stationarity of stationary subsets of A. We chose z so that $s \cap z$ was stationary in $V^{\mathbb{P}*\mathbb{Q}}$, hence $s \cap z$ (and so *a fortiori* s itself) is stationary in $V^{\mathbb{P}*\mathbb{Q}*\mathbb{R}}$. ☐

We are now in a position to apply MM^+ to the stationary-preserving forcing $\mathbb{P}*\mathbb{Q}*\mathbb{R}$ and the name s. We define some dense sets in $\mathbb{P}*\mathbb{Q}*\mathbb{R}$ (here $\alpha < \aleph_1$).

$$
\begin{aligned}
D_0 &= \{ (p,q,r) \ : \ \text{“}p \text{ decides whether } z \text{ is } x_0,\ x_1 \text{ or } y\text{”}. \} \\
D_1^\alpha &= \{ (p,q,r) \ : \ \alpha < \sup(r) \} \\
D_2^\alpha &= \{ (p,q,r) \ : \ \alpha \subseteq \operatorname{dom}(q), \exists \beta \in \operatorname{dom}(q)\ q(\beta) \in \lambda - \sup(q\text{“}\alpha \cap \lambda) \} \\
D_3^\alpha &= \{ (p,q,r) \ : \ \alpha \subseteq \operatorname{dom}(q), \exists \beta \in \operatorname{dom}(q)\ q(\beta) \in \kappa - \sup(q\text{“}\alpha \cap \kappa) \}
\end{aligned}
$$

Applying MM^+ to $\mathbb{P}*\mathbb{Q}*\mathbb{R}$ and $\{D_0\} \cup \{ D_i^\alpha \ : \ \alpha < \aleph_1, 1 \leq i \leq 3 \}$ we get a filter F on $\mathbb{P}*\mathbb{Q}*\mathbb{R}$ which meets all these dense sets and which realises s as a stationary set (we will denote the realisation of s by s^*).

Let $P = \bigcup \{ p \ : \ (p,q,r) \in F \}$ and define Q, R similarly.

Claim. *$P \in \mathbb{P}$ and $P \Vdash_{\mathbb{P}}$ “$S \cap [\operatorname{range}(Q) \cap \lambda]^{\aleph_0}$ is stationary”*

Proof. Since $\operatorname{dom}(p) = \sup(\operatorname{range}(q) \cap \kappa)$ for all the conditions $(p,q,r) \in F$, $\operatorname{dom}(P) = \sup(\operatorname{range}(Q) \cap \kappa)$. Let α denote $\operatorname{dom}(P)$, then we met enough dense sets to guarantee that $\operatorname{cf}(\alpha) = \aleph_1$. We also met enough sets to ensure that $|\operatorname{range}(Q) \cap \lambda| = \aleph_1$. From P we can read off X_0^*, X_1^* and Y^* which are subsets of α.

We need to check that P is a condition. The point here is that R is a closed and unbounded subset of $\aleph_1$, and so $\{ \sup(Q\text{“}\alpha \cap \kappa) \ : \ \alpha \in R \}$ is a closed unbounded subset of one of the sets X_0^*, X_1^* or Y^*. This guarantees that at least one of the sets X_i^* is not stationary in α, and hence that $P \in \mathbb{P}$.

Since s^* is the realisation of s, for each $\beta \in s^*$ we have $P \Vdash Q\text{“}\beta \cap \lambda \in S$. Let $T = \{ Q\text{“}\beta \cap \lambda \ : \ \beta \in s^* \}$, then clearly

$$V \models \text{“}T \text{ is stationary in } [\operatorname{range}(Q) \cap \lambda]^{\aleph_0}\text{”}.$$

Since $\mathbb{P}$ is countably closed it is proper, so that

$$V^{\mathbb{P}} \models \text{“}T \text{ is stationary in } [\operatorname{range}(Q) \cap \lambda]^{\aleph_0}\text{”}.$$

Finally $P \Vdash T \subseteq S \cap [\operatorname{range}(Q) \cap \lambda]^{\aleph_0}$, so that the claim is proved. ☐

This concludes the proof that $\operatorname{Refl}^*([\lambda]^{\aleph_0})$ holds in $V^{\mathbb{P}}$. ☐

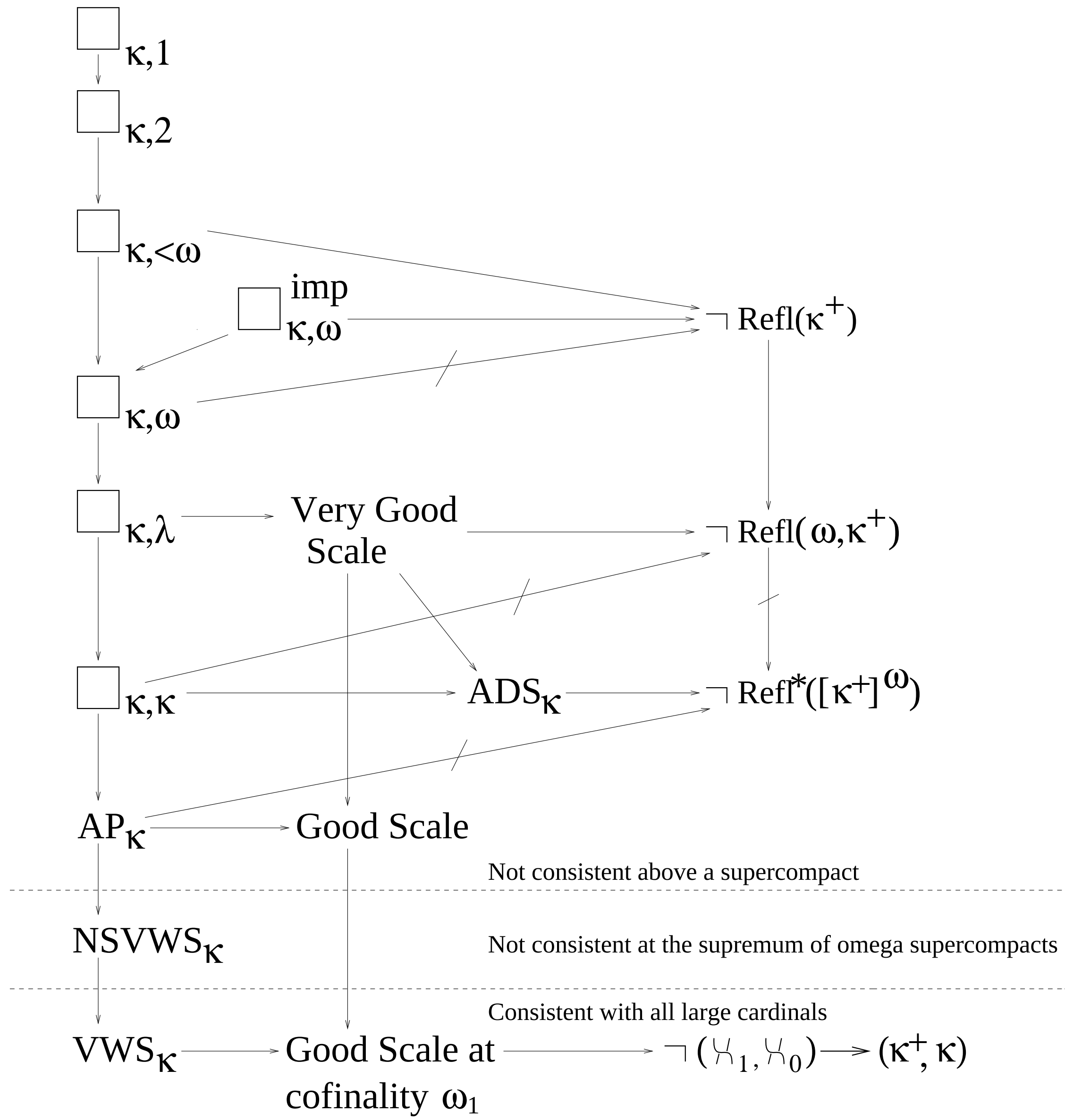
Remark 13.1. Larson [29] has independently used quite similar methods to study the relationship between various reflection principles for ordinals and countable sets of ordinals. In particular Theorem 5.5 of [29] and Theorem 23 are proved in the same way, though neither one literally subsumes the other.

14. A picture and some problems

The following picture indicates how some of the principles considered in this paper are related to each other, in the case when κ is a strong limit cardinal of cofinality ω. For the benefit of readers of [13] we have indicated the relationship of the Not So Very Weak Square principle to the other principles.

There are many interesting questions still open. For example

1. How much of $\square_\kappa$ is needed in the various classical applications of square?
2. Does VGS_κ imply $\square^*_\kappa$?
3. Does $\square_{\kappa,<\kappa}$ imply VGS_κ?



<u>κ strong limit of cofinality ω</u>

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