$\square$ ON THE SINGULAR CARDINALS

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Abstract. We give upper and lower bounds for the consistency strength of the failure of a combinatorial principle introduced by Jensen, *Square on singular cardinals.*

A combinatorial principle of great importance in set theory is the *Global $\square$ principle* of Jensen [6]:

Global $\square$: There exists $\langle C_\alpha \mid \alpha \text{ a singular ordinal } \rangle$ such that for each α, C_α is a closed unbounded subset of α of ordertype less than α, the limit points of C_α are singular ordinals, and $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ whenever $\bar\alpha$ is a limit point of C_α.

A weakening of Global $\square$ is the following: We say that $\square$ *holds on the singular cardinals* if and only if there exists $\langle C_\alpha \mid \alpha \text{ a singular cardinal } \rangle$ such that for each α, C_α is a closed unbounded subset of $\mathrm{Card} \cap \alpha$ of ordertype less than α, the limit points of C_α are singular cardinals, and $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ whenever $\bar\alpha$ is a limit point of C_α.

Jensen observed that Global $\square$ is equivalent to the conjunction of $\square$ on the singular cardinals together with his well-known principles $\square_\kappa$ for all uncountable cardinals κ. The point is that if α is a singular ordinal then either α is a singular cardinal or α is a limit ordinal with $\kappa < \alpha < \kappa^+$ for a unique cardinal κ: the fragment of Global $\square$ which assigns a singularising club set C_α to each limit α with $\kappa < \alpha < \kappa^+$ corresponds to $\square_\kappa$.

$\square$ principles are typically used as witnesses to various forms of incompactness or non-reflection: for example if $\square_\kappa$ holds then there exist κ^+-Aronszajn trees and non-reflecting stationary subsets of κ^+. Since large cardinal axioms embody principles of compactness and reflection, it is not surprising that there is some tension between $\square$ and large cardinals, and this has been extensively investigated. We should distinguish here between the questions "To what extent are large cardinals

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compatible with $\Box$ principles?" and "What is the consistency strength of the failure of $\Box$ principles?"

The situation for $\Box_\kappa$ is fairly well understood. Jensen [11] showed that if κ is *subcompact* (a large cardinal property a little stronger than 1-extendibility) then $\Box_\kappa$ fails; Burke [1] gave a rather similar argument using generic elementary embeddings. Cummings and Schimmerling [2] gave a forcing proof that Global $\Box$ is consistent with the existence of a 1-extendible cardinal. Foreman and Magidor [3] identified a natural weakening of $\Box_\kappa$ (the Very Weak Square) which is consistent with all standard large cardinals; the same paper discusses weakenings of $\Box_\kappa$ (based on work of Baumgartner) where the club sets only have to exist at points of certain cofinalities, and which are again consistent with large cardinals. Friedman [4] studied the consistency of various L-like combinatorial principles, including squares on restricted cofinalities, with very strong large cardinal axioms; this paper also contains a discussion of the relation between subcompactness and other strong axioms.

As for consistency strength, it is known that the failure of $\Box_\lambda$ for λ regular and uncountable is equiconsistent with the existence of a Mahlo cardinal. Jensen [6] showed that if $\Box_\lambda$ fails then λ^+ is Mahlo in L, and Solovay showed that collapsing a Mahlo cardinal to λ^+ by a Lévy collapse yields a model in which $\Box_\lambda$ fails.

Failure of $\Box_\lambda$ for λ singular is stronger and rather more mysterious; an unpublished forcing construction of Zeman gives a measurable subcompact cardinal as an upper bound, while at the lower end Steel [12] has shown that if κ is a singular strong limit cardinal and $\Box_\kappa$ fails then $\mathrm{AD}^{L(\mathbb{R})}$ holds.

The lower bounds in consistency strength are proved using inner models, so it is important to know that $\Box$ principles are true in these models. Global $\Box$, and hence $\Box$ on the singular cardinals, holds in L [6]. Welch [13] showed that Global $\Box$ holds in the Dodd-Jensen core model, and Wylie [14] showed that Global $\Box$ holds in Jensen's core model with measures of order zero. Jensen [7] showed that $\forall\kappa\ \Box_\kappa$ holds in the core model below zero-pistol, and Zeman [15] showed that Global $\Box$ holds in this model. Schimmerling and Zeman [11] showed that $\forall\kappa\ \Box_\kappa$ holds in all extender models which are currently known to exist, and Zeman [16] showed that Global $\Box$ holds in all such models.

In this paper we determine a lower bound for the consistency strength of a failure of $\Box$ on the singular cardinals, and also analyse the large cardinal hypotheses required to refute this principle.

Theorem 1. *If it is consistent with ZFC that $\Box$ on the singular cardinals up to α fails for some cardinal α, then it is consistent with ZFC that there is an inaccessible cardinal which is a stationary limit of cardinals of uncountable Mitchell order.*

To properly state a global result, we make use of Gödel-Bernays class theory GB:

Theorem 2. *If it is consistent with GB that $\Box$ on the singular cardinals fails, then it is consistent with GB that the class of cardinals of uncountable Mitchell order is stationary.*

In order to state our last theorem, we need some definitions. We say that a cardinal κ is *almost inaccessibly hyperstrong* iff it is the critical point of an elementary embedding $j : V \to M$ with $V_\lambda \subseteq M$ for some M-inaccessible $\lambda > j(\kappa)$. To clarify where this axiom sits in the large cardinal hierarchy we consider some more large cardinal properties:

- κ is *inaccessibly hyperstrong* iff it is the critical point of an elementary embedding $j : V \to M$ with $V_\lambda \subseteq M$ for some V-inaccessible $\lambda > j(\kappa)$.
- I_κ *holds* if and only if there is an inaccessible cardinal $\gamma > \kappa$ such that κ is γ-supercompact.
- J_κ *holds* if and only if there is an inaccessible cardinal $\gamma > \kappa$ such that κ is *$< \gamma$-supercompact.* That is there is $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \gamma$ and ${}^{<\gamma}M \subseteq M$.

Lemma 3.

(1) *If κ is inaccessibly hyperstrong then there are unboundedly many $\mu < \kappa$ such that I_μ.*
(2) *If I_κ holds then there are unboundedly many almost inaccessibly hyperstrong $\mu < \kappa$.*
(3) *If κ is almost inaccessibly hyperstrong then there are unboundedly many $\mu < \kappa$ such that J_μ.*

Proof. Suppose first that κ is inaccessibly hyperstrong and fix $j : V \to M$ and V-inaccessible $\lambda > j(\kappa)$ with $V_\lambda \subseteq M$. Let $\bar{\lambda}$ be the least inaccessible greater than κ. Then $j(\bar{\lambda}) \leq \lambda$ and $j``\bar{\lambda}$ is bounded in $j(\bar{\lambda})$, so $j``\bar{\lambda} \in M$ and thus κ is $\bar{\lambda}$-supercompact. Since $\bar{\lambda} < j(\kappa) < \lambda$ we see that I_κ holds in M, so as usual there are many $\mu < \kappa$ such that I_μ holds in V.

Now suppose that I_κ holds, let γ be the least inaccessible greater than κ and fix $j : V \to M$ such that $j(\kappa) > \gamma$ and ${}^{\gamma}M \subseteq M$. Let $\eta = \sup j``\gamma$, let E be the (long) (κ, η) extender for j, and note that since $j \restriction V_\gamma \in M$, also $E \in M$. Let $k : M \to N = Ult(M, E)$ be the ultrapower map, then as usual we may argue that $V_\eta^M = V_\eta^N$ and $j \restriction V_\gamma = k \restriction V_\gamma$. Since k is continuous at γ and j is not, $k(\kappa) = j(\kappa) < k(\gamma) = \eta < j(\gamma)$. So $k(\gamma)$ is inaccessible in N (but not in M). It is now easy to see that κ is almost inaccessibly hyperstrong in M.

Finally let κ be almost inaccessibly hyperstrong and fix $j : V \to M$ and M-inaccessible $\lambda > j(\kappa)$ with $V_\lambda \subseteq M$. Let γ be the least inaccessible greater than κ, so that $\gamma < j(\kappa) < \lambda$ and $j(\gamma) \leq \lambda$. Then for every $\delta < \gamma$ we have $j``\delta \in M$, and if U_δ is the induced supercompactness measure then $\langle U_\delta : \delta < \gamma \rangle$ forms a system of measures whose limit ultrapower witnesses that κ is $< \gamma$-supercompact. Of course $\langle U_\delta : \delta < \gamma \rangle \in M$ and so easily J_κ holds in M. $\square$

Theorem 4.

(1) *If κ has the property I_κ then $\square$ fails on the singular cardinals (and in fact it fails on the singular cardinals less than γ, where γ is the least inaccessible greater than κ). (Due independently to Dickon Lush, see the following Remark).*
(2) *There is a class forcing extension of V in which all almost inaccessibly hyperstrong cardinals retain that property and $\square$ holds on the singular cardinals.*

Remark 1. *The referee pointed out that Dickon Lush, a former student of Jensen, had proved Theorem 4 part 1 and a slightly weaker form of Theorem 4 part 2 in some work which was never written up.*

Jensen informed the authors that he believes that Lush proved the following results:

(1) *If there is a cardinal κ such that I_κ then $\square$ on the singular cardinals fails to hold.*
(2) *It is consistent that $\square$ holds on the singular cardinals and there is a supercompact cardinal.*

We will prove Theorem 1 using Mitchell's core model for sequences of measures [8, 9, 10]. This is an inner model K_M which is defined on the assumption that there is no inner model in which there is κ such that $o(\kappa) = \kappa^{++}$. We list the salient properties of K_M:

Fact 5.

- *Global $\square$ holds in K_M.*
- *If λ is a singular cardinal of uncountable cofinality, then either λ is singular in K_M or $K_M \models o(\lambda) \geq cf^V(\lambda)$.*

Proof of Theorem 1.

We will prove that if there is no inner model of “there is a stationary limit of cardinals of uncountable Mitchell order”, then $\Box$ holds on the singular cardinals up to α for every cardinal α. The argument is of the same general form as Jensen's argument [6] that if $\Box_{\omega_1}$ fails then ω_2 is Mahlo in L.

By our assumption there is no inner model of “there is κ such that $o(\kappa) = \kappa^{++}$”, and so the Mitchell core model K_M exists. If κ is V-inaccessible then by our assumption there is a club $C \subseteq \kappa$ in K_M such that $o^{K_M}(\alpha) < \omega_1^{K_M}$ for all $\alpha \in C$, and so *a fortiori* $o^{K_M}(\alpha) < \omega_1^V$ for all $\alpha \in C$.

We establish $\Box$ on the singular cardinals up to α by induction on the cardinal α. We may assume that α is a limit cardinal, else the result follows immediately by induction. If α is singular, then let C_α be any closed unbounded subset of Card$\cap\alpha$ whose ordertype is less than its minimum and whose successor elements are successor cardinals. For limit points $\bar{\alpha}$ of C_α, define $C_{\bar{\alpha}}$ to be $C_\alpha \cap \bar{\alpha}$. By induction we can choose $\Box$ sequences on the singular cardinals inside open intervals determined by adjacent elements of $C_\alpha \cup \{0\}$, thereby obtaining a $\Box$ sequence on all singular cardinals up to α.

Now suppose that α is inaccessible. There is a closed unbounded subset C of α consisting of cardinals whose Mitchell order in K_M is countable in V. By fact 5, each singular cardinal in C of uncountable cofinality is singular in K_M.

By fact 5, $\Box$ on singular cardinals holds in K_M, so there is in K_M

$$\langle C_\beta \mid \beta < \alpha,\ \beta \text{ a singular cardinal in } K_M \rangle$$

a $\Box$ sequence on the singular cardinals of K_M less than α. For each cardinal $\beta \in C$ which is singular in K_M, define D_β to be $C_\beta \cap \text{Card}$, if this is cofinal in β, and otherwise define D_β to be a set of successor cardinals cofinal in β of ordertype ω. Note that if $\beta \in C$ is a cardinal which is singular in K_M and $\bar{\beta}$ is a limit point of D_β then $\bar{\beta}$ is a cardinal which is singular in K_M and $D_{\bar{\beta}}$ equals $D_\beta \cap \bar{\beta}$. If β is a singular cardinal in C which is regular in K_M then β has cofinality ω and we again choose D_β to be a set of successor cardinals cofinal in β of ordertype ω. The result is a $\Box$ sequence on the singular cardinals in C. As in the singular case, we can extend this to a $\Box$ sequence on all singular cardinals less than α using $\Box$ sequences on the singular cardinals inside open intervals determined by adjacent elements of $C \cup \{0\}$.

The proof of Theorem 2 is similar. Assume that the class of cardinals of uncountable Mitchell order is non-stationary. Then using a closed unbounded class of cardinals of countable Mitchell order, we can construct a $\Box$ sequence on the singular cardinals, provided that for any cardinal α there is a $\Box$ sequence on the singular cardinals up to α; but if the latter were to fail, then by Theorem 1 there would be an inaccessible cardinal which is a stationary limit of cardinals of uncountable Mitchell order, more than enough to imply the consistency of "the class of cardinals of uncountable Mitchell order is stationary".

Proof of Theorem 4.
1) Let κ satisfy I_κ and let γ be the least inaccessible greater than κ. Fix $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \gamma$ and ${}^\gamma M \subseteq M$. Let $S \subseteq \gamma$ be a stationary set of cardinals of cofinality ω, and let $\eta = \sup j``\gamma$. Since $j``S \subseteq j(S) \cap \eta$ and j is continuous at points of cofinality ω, it is routine to check that $j(S) \cap \eta$ is stationary in η, and this statement is downwards absolute to M. Since γ is a limit of cardinals and j is elementary, η is a limit of M-cardinals and so η is an M-cardinal; what is more $j``\gamma \in M$ and $\gamma < j(\kappa) < \eta$ so η is a *singular* cardinal in M. So by elementarity the stationarity of S reflects at some singular cardinal μ less than γ.

This is enough to refute $\Box$ on the singular cardinals less than γ, by the standard mechanism for generating non-reflecting sets from $\Box$ sequences: if $\langle C_\alpha \mid \alpha < \gamma,\ \alpha \text{ a singular cardinal}\rangle$ is such a $\Box$ sequence, then by Fodor's lemma there is an ordinal η such that the set S of cardinals $\alpha < \gamma$ with cofinality ω and C_α of ordertype η is stationary. Then for every singular cardinal $\mu < \gamma$ it follows from the coherence of the club sets in the $\Box$ sequence that $\lim(C_\mu)$ meets S in at most one point, so that S does not reflect at any singular cardinal $\mu < \gamma$.

We note that assuming I_κ we actually established the following reflection principle: "every stationary subset of $\gamma \cap \mathrm{cof}\,(< \kappa)$ reflects at singular cardinals of arbitrarily large cofinalities less than κ". So I_κ will also be incompatible with some weakened versions of $\Box$ on the singular cardinals.

2) It remains to show that we can force $\Box$ on singular cardinals while preserving all the almost inaccessibly hyperstrong cardinals. Before adding $\Box$ we will perform a preparation step to force some instances of GCH; this will make the iteration for $\Box$ better behaved.

The preparation forcing will be a class reverse Easton iteration $\mathbb{R}$, in which we force with the standard collapsing poset $Coll(\lambda^+, 2^\lambda)$ for all λ which are singular limits of inaccessibles. Standard arguments [5] show that ZFC is preserved. We claim that in the extension

- For every μ which is a singular limit of inaccessibles, $2^\mu = \mu^+$.

- All almost inaccessibly hyperstrong cardinals from the ground model are preserved.

The first claim is straightforward; by standard arguments if μ is a singular limit of inaccessibles in the extension then it is a singular limit of inaccessibles in the ground model, and the iteration $\mathbb{R}$ then forces (by design) that $2^\mu = \mu^+$. For the second claim let the embedding $j : V \to M$ be a witness that κ is almost inaccessibly hyperstrong in V. By the usual extender arguments we may assume that

$$M = \{j(f)(a) : \text{Dom } (f) = H_\lambda, a \in H_{\lambda^*}\}$$

where λ is the least inaccessible greater than κ and $\lambda^* = \sup j[\lambda] = j(\lambda)$ is the least M-inaccessible greater than $j(\kappa)$.

Let G be $\mathbb{R}$-generic, and break up $\mathbb{R}$ as $\mathbb{R}_0 * \dot{\mathbb{R}}_1 * \dot{\mathbb{R}}_2$ where $\mathbb{R}_0$ is the part of the iteration up to stage κ, $\mathbb{R}_1$ is the part between κ and $j(\kappa)$, and $\mathbb{R}_2$ is the part above $j(\kappa)$. Break up G in the corresponding way as $G_0 * G_1 * G_2$. By the agreement between V and M there are no inaccessible cardinals between κ and λ so that $\mathbb{R}_1 * \dot{\mathbb{R}}_2$ is (λ, ∞)-distributive (i.e., does not add new λ-sequences of ordinals) in $V[G_0]$.

We now lift j as follows. Since conditions in G_0 have bounded supports $j``G_0 \subseteq G_0 * G_1$, and so we may lift to get $j : V[G_0] \to M[G_0 * G_1]$. Since $\mathbb{R}_1 * \dot{\mathbb{R}}_2$ is $(\lambda, \infty)$-distributive and $|H_\lambda| = \lambda$, $j``G_1 * G_2$ generates a generic filter H for $j(\mathbb{R}_1 * \dot{\mathbb{R}}_2)$ over $M[G_0 * G_1]$. Hence we may lift again to get in $V[G]$ an embedding $j : V[G] \to M[G_0 * G_1 * H]$.

To finish we note that there are no inaccessible cardinals between $j(\kappa)$ and λ^*, so that easily $\mathbb{R}_2$ is (λ^*, ∞)-distributive in $V[G_0 * G_1]$. It is now routine to check that $V[G]$ and $M[G_0 * G_1 * H]$ agree up to λ^*, so that κ is still almost inaccessibly hyperstrong in $V[G]$. To lighten the notation we now replace our original ground model by the generic extension of that ground model by $\mathbb{R}$, and write V for this generic extension.

Working over the new V we perform a reverse Easton iteration $\mathbb{P}$ where at each inaccessible stage α, a $\square$ sequence on the singular cardinals less than α is added via the forcing $\mathbb{Q}_\alpha$ whose conditions are partial $\square$ sequences on the singular cardinals less than or equal to some cardinal less than α, ordered by end-extension. For any cardinal $\bar{\alpha} < \alpha$, any condition in $\mathbb{Q}_\alpha$ can be extended to have length at least $\bar{\alpha}$; this is proved by induction on $\bar{\alpha}$, using the existence of generic objects at inaccessibles less than α to handle the case of inaccessible $\bar{\alpha}$, and the same idea as in the singular cardinal limit step of Theorem 1 to handle the case of singular $\bar{\alpha}$. Standard arguments show that $\mathbb{Q}_\alpha$, and indeed the entire iteration at and above stage α, is $< \alpha$-strategically closed for each inaccessible α. Similar arguments for forcing a Global $\Box$ sequence are given in some detail in [2]; see [1], [4], and [3] for more on forcing to add $\Box$-sequences of various sorts.

We claim that $\mathbb{P}$ preserves cardinals and cofinalities. For each regular κ we may factor $\mathbb{P}$ as $\mathbb{P}(\leq \kappa) * \mathbb{P}(> \kappa)$. The cardinal arithmetic which we arranged in the preparation step implies that $\mathbb{P}(\leq \kappa)$ has a dense set of size κ, so that is has κ^+-c.c. The desired preservation of cardinals and cofinalities then follows from the fact that $\mathbb{P}(> \kappa)$ is forced to be $< \kappa^+$-strategically closed.

We need a certain homogeneity fact about $\mathbb{Q}_\gamma$ for γ inaccessible. Similar arguments will work for other posets to add various kinds of $\Box$-sequences.

Lemma 6. *Let $p, q \in \mathbb{Q}_\gamma$ be conditions with the same domain. Then there is an isomorphism between $\{r \in \mathbb{Q}_\gamma : r \leq p\}$ and $\{r \in \mathbb{Q}_\gamma : r \leq q\}$.*

Proof. Let the common domain of p and q be the set of all singular cardinals less than or equal to some cardinal α.

Let $r \leq p$, so that r is a $\Box$ sequence on the singular cardinals less than or equal to some cardinal β. We define a condition r^* with the same domain as r and extending q as follows:

(1) $r^*_\eta = q_\eta$ for $\eta \leq \alpha$.
(2) If $\eta > \alpha$ and $\lim(r_\eta) \cap (\alpha + 1) = \emptyset$ then $r^*_\eta = r_\eta$.
(3) Otherwise let $\zeta = \sup(\lim(r_\eta) \cap (\alpha + 1))$ and note that ζ is a limit point of r_η, and hence is a singular cardinal less than or equal to α: in this case define $r^*_\eta = q_\zeta \cup (r_\eta \setminus \zeta)$.

Similarly if $r \leq q$ we define $r_* \leq p$ by replacing q by p in the definition above. It is easy to see that the maps $r \mapsto r^*$ and $r \mapsto r_*$ are mutually inverse and set up an isomorphism between $\{r \in \mathbb{Q}_\gamma : r \leq p\}$ and $\{r \in \mathbb{Q}_\gamma : r \leq q\}$. $\Box$

The following corollary is standard.

Lemma 7. *If $p \in \mathbb{Q}_\beta$ and G is $\mathbb{Q}_\beta$-generic there is a generic G^* such that $V[G^*] = V[G]$ and $p \in G^*$.*

Suppose that κ is almost inaccessibly hyperstrong, witnessed by $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $V_\lambda \subseteq M$, where $\lambda > j(\kappa)$ and λ is M-inaccessible. Let $\bar{\lambda}$ be the least inaccessible greater than κ, so that without loss of generality we may assume that $\lambda = j(\bar{\lambda})$.

We may assume that every element of M is of the form $j(f)(a)$ where f has domain $V_{\bar{\lambda}}$ and a is an element of $V_{\lambda'}$, where λ' is the supremum of $j[\bar{\lambda}]$. Otherwise, we can replace M by the transitive collapse $\bar{M}$ of the class H of all such $j(f)(a)$'s and j by $\pi \circ j$, where $\pi : H \simeq \bar{M}$. In particular, we can assume that $\lambda' = \lambda$, that is to say j is continuous at $\bar{\lambda}$, and every element of M is of the form $j(f)(a)$ where f has domain V_μ for some $\mu < \bar{\lambda}$ and a is an element of V_λ.

We show that in $V[G]$ there is a $j(\mathbb{P}) = \mathbb{P}^*$ generic G^* over M which contains $j[G]$ as a subclass. It then follows that j can be definably lifted in $V[G]$ to $j^* : V[G] \to M[G^*]$, where $V[G]$ and $M[G^*]$ have the same sets of rank less than λ, and therefore κ is almost inaccessibly hyperstrong in $V[G]$.

Define $G^*_{j(\kappa)} =$ (G^* below stage $j(\kappa)$ of the $\mathbb{P}^*$ iteration) to be the same as $G_{j(\kappa)}$. The generic object $G(\kappa)$ is a condition in $j(\mathbb{Q}_\kappa)$, so by Lemma 7 we may obtain $G^*(j(\kappa)) =$ (G^* at stage $j(\kappa)$ of the $\mathbb{P}^*$ iteration) by altering $G(j(\kappa))$ to get a condition extending $G(\kappa)$. Thus we obtain a lifting of j to an elementary embedding $j_0^* : V[G_{\kappa+1}] \to M[G^*_{j(\kappa)+1}]$.

λ is the least nontrivial stage of the $\mathbb{P}^*$ iteration past $j(\kappa)$. We define $G^*(\lambda)$ to consist of all conditions extended by some condition of the form $j_0^*(p)$ where p belongs to $G(\bar{\lambda})$, the generic chosen by G at stage $\bar{\lambda}$ in the $\mathbb{P}$-iteration. We claim that $G^*(\lambda)$ so defined is indeed generic over $M[G^*_\lambda]$ for $\mathbb{P}^*(\lambda)$. Suppose that D is open dense on $\mathbb{P}^*(\lambda)$ and belongs to $M[G^*_\lambda]$. Then D is of the form $j(f)(a)^{G^*_\lambda}$ where a belongs to V_λ and $\mathrm{dom}(f) = V_\mu$ for some $\mu < \bar{\lambda}$. Now using the $\bar{\lambda}$-distributivity of the forcing $\mathbb{P}(\bar{\lambda})$, choose p in $G(\bar{\lambda})$ that meets all open dense subsets of $\mathbb{P}(\bar{\lambda})$ of the form $f(\bar{a})^{G_{\bar{\lambda}}}$, $\bar{a}$ in V_μ. Then $j_0^*(p)$ belongs to $G^*(\lambda)$ and meets all open dense subsets of $\mathbb{P}^*(\lambda)$ of the form $j(f)(b)^{G^*_\lambda}$, b in $V_{j(\mu)}$. In particular, $j_0^*(p)$ meets $j(f)(a)^{G^*_\lambda} = D$. Thus we have demonstrated the genericity of $G^*(\lambda)$ and can extend j_0^* to an elementary embedding $j_1^* : V[G_{\bar{\lambda}+1}] \to M[G^*_{\lambda+1}]$.

Finally, as the iteration $\mathbb{P}$ strictly above stage $\bar{\lambda}$ is $\bar{\lambda}^+$-distributive, it is easy to use the previous argument to extend the embedding j_1^* to all of $V[G]$, taking G^* strictly above stage λ to consist of all conditions extended by some condition of the form $j_1^*(p)$ where p belongs to G strictly above stage $\bar{\lambda}$. This proves Theorem 4.

Remark 2. *The referee pointed out that we do not actually need the final embedding $j^* : V[G] \to M[G^*]$, since it will suffice to have $j' : V[G] \to N$ where $j'(\bar{\lambda}) = \lambda$ and the models N and $V[G]$ agree to rank λ. Such a j' can be built from j_1^* using the usual extender techniques.*

We finish with the natural open question:
Question: What is the right upper bound for the consistency strength of "□ on singular cardinals fails"? Is Theorem 1 optimal?

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