“Identity Crises and Strong Compactness”

by

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Abstract: Combining techniques of the first author and Shelah with ideas of Magidor, we show how to get a model in which, for fixed but arbitrary finite n, the first n strongly compact cardinals $\kappa_1, \ldots, \kappa_n$ are so that κ_i for $i = 1, \ldots, n$ is both the i^{th} measurable cardinal and κ_i^+ supercompact. This generalizes an unpublished theorem of Magidor and answers a question of Apter and Shelah.

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§0 Introduction and Preliminaries

As is well-known, the notion of strong compactness is a singularity in the hierarchy of large cardinals. The fundamental work of Magidor [Ma] shows that the least strongly compact cardinal κ can either be the least supercompact cardinal or the least measurable cardinal, in which case κ isn't even 2^κ supercompact. A generalization of this result by Kimchi and Magidor [KiM] shows that the (possibly proper) classes of supercompact and strongly compact cardinals can coincide except at measurable limit points or that the first n (for $n \in \omega$) strongly compact cardinals can be the first n measurable cardinals.

The purpose of this paper is to show that the techniques of [AS97a] and [AS97b] can be combined with unpublished ideas of Magidor to produce a model in which the first n (for $n \in \omega$) strongly compact cardinals are not only the first n measurable cardinals, but each is a little supercompact. Specifically, we prove the following.

Theorem 1. *Con(ZFC + There are $n \in \omega$ supercompact cardinals) $\Longrightarrow$ Con(ZFC + The first n strongly compact cardinals $\kappa_1, \ldots, \kappa_n$ are the first n measurable cardinals + $2^{\kappa_i} = \kappa_i^{++}$ for $i = 1, \ldots, n$ + Each κ_i is κ_i^+ supercompact for $i = 1, \ldots n$).*

A bit of history is perhaps in order now. As was just noted, in the early 1970s, Magidor in [Ma] showed the consistency, relative to the existence of a strongly compact cardinal, of the least strongly compact cardinal being the least measurable cardinal. In the spring of 1983, Woodin, in response to a question put to him by the first author, showed the consistency (see [CW]), relative to the existence of a cardinal κ which is κ^{+3} supercompact, of the least measurable cardinal κ being so that $2^\kappa = \kappa^{++}$ and κ is κ^+ supercompact. In the mid 1980s, Kimchi and Magidor did the work of [KiM]. In late 1992, Shelah and the first author began the research leading to the results of [AS97a] and

[AS97b]. The main theorem of [AS97a] showed, roughly speaking, the relative consistency of the classes of strongly compact and supercompact cardinals coinciding level by level, except where explicitly prohibited by ZFC. This strengthened the work of [KiM]. The main theorem of [AS97b] showed that Menas' result of [Me] that the least measurable limit κ of either strongly compact or supercompact cardinals isn't 2^κ supercompact is best possible by constructing, starting from a supercompact limit of supercompact cardinals, a model in which the least measurable limit κ of both strongly compact and supercompact cardinals is so that $2^\kappa = \kappa^{++}$ and κ is κ^+ supercompact. The forcing conditions of [AS97b] were generalizations of the forcing conditions of [AS97a], and both provided, among other things, an alternate way of proving Woodin's aforementioned 1983 theorem. This still left open the question of combining Woodin's results with the results of Magidor and Kimchi and Magidor, i.e., obtaining a model in which the least measurable cardinal κ is both the least strongly compact cardinal and is κ^+ supercompact, or in general, obtaining a model in which the first n measurable cardinals $\kappa_1, \ldots, \kappa_n$ (for $n \in \omega$) are the first n strongly compact cardinals, with each measurable cardinal κ_i being κ_i^+ supercompact. This question went unresolved for a number of years, despite several attempts at solving it made by Shelah and the first author. Then, during the January 7-13, 1996 meeting in Set Theory held at the Mathematics Research Institute, Oberwolfach, the first author proved (see [A97a]) Theorem 1 for $n = 1$. His proof, however, was non-iterative, and the question remained of proving Theorem 1 for arbitrary finite n. This question was finally resolved by the two authors of this paper during their stay as Research in Pairs fellows at the Mathematics Research Institute, Oberwolfach, June 8-21, 1997.

We take the opportunity here to make two remarks about Theorem 1. The first is that there is nothing special about each κ_i being κ_i^+ supercompact in Theorem 1. In fact, each κ_i can be κ_i^{++}, κ_i^{+++}, κ_i^{+4}, etc. supercompact, so long as $2^{\kappa_i} > \kappa_i^{++}$, $2^{\kappa_i} > \kappa_i^{+++}$, $2^{\kappa_i} > \kappa_i^{+4}$, etc. After completing the proof of Theorem 1, interested readers are invited to look at the statement of Theorem 3 of [AS97b] in order to determine for themselves what variants are possible. The second is that no proof is currently known, in both Theorem 1 and the corresponding result of [KiM], when n is infinite. This will be discussed further at the end of the paper.

The structure of this paper is as follows. Section 0 contains our introductory comments and preliminary remarks concerning notation, terminology, etc. Section 1 contains a discussion of a certain modification of the main forcing notion of [AS97a] (given at the end of [AS97b]) that will be critical in the proof of Theorem 1. Section 1 also contains a discussion of the main forcing notion of [AS97b], which can be used as an alternative in the proof of Theorem 1. Section 2 shows how the forcing notions discussed in Section 1 can be used to force a supercompact cardinal to have a certain special kind of supercompactness embedding that will be key in the proof of Theorem 1. Section 3 then gives a proof of Theorem 1 for $n = 1$. Section 4 contains a proof of Theorem 1 for arbitrary finite n. Section 5 has our concluding remarks.

We give now some preliminary information concerning notation and terminology. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. For $\alpha < \beta$ ordinals, $[\alpha, \beta], [\alpha, \beta), (\alpha, \beta]$, and (α, β) are as in standard interval notation. If x is a set, then $\mathrm{TC}(x)$ is the transitive closure of x.

When forcing, $q \geq p$ will mean that q is stronger than p. For P a partial ordering, φ a formula in the forcing language with respect to P, and $p \in P$, $p \| \varphi$ will mean p decides φ. For G V-generic over P, we will use both $V[G]$ and V^P to indicate the universe obtained by forcing with P. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$, especially when x is some variant of the generic set G, or x is in the ground model V.

If κ is a cardinal and P is a partial ordering, P is κ-closed if given a sequence $\langle p_\alpha : \alpha < \kappa \rangle$ of elements of P so that $\beta < \gamma < \kappa$ implies $p_\beta \leq p_\gamma$ (an increasing chain of length κ), then there is some $p \in P$ (an upper bound to this chain) so that $p_\alpha \leq p$ for all $\alpha < \kappa$. P is $< \kappa$-closed if P is δ-closed for all cardinals $\delta < \kappa$. P is κ-directed closed if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha : \alpha < \delta \rangle$ of elements of P (where $\langle p_\alpha : \alpha < \delta \rangle$ is directed if for every two distinct elements $p_\rho, p_\nu \in \langle p_\alpha : \alpha < \delta \rangle$, p_ρ and p_ν have a common upper bound p_σ) there is an upper bound $p \in P$. P is κ-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, then player II has a strategy which ensures the game can always be continued. Note that if P is κ-strategically closed and $f : \kappa \to V$ is a function in V^P, then $f \in V$. P is $< \kappa$-strategically closed if P is δ-strategically closed for all cardinals $\delta < \kappa$. P is $\prec \kappa$-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, then player II has a strategy which ensures the game can always be continued. Note that trivially, if P is $< \kappa$-closed, then P is $< \kappa$-strategically closed and $\prec \kappa$-strategically closed. The converse of both of these facts is false.

The usual partial ordering for adding λ Cohen subsets to a regular cardinal κ will be written as $\mathrm{Add}(\kappa, \lambda)$. Standard arguments show $\mathrm{Add}(\kappa, \lambda)$ is κ-directed closed. See [J], Lemmas 19.7 and 19.8, pages 181-182, for further details.

We mention that we are assuming complete familiarity with the notions of measurability, strong compactness, and supercompactness. Interested readers may consult [SRK], [Ka], or [KaM] for further details. We note only that all elementary embeddings witnessing the λ supercompactness of κ are presumed to come from some fine, κ-complete, normal ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$, and all elementary embeddings witnessing the λ strong compactness of κ are presumed to come from some fine, κ-complete ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda)$. An equivalent definition for κ being λ strongly compact is that there is an elementary embedding $j : V \to M$ having critical point κ so that for any $x \subseteq M$ with $|x| \le \lambda$, there is some $y \in M$ such that $x \subseteq y$ and $M \models$ "$|y| < j(\kappa)$".

Finally, we mention that since ideas and notions from [AS97a] and [AS97b] are used throughout the course of this paper, it would be most helpful to readers if copies of these papers were kept close at hand. In fact, at many instances during our exposition, we will refer to proofs not given here but found in either [AS97a] or [AS97b]. These papers, however, need not be read in their entirety to follow this paper. In order to facilitate the understanding of readers, though, we will keep as much as possible to the notations and terminology of [AS97a] and [AS97b].

§1 The Forcing Notions of [AS97a] and [AS97b]

Fix $\gamma < \delta < \lambda$, $\lambda > \delta^+$ regular cardinals in our ground model V, with δ inaccessible and λ either inaccessible or the successor of a cardinal of cofinality $> \delta$. We recall now the partial orderings $P^0_{\delta,\lambda}$ and $P^2_{\delta,\lambda}[S]$ of [AS97a] and [AS97b] and both the version of $P^1_{\delta,\lambda}[S]$

given in Section 4 of [AS97b] (which is a modification of the partial ordering $P^1_{\delta,\lambda}[S]$ of [AS97a]) and the version of $P^1_{\delta,\lambda}[S]$ of [AS97b] used in the proof of Theorem 3 of that paper.

We assume GCH holds for all cardinals $\kappa \geq \delta$. As in Section 4 of [AS97b], the first notion of forcing $P^0_{\delta,\lambda}$ is just the standard notion of forcing for adding a non-reflecting stationary set of ordinals of cofinality γ to λ. Specifically, $P^0_{\delta,\lambda} = \{p :$ For some $\alpha < \lambda$, $p : \alpha \to \{0,1\}$ is a characteristic function of S_p, a subset of α not stationary at its supremum nor having any initial segment which is stationary at its supremum, so that $\beta \in S_p$ implies $\beta > \delta$ and $\text{cof}(\beta) = \gamma\}$, ordered by $q \geq p$ iff $q \supseteq p$ and $S_p = S_q \cap \sup(S_p)$, i.e., S_q is an end extension of S_p. It is well-known that for G V-generic over $P^0_{\delta,\lambda}$ (see [B] or [KiM]), in $V[G]$, since GCH holds in V for all cardinals $\kappa \geq \delta$, a non-reflecting stationary set $S = S[G] = \cup\{S_p : p \in G\} \subseteq \lambda$ of ordinals of cofinality γ has been introduced, the bounded subsets of λ are the same as those in V, and cardinals, cofinalities, and GCH at cardinals $\kappa \geq \delta$ have been preserved. It is also virtually immediate that $P^0_{\delta,\lambda}$ is γ-directed closed, and it can be shown (see [B], Lemma 4.15, page 436 or [KiM]) that $P^0_{\delta,\lambda}$ is $\prec \lambda$-strategically closed.

Work now in $V_1 = V^{P^0_{\delta,\lambda}}$, letting $\dot{S}$ be a term always forced to denote the above set S. $P^2_{\delta,\lambda}[S]$ is the standard notion of forcing for introducing a club set C which is disjoint to S (and therefore makes S non-stationary). Specifically, $P^2_{\delta,\lambda}[S] = \{p :$ For some successor ordinal $\alpha < \lambda$, $p : \alpha \to \{0,1\}$ is a characteristic function of C_p, a club subset of α, so that $C_p \cap S = \emptyset\}$, ordered by $q \geq p$ iff C_q is an end extension of C_p. It is again well-known (see [MS]) that for H V_1-generic over $P^2_{\delta,\lambda}[S]$, a club set $C = C[H] = \cup\{C_p : p \in H\} \subseteq \lambda$ which is disjoint to S has been introduced, the bounded subsets of λ are the same as those in V_1, and cardinals, cofinalities, and GCH for cardinals $\kappa \geq \delta$ have been preserved.

The following lemma is proven in both [AS97a] and [AS97b].

LEMMA 1 (LEMMA 1 OF [AS97A] AND [AS97B]). $\Vdash_{P^0_{\delta,\lambda}}$ "$\clubsuit(\dot{S})$", *i.e.,* $V_1 \models$ "*There is a sequence* $\langle x_\alpha : \alpha \in S \rangle$ *so that for each* $\alpha \in S$, $x_\alpha \subseteq \alpha$ *is cofinal in* α, *and for any* $A \in [\lambda]^\lambda$, $\{\alpha \in S : x_\alpha \subseteq A\}$ *is stationary*".

We fix now in V_1 a $\clubsuit(S)$ sequence $X = \langle x_\alpha : \alpha \in S \rangle$. We are ready to define in V_1 in the same manner as was done in Section 4 of [AS97b] the partial ordering $P^1_{\delta,\lambda}[S]$. First, since each element of S has cofinality γ, each $x \in X$ can be assumed to be so that order type$(x) = \gamma$. Then, $P^1_{\delta,\lambda}[S]$ is defined as the set of all 4-tuples $\langle w, \alpha, \bar{r}, Z \rangle$ satisfying the following properties.

1. $w \in [\lambda]^{<\delta}$.
2. $\alpha < \delta$.
3. $\bar{r} = \langle r_i : i \in w \rangle$ is a sequence of functions from α to $\{0, 1\}$, i.e., a sequence of subsets of α.
4. $Z \subseteq \{x_\beta : \beta \in S\}$ is a set so that if $z \in Z$, then for some $y \in [w]^\gamma$, $y \subseteq z$ and $z - y$ is bounded in the β so that $z = x_\beta$. In other words, for every $x_\beta \in Z$, $w \cap x_\beta$ is cobounded in x_β.

As in [AS97a], the definition of Z implies $|Z| < \delta$.

The ordering on $P^1_{\delta,\lambda}[S]$ is given by $\langle w^1, \alpha^1, \bar{r}^1, Z^1 \rangle \leq \langle w^2, \alpha^2, \bar{r}^2, Z^2 \rangle$ iff the following hold.

1. $w^1 \subseteq w^2$.
2. $\alpha^1 \leq \alpha^2$.

3. If $i \in w^1$, then $r_i^1 \subseteq r_i^2$.

4. $Z^1 \subseteq Z^2$.

5. If $z \in Z^1 \cap [w^1]^\gamma$ and $\alpha^1 \leq \alpha < \alpha^2$, then $|\{i \in z : r_i^2(\alpha) = 0\}| = |\{i \in z : r_i^2(\alpha) = 1\}| = \gamma$.

The intuition behind the definition of $P^1_{\delta,\lambda}[S]$ just given is essentially the same as in [AS97a]. Specifically, we wish to be able simultaneously to make $2^\delta = \lambda$, destroy the measurability of δ, and be able to resurrect the $< \lambda$ supercompactness of δ if necessary. $P^1_{\delta,\lambda}[S]$ has been designed so as to allow us to do all of these things.

The proof that $V_1^{P^1_{\delta,\lambda}[S]} \models$ "δ is non-measurable" is as in Lemma 3 of [AS97a]. In particular, the argument of Lemma 3 of [AS97a] will show that δ can't carry a γ-additive uniform ultrafilter. We can then carry through the proof of Lemma 4 of [AS97a] to show $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$ is equivalent to $\text{Add}(\lambda, 1) * \dot{\text{Add}}(\delta, \lambda)$. The proofs of Lemma 5 of [AS97a] and Lemma 6 of [AS97b] will then show $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ preserves cardinals and cofinalities, is λ^+-c.c., is $< \delta$-strategically closed, and is so that $V^{P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]} \models$ "$2^\kappa = \lambda$ for every cardinal $\kappa \in [\delta, \lambda)$".

Although the above definition of $P^1_{\delta,\lambda}[S]$ (henceforth to be referred to as the "simpler form") is perfectly adequate for our purposes, as mentioned at the end of [AS97b], it will not suffice to prove Theorem 3 of [AS97b]. In order to do this, a more complicated form of $P^1_{\delta,\lambda}[S]$ is required. Since this version of $P^1_{\delta,\lambda}[S]$ will also work for Theorem 1, for completeness, we recall its definitions and properties here. First, we fix $\delta < \lambda$ as before. We then as was done in [AS97b] redefine $P^0_{\delta,\lambda}$ as the partial ordering which adds a non-reflecting stationary set of ordinals S of cofinality δ to λ. The definition of $P^2_{\delta,\lambda}[S]$ remains the same. Having fixed a $\clubsuit(S)$ sequence as before, $P^1_{\delta,\lambda}[S]$ is then the set of all 5-tuples $\langle w, \alpha, \bar{r}, Z, \Gamma\rangle$ satisfying the following properties.

1. $w \subseteq \lambda$ is so that $|w| = \delta$.

2. $\alpha < \delta$.

3. $\bar{r} = \langle r_i : i \in w\rangle$ is a sequence of functions from α to $\{0, 1\}$, i.e., a sequence of subsets of α.

4. Z is a function so that:

a) $\mathrm{dom}(Z) \subseteq \{x_\beta : \beta \in S\}$ and $\mathrm{range}(Z) \subseteq \{0, 1\}$.

b) If $z \in \mathrm{dom}(Z)$, then for some $y \in [w]^\delta$, $y \subseteq z$ and $z - y$ is bounded in the β so that $z = x_\beta$.

5. Γ is a function so that:

a) $\mathrm{dom}(\Gamma) = \mathrm{dom}(Z)$.

b) If $z \in \mathrm{dom}(\Gamma)$, then $\Gamma(z)$ is a closed, bounded subset of α such that if γ is inaccessible, $\gamma \in \Gamma(z)$, and β is the γ^{th} element of z, then $\beta \in w$, and for some $\beta' \in \beta \cap w \cap z$, $r_{\beta'}(\gamma) = Z(z)$.

Note that the definitions of Z and Γ imply $|\mathrm{dom}(Z)| = |\mathrm{dom}(\Gamma)| \le \delta$.

The ordering on $P^1_{\delta,\lambda}[S]$ is given by $\langle w^1, \alpha^1, \bar{r}^1, Z^1, \Gamma^1\rangle \le \langle w^2, \alpha^2, \bar{r}^2, Z^2, \Gamma^2\rangle$ iff the following hold.

1. $w^1 \subseteq w^2$.

2. $\alpha^1 \le \alpha^2$.

3. If $i \in w^1$, then $r^1_i \subseteq r^2_i$ and $|\{i \in w^1 : r^2_i \restriction (\alpha_2 - \alpha_1)$ is not constantly $0\}| < \delta$.

4. $Z^1 \subseteq Z^2$.

5. $\mathrm{dom}(\Gamma^1) \subseteq \mathrm{dom}(\Gamma^2)$.

6. If $z \in \mathrm{dom}(\Gamma^1)$, then $\Gamma^1(z)$ is an initial segment of $\Gamma^2(z)$ and $|\{z \in \mathrm{dom}(\Gamma^1) : \Gamma^1(z) \neq \Gamma^2(z)\}| < \delta$.

The intuition behind the above definition of $P^1_{\delta,\lambda}[S]$ is the same as in [AS97b]. If δ is measurable, then δ must carry a normal measure. The forcing $P^1_{\delta,\lambda}[S]$ has specifically been designed to destroy this fact. (See Lemma 3 of [AS97b] for a proof.) It has been designed, however, to destroy the measurability of δ "as lightly as possible", making little damage, assuming δ is $< \lambda$ supercompact. Specifically, if δ is $< \lambda$ supercompact, then the non-reflecting stationary set S, having been added to λ, does not kill the $< \lambda$ supercompactness of δ by itself. The additional forcing $P^1_{\delta,\lambda}[S]$ is necessary to do the job and has been designed so as not only to destroy the $< \lambda$ supercompactness of δ but to destroy the measurability of δ as well. The forcing $P^1_{\delta,\lambda}[S]$, however, has been designed so that if necessary, we can resurrect the $< \lambda$ supercompactness of δ by forcing further with $P^2_{\delta,\lambda}[S]$.

We conclude this section by noting that there are additional properties of the more complicated version of $P^1_{\delta,\lambda}[S]$ that will be relevant to our work. These will be discussed in more detail in later sections.

§2 A Supercompact Cardinal with a Special Kind of Embedding

In this section, we force and construct a supercompact cardinal possessing a special sort of supercompactness embedding. Such cardinals will be critical in the proof of Theorem 1. Specifically, we prove the following.

Lemma 2. *Suppose $V \models$ "ZFC + GCH + κ is supercompact". There is then a partial ordering $P^{\kappa,0} \in V$ so that $V^{P^{\kappa,0}} \models$ "κ is supercompact + $2^\kappa = \kappa^{++}$". In addition, there is an elementary embedding $j^* : V^{P^{\kappa,0}} \to M^{j^*(P^{\kappa,0})}$ definable in $V^{P^{\kappa,0}}$ witnessing the κ^+ supercompactness of κ so that $M^{j^*(P^{\kappa,0})} \models$ "κ isn't measurable".*

Proof of Lemma 2: Fix $f : \kappa \to V_\kappa$ a Laver function [L], i.e., f is so that for every x and every $\lambda \geq |\mathrm{TC}(x)|$, there is a λ supercompact ultrafilter $\mathcal{U}_{\lambda,x}$ with associated embedding $j_{\mathcal{U}_{\lambda,x}} : V \to M$ so that $j_{\mathcal{U}_{\lambda,x}}(f)(\kappa) = x$. Also, let $\langle \delta_\alpha : \alpha \leq \kappa \rangle$ enumerate the inaccessibles $\leq \kappa$, and let $\gamma < \delta_0$ be a fixed but arbitrary regular cardinal.

As Laver does in [L], we define now simultaneously an Easton support iteration $P^{\kappa,0} = \langle \langle P^*_\alpha, \dot{Q}^*_\alpha \rangle : \alpha \leq \kappa \rangle$ and a sequence of ordinals $\langle \rho_\alpha : \alpha < \kappa \rangle$, where $\rho_\lambda = \bigcup_{\alpha<\lambda} \rho_\alpha$ if λ is a limit ordinal. We use here in our definition the simpler form of $P^1_{\delta,\lambda}[S]$ of Section 1 defined using γ and the associated $P^0_{\delta,\lambda}$ and $P^2_{\delta,\lambda}[S]$ and indicate at the end of the section the modifications needed when the more complicated form of $P^1_{\delta,\lambda}[S]$ is used. Specifically, the definition has P_0 being trivial with $\rho_0 = 0$, and $P^*_{\alpha+1} = P^*_\alpha * \dot{Q}^*_\alpha$, where $\Vdash_{P_\alpha}$ "$\dot{Q}^*_\alpha$ is trivial" and $\rho_{\alpha+1} = \rho_\alpha$ unless one of the following holds:

1. If for all $\beta < \alpha$, $\rho_\beta < \alpha$ and $\delta_\alpha < \kappa$ is so that $V \models$ "δ_α isn't δ^+_α supercompact", then $P^*_{\alpha+1} = P^*_\alpha * \dot{Q}^*_\alpha$, where $\dot{Q}^*_\alpha$ is a term for $P^0_{\delta_\alpha,\delta^{++}_\alpha} * P^1_{\delta_\alpha,\delta^{++}_\alpha}[\dot{S}_{\delta^{++}_\alpha}]$, and $\dot{S}_{\delta^{++}_\alpha}$ is a term for the non-reflecting stationary subset of δ^{++}_α introduced by $P^0_{\delta_\alpha,\delta^{++}_\alpha}$. If $f(\alpha)$ is an ordinal and $f(\alpha) > \rho_\alpha$, then $\rho_{\alpha+1} = f(\alpha)$. If this condition on $f(\alpha)$ doesn't hold, then $\rho_{\alpha+1} = \rho_\alpha$.

2. If for $\beta < \alpha$, $\rho_\beta < \alpha$ and $\delta_\alpha \leq \kappa$ is so that $V \models$ "δ_α is δ^+_α supercompact", then $P^*_{\alpha+1} = P^*_\alpha * \dot{Q}^*_\alpha$, where $\dot{Q}^*_\alpha$ is a term for $P^0_{\delta_\alpha,\delta^{++}_\alpha} * (P^1_{\delta_\alpha,\delta^{++}_\alpha}[\dot{S}_{\delta^{++}_\alpha}] \times P^2_{\delta_\alpha,\delta^{++}_\alpha}[\dot{S}_{\delta^{++}_\alpha}])$. If

$f(\alpha)$ is an ordinal and $f(\alpha) > \rho_\alpha$, then $\rho_{\alpha+1} = f(\alpha)$. If this condition on $f(\alpha)$ doesn't hold, then $\rho_{\alpha+1} = \rho_\alpha$.

Suppose now $j : V \to M$ is an embedding witnessing the κ^+ supercompactness of κ so that $M \models$ "κ isn't κ^+ supercompact". Lemma 9 of [AS97b] shows that if $P^{\kappa,0} = P^*_\kappa * \dot{Q}^*_\kappa = P^*_\kappa * (\dot{P}^0_{\kappa,\kappa^{++}} * (P^1_{\kappa,\kappa^{++}}[\dot{S}_{\kappa^{++}}] \times P^2_{\kappa,\kappa^{++}}[\dot{S}_{\kappa^{++}}]))$ were an iteration as defined in the proofs of Theorem 1 or Theorem 3 of [AS97b], then $j : V \to M$ extends to $j^* : V^{P^{\kappa,0}} \to M^{j^*(P^{\kappa,0})}$ witnessing the κ^+ supercompactness of κ in a manner definable in $V^{P^{\kappa,0}}$. The type of iteration used in the proofs of Theorem 1 or Theorem 3 of [AS97b], however, is essentially the one just described here. The only real difference is that here, we use a Laver function to "space out" the iteration at successor stages below κ. At stage $\kappa + 1$ in V, however, the partial ordering used in the iteration is $P^0_{\kappa,\kappa^{++}} * (P^1_{\kappa,\kappa^{++}}[\dot{S}_{\kappa^{++}}] \times P^2_{\kappa,\kappa^{++}}[\dot{S}_{\kappa^{++}}])$, and at stage $\kappa + 1$ in M, the partial ordering used in the iteration is $P^0_{\kappa,\kappa^{++}} * P^1_{\kappa,\kappa^{++}}[\dot{S}_{\kappa^{++}}]$. These occurrences at stage $\kappa+1$ in V and M in conjunction with the definition of $P^{\kappa,0}$ will then allow the arguments of Lemma 9 of [AS97b] to go through to yield that j extends to $j^* : V^{P^{\kappa,0}} \to M^{j^*(P^{\kappa,0})}$. Note that since $P^0_{\kappa,\kappa^{++}} * P^1_{\kappa,\kappa^{++}}[\dot{S}_{\kappa^{++}}]$ is used at stage $\kappa+1$ in M, Lemma 3 of [AS97a] and Lemma 8 of [AS97b] show that $M^{j^*(P^{\kappa,0})} \models$ "κ isn't measurable".

We show now that $V^{P^{\kappa,0}} \models$ "κ is supercompact". To do this, we give an argument similar to the one given in the proof of Lemma 2 of [A∞]. Specifically, let $\gamma > \kappa^{++}$ be an arbitrary cardinal, and let $\lambda > 2^{[\gamma]^{<\kappa}}$ be a cardinal so that for some embedding $k : V \to M$ witnessing the λ supercompactness of κ, $k(f)(\kappa) = \lambda$. By the definition of $P^{\kappa,0}$ and the properties of k, $k(P^{\kappa,0}) = (P^*_\kappa * (P^0_{\kappa,\kappa^{++}} * (P^1_{\kappa,\kappa^{++}}[\dot{S}_{\kappa^{++}}] \times P^2_{\kappa,\kappa^{++}}[\dot{S}_{\kappa^{++}}]))) * \dot{Q}^* = (P^*_\kappa * \dot{Q}^*_\kappa) * \dot{Q}^* = P^{\kappa,0} * \dot{Q}^* = P^{\kappa,0} * \dot{R}^* * \dot{Q}^*_{k(\kappa)}$, where $\dot{R}^*$ is a term for the $M^{P^{\kappa,0}}$ partial ordering $P^*_{k(\kappa)}/P^{\kappa,0}$. By the definition of $P^{\kappa,0}$, in M, $\Vdash_{P^{\kappa,0}}$ "The field of $\dot{Q}$ is composed of cardinals $> \lambda$". Further, by the definition of $P^{\kappa,0}$ and the fact $M^\lambda \subseteq M$, it is true that in V and M, $\Vdash_{P^{\kappa,0}}$ "Both $\dot{R}^*$ and $\dot{R}^* * \dot{Q}^*_{k(\kappa)}$ are λ-strategically closed and $\lambda > 2^{[\gamma]^{<\kappa}}$". And, by our earlier remarks, in both V and M, $\Vdash_{P^*_\kappa}$ "$\dot{Q}^*_\kappa$ is forcing equivalent to $\dot{\text{Add}}(\kappa^{++}, 1) * \dot{\text{Add}}(\kappa, \kappa^{++})$, a κ-directed closed partial ordering having size κ^{++}". Therefore, $V^{P^*_\kappa * \dot{Q}^*_\kappa} = V^{P^{\kappa,0}} \models$ "$2^\kappa = \kappa^{++}$", and the standard arguments (see, e.g., Lemma 2 of [A∞]) in turn show that $M^{P^{\kappa,0} * \dot{R}^*}$ remains λ-closed with respect to $V^{P^{\kappa,0} * \dot{R}^*}$ and that if $G_0 * G_1$ is V-generic over $P^*_\kappa * \dot{Q}^*_\kappa = P^{\kappa,0}$ and G_2 is $V[G_0][G_1]$-generic over R^*, in $V[G_0][G_1][G_2]$, we can find a master condition q extending each $p \in k''G_1$. If G_3 is $V[G_0][G_1][G_2]$-generic over $Q_{k(\kappa)}$ so that $q \in G_3$, in $V[G_0][G_1][G_2][G_3]$, there is an elementary embedding $k^* : V[G_0][G_1] \to M[G_0][G_1][G_2][G_3]$ extending k. Since in V, $\Vdash_{P^{\kappa,0}}$ "$\dot{R}^* * \dot{Q}^*_{k(\kappa)}$ is λ-strategically closed", $V[G_0][G_1] \models$ "κ is γ supercompact". This proves Lemma 2.

◻ Lemma 2

When the more complicated version of $P^1_{\delta,\lambda}[S]$ and the associated versions of $P^0_{\delta,\lambda}$ and $P^2_{\delta,\lambda}[S]$ are employed as the building blocks of P^*, instead of working with δ^{++}_α, we use δ^{+++}_α, i.e., at each non-trivial stage in our iteration, we force with either $P^0_{\delta_\alpha, \delta^{+++}_\alpha} * P^1_{\delta_\alpha, \delta^{+++}_\alpha}[\dot{S}_{\delta^{+++}_\alpha}]$ or $P^0_{\delta_\alpha, \delta^{+++}_\alpha} * (P^1_{\delta_\alpha, \delta^{+++}_\alpha}[\dot{S}_{\delta^{+++}_\alpha}] \times P^2_{\delta_\alpha, \delta^{+++}_\alpha}[\dot{S}_{\delta^{+++}_\alpha}])$. This is since by Lemma 6 of [AS97b], both of the just mentioned partial orderings will collapse δ^+_α. Except for this difference, however, the proof of Lemma 2 is the same as before, making the appropriate references to Lemmas 6 and 8 of [AS97b] as necessary.

In conclusion to this section, we note that if we assume that κ has no inaccessible cardinals above it, no use of the Laver function f is needed in the definition of $P^{\kappa,0}$. At each $\delta_\alpha < \kappa$ which isn't δ^+_α supercompact, we can force as in Case 1 of the definition of

$P^{\kappa,0}$, and at each $\delta_\alpha \le \kappa$ which is δ^+_α supercompact, we can force as in Case 2 of the definition of $P^{\kappa,0}$. We leave it to any interested readers to verify that the proof of Lemma 2 becomes simpler under these circumstances. It is only when there are large enough cardinals above κ that the use of the Laver function f is required in the definition of $P^{\kappa,0}$.

§3 The Case $n = 1$

We present in this section a proof of Theorem 1 when $n = 1$. We assume that $V \models$ "ZFC + κ is supercompact". By Lemma 2, we also assume that $V \models$ "$2^\kappa = \kappa^{++}$" and that there is a κ^+ supercompactness embedding $k^*_0 : V \to M^*$ generated by a κ^+ supercompact ultrafilter over $P_\kappa(\kappa^+)$ so that $M^* \models$ "κ isn't measurable". Further, we assume for the time being that there are no measurable cardinals in V above κ.

Fix now an arbitrary regular cardinal $\gamma < \kappa$. Let $\langle \delta_\alpha : \alpha < \kappa \rangle$ this time enumerate the measurables $< \kappa$. The partial ordering $P^{\kappa,1}$ we use in the proof of Theorem 1 when $n = 1$ is the Easton support iteration $\langle \langle P^\kappa_\alpha, \dot{Q}^\kappa_\alpha \rangle : \alpha < \kappa \rangle$, where P^κ_0 is trivial and $\Vdash_{P^\kappa_\alpha}$ "$\dot{Q}^\kappa_\alpha$ adds a non-reflecting stationary set of ordinals of cofinality γ to δ_α".

Lemma 3. $V^{P^{\kappa,1}} \models$ *"No cardinal $\delta < \kappa$ is measurable".*

Proof of Lemma 3: Let $\delta < \kappa$ be so that $V \models$ "δ is measurable". It must therefore be the case that $\delta = \delta_\alpha$ for some $\alpha < \kappa$. This allows us to write $P^{\kappa,1} = P^\kappa_\alpha * \dot{Q}^\kappa_\alpha * \dot{R} = P^\kappa_{\alpha+1} * \dot{R}$.

By the definition of $P^{\kappa,1}$ and the fact that any stationary subset of a measurable (or weakly compact) cardinal must reflect, $V^{P^\kappa_{\alpha+1}} \models$ "δ isn't measurable since there is $S \subseteq \delta$ which is a non-reflecting stationary set of ordinals of cofinality γ". Since by the definition of $P^{\kappa,1}$, $\Vdash_{P^\kappa_{\alpha+1}}$ "$\dot{R}$ is δ'-strategically closed for δ' the least inaccessible above δ", $V^{P^\kappa_{\alpha+1} * \dot{R}} = V^{P^{\kappa,1}} \models$ "$S \subseteq \delta$ is a non-reflecting stationary set of ordinals of cofinality γ, so δ isn't measurable". Thus, $V^{P^{\kappa,1}} \models$ "No V-measurable cardinal $\delta < \kappa$ is measurable".

The proof of Lemma 3 will therefore be complete once we have shown there is no cardinal $\delta < \kappa$ so that $\Vdash_{P^{\kappa,1}}$ "δ is measurable".

To do this, we give an argument similar to the one found in the last part of Lemma 8 of [A97b], which in turn is essentially the same as the arguments given in Theorem 2.1.5 of [H] and Theorem 2.5 of [KiM]. Assume that $V^{P^{\kappa,1}} \models$ "δ is measurable". Since we have just shown that no V-cardinal is measurable in $V^{P^{\kappa,1}}$, we can write $P^{\kappa,1} = P^\kappa_\zeta * \dot{R}$, where $\delta \notin \text{field}(P^\kappa_\zeta)$ and $\Vdash_{P^\kappa_\zeta}$ "$\dot{R}$ is δ'-strategically closed for δ' the least inaccessible above δ". Thus, $\Vdash_{P^\kappa_\zeta}$ "δ is measurable" iff $\Vdash_{P^{\kappa,1}}$ "δ is measurable", so we show without loss of generality that $\Vdash_{P^\kappa_\zeta}$ "δ isn't measurable".

Note now that since $V^{P^\kappa_\zeta} \models$ "δ is Mahlo", $V \models$ "δ is Mahlo". Next, let $p \in P^\kappa_\zeta$ be so that $p \Vdash$ "$\dot{\mu}$ is a measure over δ". We show there is some $q \geq p$, $q \in P^\kappa_\zeta$ so that for every $X \in (\wp(\delta))^V$, $q \|$ "$X \in \dot{\mu}$". To do this, we build in V a binary tree $\mathcal{T}$ of height δ, assuming no such q exists. The root of our tree is $\langle p, \delta \rangle$. At successor stages $\beta+1$, assuming $\langle r, X \rangle$ is on the β^{th} level of $\mathcal{T}$, $r \geq p$, and $X \subseteq \delta$, $X \in V$ is so that $r \Vdash$ "$X \in \dot{\mu}$", we let $X = X_0 \cup X_1$ be such that $X_0, X_1 \in V$, $X_0 \cap X_1 = \emptyset$, and for $r_0 \geq r$, $r_1 \geq r$ incompatible, $r_0 \Vdash$ "$X_0 \in \dot{\mu}$" and $r_1 \Vdash$ "$X_1 \in \dot{\mu}$". We can do this by our hypothesis of the non-existence of a $q \in P^\kappa_\zeta$ as mentioned earlier. We place both $\langle r_0, X_0 \rangle$ and $\langle r_1, X_1 \rangle$ in $\mathcal{T}$ at height $\beta + 1$ as the successors of $\langle r, X \rangle$. At limit stages $\lambda < \delta$, for each branch $\mathcal{B}$ in $\mathcal{T}$ of height $\leq \lambda$, we take the intersection of all second coordinates of elements along $\mathcal{B}$. The result is a partition of δ into $\leq 2^\lambda$ many sets, so since δ is Mahlo in V, $2^\lambda < \delta$, i.e., the partition is into $< \delta$ many sets. Since $V^{P^\kappa_\zeta} \models$ "δ is measurable", there is at least one element Y of this partition resulting from a branch of height λ and a condition $s \geq p$ so that $s \Vdash$ "$Y \in \dot{\mu}$". For all such

Y, we place a pair of the form $\langle s, Y\rangle$ into $\mathcal{T}$ at level λ as the successor of each element of the branch generating Y.

Work now in $V^{P^\kappa_\zeta}$. Since δ is measurable in $V^{P^\kappa_\zeta}$, $V^{P^\kappa_\zeta} \models$ "δ is weakly compact". By construction, $\mathcal{T}$ is a tree having δ levels so that each level has size $< \delta$. Thus, by the weak compactness of δ in $V^{P^\kappa_\zeta}$, we can let $\mathcal{B} = \langle\langle r_\beta, X_\beta\rangle : \beta < \delta\rangle$ be a branch of height δ through $\mathcal{T}$. If we define for $\beta < \delta$ $Y_\beta = X_\beta - X_{\beta+1}$, then since $\langle X_\beta : \beta < \delta\rangle$ is so that $0 \le \beta < \rho < \delta$ implies $X_\beta \supseteq X_\rho$, for $0 \le \beta < \rho < \delta$, $Y_\beta \cap Y_\rho = \emptyset$. Since by the construction of $\mathcal{T}$, at level $\beta + 1$, the two second coordinate portions of the successor of $\langle r_\beta, X_\beta\rangle$ are $X_{\beta+1}$ and Y_β, for the s_β so that $\langle s_\beta, Y_\beta\rangle$ is at level $\beta + 1$ of $\mathcal{T}$, $\langle s_\beta : \beta < \delta\rangle$ must form in $V^{P^\kappa_\zeta}$ an antichain of size δ in P^κ_ζ.

In $V^{P^\kappa_\zeta}$, P^κ_ζ is embeddable as a subordering of the Easton support product $\prod_{\alpha<\zeta} Q^\kappa_\alpha$ as calculated in $V^{P^\kappa_\zeta}$. As $V^{P^\kappa_\zeta} \models$ "δ is Mahlo", this immediately implies that $V^{P^\kappa_\zeta} \models$ "P^κ_ζ is δ-c.c.", contradicting that $\langle s_\beta : \beta < \delta\rangle$ is in $V^{P^\kappa_\zeta}$ an antichain of size δ. Thus, there is some $q \ge p$ so that for every $X \in (\wp(\delta))^V$, $q \|$"$X \in \dot{\mu}$", i.e., δ is measurable in V. This contradiction proves Lemma 3.

$\square$ Lemma 3

Lemma 4. $V^{P^{\kappa,1}} \models$ "*κ is both strongly compact and κ^+ supercompact*".

Proof of Lemma 4: The proof of Lemma 4 heavily uses unpublished ideas of Magidor (which don't even appear in the circulated manuscript of [KiM]). Let $\lambda > 2^{[\kappa^+]^{<\kappa}} = 2^{\kappa^+} = 2^\kappa = \kappa^{++}$ be an arbitrary cardinal, and let $k_1 : V \to M$ be an embedding witnessing the λ supercompactness of κ. λ has been chosen large enough so that any ultrafilter over $P_\kappa(\kappa^+)$ present in V is an element of M, so we may assume by the remarks in the first paragraph of this section that $k_2 : M \to N$ is an embedding witnessing the κ^+ supercompactness of κ definable in M so that $N \models$ "κ isn't measurable". It is easily verifiable using the embedding definition of λ strong compactness given in Section 0 that $j = k_2 \circ k_1$ is so that $j : V \to N$ is a λ strongly compact embedding that also witnesses the κ^+ supercompactness of κ. We show that j extends to $j^* : V^{P^{\kappa,1}} \to N^{j^*(P^{\kappa,1})}$, thus proving Lemma 4.

To do this, write $j(P^{\kappa,1})$ as $P^{\kappa,1} * \dot{Q}^\kappa * \dot{R}^\kappa$, where $\dot{Q}^\kappa$ is a term for the portion of $j(P^{\kappa,1})$ between κ and $k_2(\kappa)$ and $\dot{R}^\kappa$ is a term for the rest of $j(P^{\kappa,1})$, i.e., the part above $k_2(\kappa)$. Note that since $N \models$ "κ isn't measurable", $\kappa \notin \text{field}(\dot{Q}^\kappa)$. Also, since $M \models$ "κ is measurable", by elementarity, $N \models$ "$k_2(\kappa)$ is measurable". Thus, the field of $\dot{Q}^\kappa$ is composed of all N-measurable cardinals in the interval $(\kappa, k_2(\kappa)]$ (so $k_2(\kappa) \in \text{field}(\dot{Q}^\kappa)$), and the field of $\dot{R}^\kappa$ is composed of all N-measurable cardinals in the interval $(k_2(\kappa), k_2(k_1(\kappa)))$.

Let G_0 be V-generic over $P^{\kappa,1}$. We construct in $V[G_0]$ an $N[G_0]$-generic object G_1 over Q^κ and an $N[G_0][G_1]$-generic object G_2 over R^κ. Since $P^{\kappa,1}$ is an Easton support iteration of length κ with no forcing done at stage κ, the construction of G_1 and G_2 automatically guarantees that $j''G_0 \subseteq G_0 * G_1 * G_2$, meaning that $j : V \to N$ extends to $j^* : V[G_0] \to N[G_0][G_1][G_2]$.

To build G_1, note that since k_2 can be assumed to be generated by an ultrafiler $\mathcal{U}$ over $(P_\kappa(\kappa^+))^M = (P_\kappa(\kappa^+))^V$, and since in both V and M, $2^{\kappa^+} = 2^\kappa = \kappa^{++}$, $|k_2(\kappa^{++})| = |k_2(2^\kappa)| = |\{f : f : P_\kappa(\kappa^+) \to \kappa^{++} \text{ is a function}\}| = |[\kappa^{++}]^{\kappa^+}| = \kappa^{++}$. Thus, as $N[G_0] \models$ "$|Q^\kappa| = k_2(2^\kappa)$", we can let $\langle D_\alpha : \alpha < \kappa^{++} \rangle$ enumerate in $V[G_0]$ the dense open subsets of Q^κ present in $N[G_0]$. Since the κ^+ closure of N with respect to either M or V implies the least element of the field of Q^κ is $> \kappa^{++}$, the definition of Q^κ as the Easton support iteration which adds a non-reflecting stationary set of ordinals of cofinality γ to each

$N[G_0]$-measurable cardinal in the interval $(\kappa, k_2(\kappa)]$ implies that $N[G_0] \models$ "Q^κ is $\prec \kappa^{++}$-strategically closed". By the fact the standard arguments show that forcing with the κ-c.c. partial ordering $P^{\kappa,1}$ preserves that $N[G_0]$ remains κ^+ closed with respect to either $M[G_0]$ or $V[G_0]$, Q^κ is $\prec \kappa^{++}$-strategically closed in both $M[G_0]$ and $V[G_0]$.

We can now construct G_1 in either $M[G_0]$ or $V[G_0]$ as follows. Player I picks $p_\alpha \in D_\alpha$ extending $\sup(\langle q_\beta : \beta < \alpha \rangle)$ (initially, q_{-1} is the empty condition) and player II responds by picking $q_\alpha \geq p_\alpha$ (so $q_\alpha \in D_\alpha$). By the $\prec \kappa^{++}$-strategic closure of Q^κ in both $M[G_0]$ and $V[G_0]$, player II has a winning strategy for this game, so $\langle q_\alpha : \alpha < \kappa^{++} \rangle$ can be taken as an increasing sequence of conditions with $q_\alpha \in D_\alpha$ for $\alpha < \kappa^{++}$. Clearly, $G_1 = \{p \in Q^\kappa : \exists \alpha < \kappa^{++} [q_\alpha \geq p]\}$ is our $N[G_0]$-generic object over Q^κ.

It remains to construct in $V[G_0]$ the desired $N[G_0][G_1]$-generic object G_2 over R^κ. To do this, we first note that as $\lambda > 2^\kappa$, $M \models$ "κ is measurable". This means we can write $k_1(P^{\kappa,1})$ as $P^{\kappa,1} * \dot{S}^\kappa * \dot{T}^\kappa$, where $\Vdash_{P^{\kappa,1}}$ "$\dot{S}^\kappa$ adds a non-reflecting stationary set of ordinals of cofinality γ to κ", and $\dot{T}^\kappa$ is a term for the rest of $k_1(P^{\kappa,1})$. Since we have assumed $V \models$ "No cardinal $\delta > \kappa$ is measurable", the λ closure of M with respect to V implies $M \models$ "No cardinal $\delta \in (\kappa, \lambda]$ is measurable". Thus, the field of $\dot{T}^\kappa$ is composed of all M-measurable cardinals in the interval $(\lambda, k_1(\kappa))$, which implies that in M, $\Vdash_{P^{\kappa,1} * \dot{S}^\kappa}$ "$\dot{T}^\kappa$ is $\prec \lambda^+$-strategically closed". Further, since we can assume λ is regular, $|[\lambda]^{<\kappa}| = \lambda$, and $2^\lambda = \lambda^+$ (our ground model V is constructed by forcing over a model of GCH using a set partial ordering), and since, as before, k_1 can be assumed to be generated by an ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda)$, $|k_1(\lambda^+)| = |k_1(2^\lambda)| = |2^{k_1(\lambda)}| = |\{f : f : P_\kappa(\lambda) \to \lambda^+ \text{ is a function}\}| = |[\lambda^+]^\lambda| = \lambda^+$.

Work until otherwise specified in M. Consider the "term forcing" partial ordering T^* (see [C], Section 1.5, p. 8) associated with $\dot{T}^\kappa$, i.e., $\tau \in T^*$ iff τ is a term in the forcing language with respect to $P^{\kappa,1} * \dot{S}^\kappa$ and $\Vdash_{P^{\kappa,1} * \dot{S}^\kappa}$ "$\tau \in \dot{T}^\kappa$", ordered by $\tau \geq \sigma$ iff $\Vdash_{P^{\kappa,1} * \dot{S}^\kappa}$ "$\tau \geq \sigma$". Clearly, $T^* \in M$. Also, since $\Vdash_{P^{\kappa,1} * \dot{S}^\kappa}$ "$\dot{T}^\kappa$ is $\prec \lambda^+$-strategically closed", it can easily be verified that T^* itself is $\prec \lambda^+$-strategically closed in M and, since $M^\lambda \subseteq M$, in V as well. Therefore, as $\Vdash_{P^{\kappa,1} * \dot{S}^\kappa}$ "$|\dot{T}^\kappa| = k_1(\lambda)$ and $2^{k_1(\lambda)} = (k_1(\lambda))^+ = k_1(\lambda^+)$", we can assume without loss of generality that in M, $|T^*| = k_1(\lambda)$. This means we can let $\langle D_\alpha : \alpha < \lambda^+ \rangle$ enumerate in V the dense open subsets of T^* present in M and argue as before to construct in V an M-generic object H_2 over T^*.

Note now that since N can be assumed to be given by an ultrapower of M via a normal measure $\mathcal{U} \in M$ over $(P_\kappa(\kappa^+))^M$, Fact 2 of Section 1.2 of [C] tells us that $k_2'' H_2$ generates an N-generic object G_2^* over $k_2(T^*)$. By elementariness, $k_2(T^*)$ is the term forcing whose elements are names for elements of $k_2(\dot{T}^\kappa) = \dot{R}^\kappa$ in $N^{k_2(P^{\kappa,1} * \dot{S}^\kappa)}$. Therefore, since G_2^* is N-generic over $k_2(T^*)$, and since $G_0 * G_1$ is $k_2(P^{\kappa,1} * \dot{S}^\kappa)$-generic over N, Fact 1 of Section 1.5 of [C] tells us that for $G_2 = \{i_{G_0 * G_1}(\tau) : \tau \in G_2^*\}$, G_2 is $N[G_0][G_1]$-generic over R^κ. As G_0 is a set of conditions in an Easton support iteration of length κ in which a direct limit was taken at κ, each condition in G_0 has a support which is bounded in κ. It follows that in $V[G_0]$, $j : V \to N$ extends to $j^* : V[G_0] \to N[G_0][G_1][G_2]$. This proves Lemma 4.

$\square$ Lemma 4

Since $V \models$ "$|P^{\kappa,1}| = \kappa$", $V^{P^{\kappa,1}} \models$ "$2^\kappa = \kappa^{++}$". Thus, Lemmas 3 and 4 complete the proof of Theorem 1 when $n = 1$.

$\square$ Theorem 1 ($n = 1$)

We remark that Magidor's unpublished ideas mentioned at the beginning of the proof of Lemma 4 were used in the context of a ground model V so that $V \models$ "GCH + κ is supercompact". The partial ordering used in this situation was, as now, the $P^{\kappa,1}$ of Lemma 4. The embedding k_1 in this circumstance was as described above, but the embedding k_2 was generated by a normal measure $\mathcal{U} \in M$ concentrating on non-measurable cardinals. The proof given in Lemma 4 then went through in this situation as well to show $j = k_2 \circ k_1$ extends, using that although N is only κ closed with respect to V, GCH gives fewer dense open sets to meet, i.e., in the construction of G_1, only κ^+ instead of κ^{++} dense open subsets of Q^κ have to be met.

In conclusion to this section, we note that the first author's non-iterative proof of Theorem 1 for the case $n = 1$ [A97a] used Magidor's notion of iterated Prikry forcing [Ma] to destroy all measurable cardinals found below the cardinal κ produced in Theorem 3 of [AS97b], thereby requiring an initial assumption of a supercompact limit of supercompact cardinals. At the Oberwolfach meeting at which this proof was discovered, Magidor and Woodin independently of one another told both authors a (non-iterative) proof of Theorem 1 for the case $n = 1$ could be given using Radin forcing, starting from only one supercompact cardinal. Neither Woodin's nor Magidor's proof has been published (and both proofs seem unlikely to be published anywhere in the foreseeable future), but our methods here and the methods of [A97a] provide another non-iterative proof starting from only one supercompact cardinal for the case $n = 1$.

An outline of this proof is as follows: First, start with a ground model V so that $V \models$ "GCH + κ is supercompact". Next, force using the partial ordering $P^{\kappa,0}$ of Lemma 2 to preserve the supercompactness of κ, make $2^\kappa = \kappa^{++}$, and construct a κ^+ supercompact embedding $j^* : V^{P^{\kappa,0}} \to M^{j^*(P^{\kappa,0})}$ so that $M^{j^*(P^{\kappa,0})} \models$ "κ isn't measurable". Finally, force over $V^{P^{\kappa,0}}$ using Q^*, Magidor's notion of iterated Prikry forcing of [Ma], to destroy all measurable cardinals below κ. Magidor's arguments of [Ma] yield $V^{P^{\kappa,0} * \dot{Q}^*} \models$ "κ is both strongly compact and the least measurable cardinal", and the exact same argument as given in the Lemma of [A97a] shows $V^{P^{\kappa,0} * \dot{Q}^*} \models$ "κ is κ^+ supercompact". Also, since $\Vdash_{P^{\kappa,0}}$ "$|\dot{Q}^*| = \kappa$", $V^{P^{\kappa,0} * \dot{Q}^*} \models$ "$2^\kappa = \kappa^{++}$".

The advantage of the non-iterative proof just given and the earlier non-iterative proofs previously mentioned is that the large cardinal structure above κ can be arbitrary in any of these proofs. The proof of Lemma 4 and the proofs to be given in the next section require severe restrictions on the large cardinal structure of the universe. We will comment more upon this in the concluding remarks of the paper.

§4 The Case of Arbitrary Finite n

In this section, we give a proof of Theorem 1 for arbitrary finite n.

Proof of Theorem 1: Let $V_0 \models$ "ZFC + GCH + $\kappa_1 < \kappa_2 < \cdots < \kappa_n$ are the first n (for $n \in \omega$) supercompact cardinals + No cardinal $\lambda > \kappa_n$ is measurable". Let $P^* \in V_0$ be a partial ordering so that $V = V_0^{P^*} \models$ "Each κ_i for $i = 1, \ldots, n$ is indestructible under κ_i-directed closed forcing". The existence of this sort of generalized version of Laver's partial ordering of [L] is easy to show and is found in many places, e.g., [A83], [A98], or [CFM]. Note that since as in [L], P^* can be defined as an iteration so that for $P^*_{\kappa_i}$ the portion of P^* up through stage κ_i, $|P^*_{\kappa_i}| = \kappa_i$, and since also we can assume that the portion of P^* defined beyond stage κ_i is at least λ_i-directed closed, where for the rest of this section, λ_i is the least inaccessible above κ_i, $V \models$ "$2^{\kappa_i} = \kappa_i^+$ for $i = 1, \ldots, n$". Also, by the Lévy-Solovay results [LS], since $V_0 \models$ "No cardinal $\lambda > \kappa_n$ is measurable", $V \models$ "No cardinal $\lambda > \kappa_n$ is measurable" as well.

We take now V as our ground model and let $\kappa_0 = \omega$. P will be defined as the cartesian product $\prod_{1 \le i \le n} P^{\kappa_i}$ where P^{κ_i} for $i = 1, \ldots, n$ can be defined in two ways, depending upon whether the simpler or more complicated version of the partial ordering $P^1_{\delta,\lambda}[S]$ is used. If the simpler version of $P^1_{\delta,\lambda}[S]$ is used, then $P^{\kappa_i} = P^{\kappa_i,0} * \dot{P}^{\kappa_i,1}$, where $P^{\kappa_i,0}$ is defined as in Section 2, using only those inaccessibles in the interval $(\kappa_{i-1}, \kappa_i]$ satisfying GCH in its field and fixing κ_{i-1} as the cofinality of the non-reflecting stationary sets added by each $P^0_{\delta,\lambda}$ (since $V \models$ "$2^{\kappa_i} = \kappa_i^+$", reflection shows that unboundedly many cardinals in (κ_{i-1}, κ_i) will be in the field of $P^{\kappa_i,0}$), and $P^{\kappa_i,1}$ also adds non-reflecting stationary sets of ordinals of cofinality κ_{i-1} to every $V^{P^{\kappa_i,0}}$-measurable cardinal in the interval (κ_{i-1}, κ_i). If the more complicated version of $P^1_{\delta,\lambda}[S]$ is used, then $P^{\kappa_i,0}$ is defined as in Section 2, using only those inaccessibles in the interval $(\kappa_{i-1}, \kappa_i]$ in its field satisfying GCH, and $P^{\kappa_i,1}$ is as just defined when the simpler version of $P^1_{\delta,\lambda}[S]$ is used.

For $i = 1, \ldots, n$, write $P = P_i \times P^{\kappa_i} \times P^i$, where $P_i = \prod_{1 \le j \le i-1} P^{\kappa_j}$ and $P^i = \prod_{i+1 \le j \le n} P^{\kappa_j}$. When the simpler version of $P^1_{\delta,\lambda}[S]$ is used, it easily follows that P^i is κ_i-directed closed. This is since by the remarks in the middle of p. 108 of [AS97a], each $P^1_{\delta,\lambda}[S]$ used in the definition of each $P^{\kappa_j,0}$ for $j = i+1, \ldots, n$ is at least κ_i-directed closed, so as the cofinalities of the ordinals present in the non-reflecting stationary sets added by $P^{\kappa_j,1}$ for $j = i+1, \ldots, n$ are at least κ_i, P^{κ_j} for $j = i+1, \ldots, n$ and $P^i = \prod_{i+1 \le j \le n} P^{\kappa_j}$ are all κ_i-directed closed. Also, when the more complicated version of $P^1_{\delta,\lambda}[S]$ is used, it is the case that a V-generic object for P^i is V-generic over a κ_i-directed closed partial ordering. This follows by the argument of Lemma 14 of [AS97b] combined with the fact that any partial ordering of the form $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$ used in the definition of $P^{\kappa_j,0}$ for $j = i+1, \ldots, n$ is κ_i-directed closed. (See Lemma 4 of [AS97b] for a proof.) Thus, by the indestructibility properties of κ_i and the fact P^i is λ_i-strategically closed, $V^{P^i} \models$ "κ_i is supercompact and $2^{\kappa_i} = \kappa_i^+$".

By the definition of $P^{\kappa_{i+1}}$, $V^{P^i} \models$ "No cardinal $\delta \in (\kappa_i, \kappa_{i+1})$ is measurable", for $i = 1, \ldots, n-1$. When $i = n$, we take (κ_i, κ_{i+1}) as all ordinals $> \kappa_i$ and P^i as being trivial, so our initial assumptions on V once more give us $V^{P^i} \models$ "No cardinal $\delta \in (\kappa_i, \kappa_{i+1})$ is measurable". Therefore, the arguments of Lemmas 2 - 4 apply to show $V^{P^i \times P^{\kappa_i}} \models$ "κ_i is $< \kappa_{i+1}$ strongly compact, κ_i is κ_i^+ supercompact, $2^{\kappa_i} = \kappa_i^{++}$, and no cardinal $\delta \in (\kappa_{i-1}, \kappa_i)$ is measurable", taking when $i = n$ "$< \kappa_i$ strongly compact" as meaning "δ strongly compact for all cardinals $\delta > \kappa_n$", i.e., as meaning fully strongly compact. Since by the definition of P_i, $|P_i| < \lambda_{i-1}$, the results of [LS] tell us $V^{P^i \times P^{\kappa_i} \times P_i} = V^P \models$ "κ_i is $< \kappa_{i+1}$ strongly compact, κ_i is κ_i^+ supercompact, $2^{\kappa_i} = \kappa_i^{++}$, and no cardinal $\delta \in (\kappa_{i-1}, \kappa_i)$ is measurable". Therefore, since a result of DiPrisco [DH] tells us that if δ is $< \gamma$ strongly compact and γ is ρ strongly compact, then δ is ρ strongly compact, and since we are working with finitely many cardinals $\kappa_1, \ldots, \kappa_n$, we can apply finitely often the result of [DH] to infer $V^P \models$ "κ_i is strongly compact, κ_i is κ_i^+ supercompact, $2^{\kappa_i} = \kappa_i^{++}$, and no cardinal $\delta \in (\kappa_{i-1}, \kappa_i)$ is measurable". This completes the proof of Theorem 1 for arbitrary finite n.

□ Theorem 1

§5 Concluding Remarks

In conclusion to this paper, we note that for the moment, we are restricted to proving Theorem 1, as was Magidor, only to finite values of n. The proof of Theorem 1, and Magidor's original proof of the consistency of the first $n \in \omega$ strongly compact cardinals being the first n measurable cardinals, both heavily use that the ground model contains only finitely many supercompact cardinals and no measurable cardinals beyond their supremum. This is evident in the proof of Lemma 4, its equivalent form in Magidor's original proof, and in the proof given in Section 4. Although it is possible to give alternate proofs, in both our situation and Magidor's, by composing supercompact embeddings either with embeddings generated by normal measures over measurable cardinals of Mitchell order 0 in Magidor's case or with a κ^+ supercompact embedding as constructed in Lemma 2 in our case, the proofs still require the initial assumption of only finitely many supercompact cardinals with no measurable cardinals above their supremum. (See the end of [A95] for a further discussion of this problem.) Thus, the prime question of the relative consistency of the first ω measurable and strongly compact cardinals coinciding, with or without any additional degrees of supercompactness, remains open.

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