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A SCATTERING OF ORDERS

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Abstract. A linear ordering is *scattered* if it does not contain a copy of the rationals. Hausdorff characterised the class of scattered linear orderings as the least family of linear orderings that includes the class $\mathcal{B}$ of well-orderings and reversed well-orderings, and is closed under lexicographic sums with index set in $\mathcal{B}$.

More generally, say that a partial ordering is *κ-scattered* if it does not contain a copy of any κ-dense linear ordering. We prove analogues of Hausdorff's result for κ-scattered linear orderings, and for κ-scattered partial orderings satisfying the finite antichain condition.

We also study the $\mathbb{Q}_\kappa$*-scattered* partial orderings, where $\mathbb{Q}_\kappa$ is the saturated linear ordering of cardinality κ, and a partial ordering is $\mathbb{Q}_\kappa$*-scattered* when it embeds no copy of $\mathbb{Q}_\kappa$. We classify the $\mathbb{Q}_\kappa$-scattered partial orderings with the finite antichain condition relative to the $\mathbb{Q}_\kappa$-scattered linear orderings. We show that in general the property of being a $\mathbb{Q}_\kappa$-scattered linear ordering is not absolute, and argue that this makes a classification theorem for such orderings hard to achieve without extra set-theoretic assumptions.

1. Introduction

The research described in this paper was motivated by a classical theorem of Hausdorff [6] about linear orderings. A linear ordering L is said to be *scattered* if and only if L does not embed a copy of the rationals; Hausdorff proved a structure theorem which analyses the class of scattered linear orderings. The definition of the *reverse* of an ordering is given below in Section 2.1, and the definition of *lexicographic sum* appears in Section 2.5.

Theorem (Hausdorff's classification theorem)**.** *Let $\mathcal{B}$ be the class of well-orderings and reverse well-orderings. The class of scattered linear orderings is the least class which contains $\mathcal{B}$, and is closed under lexicographic sums with index set lying in $\mathcal{B}$.*

This is a satisfying result because:

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- The objects in the "base class" $\mathcal{B}$ are very simple, and are readily seen to be scattered linear orderings.
- The result gives a stratification of the class of scattered linear orderings. Let $\mathcal{S}_0 = \mathcal{B}$, let $\mathcal{S}_{\alpha+1}$ be the class of lexicographic sums of elements of $\mathcal{S}_\alpha$ with index set in $\mathcal{B}$, and for λ limit let $\mathcal{S}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{S}_\alpha$; then $\bigcup_\alpha \mathcal{S}_\alpha$ is the class of scattered orderings. This means that one can prove results about scattered sets by induction on their complexity, where the complexity of a scattered set L is the least α with $L \in \mathcal{S}_\alpha$.

If we aim to generalise Hausdorff's theorem, then a natural approach is to try replacing linear orderings by some more general class of partial orderings. It turns out that a natural class of posets to use here is the *FAC (finite antichain condition)* posets, that is to say the posets in which every antichain is finite. Abraham and Bonnet [1] gave a structure theorem in the style of Hausdorff's result for scattered FAC posets, where a poset P is *scattered* if there is no chain in P isomorphic to the rationals.

Theorem (Abraham and Bonnet)**.** *Let $\mathcal{B}'$ be the class of WQO posets and reversals of WQO posets. The class of scattered FAC posets is the least class which contains $\mathcal{B}'$, is closed under lexicographic sums with index set lying in $\mathcal{B}'$, and is closed under augmentation.*

Here a *WQO (well quasi-ordered)* poset is exactly a well-founded FAC poset; the WQO posets are analogous to the well-orderings in the Hausdorff theorem. An *augmentation* of a poset P is a poset P' with the same underlying set as P and "more relations", that is $p \leq_P q \implies p \leq_{P'} q$ for all $p, q \in P$. The intuition here is that augmentation is needed because a lexicographic sum of posets has a "block structure"; Abraham and Bonnet [1, Section 4] give an example to show that augmentation is necessary in their result.

Džamonja and Thompson [4] considered another kind of generalisation, by varying the notion of "scattered". Given an infinite cardinal κ, say that a poset P is *κ-scattered* if there is no chain in P which is κ-dense; assuming in addition that $\kappa^{<\kappa} = \kappa$, say that a poset P is *$\mathbb{Q}_\kappa$-scattered* if there is no chain in P isomorphic to the unique saturated linear ordering of cardinality κ. When $\kappa = \omega$ these notions coincide, but as we will see they are quite different even for $\kappa = \aleph_1$.

Džamonja and Thompson studied these scatteredness properties for the classes of linear orderings and FAC partial orderings. They introduced the notion of *FAC weakening* (dropping some relations in an FAC poset but maintaining FAC) and considered the class which is the closure of linear orders with no decreasing sequence of length κ under inversions, lexicographic sums and FAC weakenings. They proved that this class includes the class of all $\mathbb{Q}_\kappa$-scattered FAC posets, and is included in the class of κ-scattered FAC posets.

Building on this previous work, in this paper we largely complete our understanding of κ-scattered orders, and give limitations on the extent to which one is able to understand $\mathbb{Q}_\kappa$-scattered orders. In particular, we answer a number of open questions from [4], After some preliminaries:

- In Section 3 we give a structure theorem along the same lines as the Hausdorff theorem for κ-scattered linear orderings. We put in the "base class" all linear orderings of size less than κ, so that our structure theorem is proved relative to this class. It seems reasonable to us to treat the class of linear orderings of size less than κ as a kind of "black box", since they are all κ-scattered for a trivial reason.

- In Section 4 we study the κ-*fat* partial orderings, that is those P which are not antichains and are such that when $a < b$ there are at least κ many c with $a < c < b$. We show that every κ-fat FAC poset contains a κ-dense chain, and give an example to show that this is not true in general if we weaken the FAC hypothesis to "every antichain is countable".
- In Section 5 we study the behaviour of FAC posets satisfying some form of scatteredness under augmentation; this is a technical issue which is important for the structure theorems of subsequent sections. Bonnet and Pouzet [3] showed that an augmentation of a scattered FAC poset is scattered, and Džamonja and Thompson extended this result to show that an augmentation of a κ-scattered (resp. $\mathbb{Q}_\kappa$-scattered) FAC poset is κ-scattered (resp. $\mathbb{Q}_\kappa$-scattered). We give an alternative proof of the easier κ-scattered case using the results of Section 4.
- In Section 6 we prove a general result about the class $\mathcal{G}$ of FAC posets P such that all chains of P lie in some class $\mathcal{G}_0$. We show under quite weak hypotheses that every element of $\mathcal{G}$ can be built from WQO posets and elements of $\mathcal{G}_0$ by a certain recipe; under stronger hypotheses the class of posets built according to this recipe is exactly $\mathcal{G}$.
- In Section 7 we use the results of Sections 3, 5 and 6 to prove structure theorems for the classes of κ-scattered FAC posets and $\mathbb{Q}_\kappa$-scattered FAC posets. As a corollary we get a structure theorem for countable FAC posets. The structure theorem for κ-scattered FAC posets has the same "black box" as the result from Section 3 for κ-scattered linear orders, namely the class of linear orderings of size less than κ. The result for $\mathbb{Q}_\kappa$-scattered FAC posets is slightly less satisfying, in that the "black box" here is the more mysterious class of $\mathbb{Q}_\kappa$-scattered linear orders.
- In the concluding Section 8 we observe that, by the results of Section 3, the property of being a κ-scattered linear ordering is upwards absolute to cardinal preserving extensions. We then argue that assuming CH there is a $\mathbb{Q}_{\aleph_1}$-scattered partial ordering whose $\mathbb{Q}_{\aleph_1}$-scatteredness is not absolute to some cardinal-preserving generic extension with the same reals (and hence the same $\mathbb{Q}_{\aleph_1}$). This suggests that no structure theorem of the sort which we proved for κ-scattered linear orderings can hold for $\mathbb{Q}_\kappa$-scattered linear orderings. In particular it implies that we should probably be content with the result from Section 7 in which $\mathbb{Q}_\kappa$-scattered FAC posets are classified relative to the $\mathbb{Q}_\kappa$-scattered linear orderings.

2. Preliminaries

2.1. **Notation and basic definitions.** If P is a partial ordering and $p, q \in P$ we will write $p \perp_P q$ for "p is incomparable with q in P", and $p \parallel_P q$ for "p is comparable with q in P". We denote by $p^{\perp}_P$ the set of q such that $q \perp_P p$, and by $p^{\parallel}_P$ the set of q such that $q \parallel_P p$. We will usually omit the subscripts unless there is some possibility of confusion.

We will denote by P^* the *reversal of* P, that is the poset with the same underlying set as P and the relation $p \leq_{P^*} q \iff q \leq_P p$ for all p, q.

Let P be a poset and let $X, Y \subseteq P$. Then:

(1) X is an *initial segment of* P (resp. a *final segment of* P) if and only if for all $b \in X$ and all $a \leq b$ (resp. $a \geq b$) we have $a \in X$.
(2) X is *cofinal in* P (resp. *coinitial in* P) if and only if for every $a \in P$ there is $b \in X$ such that $a \leq b$ (resp. $a \geq b$).
(3) If $a, b \in P$ then $(a, b)_P = \{p \in P : a < p < b\}$.
(4) X is *convex in* P if and only if $(a, b)_P \subseteq X$ for all $a, b \in X$.
(5) $X < Y$ if and only if $x < y$ for all $x \in X$ and $y \in Y$. Similarly if $a \in P$ then $X < a$ (resp. $a < X$) if and only if $x < a$ (resp. $a < x$) for all $x \in X$.

2.2. **Dense and scattered orderings.** Let κ be an infinite cardinal.

Definition 2.1. A linear ordering L is *κ-dense* if and only $|L| > 1$, and for every $a, b \in L$ with $a < b$ the interval (a, b) has cardinality at least κ.

It is easy to see that any κ-dense ordering has a subordering of cardinality κ which is κ-dense. Such a subordering will have the property that every open interval (a, b) has cardinality exactly κ. The following rather trivial fact about κ-dense linear orderings will be useful.

Lemma 2.2. *Let L be κ-dense, and let $L = L_0 \cup L_1$ where the L_i are disjoint. Then either L_0 is κ-dense or some interval of L_1 is κ-dense, and* vice versa.

We will only apply the term "κ-dense" to *linear orderings*, reserving a different term for partial orderings which satisfy the obvious generalisation.

Definition 2.3. A poset P is *κ-fat* if and only if it is not an antichain, and for every $a, b \in P$ with $a < b$ the interval (a, b) has cardinality at least κ.

Definition 2.4. A poset P is *κ-scattered* if and only if there is no subset $L \subseteq P$ such that the restriction of P to L is a κ-dense linear ordering. A poset is *scattered* if and only if it is $\aleph_0$-scattered, or equivalently it does not embed a copy of $\mathbb{Q}$.

Definition 2.5. A linear ordering L is *σ-scattered* if and only if there exist $L_n \subseteq L$ for $n \in \omega$ such that $L = \bigcup_n L_n$, and the restriction of L to each L_n is scattered.

2.3. **Saturated linear orderings.** Let κ be a regular cardinal. Recall that in model theory a model $\mathcal{M}$ is said to be *κ-saturated* if and only if $\mathcal{M}$ realises all types over subsets of size less than κ. In linear orderings this property can be stated in a very simple form: L is κ-saturated if and only if for any two sets $A, B \subseteq L$ of cardinality less than κ, if $A < B$ there is $x \in L$ such that $A < x < B$.

Before the development of model theory, Hausdorff [6] studied κ-saturated linear orderings and proved the following facts: from a modern perspective these are special cases of general model-theoretic results.

- If $\kappa^{<\kappa} = \kappa$ there is a κ-saturated linear ordering of cardinality κ
- Any two κ-saturated linear orderings of cardinality κ are isomorphic.
- A κ-saturated linear ordering of cardinality κ is *universal*, in the sense that it contains copies of every linear ordering of cardinality κ.

In the case that a κ-saturated linear ordering of size κ exists, we will refer to it as $\mathbb{Q}_\kappa$. In a context where $\mathbb{Q}_\kappa$ exists, we define:

Definition 2.6. A poset P is *$\mathbb{Q}_\kappa$-scattered* if and only if there is no subset $L \subseteq P$ such that the restriction of P to L is isomorphic to $\mathbb{Q}_\kappa$.

Our main interest will be in the case $\kappa = \aleph_1$, and so we collect some easy facts about $\mathbb{Q}_{\aleph_1}$. We recall that if M is a transitive model of ZFC set theory, then an *outer model of* M is a transitive model of ZFC which contains M and has the same ordinals of M; for example the "generic extensions" of M produced by forcing are outer models of M.

Lemma 2.7. *Assume that $\mathbb{Q}_{\aleph_1}$ exists. Then*

(1) *CH holds.*
(2) *If W is an outer model of V with no new reals, then $\mathbb{Q}^V_{\aleph_1}$ is $\aleph_1$-saturated in W.*
(3) *If W is an outer model of V with a new real, then $\mathbb{Q}^V_{\aleph_1}$ is not $\aleph_1$-saturated in W.*

Proof. We take each claim in turn.

(1) Fix an order-embedding $f : \mathbb{Q} \to \mathbb{Q}_{\aleph_1}$. Given $x \in \mathbb{Q}_{\aleph_1}$ and $r \in \mathbb{R}$, we will say that x *codes* r if and only if $\{q : f(q) \leq x\} = \{q : q \leq r\}$. Clearly every x codes at most one real r, and it follows immediately from saturation that every real r is coded by at least one x. Since $\mathbb{Q}_{\aleph_1}$ has size $\aleph_1$, it follows immediately that CH holds.
(2) If W is an outer model of V with no new reals, then $\aleph_1^V = \aleph_1^W$ and there are no new countable subsets of ω_1. It follows easily $\mathbb{Q}^V_{\aleph_1}$ is still $\aleph_1$-saturated in W.
(3) Suppose that r is a new real. If $\mathbb{Q}^V_{\aleph_1}$ were still $\aleph_1$-saturated in W then we could find $x \in \mathbb{Q}^V_{\aleph_1}$ which codes r, but then we could work in V to recover r from f and x.

□

We would like to thank Martin Goldstern for pointing out the coding method used in the proof. Similar arguments show that $\mathbb{Q}_\kappa$ exists if and only if $\kappa^{<\kappa} = \kappa$, and is absolute to exactly those extensions with the same bounded subsets of κ.

It will be useful to have a concrete realisation of $\mathbb{Q}_{\aleph_1}$. The following example is due to Sierpiński [8].

Fact 2.8. *Let L be the set of functions f from ω_1 to 2, such that there is $\alpha < \omega_1$ with $f(\alpha) = 1$ and $f(\beta) = 0$ for all $\beta > \alpha$. Order L with the lexicographic ordering. Then L is $\aleph_1$-saturated. If in addition CH holds then $|L| = \aleph_1$, so that we may take $\mathbb{Q}_{\aleph_1} = L$.*

2.4. **FAC and WQO posets.** Two standard classes of posets will be particularly important.

Definition 2.9. Let P be a poset.

(1) P is *FAC* (finite antichain condition) if and only if every antichain is finite.
(2) P is *WQO* (well quasi-ordered) if and only if P is FAC and well-founded.

Intuitively FAC posets are "close to being linear orderings" and WQO posets are "close to being well-orderings". We recall some useful facts about FAC and WQO posets.

Lemma 2.10. *The following are equivalent for a poset P:*

(1) *P is WQO.*

(2) *For any ω-sequence $\langle p_i : i < \omega\rangle$ of elements of P, there exists an increasing sequence $\langle i_n : n < \omega\rangle$ such that $m < n \implies p_{i_m} \leq p_{i_n}$ for all m, n.*
(3) *The set of initial segments of P is well-founded under inclusion.*

Proof. (1) implies (2): Colour the pairs (i, j) with $i < j$ in the following way: (i, j) is red if p_i and p_j are incomparable, green if $p_i > p_j$ and blue if $p_i \leq p_j$. By Ramsey's theorem there is an infinite homogeneous set, and by hypothesis there are no infinite red-homogeneous or green-homogeneous sets.

(2) implies (3): Suppose for a contradiction that $\langle A_i : i < \omega\rangle$ is a sequence of initial segments such that $A_{i+1} \subsetneq A_i$ for all i. Choose $p_i \in A_i - A_{i+1}$ for each i, and appeal to Lemma 2.10 to find $i < j$ such that $p_i \leq p_j$. Then $p_i \in A_j$ since A_j is downward closed, and $A_j \subseteq A_{i+1}$ because $i + 1 \leq j$, but $p_i \notin A_{i+1}$. This is a contradiction

(3) implies (1): Suppose for a contradiction that $\langle p_n : n \in \omega\rangle$ is either a strictly deceasing sequence or a 1-1 enumeration of an infinite antichain. In either case the sequence of sets $A_n = \{p : \exists m \geq n\ p \leq p_m\}$ is a strictly decreasing sequence of initial segments, contradicting the assumption that the set of initial segments is well-founded under inclusion. □

Observe that if Q is an FAC poset then the set of antichains of Q is well-founded under reverse inclusion. We will use this to define the *antichain rank* on the set of antichains.

Definition 2.11. Let Q be an FAC poset and denote by $A(Q)$ the set of antichains of Q.

(1) For $A \in A(Q)$, $\rho_Q(A)$ is the rank of A in $(A(Q), \supsetneq)$.
(2) $\rho(Q) = \rho_Q(\emptyset)$.

If Q is an FAC poset and $q \in Q$, then $q^\perp$ is also an FAC poset, and $\rho(q^\perp) < \rho(Q)$. We will use this to power several inductive arguments. We note that only the empty ordering has rank zero, and that the linear orderings are exactly the orderings of rank one.

2.5. **Basic constructions.** We will build complex posets out of simpler ones using *lexicographic sums.* If Q is a poset and $\langle P_q : q \in Q\rangle$ is a Q-indexed family of posets then the lexicographic sum is obtained as follows. We form the set of all pairs (q, r) with $q \in Q$ and $r \in P_q$, and then order them by stipulating that $(q_0, r_0) \leq (q_1, r_1)$ if and only if either $q_0 < q_1$ in Q, or $q_o = q_1$ and $r_0 \leq r_1$ in P_{q_0}. Intuitively we are replacing each $q \in Q$ by a copy of P_q, and then putting all elements of P_{q_0} below all elements of P_{q_1} when $q_0 < q_1$.

Another basic operation on posets is *augmentation.* Given two partial orderings $\leq$ and $\leq'$ on a set X, let $P = (X, \leq)$ and $P' = (X, \leq')$.

- P' is an *augmentation of* P if and only if $x \leq y \implies x \leq' y$ for all $x, y \in X$.
- P' is a *linearisation* of P if and only if P' is an augmentation of P, and P' is a linear order.

Szpilrajn [10] proved that every poset has a linearisation. Similarly every well-founded poset has a well-founded linearisation.

We will often be in the following situation: we have a poset P and a partition of P into disjoint sets X_i for i in some index set I. Given a partial ordering $\preceq$ of I, we wish to show that P is an augmentation of the lexicographic sum over $(I, \prec)$

of P_i, where P_i is the restriction of P to X_i. For this to be true it is necessary and sufficient that if $i \prec j$ then $x <_P y$ for all $x \in X_i$, $y \in X_j$.

The following lemma is straightforward:

Lemma 2.12. *Let $\mathcal{C}$ be any of the following classes of posets: FAC, WQO, κ-scattered, $\mathbb{Q}_\kappa$-scattered. Then every lexicographic sum of elements of $\mathcal{C}$ is an element of $\mathcal{C}$.*

2.6. **Cofinal and co-initial sets.** It is a standard fact that any poset P has a cofinal well-founded subset. To see this we just build a sequence of elements $p_\alpha \in P$ with ordinal indices α; if $\{p_\alpha : \alpha < \beta\}$ is not cofinal then we choose p_β so that $p_\beta \not\leq p_\alpha$ for $\alpha < \beta$, and if $\{p_\alpha : \alpha < \beta\}$ is cofinal then we halt the construction. We see that $p_\alpha < p_\beta \implies \alpha < \beta$ for all α and β, so that the set of p_α's is well-founded. Since the reversed poset P^* also has a well-founded cofinal set, P has a co-initial set which is well-founded in the reverse ordering.

When P happens to be a linear ordering, this argument allows us to write P as the lexicographic sum of its restrictions to some convex sets, indexed along a set of the form $\gamma^* + \delta$ for some ordinals γ, δ. This description of a linear ordering is often useful.

Since FAC posets are "close to linear", we can hope for a similar decomposition of an FAC poset. Abraham and Bonnet [1] proved such a decomposition result; the following Lemma summarises Lemma 3.1 in [1] (see also [5, §9.9.1]) together with the discussion immediately following that Lemma. We have included a proof sketch.

Lemma 2.13. *Let P be an FAC poset. Then there exist an ordinal ζ and $p_\alpha \in P$, $Z_\alpha \subseteq P$ for $\alpha < \zeta$ such that:*

1. *$\{p_\alpha : \alpha < \zeta\}$ is a WQO subset of P.*
2. *Z_α is a convex set with maximum element p_α. The Z_α are disjoint and form a partition of P.*
3. *P is an augmentation of the lexicographic sum of the Z_α along the index set $(\zeta, \preceq)$, where $\alpha \preceq \beta \iff p_\alpha \leq p_\beta$.*

Proof Sketch. Choose p_β so that $p_\beta \not\leq p_\alpha$ for $\alpha < \beta$, and additionally (using Lemma 2.10) so that the initial segment $\{p_\alpha : \alpha < \beta, p_\alpha < p_\beta\}$ of the WQO $\{p_\alpha : \alpha < \beta\}$ is minimal under inclusion. Then let Z_β be the set of $p \in P$ such that β is minimal with $p \leq p_\beta$. Verify that if $p_\alpha < p_\beta$ then $Z_\alpha < Z_\beta$. □

3. Analysis of κ-scattered linear orderings

Throughout this section let κ be a fixed infinite cardinal. We will give a structure theorem for κ-scattered linear orderings which parallels the classical Hausdorff theorem for scattered orderings. In the Hausdorff theorem the "basic orderings" are just well-orderings and converse well-orderings, but we need a larger class of basic linear orderings.

Definition 3.1. Let $\mathcal{BL}_\kappa$ be the class of all linear orderings D such that:

- $|D| < \kappa$, or
- D is a well-ordering or a converse well-ordering.

We note that it is reasonable to put all the linear orderings of size less than κ into our basic class. While the class of all such orderings is probably very complex, they are all trivially κ-scattered and we should not expect our classification to say very much about them.

Definition 3.2. Let $\mathcal{L}_\kappa$ be the least class of linear orderings which contains $\mathcal{BL}_\kappa$, and is closed under lexicographic sums with index set in $\mathcal{BL}_\kappa$.

We will need some elementary properties of the class $\mathcal{L}_\kappa$.

Lemma 3.3. *The class $\mathcal{L}_\kappa$ consists of κ-scattered orderings, and is closed under reversals, restrictions, and lexicographic sums with index set in $\mathcal{L}_\kappa$.*

Proof. We will stratify $\mathcal{L}_\kappa$ and then use induction on the strata. Let $\mathcal{L}^0_\kappa = \mathcal{BL}_\kappa$, let $\mathcal{L}^{\alpha+1}_\kappa$ be the class of all lexicographic sums of elements of $\mathcal{L}^\alpha_\kappa$ with index set in $\mathcal{BL}_\kappa$, and let $\mathcal{L}^\lambda_\kappa = \bigcup_{\alpha<\lambda} \mathcal{L}^\alpha_\kappa$ for λ limit. It is easy to see that the classes $\mathcal{L}^\alpha_\kappa$ form an increasing sequence with union $\mathcal{L}_\kappa$. We may then verify by a routine induction that $\mathcal{L}^\alpha_\kappa$ consists of κ-scattered orderings, and is closed under reversals and restrictions.

To finish we will show by induction on α that $\mathcal{L}_\kappa$ is closed under lexicographic sums with index set in $\mathcal{L}^\alpha_\kappa$. This is immediate by definition for $\alpha = 0$, and limit stages are easy. Suppose we have closure for sums with index sets in $\mathcal{L}^\alpha_\kappa$, and consider a lexicographic sum $\sum_{a\in L} M_a$ where $L \in \mathcal{L}^{\alpha+1}_\kappa$, and $M_a \in \mathcal{L}_\kappa$ for all a.

By the definition of $\mathcal{L}^{\alpha+1}_\kappa$, we may represent L as a lexicographic sum $\sum_{b\in L'} L_b$, where $L' \in \mathcal{BL}_\kappa$ and $L_b \in \mathcal{L}^\alpha_\kappa$. For each $b \in L'$, the set $I_b = \{(b, x) : x \in L_b\}$ is a convex subset of L which is order-isomorphic to L_b, in particular $I_b \in \mathcal{L}^\alpha_\kappa$. Let $N_b = \sum_{a\in I_b} M_a$, then N_b is a convex subset of $\sum_{a\in L} M_a$. Also N_b is a lexicographic sum of elements of $\mathcal{L}_\kappa$ with index set in $\mathcal{L}^\alpha_\kappa$, so that $N_b \in \mathcal{L}_\kappa$ by our induction hypothesis. Clearly $\sum_{a\in L} M_a$ is isomorphic to $\sum_{b\in L'} N_b$, a lexicographic sum of elements of $\mathcal{L}_\kappa$ with index set in $\mathcal{BL}_\kappa$, and so $\sum_{a\in L} M_a \in \mathcal{L}_\kappa$. $\square$

We will soon show that $\mathcal{L}_\kappa$ is exactly the class of κ-scattered orderings, but before that we develop some machinery. The arguments here are parallel to those in one of the standard proofs of Hausdorff's Theorem.

Definition 3.4. For a linear order L, we define an equivalence relation E^L by letting aE^Lb if and only if the open interval bounded by a and b has cardinality less than κ.

We will often omit the superscript L.

Lemma 3.5. *Let L be a linear ordering. Then*

1. *L is κ-dense if and only if $|L| > 1$ and E is the identity relation on L.*
2. *If L forms a single equivalence class under E, then $L \in \mathcal{L}_\kappa$.*

Proof. The first claim is immediate from the definition of the relation E. For the second claim let L form a single E-class. As is true for any linear ordering, we may find ordinals γ and δ such that L can be written as a lexicographic sum of points and bounded intervals indexed by $\gamma^* + \delta$; since L forms a single E-class each term in this sum is a linear ordering of size less than κ. It follows immediately from the closure properties of $\mathcal{L}_\kappa$ that $L \in \mathcal{L}_\kappa$. $\square$

We note that each equivalence class of E is convex, so that the quotient L/E can naturally be given the structure of a linear ordering. The operation which maps L to L/E will play an important role for us, analogous to that of the Cantor-Bendixson derivative in the proof of the Perfect Set Theorem. As one would expect from this analogy, we will need to iterate this operation.

Definition 3.6. Given a linear ordering L we define by recursion equivalence relations E_α on L, which will have convex equivalence classes and will increase as α increases. We denote by L_α the quotient L/E_α with the natural linear ordering, and by $[a]_\alpha$ the class of a in the relation E_α.

- E_0 is the identity relation.
- $aE_{\alpha+1}b$ if and only if $[a]_\alpha E^{L_\alpha}[b]_\alpha$.
- For λ limit, $aE_\lambda b$ if and only if there is $\alpha < \lambda$ such that $aE_\alpha b$.

Note in particular that $L_{\alpha+1}$ is isomorphic to the quotient of L_α by E^{L_α}.

Definition 3.7. If L is a linear ordering, $\lambda(L)$ is the least ordinal α such that $E_\alpha = E_{\alpha+1}$.

It is easy to see that $\lambda(L)$ exists and $\lambda(L) < |L|^+$.

Lemma 3.8. *If L is κ-scattered then $|L_{\lambda(L)}| = 1$.*

Proof. Let $\alpha = \lambda(L)$, and suppose for contradiction that $|L_\alpha| > 1$. Since $E_\alpha = E_{\alpha+1}$, E^{L_α} is the identity relation on L_α. Since L_α has more than one point, L_α is a κ-dense linear ordering. Choosing representative elements for each class in L_α, and recalling that the classes are convex sets in L, we obtain an order preserving map from L_α into L. This contradicts the assumption that L is κ-scattered. □

We need one more easy technical fact.

Lemma 3.9. *Let L be a linear ordering. Then*

(1) *If M is a convex set in L, E^M_α is the restriction of E^L_α to M.*
(2) *If M is an equivalence class of E_α then $\lambda(M) \leq \alpha$.*

Proof. The first claim follows by a straightforward induction on α. For the second claim observe that M is convex, hence by the first claim $|M_\alpha| = 1$ and so by definition $\lambda(M) \leq \alpha$. □

We can now give the promised analysis of κ-scattered linear orderings.

Theorem 3.10. *Let L be a linear ordering. Then the following are equivalent:*

(1) *L is κ-scattered.*
(2) *$L \in \mathcal{L}_\kappa$.*
(3) *$|L_{\lambda(L)}| = 1$.*

Proof. Lemma 3.3 implies that orderings in $\mathcal{L}_\kappa$ are κ-scattered, and Lemma 3.8 implies that L being κ-scattered implies $|L_{\lambda(L)}| = 1$. To finish we show by induction on $\lambda(L)$, for all linear orders L simultaneously, that if $|L_{\lambda(L)}| = 1$ then L is κ-scattered. If $\lambda(L) = 0$ then $|L| = 1$ and there is nothing to do.

Suppose that $\lambda(L)$ is a successor ordinal, say $\lambda(L) = \alpha + 1$. The key points are that we can view L as the lexicographic sum of the E_α-classes with index ordering L_α, and that since $|L_{\alpha+1}| = 1$ any two points of L_α are equivalent in E^{L_α}.

By Lemma 3.9, if M is a E_α-class then $\lambda(M) \leq \alpha$, and so by induction $M \in \mathcal{L}_\kappa$. By Lemma 3.5 $L_\alpha \in \mathcal{L}_\kappa$. By definition the class $\mathcal{L}_\kappa$ is closed under lexicographic sums, so $L \in \mathcal{L}_\kappa$.

Finally suppose that $\lambda(L)$ is a limit ordinal, say $\lambda(L) = \lambda$. Let $a \in L$ be arbitrary and let A_γ be the E_γ-class of a; we note that A_γ is convex, A_γ increases with γ and $L = \bigcup_{\gamma<\lambda} A_\gamma$.

By Lemma 3.9 $\lambda(A_\gamma) \leq \gamma$, so by induction $A_\gamma \in \mathcal{L}_\kappa$. Now let $L_\gamma = \{b < a : b \in A_{\gamma+1} \setminus A_\gamma\}$ and $R_\gamma = \{b > a : b \in A_{\gamma+1} \setminus A_\gamma\}$. Since $\mathcal{L}_\kappa$ is closed under restriction, each of these sets is in $\mathcal{L}_\kappa$. If $\gamma < \delta < \lambda$ then $L_\delta < L_\gamma < A_0 = \{a\} < R_\gamma < R_\delta$; so L is the lexicographic sum indexed by $\lambda^* + 1 + \lambda$ of orderings in $\mathcal{L}_\kappa$, hence $L \in \mathcal{L}_\kappa$. □

Theorem 3.10 can be used to prove analogues of classical facts about scattered posets. We give some easy examples.

Corollary 3.11. *Let $\lambda = cf(\lambda) \geq \kappa$, and let S_λ be the class of linear orderings of size less than λ. Then $\mathcal{L}_\kappa \cap S_\lambda$ is the least class that contains $\mathcal{BL}_\kappa \cap S_\lambda$ and is closed under lexicographic sums with index set in $\mathcal{BL}_\kappa \cap S_\lambda$.*

Proof. Since λ is regular, forming the closure of $\mathcal{BL}_\kappa \cap S_\lambda$ under lexicographic sums with index in this class only generates elements of S_λ, and clearly everything so generated is in $\mathcal{L}_\kappa$. Conversely if we take an element of $\mathcal{L}_\kappa \cap S_\lambda$ and analyse it by Theorem 3.10, only elements of $\mathcal{BL}_\kappa \cap S_\lambda$ appear in this analysis. □

The next result is a generalisation of a result of Fraïssé [5, §5.3.2]

Corollary 3.12. *Let L be a κ-scattered linear ordering of cardinality κ. Then there is an ordinal less than κ^+ which does not embed into L.*

Proof. Applying Corollary 3.11 with $\lambda = \kappa^+$, $\mathcal{L}_\kappa \cap S_{\kappa^+}$ is the least class that contains $\mathcal{BL}_\kappa \cap S_{\kappa^+}$ and is closed under lexicographic sums with index set in $\mathcal{BL}_\kappa \cap S_{\kappa^+}$. A routine induction shows that for every $L \in \mathcal{L}_\kappa \cap S_{\kappa^+}$ there is an ordinal $\gamma < \kappa^+$ such that γ does not embed into L. □

A classical result by Laver [7] states that the class of σ-scattered linear orderings is well-quasi-ordered (actually better-quasi-ordered) under embeddability. We consider the question of whether this kind of result can be extended to cover κ-scattered linear orderings for κ uncountable. The situation depends on the value of κ.

It follows from Theorem 3.10, together with the closure properties of σ-scattered orderings, that every $\aleph_1$-scattered linear ordering is σ-scattered. So it follows from Laver's theorem that the class of $\aleph_1$-scattered linear orderings is better-quasi-ordered under embeddability. We note that the class of σ-scattered orderings contains some $\aleph_1$-dense linear orderings, for example the set of all finite sequences from ω_1 with the reverse lexicographic ordering, so it is properly larger than the class of $\aleph_1$-scattered linear orderings.

The situation is different for the class of $\aleph_2$-scattered linear orderings, which trivially includes the class of all orderings of cardinality $\aleph_1$. It is a standard fact that if κ is uncountable then there are 2^κ pairwise non-embeddable linear orderings. Stronger results along these lines are known: for example Todorčević [11] gave an elegant construction of a class of 2^κ pairwise non-embeddable rigid κ-dense linear orderings.

4. κ-fat FAC partial orders

As we will see in Theorem 4.2, in general κ-fat posets need not embed κ-dense linear orderings. But the situation is better for κ-fat FAC posets.

Theorem 4.1. *If P is a κ-fat FAC poset then P embeds a κ-dense linear ordering.*

Proof. We will proceed by induction on the rank of P. There are two cases to consider:
Case one: There is $p \in P$ such that $p^\perp$ is not an antichain. In this case, by the remarks after Definition 2.11, $p^\perp$ is a κ-fat FAC poset of lower rank, which embeds a κ-dense linear ordering by induction.
Case two: For every $p \in P$, $p^\perp$ is an antichain.

Claim. Distinct maximal antichains are disjoint.

Let C, D be distinct maximal antichains. Suppose for a contradiction that there is $p \in C \cap D$. Choose $q \in D - C$. By the maximality of C, q is comparable with some element r of C. Since q, p are both in the antichain D, $q \perp p$ and so $r \neq p$. Since C is an antichain $r \perp p$. So $p^\perp$ is not an antichain, a contradiction.

Claim. If C and D are distinct maximal antichains, every element of C is compatible with every element of D.

If not there are $c \in C$ and $d \in D$ with $c \perp d$. We can extend the antichain $\{c, d\}$ to a maximal antichain E, then E meets C but $E \neq C$. This contradicts the last claim.

Claim. If C and D are distinct maximal antichains then either every element of C is less than every element of D or vice versa.

Suppose that $c_1 < d < c_2$ with $c_1, c_2 \in C$ and $d \in D$. Then c_1 is comparable with c_2, but this is impossible since C is an antichain. Our claim now follows from the preceding one.

Claim. P embeds a κ-dense linear ordering.

By the previous claim, P is the lexicographic sum of a linearly ordered set of antichains, say the order type is L. Since P is FAC, each antichain is finite. Since κ is infinite the linear order L is κ-dense. Clearly P embeds L. □

Without the FAC the situation is different.

Theorem 4.2. *There is an $\aleph_1$-fat poset with no uncountable chain and no uncountable antichain.*

Proof. We start by recalling a standard example (due to Sierpiński [9]) of an uncountable poset with no uncountable chains or antichains. Let r_α for $\alpha < \omega_1$ be distinct real numbers. Consider the well-founded partial order $\prec$ on ω_1 in which $\alpha \prec \beta$ if and only if both $\alpha < \beta$ and $r_\alpha < r_\beta$; it follows readily from the separability of $\mathbb{R}$ that there are no uncountable chains or antichains in this partial ordering. It is immediate from the definition that $\{\alpha : \alpha \prec \beta\}$ is countable for each β.

For our purposes we must thin out $(\omega_1, \prec)$ slightly. Let X be the set of β for which $\{\alpha : \beta \prec \alpha\}$ is countable, then we claim that X is uncountable. For if not we may build an ω_1-sequence $\langle \beta_\gamma : \gamma < \omega_1 \rangle$ of pairwise incomparable elements of X, by choosing at step γ some $\beta_\gamma \in X$ which does not lie in the countable set $\{\beta : \exists \delta < \gamma \ \beta \parallel \beta_\delta\}$. Now let $P = (\omega_1 - X, \prec)$; then P is an uncountable well-founded poset, P has no uncountable chain or antichain, and in addition for every $p \in P$ there are uncountably many $q \in P$ with $q \succ p$.

We now define a poset Q. The elements of Q are finite sequences from P, and the ordering on Q is the following "inverse lexicographic ordering". Given $x = (q_0, \ldots q_{m-1})$ and $y = (r_0, \ldots, r_{n-1})$, $x < y$ if and only if

- there is i such that $q_i \neq r_i$, and $q_i \prec r_i$ for the least such i, or
- $m > n$ and $q_i = r_i$ for every $i < n$.

Claim. Q is $\aleph_1$-fat.

Let $x = (q_0, \dots q_{m-1})$ and $y = (r_0, \dots, r_{n-1})$ with $x < y$. There are two cases to consider. If x and y disagree at some point, then all sequences of the form $w = y \frown p$ are such that $x < w < y$. If on the other hand x properly extends y, then all sequences of the form $w = y \frown p$ with $p \succ q_n$ are such that $x < w < y$. In either case our choice of P assures us that there are $\aleph_1$ many possibilities for w.

Claim. Q has no uncountable chain.

Suppose for a contradiction that $\langle x_\alpha : \alpha < \omega_1 \rangle$ is a sequence of distinct pairwise comparable elements. Without loss of generality all the x_α have the same length n, say $x_\alpha = (p^\alpha_0, \dots p^\alpha_{n-1})$. Let $C_0 = \{p^\alpha_0 : \alpha \in \omega_1\}$, then C_0 must form a chain in P, since the order on Q is lexicographic. Since P has only countable chains, C_0 is countable, and we may find an uncountable $A_0 \subseteq \omega_1$ and a fixed $c_0 \in C_0$ such that $p^0_\alpha = c_0$ for all $\alpha \in A_0$. Repeating this argument we find uncountable sets $A_0 \supseteq \dots \supseteq A_{n-1}$ and $c_0, \dots c_{n-1}$ such that $p^\alpha_j = c_j$ for all $\alpha \in A_j$. So $x_\alpha = (c_0, \dots c_{n-1})$ for all $\alpha \in A_{n-1}$, contradicting our assumption that the x_α are distinct.

A very similar argument shows that Q has no uncountable antichain, concluding the proof. □

5. Augmentations of κ-scattered and $\mathbb{Q}_\kappa$-scattered posets

An important step in the proof of Abraham and Bonnet's structure theorem for scattered FAC posets [1] is an argument that the class of scattered FAC posets is closed under augmentation. Džamonja and Thompson [4] proved that a similar result holds for κ-scattered and $\mathbb{Q}_\kappa$-scattered posets. The following fact records part of the information from [4, Lemma 2.11].

Fact 5.1. *Let P be an FAC poset.*

1. *If P is κ-scattered then all augmentations of P are κ-scattered.*
2. *If P is $\mathbb{Q}_\kappa$-scattered then all augmentations of P are $\mathbb{Q}_\kappa$-scattered.*

Theorem 4.1 can be used to give an alternative proof of part (1) from Fact 5.1, and we record this proof here.

Alternative proof of (1). We start by reducing to a simpler case. Suppose that there is a FAC κ-scattered poset P with an augmentation P' that embeds a κ-dense linear ordering. Then we may further augment P' to obtain a linear ordering P'' with a subset L which is κ-dense in the ordering of P''. The class of FAC κ-scattered posets is closed under restriction, so the restriction of P to L is a κ-scattered FAC poset which has a κ-dense linearisation. It will therefore suffice to prove that every linearisation of a FAC κ-scattered poset is not κ-dense.

Suppose for a contradiction that P is an FAC κ-scattered poset and that P' is a κ-dense linearisation of P. By Theorem 4.1 the poset P is not κ-fat, so we may find $a^0_0 < a^1_0$ in P such that the interval $I_0 = (a^0_0, a^1_0)_P$ has size less than κ. Since P' is κ-dense the interval $J_0 = (a^0_0, a^1_0)_{P'}$ is a κ-dense linear ordering; so $J_0 - I_0$ is also such an ordering. We note that every element of $J_0 - I_0$ must be incomparable with at least one of a^0_0, a^1_0 in P, for if it were comparable with both it would have to lie between them in P and so would belong to I_0. Appealing to Lemma 2.2 we may find $K_0 \subseteq J_0 - I_o$ and $i_0 \in 2$ such that the restriction of P' to K_0 is κ-dense, and every element of K_0 is incomparable with $a_0^{i_0}$ in P.

Let P_1 be the restriction of P to K_0, and let P_1' be the restriction of P' to this set. Now P_1' is a κ-scattered FAC poset with a κ-dense linearisation so we may appeal to Theorem 4.1 again, and find $a_1^0 < a_1^1$ in K_0 such that $I_1 = (a_1^0, a_1^1)_{P_1}$ has size less than κ. Repeating the argument we will choose inductively a_n^0, a_n^1, I_n, J_n, K_n, i_n, P_n, P_n', such that:

- $I_n = (a_n^0, a_n^1)_{P_n}$, I_n has size less than κ.
- $J_n = (a_n^0, a_n^1)_{P_n'}$, J_n is a κ-dense linear ordering.
- Every element of $J_n - I_n$ is incomparable with at least one of a_n^0, a_n^1 in P.
- $K_n \subseteq J_n - I_n$, and every element of K_n is incomparable with $a_n^{i_n}$ in P.
- P_{n+1} is the restriction of P to K_n, and P_{n+1} is a κ-scattered FAC poset.
- P_{n+1}' is the restriction of P' to K_n, and P_{n+1}' is a κ-dense linear order.

To finish we choose i such that $i_n = i$ for infinitely many n, and let $A = \{n : i_n = i\}$. Suppose that $m, n \in A$ with $m < n$. Then by construction $a_n^i \in K_m$, a_n^i is incomparable with $a_m^{i_m}$, and $a_m^{i_m} = a_m^i$. It follows that $\{a_n^i : n \in A\}$ is an infinite antichain in P, a contradiction since P is an FAC poset. This contradiction concludes the proof. □

Using an example from a previous section, we may see that the FAC hypothesis in Fact 5.1 is crucial.

Theorem 5.2. *There is a uncountable poset Q such that*

(1) *Q has no uncountable chains or antichains (in particular Q is $\aleph_1$-scattered).*
(2) *Q has an $\aleph_1$-dense linearisation with no strictly decreasing ω_1-sequence.*

Proof. We take as Q the poset from Theorem 4.2. We recall that we started with a poset P of size $\aleph_1$ such that

- P has no uncountable chain or antichain.
- P is well-founded.
- For every $p \in P$ there are countably many q with $q < p$, and uncountably many q with $q > p$.

We then defined Q to be the set of finite sequences from P, ordered by the inverse lexicographic ordering.

We will now linearise P, in a judicious fashion. From the properties of P it follows that each element of P has countable rank, and the set of elements of a fixed rank is countable, so we may find P' which is a linearisation of P with order type ω_1. This induces a linearisation Q' of Q, whose order type is that of the finite sequences from ω_1 under the inverse lexicographic ordering. It is now routine to check, by arguments like those of Theorem 4.2, that Q' is $\aleph_1$-dense and has no decreasing ω_1-chain. □

6. Forbidden linear orders in FAC posets

In this section we will prove a rather general theorem about classes of FAC posets defined by forbidding certain linear orderings. To be a bit more precise, given a class $\mathcal{G}_0$ of linear orderings, we will consider the class $\mathcal{G}$ of FAC posets in which every chain belongs to $\mathcal{G}_0$. We will prove that under mild assumptions on $\mathcal{G}_0$, every element of $\mathcal{G}$ can be built up from members of $\mathcal{G}_0$.

Definition 6.1. A class $\mathcal{G}_0$ of linear orderings is *reasonable* if and only if $\mathcal{G}_0$ contains a nonempty ordering and is closed under reversals and restrictions.

Definition 6.2. Given a reasonable class $\mathcal{G}_0$ of linear orderings, the *closure* $cl(\mathcal{G}_0)$ of $\mathcal{G}_0$ is the least class of posets which contains $\mathcal{G}_0$ and is closed under the operations:

- Lexicographic sum with index set either a WQO poset, the inverse of a WQO poset, or an element of $\mathcal{G}_0$.
- Augmentation.

It is easy to see that $cl(\mathcal{G}_0)$ consists of FAC posets and is closed under restrictions and reversals.

The following technical lemma records another useful closure property of $cl(\mathcal{G}_0)$.

Lemma 6.3. *Let Q be a poset, and suppose that there is $q \in Q$ be such that $q^{\parallel}$ and $q^{\perp}$ are both in $cl(\mathcal{G}_0)$. Then $Q \in cl(\mathcal{G}_0)$.*

Proof. Let $X_0 = q^{\parallel}$ and $X_1 = q^{\perp}$. Now we form a lexicographic sum Q' of the orderings X_i, ordering the indices so that 0 is incomparable with 1. Since the X_i both lie in $cl(\mathcal{G}_0)$ and finite posets are trivially WQO, $Q' \in cl(\mathcal{G}_0)$. Clearly Q is an augmentation of Q', so $Q \in cl(\mathcal{G}_0)$. □

Theorem 6.4. *Let $\mathcal{G}$ be the class of FAC posets such that every chain is in $\mathcal{G}_0$. Then $\mathcal{G} \subseteq cl(\mathcal{G}_0)$.*

Proof. We will proceed by induction on the antichain rank of an FAC poset $P \in \mathcal{G}$. We note that since $\mathcal{G}_0$ is closed under restriction, $\mathcal{G}_0 \subseteq \mathcal{G}$.

Let P be an FAC poset, and assume that $Q \in cl(\mathcal{G}_0)$ for every FAC $Q \in \mathcal{G}$ with $\rho(Q) < \rho(P)$. In particular $p^{\perp} \in cl(\mathcal{G}_0)$ for every $p \in P$, a fact which will play a crucial role at several points.

We define a binary relation $\equiv$ on P by stipulating that $p \equiv q$ if and only if:

- p is incomparable with q, or
- $p \leq q$ and $(p, q) \in cl(\mathcal{G}_0)$, or
- $q \leq p$ and $(q, p) \in cl(\mathcal{G}_0)$.

Claim. $\equiv$ is an equivalence relation.

Clearly $\equiv$ is reflexive and symmetric, so we check only that it is transitive. Let $a \equiv b$ and $b \equiv c$, where we may as well assume that a, b, c are distinct. There are four cases to check:

Case 1: $a < b$ and $b < c$. Let $Q = (a, c)$. Then (a, b) and (b, c) are in $cl(\mathcal{G}_0)$ by definition. $b^{\perp} \in cl(\mathcal{G}_0)$ by our assumption on P, and $b^{\perp} \cap Q \in cl(\mathcal{G}_0)$ because $cl(\mathcal{G}_0)$ is closed under restriction. Applying Lemma 6.3 we see that $Q \in cl(\mathcal{G}_0)$, and so by definition $a \equiv c$.

Case 2: $a < b$ and $b \perp c$. If $a \perp c$ we are done, so we assume that $a < c$. Let $X_0 = (a, c) \cap (a, b)$, and $X_1 = (a, c) - (a, b)$. Then $X_0 \in cl(\mathcal{G}_0)$ because $(a, b) \in cl(\mathcal{G}_0)$, and $X_1 \in cl(\mathcal{G}_0)$ because $X_1 \subseteq b^{\perp}$. Finally (a, c) is an augmentation of the lexicographic sum of X_0, X_1 in which $0, 1$ are incomparable.

Case 3: $a \perp b$ and $b < c$. This is exactly like the previous case.

Case 4: $a \perp b$ and $b \perp c$. If $a \perp c$ we are done, so we may as well assume that $a < c$. Then $(a, c) \subseteq b^{\perp}$, and again we are done.

Claim. The equivalence classes of $\equiv$ are convex.

Let $a < b$ with $a \equiv b$. Then for every $c \in (a, b)$ we have $(a, c) \subseteq (a, b)$, and so $a \equiv c$ since $cl(\mathcal{G}_0)$ is closed under restriction.

Claim. Each equivalence class is in $cl(\mathcal{G}_0)$.

Let C be such a class, and let $c \in C$. By Lemma 6.3 and the fact that $c^\perp \in cl(\mathcal{G}_0)$, it will suffice to show that $\{d \in C : d > c\}$ and $\{d \in C : d < c\}$ are both in $cl(\mathcal{G}_0)$.

In fact we will just argue that the set $Y = \{d \in C : d > c\}$ is in $cl(\mathcal{G}_0)$, the argument for $\{d \in C : d < c\}$ will be symmetric. Since Y is an FAC poset we may appeal to Lemma 2.13 and choose an ordinal ζ and $d_\alpha \in Y$, $Z_\alpha \subseteq (c, d_\alpha)_P \subseteq Y$ for $\alpha < \zeta$ such that

(1) $\{d_\alpha : \alpha < \zeta\}$ is a WQO subset of Y.
(2) Each Z_α is convex with maximum element d_α, and the Z_α form a partition of Y.
(3) Y is an augmentation of the lexicographic sum of the Z_α for $\alpha < \zeta$, with indices ordered by $\alpha < \beta \iff d_\alpha < d_\beta$.

Since C is an equivalence class, $(c, d_\alpha)_P \in cl(\mathcal{G}_0)$, and so $Z_\alpha \in cl(\mathcal{G}_0)$. It follows from the closure properties of $cl(\mathcal{G}_0)$ that $Y \in cl(\mathcal{G}_0)$.

Claim. If C and D are distinct equivalence classes, then either every element of C is less than every element of D or vice versa.

Since incomparable elements are equivalent, every $c \in C$ is comparable with every $d \in D$. Suppose for a contradiction that we have $c_1 < d < c_2$ with $c_1, c_2 \in C$ and $d \in D$. Since classes are convex we have $d \in C$, a contradiction since C and D are disjoint.

Claim. $P \in cl(\mathcal{G}_0)$.

The equivalence classes are linearly ordered in some order type L, and P is an L-indexed sum of equivalence classes. In particular (choosing a point from each class) P contains a copy of L. Since every chain of P lies in $\mathcal{G}_0$, $L \in \mathcal{G}_0$. Finally since each class is in $cl(\mathcal{G}_0)$, and $cl(\mathcal{G}_0)$ is closed under lexicographic sums with index set in $\mathcal{G}_0$, we see that $P \in cl(\mathcal{G}_0)$. $\square$

Theorem 6.4 tells us that every element of $\mathcal{G}$ is built up from elements of $\mathcal{G}_0$ by a certain recipe. It does *not* in general guarantee that all posets built up according to this recipe are in $\mathcal{G}$, because for example $\mathcal{G}$ may not be closed under augmentations.

The following result may appear slightly *ad hoc*, but is well-adapted for use in proving some classification results in the following section.

Corollary 6.5. *Let $\mathcal{G}_0$ and $\mathcal{G}$ be as above, and assume in addition that:*

(1) *$\mathcal{G}_0$ contains all well-orderings, and is closed under lexicographic sums with index set in $\mathcal{G}_0$.*
(2) *$\mathcal{G}$ is closed under augmentations.*

Then $\mathcal{G} = cl(\mathcal{G}_0)$.

Proof. The extra closure assumptions on $\mathcal{G}_0$ easily imply that $\mathcal{G}$ is closed under lexicographic sums with index sets that are WQO's, converse WQO's or elements of $\mathcal{G}$. Since we also assumed that $\mathcal{G}$ is closed under augmentation, $\mathcal{G}$ is closed under all the operations which are used to build $cl(\mathcal{G}_0)$. Since $\mathcal{G}_0 \subseteq \mathcal{G}$ it follows that $cl(\mathcal{G}_0) \subseteq \mathcal{G}$, and hence by Theorem 6.4 that $cl(\mathcal{G}_0) = \mathcal{G}$. $\square$

7. Analysis of κ-scattered FAC partial orderings

In this section we prove results which characterise the κ-scattered FAC posets and the $\mathbb{Q}_\kappa$-scattered posets, and derive a structure theorem for countable FAC posets. The results are more satisfactory in the κ-scattered case, in the $\mathbb{Q}_\kappa$-scattered case we only achieve a classification relative to the class of $\mathbb{Q}_\kappa$-scattered linear orderings. The results of Section 8 will suggest that probably this is all we should hope for.

Definition 7.1. Let $\mathcal{BP}_\kappa$ be the class of posets P such that P is either a WQO poset, the reverse of a WQO poset, or a linear ordering of cardinality less than κ.

Definition 7.2. Let $\mathcal{P}_\kappa$ be the least class of posets such that

- $\mathcal{P}_\kappa$ contains $\mathcal{BP}_\kappa$.
- $\mathcal{P}_\kappa$ is closed under lexicographic sums with index set in $\mathcal{BP}_\kappa$.
- $\mathcal{P}_\kappa$ is closed under augmentation.

Theorem 7.3. *$\mathcal{P}_\kappa$ is the class of κ-scattered FAC posets.*

Proof. We will use the general result of Theorem 6.4 to make a preliminary analysis of κ-scattered FAC posets in terms of the class of κ-scattered linear orderings. We will then use our structure theory for κ-scattered linear orderings from Theorem 3.10 to refine this analysis and obtain the claimed result.

As in Section 3, we denote by $\mathcal{L}_\kappa$ the class of κ-scattered linear orderings. Let $\mathcal{G}$ be the class of κ-scattered FAC posets, so that $\mathcal{G}$ is exactly the class of FAC posets where every chain is in $\mathcal{L}_\kappa$. The class $\mathcal{L}_\kappa$ contains all well-orderings and is closed under reversal, restrictions, and ordered sums with index set in $\mathcal{L}_\kappa$. By Fact 5.1 the class $\mathcal{G}$ is closed under augmentations.

Appealing to Corollary 6.5 we obtain a preliminary version of the structure theorem: $\mathcal{G}$ is the least class which:

- Contains $\mathcal{L}_\kappa$.
- Is closed under lexicographic sums whose index sets are either WQO posets, converse WQO posets or elements of $\mathcal{L}_\kappa$.
- Is closed under augmentations.

Now we appeal to the structure theory for $\mathcal{L}_\kappa$ from Theorem 3.10. From that theorem it follows immediately that $\mathcal{L}_\kappa \subseteq \mathcal{P}_\kappa$. What is more an easy induction shows that $\mathcal{P}_\kappa$ is closed under all lexicographic sums with index set in $\mathcal{L}_\kappa$. From the preliminary structure theorem which we just stated, it follows that $\mathcal{G} = \mathcal{P}_\kappa$. □

As a corollary of Theorem 7.3 we get a structure theorem for countable FAC posets.

Theorem 7.4. *Let $\mathcal{BC}$ be the class of all countable posets which are either WQO, reverse WQO, or linear orders. Let $\mathcal{C}$ be the least class of posets which contains $\mathcal{BC}$, is closed under lexicographic sums with index set in $\mathcal{BC}$, and is closed under augmentation. Then $\mathcal{C}$ is the class of countable FAC posets.*

Proof. It is easy to see that every element of $\mathcal{C}$ is countable and FAC. Conversely every countable FAC poset is trivially $\aleph_1$-scattered, so we can apply the structure theory of Theorem 7.3; clearly only countable posets are used in the decomposition of a countable FAC poset so that every countable FAC poset is in $\mathcal{C}$. □

Turning to $\mathbb{Q}_\kappa$-scattered FAC posets we get a similar structure theory.

Definition 7.5. Let $\mathcal{BP}^*_\kappa$ be the class of posets P such that P is either a WQO poset, the reverse of a WQO poset, or a $\mathbb{Q}_\kappa$-scattered linear ordering.

Definition 7.6. Let $\mathcal{P}^*_\kappa$ be the least class of posets such that

- $\mathcal{P}^*_\kappa$ contains $\mathcal{BP}^*_\kappa$.
- $\mathcal{P}^*_\kappa$ is closed under lexicographic sums with index set in $\mathcal{BP}^*_\kappa$.
- $\mathcal{P}^*_\kappa$ is closed under augmentation.

By an argument exactly parallel to that for the preliminary step in the proof of Theorem 7.3, we obtain:

Theorem 7.7. *$\mathcal{P}^*_\kappa$ is the class of $\mathbb{Q}_\kappa$-scattered FAC posets.*

Theorem 7.7 is in some ways less satisfying than Theorem 7.3, because we are missing the kind of detailed analysis of $\mathbb{Q}_\kappa$-scattered linear orderings which Theorem 7.3 affords for the κ-scattered linear orderings. In the following section we will argue that no such analysis is possible.

8. Absoluteness

In this section we prove a consistency result which puts a limit on the possibility of analyzing the $\mathbb{Q}_{\aleph_1}$-scattered linear orderings. To be more precise we will show that under CH there is a $\mathbb{Q}_{\aleph_1}$-scattered linear ordering, whose $\mathbb{Q}_{\aleph_1}$-scatteredness is effaced by some forcing extension which does not add reals.

We start by noting that the property of being a scattered linear ordering is upwards absolute. This is straightforward, and in fact we will give two easy proofs.
Proof one: Let L be a scattered ordering. Fix an enumeration (q_n) of the rationals, and let T be the subtree of ${}^{<\omega}L$ consisting of sequences s such that $q_i < q_j \iff s(i) <_L s(j)$ for all $i, j \in dom(s)$. It is clear that the definition of T from P is upwards absolute, and that L is scattered if and only if T has no infinite branch. By standard arguments this is equivalent in turn to the upwards absolute statement that there is an ordinal rank function on T.
Proof two: Let L be a scattered linear ordering. By the Hausdorff analysis of scattered orderings, L belongs to the closure of the class of ordinals under the operations of lexicographic sum and reversal. This is clearly upwards absolute.

Using Theorem 3.10, the argument of Proof two extends to $\aleph_1$-scattered linear orderings. That is to say we get

Theorem 8.1. *Let κ be an infinite cardinal. Then the property of being a κ-scattered linear ordering is upwards absolute to cardinal preserving extensions.*

A priori the situation for $\mathbb{Q}_{\aleph_1}$-scattered orderings is not so clear. To start with the definition of $\mathbb{Q}_{\aleph_1}$ is not upwards absolute; however as we saw in Lemma 2.7 the definition of $\mathbb{Q}_{\aleph_1}$ is absolute to extensions with the same reals, so we will consider whether the property of being $\mathbb{Q}_{\aleph_1}$-scattered is absolute to such extensions. The argument of Proof one above hinged on the countability of $\mathbb{Q}$, so does not generalise. Theorem 8.1 suggests that if we can give some reasonable analysis of $\mathbb{Q}_{\aleph_1}$-scattered orderings along the lines of Hausdorff's theorem, then the property of being $\mathbb{Q}_{\aleph_1}$-scattered should be absolute to extensions with the same reals. So the following Theorem suggests that no such analysis can be given, at least not without extra assumptions.

Theorem 8.2. *Let CH hold. Then there exist a $\mathbb{Q}_{\aleph_1}$-scattered linear ordering L and a forcing poset which adds no reals, such that L is not $\mathbb{Q}_{\aleph_1}$-scattered in the generic extension.*

Proof. We will begin by giving a rather general construction of a linear ordering from a tree T and a "lexicographic" ordering of its vertices. To be more precise, we fix a tree T of height and cardinality $\aleph_1$, together with an assignment to each node $v \in T$ of a linear ordering $<_v$ of the immediate successors of v. We assume that T has *unique limits*, that is to say if v and w are nodes of some limit height λ with the same predecessors then $v = w$.

Let $B(T)$ be the set of countable branches of T, where by *branch* we mean a downward closed linearly ordered subset of T. We define a lexicographic ordering $<_l$ of $B(T)$ in the standard way; $b <_l c$ if and only if

- EITHER b is a proper initial segment of c.
- OR Neither of b, c is a proper initial segment of the other and $v_b <_w v_c$, where w is the maximal node in $b \cap c$, v_b is the least element of b above w, and v_c is the least element of b above w.

We now define L to be the set of functions p such that $dom(p) \in B(T)$ and $rng(p) \subseteq 2$. To define the ordering on L, let $p, q \in L$ with $dom(p) = b$ and $dom(q) = c$. Then $p <_L q$ if and only if

- EITHER p and q disagree at some point of $b \cap c$, and $p(w) < q(w)$ for the least $w \in b \cap c$ such that $p(w) \neq q(w)$.
- OR p and q agree on $b \cap c$, and $b <_l c$.

It is routine to check that L is linearly ordered by $<_L$. We note also that the definitions of $B(T)$ and L are absolute to extensions with the same reals.

Lemma 8.3. *If T has an uncountable branch then L is not $\mathbb{Q}_{\aleph_1}$-scattered.*

Proof. It is immediate from the definitions of L and $<_L$ that if T has an uncountable branch then L embeds a copy of $(2^{<\omega_1}, <_{lex})$. As we saw in Fact 2.8, $(2^{<\omega_1}, <_{lex})$ is not $\mathbb{Q}_{\aleph_1}$-scattered. $\square$

Lemma 8.4. *If $<_L$ embeds a copy of ω_1, and for each v the ordering $<_v$ does not embed a copy of ω_1, then T has an uncountable branch.*

Proof. Let $\langle f_\nu : \nu < \omega_1 \rangle$ be a $<_L$-increasing sequence of elements of L.

We will construct by recursion on $\alpha < \omega_1$ a branch $b_\alpha \in B(T)$ with length α, a function $g_\alpha : b_\alpha \to 2$, and an ordinal $\gamma_\alpha < \omega_1$ which is strictly increasing with α. Our construction will satisfy the induction hypothesis that for all $\nu > \gamma_\alpha$ the branch b_α is an initial segment of $dom(f_\nu)$, and $f_\nu \restriction b_\alpha = g_\alpha$.

We begin by setting b_0 equal to the empty branch, g_0 equal to the empty function and $\gamma_0 = 0$. For $\lambda < \omega_1$ a limit ordinal, we start by setting $\gamma_\lambda = \sup_{\alpha<\lambda} \gamma_\alpha$. For $\nu \geq \gamma_\lambda$ and $\alpha < \lambda$, we have that b_α is an initial segment of $dom(f_\nu)$ and $g_\alpha = f_\nu \restriction b_\alpha$; so we may set $b_\lambda = \bigcup_{\alpha<\lambda} b_\alpha$, $g_\lambda = \bigcup_{\alpha<\lambda} g_\alpha$, and maintain the induction hypothesis.

For the successor step, suppose we have determined b_α, g_α and γ_α. We claim that for sufficiently large $\nu > \gamma_\alpha$

- The length of $dom(f_\nu)$ is at least $\alpha + 1$.
- The point of $dom(f_\nu)$ on level α is independent of ν.
- The value of $f_\nu(w)$ is independent of ν.

The first point is easy since $f_\nu \restriction \alpha = g_\alpha$. The second point follows from the unique limit property for α limit; if on the other hand $\alpha = \beta + 1$ and w is the point of level β in b_α, then it follows from the fact that the ordering $<_w$ does not embed ω_1. In either case the third point is then immediate. It is now clear that we can choose suitable values for $\gamma_{\alpha+1}$, $b_{\alpha+1}$, and $g_{\alpha+1}$, □

To finish the proof we construct a suitable T and L. Fix S a stationary and co-stationary subset of ω_1, together with an injective map $\alpha \mapsto r_\alpha$ from S to $\mathbb{R}$. Let T be the set of closed and bounded subsets of S, and let the ordering on T be end-extension; we note that the elements of T are exactly the conditions in the standard forcing poset $CUB(S)$ [2] for shooting a club set through the stationary set S, and the ordering on T is the reverse of the ordering on that poset. The immediate successors of a node $c \in T$ are the sets of form $c \cup \{\alpha\}$ where $\alpha \in S$ and $\alpha > \max(c)$; we order them by ruling that $c \cup \{\alpha\} <_c c \cup \{\beta\}$ if and only if $r_\alpha < r_\beta$.

Since S is co-stationary T has no uncountable branch, so the ordering L does not embed a copy of ω_1, and *a fortiori* L is $\mathbb{Q}_{\aleph_1}$-scattered. If we now force with $CUB(S)$ then in the extension there are no new reals, but S now contains a club set, so T has an uncountable branch and L is no longer $\mathbb{Q}_{\aleph_1}$-scattered. □

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