MORE ON FULL REFLECTION BELOW $\aleph_\omega$

JAMES CUMMINGS AND DORSHKA WYLIE

ABSTRACT. Jech and Shelah [2] studied full reflection below $\aleph_\omega$, and produced a model in which the extent of full reflection is maximal in a certain sense. We produce a model in which full reflection is maximised in a different direction.

1. Introduction

Recall that if κ is an uncountable regular cardinal and S is a stationary subset of κ, then S *reflects at an ordinal* $\alpha < \kappa$ if and only if $cf(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α. We will call the set of $\alpha < \kappa$ to which S reflects the *trace of* S and will denote it by $Tr(S)$.

The phenomenon of stationary reflection has been extensively studied. It plays an important role in combinatorial set theory, and is closely tied to the theory of large cardinals.

Given S and T which are both stationary subsets of κ, S *reflects fully in* T *($S < T$)* if and only if S reflects at almost every point of T, that is to say there is $C \subseteq \kappa$ club such that S reflects at every point of $C \cap T$. The relation $<$ is known [3] to be a well-founded strict partial ordering on the stationary subsets of κ, whose height measures the degree of Mahloness of the cardinal κ.

We will study the following relation between stationary subsets of an uncountable regular cardinal κ:

Definition 1. *$S <^* T$ if and only if $U < T$ for every stationary $U \subseteq S$.*

Suppose that there are stationary subsets S and T of κ such that that $S <^* T$. By the Solovay splitting theorem, S can be partitioned into κ many disjoint stationary pieces S_i for $i < \kappa$. Fix for each i a club set $C_i \subseteq \kappa$ such that S_i reflects at every $\gamma \in C_i \cap T$. If C is the diagonal intersection of the C_i, and $\gamma \in C \cap T$, then $S_i \cap \gamma$ is stationary in γ for all $i < \gamma$. For each such γ we fix a club set in γ with order type $cf(\gamma)$, and observe that it meets each of the sets $S_i \cap \gamma$ for $i < \gamma$, so that $cf(\gamma) \geq |\gamma|$. But clearly $cf(\gamma) \leq |\gamma|$ for any limit γ, and we conclude that $cf(\gamma) = |\gamma|$ for almost all $\gamma \in T$.

We now distinguish two cases. If κ is a successor cardinal, say $\kappa = \mu^+$, then almost every $\gamma < \kappa$ has cardinality μ, and so almost every ordinal in T must be of cofinality μ, in particular μ must be regular. Since $\mu^+ \cap cof(\mu)$ is non-reflecting, it must also be the case that almost every ordinal in S has cofinality less than μ. If on the other hand κ is a limit cardinal then almost every $\gamma < \kappa$ is a cardinal, and it follows that almost every element of T is regular; in particular κ is weakly Mahlo.

Magidor [7] showed that consistently $S <^* T$ with $S = \omega_2 \cap cof(\omega)$, $T = \omega_2 \cap cof(\omega_1)$, starting with a weakly compact cardinal κ which is collapsed to become ω_2. This argument is easily adapted to show that if $\lambda < \kappa$ with λ regular and uncountable and κ weakly compact, there is a generic extension in which $\kappa = \lambda^+$ and $S <^* T$ for $S = \lambda^+ \cap cof(< \lambda)$, $T = \lambda^+ \cap cof(\lambda)$.

In an earlier version of this paper we falsely claimed that the relation $S <^* T$ can only hold in successors of regular cardinals. After the referee had pointed out this mistake, Magidor [9] (responding to a question from the first author) showed that using a weakly compact cardinal we may obtain a model in which κ is the least weakly Mahlo cardinal and $S <^* T$ where $S = \kappa \cap cof(\omega)$, $T = \kappa \cap REG$. In the remainder of this paper we will work in the context of successors of regular cardinals.

Jech and Shelah [2] showed the surprising fact that stationary reflection at one cardinal can be an obstacle to full reflection at a smaller cardinal. We will give a version of their argument in the next section as Lemma 2.

Given $m < n < \omega$, we define $S^n_m = \omega_n \cap cof(\omega_m)$. It will follow from Lemma 2 that if every stationary subset of S^{n+3}_{n+1} reflects at a point in S^{n+3}_{n+2}, then the relation $S^{n+2}_n <^* S^{n+2}_{n+1}$ fails. In particular the relations $S^{n+2}_n <^* S^{n+2}_{n+1}$ and $S^{n+3}_{n+1} <^* S^{n+3}_{n+2}$ can not hold simultaneously.

Jech and Shelah [2] produced by forcing a model in which $S^2_0 <^* S^2_1$, and $S^{n+2}_i <^* S^{n+2}_{n+1}$ for all $n \geq 1$ and all $i \leq n - 1$. In their model there is necessarily no stationary reflection between S^{n+2}_n and S^{n+2}_{n+1} for $n \geq 1$.

We will produce a model in which there are stationary sets $B_n \subseteq S^{n+2}_{n+1}$ such that $S^{n+2}_i <^* B_n$ for all $i \leq n$. Given the limiting result discussed above, it is clear that B_n must be co-stationary in S^{n+2}_{n+1}, but our proof will show that in a certain sense our set B_n is as large as possible. We defer the precise statement of our main theorem till the end of the next section, when we will be able to make the sense in which B_n is maximal more precise.

We have tried to make our notation standard. We conclude this introduction by reviewing a couple of definitions and facts that will play a central role.

- Let κ be an uncountable regular cardinal and let $S \subseteq \kappa$ be stationary in κ. Then we denote by $CUB(\kappa, S)$ the forcing poset whose conditions are closed bounded subsets c of κ with $c \subseteq S$, ordered by end-extension. This poset will always add a closed and unbounded subset of κ contained in S, but in general it may very well collapse cardinals or add bounded subsets of κ.
- Let κ be supercompact. Then $f : \kappa \to V_\kappa$ is a *Laver function* (or *Laver diamond*) if and only if for every cardinal λ and for every $x \in H_{\lambda^+}$ there is $j : V \to M$ witnessing κ is λ-supercompact with $j(f)(\kappa) = x$. Laver [5] showed that every supercompact cardinal has a Laver function.
- A special case of the Lévy-Solovay theorem [6] states that if κ is supercompact and $\mathbb{P}$ is a forcing poset with cardinality less than κ, then forcing with $\mathbb{P}$ preserves the supercompactness of κ.
- A result of Magidor [7] states that if κ is regular and $\mathbb{Q}$ is a κ-closed poset, then for large enough λ $\mathbb{Q}$ can be completely embedded into the Lévy collapse $Coll(\kappa, \lambda)$ in such a way that the quotient forcing is κ-closed.

2. Partial squares

One of the main ideas in Jech and Shelah's paper [2] is the use of *partial square sequences* to put bounds on the extent of stationary reflection. Square sequences were introduced by Jensen [4] in his work on the fine structure of L; they need not always exist, but a remarkable result by Shelah (which we state below as Fact 1) shows just in ZFC that many "partial square sequences" always exist.

Definition 2. *Let κ and λ be uncountable regular cardinals with $\kappa < \lambda$ and let $S \subseteq \lambda \cap cof(\kappa)$ with S stationary in λ. A* partial square on S *is a sequence $\langle C_\beta : \beta \in S\rangle$ such that*

1. *C_β is a club subset of β with order type κ for every $\beta \in S$.*
2. *For all $\beta, \gamma \in S$ and all $\alpha \in \lim(C_\beta) \cap \lim(C_\gamma)$, $C_\beta \cap \alpha = C_\gamma \cap \alpha$.*

Fact 1. *Let κ and λ be uncountable regular cardinals with $\kappa < \lambda$, and let $S \subseteq \lambda^+ \cap cof(\kappa)$ be stationary. Then S is the union of λ pairwise disjoint stationary sets, each carrying a partial square sequence.*

Jech and Shelah [2] showed that partial squares plus reflection can impose an obstacle to reflection at smaller cardinals. Before making this precise, we explain the main idea. By Fact 1 there is always a stationary subset A of S^{n+2}_n which carries a partial square. If A reflects at a point in S^{n+2}_{n+1}, then we can "pull back" the partial square in a straightforward way to obtain a partial square on a stationary set $B \subseteq S^{n+1}_n$, which can then be used to generate non-reflecting stationary sets.

Lemma 1. [2] *Let $S \subseteq \lambda \cap cof(\kappa)$ be a stationary set which carries a partial square sequence. For every regular $\mu < \kappa$ there is a stationary subset of $\lambda \cap cof(\mu)$ which does not reflect at any point of S.*

In conversation with the first author Magidor noted the following strengthened version of Lemma 1.

Lemma 2. [8] *Let $S \subseteq \lambda \cap cof(\kappa)$ be a stationary set which carries a partial square sequence. Then for every regular $\mu < \kappa$ and every stationary $A \subseteq \lambda \cap cof(\mu)$, there is a stationary set $B \subseteq A$ such that B does not reflect at any point of S.*

Proof. Let T be the set of $\delta \in S$ such that $A \cap \delta$ is stationary in δ. If T is non-stationary then fix $C \subseteq \lambda$ which is club and disjoint from T, and set $B = A \cap C$. Clearly B can only reflect at points of C, so B reflects nowhere in S.

So we may now assume that T is stationary. In this case we define $A' = \bigcup_{\delta \in T} \lim(C_\delta) \cap A$. We claim that A' is stationary: for if E is club in λ, we may first choose $\delta \in T \cap \lim(E)$, and then use the fact that $A \cap \delta$ is stationary in δ to choose a point in $\lim(C_\delta) \cap E \cap A$.

We define a function $f : A' \to \kappa$, by setting $f(\alpha) = ot(C_\delta \cap \alpha)$ for some $\delta \in S$ with $\alpha \in \lim(C_\delta)$. Since $\langle C_\delta : \delta \in S \rangle$ is a partial square sequence, the value of $f(\alpha)$ is independent of the choice of δ. We choose $B \subseteq A'$ on which f is constant, and claim that B reflects at no point of S. To see this we fix $\delta \in S$. If $\delta \notin T$ then by definition $A \cap \delta$ is non-stationary in δ, and so since $B \subseteq A$ we see that $B \cap \delta$ is non-stationary in δ. If on the other hand $\delta \in T$ then f is strictly increasing on $A' \cap \lim(C_\delta)$, so that B can meet the club set $\lim(C_\delta)$ at most once, hence $B \cap \delta$ is non-stationary in δ. □

We can now state the main theorem of this paper in a precise way.

Theorem 1. *If there exist infinitely many supercompact cardinals, then in some generic extension the following holds: there are sets $B_n \subseteq S^{n+2}_{n+1}$ for every $n < \omega$ such that*

(1) *Both B_n and $S^{n+2}_{n+1} \setminus B_n$ are stationary.*
(2) *$S^{n+2}_i <^* B_n$ for all $i \leq n$.*
(3) *$S^{n+2}_{n+1} \setminus B_n$ carries a partial square sequence.*

The B_n are maximal in the following precise sense: as we saw in the discussion following Fact 1, because $S^{n+3}_{n+1} <^* B_{n+1}$, some stationary subset of S^{n+2}_{n+1} carries a partial square. By Lemma 2 it is hopeless to demand any stationary reflection to such a set, so the most we can hope for is strong reflection to the complement.

The structure of the proof is this: we will successively collapse supercompact cardinals so that the n^{th} supercompact cardinal becomes ω_{n+2}. We will argue that in the generic extension there exist certain generic elementary embeddings which will enforce the desired stationary reflection. A problem arises here, namely that earlier stages of the construction must anticipate what happens later in the construction. A similar issue arose in the first author's construction (jointly with Matt Foreman) of a model in which ω_n has the tree property for all n with $2 \leq n < \omega$ [1]; we will resolve the issue in the same way as in [1] by using Laver functions at each stage to anticipate all possibilities for the later stages.

3. Stationary set preservation

We will require several technical lemmas on the preservation of stationary sets by forcing. The first one is (a special case of) Lemma 2.8 in reference [2], and we refer the reader to that paper for the proof. It is interesting to note that by work of Shelah we can remove the GCH assumption, and that in the harder case $m + 1 < n$ the proof uses the existence of partial squares.

Lemma 3. *Let GCH hold. Let $m < n < \omega$, let $S \subseteq S^n_m$ be stationary in ω_n, and let $\mathbb{P}$ be ω_{m+1}-closed. Then forcing with $\mathbb{P}$ preserves the stationarity of S.*

It follows from Lemma 3 that ω_{n+1}-closed forcing will preserve the stationarity of stationary subsets $T \subseteq S^{n+1}_n$ and $A \subseteq S^{n+2}_n$. The point of Lemma 4 is that the two stationary sets are preserved in a compatible way. We will use it later (in the proof of Theorem 1) in combination with Lemma 5, to preserve stationary sets in a setting where Lemma 3 can not be applied. The GCH assumption in the next result is more than is needed, but somewhat simplifies the proof.

Lemma 4. *Let GCH hold. Let $T \subseteq S^{n+1}_n$ be stationary in ω_{n+1} and let $A \subseteq S^{n+2}_n$ be stationary in ω_{n+2}. Let $\mathbb{P}$ be an ω_{n+1}-closed poset which collapses ω_{n+2}. Let $f : \omega_{n+1} \to \omega^V_{n+2}$ be a continuous and cofinal map in the generic extension. Then in the generic extension the set $\{\alpha \in T : f(\alpha) \in A\}$ is stationary in ω_{n+1}.*

Proof. For each $\alpha < \omega_{n+2}$ let $P_\alpha = [\alpha]^{<\omega_n}$, so that easily $|P_\alpha| \leq \omega_{n+1}$ for all α and also $|P_\alpha| \leq \omega_n$ for $\alpha < \omega_{n+1}$. Let $\vec{P} = \langle P_\alpha : \alpha < \omega_{n+2} \rangle$.

Let p be a condition, let $\dot{C}$ be a name for a club subset of ω_{n+1}, and let $\dot{f}$ be a name for f. Notice that by the closure of $\mathbb{P}$, $f \restriction \eta \in V$ for all $\eta < \omega_{n+1}$. Fix θ a sufficiently large regular cardinal and $<_\theta$ a well-ordering of H_θ. Below we will abuse notation by writing "H_θ" for the structure $(H_\theta, \in, <_\theta)$.

We build a continuous increasing chain $\langle M_i : i < \omega_{n+2} \rangle$ where $\{p, \dot{C}, \dot{f}, \vec{P}\} \subseteq M_0$ and for all i

(1) $M_i \prec H_\theta$.
(2) $|M_i| = \omega_{n+1} \subseteq M_i$ (so that also $M_i \cap \omega_{n+2} \in \omega_{n+2}$).
(3) $M_i \cap \omega_{n+2} < M_{i+1} \cap \omega_{n+2}$.

It is easy to see that $\langle M_i \cap \omega_{n+2} : i < \omega_{n+2} \rangle$ is increasing, continuous and cofinal in ω_{n+2}. We find $\beta < \omega_{n+2}$ such that $\beta = M_\beta \cap \omega_{n+2} \in A$, and let $M = M_\beta$. Note that for every $\alpha < \beta$ we have $P_\alpha \in M$ and so $P_\alpha \subseteq M$. We choose D club in β of order type ω_n.

The key point is that since $\bigcup_{\alpha<\beta} P_\alpha \subseteq M$, every proper initial segment of D is an element of M. Let $D^* =_{\rm def} D \cup \{D \cap \alpha : \alpha \in D\}$, so that $D^* \subseteq M$. We build another continuous increasing chain $\langle N_j : j < \omega_{n+1} \rangle$ such that $\{p, \dot{C}, \dot{f}, \vec{P}\} \cup D^* \subseteq N_0 \subseteq M$ and for every j

(1) $N_j \prec M$.
(2) $|N_j| = \omega_n \subseteq N_j$ (so that also $N_j \cap \omega_{n+1} \in \omega_{n+1}$).
(3) $N_j \cap \omega_{n+1} < N_{j+1} \cap \omega_{n+1}$.

We choose $\alpha < \omega_{n+1}$ so that $\alpha = N_\alpha \cap \omega_{n+1} \in T$, and let $N = N_\alpha$. Since $D \subseteq N \subseteq M$, and D is unbounded in $M \cap \omega_{n+2}$, we see that $\sup(N \cap \omega_{n+2}) = \sup(M \cap \omega_{n+2}) = \beta$. Note that $P_\gamma \in N$ and so $P_\gamma \subseteq N$ for all $\gamma < \alpha$. Now we choose E club in α of order type ω_n, and note that since $\bigcup_{\gamma<\alpha} P_\gamma \subseteq N$ every proper initial segment of E lies in N.

We pause to take stock of the construction so far. We have built a structure $N \prec H_\theta$ such that $p, \dot{C}, \dot{f} \in N$, $\beta = \sup(N \cap \omega_{n+2}) \in A$, $\alpha = N \cap \omega_{n+1} \in T$, and there are club sets $D \subseteq \beta$ and $E \subseteq \alpha$ of order type ω_n, such that all proper initial segments of D and of E lie in N. Enumerate D in increasing order as $\langle \delta_k : k < \omega_n \rangle$, and enumerate E in increasing order as $\langle \epsilon_k : k < \omega_n \rangle$,

Now we build a decreasing chain of conditions $p_k \in \mathbb{P}$ for $k < \omega_n$. We start by setting $p_0 = p$, and then recursively choose p_{k+1} to be the $<_\theta$-least condition such that

(1) $p_{k+1} \leq p_k$.
(2) p_{k+1} determines the value of $\min(\dot{C}) \setminus \epsilon_k$.
(3) There exist $\eta \geq \epsilon_k$, $\zeta \geq \delta_k$ and $g \in V$ such that $p_{k+1} \Vdash \dot{f} \restriction \eta + 1 = g$ and $g(\eta) = \zeta$.

At limits we take the $<_\theta$-least lower bound for the sequence so far. The key point is that $p_k \in N$ for all k. It is easy to see that if $p_k \in N$ then $p_{k+1} \in N$, since p_{k+1} is defined from parameters in N. If $k < \omega_n$ is limit then observe that the sequence $\langle p_i : i < k \rangle$ can be computed from $p, \dot{C}$ and sufficiently long initial segments of D and E, so that $\langle p_i : i < k \rangle \in N$ and hence $p_k \in N$.

Let q be a lower bound for $\langle p_i : i < \omega_n \rangle$. By construction q determines $\dot{f} \restriction \alpha$ and forces that $\dot{f}``\alpha$ is cofinal in β, so q forces that $\dot{f}(\alpha) = \beta$. Also q forces that $\dot{C} \cap \alpha$ is cofinal in α, so q forces that $\alpha \in \dot{C}$. Since $\alpha \in T$ and $\beta \in A$, q forces that $\dot{C}$ meets $\{\eta \in T : f(\eta) \in A\}$, so that this set is stationary as claimed. □

Lemma 5. *Let GCH hold and let $m < \omega$. Let $\mathbb{S}$ be an iteration with ω_m-supports such that for some stationary set $B \subseteq S^{m+1}_m$, every factor is either ω_{m+1}-closed or is of the form $CUB(\omega_{m+1}, T)$ where $S^{m+1}_{<m} \cup B \subseteq T \subseteq \omega_{m+1}$.*

Then $\mathbb{S}$ is ω_m-closed, adds no ω_m-sequences of ordinals, and preserves the stationarity of every stationary subset of B.

Proof. Every factor is ω_m-closed, and the supports have size ω_m, so easily $\mathbb{S}$ is ω_m-closed. Let $U \subseteq B$ be stationary in ω_{m+1}, and let $\dot{\tau}$ name a function from ω_m to the ordinals. Let $\dot{C}$ be a name for a club set in ω_{m+1}. We will show that any condition $p \in \mathbb{S}$ can be extended to a condition q which forces that $\dot{C}$ meets U, and also determines the value of $\dot{\tau}$.

Choose θ some very large regular cardinal and find $X \prec H_\theta$ containing everything relevant such that $\delta =_{\text{def}} X \cap \omega_{m+1} \in U$ and ${}^{<\omega_m}X \subseteq X$. Fix δ_j for $j < \omega_m$ increasing and cofinal in δ.

We now build a decreasing ω_m-sequence $\langle p_i : i < \omega_m \rangle$ of conditions in $\mathbb{S} \cap X$ such that

1. $p_0 = p$.
2. p_{i+1} determines the values of $\min(\dot{C}) \setminus \delta_i$ and $\dot{\tau}(i)$.
3. For every $j \in supp(p_i)$, where $\mathbb{Q}_j$ is of the form $CUB(\omega_{m+1}, T)$ for some T, $p_{i+1}(j)$ names a set which contains an ordinal larger than δ_i.

This is easy since $\mathbb{S}$ is ω_m-closed, and X is closed under $< \omega_m$-sequences.

We now construct a lower bound p_∞ for $\langle p_i : i < \omega_m \rangle$, defining $p_\infty(j)$ by induction on j. There is no problem with the size of the support, because $\mathbb{S}$ is an iteration with ω_m-supports. Let $j \in \bigcup_i supp(p_i)$.

If $p_\infty \restriction j$ forces that $\langle p_i(j) : i < \omega_m \rangle$ is a decreasing sequence in an ω_{m+1}-closed poset, there is no problem in choosing $p_\infty(j)$. Otherwise $p_\infty \restriction j$ forces that $\langle p_i(j) : i < \omega_m \rangle$ is a sequence of closed bounded subsets of δ, and that the sequence of elements $\max p_i(j)$ is increasing and unbounded in δ. In this case we may take $p_\infty(j)$ to be a name for $\bigcup_i p_i(j) \cup \{\delta\}$, which is a safe choice because $\delta \in U \subseteq T$. Now $p_\infty \Vdash \delta \in \dot{C} \cap U$ and p_∞ also determines every value of $\dot{\tau}$. □

4. The main construction

We start by giving an outline of the construction. Our starting hypothesis is that we have an increasing sequence $\langle \kappa_n : 0 \leq n < \omega \rangle$ of supercompact cardinals. By a suitable preparation forcing we may assume in addition that GCH holds.

After round n of the construction we will be in a situation where $\kappa_i = \omega_{i+2}$ for $i \leq n$. We will begin round $n+1$ of the construction by adding a set $B_n \subseteq S^{n+2}_{n+1}$, together with a partial square sequence on $S^{n+2}_{n+1} \setminus B_n$.

The remainder of round $n+1$ consists of a certain iteration of length κ_{n+1} with ω_{n+1}-supports. In the course of this iteration we will be shooting club sets through certain subsets of κ_n (which is now ω_{n+2}) with a view to making $S^{n+2}_{\leq n} <^* B_n$, and will also be forcing with various ω_{n+2}-directed closed posets whose definition we postpone for now. Round $n+1$ of the construction will be ω_{n+1}-directed closed, will add no $< \omega_{n+2}$-sequences and will collapse κ_{n+1} to become ω_{n+3}.

Before continuing with a more precise description of the construction, we describe the forcing poset we will use to add the sets B_n and the related partial squares.

Definition 3. *Let κ be regular. Then $\mathbb{P}(\kappa)$ is the set of functions p such that*

1. $\mathrm{dom}(p)$ *is a successor ordinal less than κ^+.*
2. $p(\beta) = 0$ *if $\beta \in dom(p)$ and β is either zero or a successor ordinal.*
3. *For every limit ordinal $\beta \in \mathrm{dom}(p)$, either $p(\beta) = 1$ or $p(\beta)$ is a closed unbounded subset of β with order type at most κ.*
4. *If $p(\beta)$ is a closed unbounded subset of β, then for every $\gamma \in \lim p(\beta)$, $p(\gamma) = p(\beta) \cap \gamma$.*

The conditions are ordered by extension.

Lemma 6. *Let κ be regular. Then*

1. *The poset $\mathbb{P}(\kappa)$ is κ-closed.*
2. *Forcing with $\mathbb{P}(\kappa)$ adds no κ-sequences of ordinals.*
3. *Let G be a $\mathbb{P}(\kappa)$-generic filter. Let $f = \cup G$, $B = \{\alpha \in \kappa^+ \cap cof(\kappa) : f(\alpha) = 1\}$, $B' = (\kappa^+ \cap cof(\kappa)) \setminus B$, and $C_\alpha = f(\alpha)$*

for $\alpha \in B'$. Then in the generic extension B and B' are both stationary in κ^+, and $\langle C_\alpha : \alpha \in B' \rangle$ is a partial square sequence.

Proof. It is easy to see that any condition has arbitrarily long extensions, and that if B' is stationary then $\langle C_\alpha : \alpha \in B' \rangle$ is a partial square sequence. To show that the poset is κ-closed, let us be given a strictly decreasing sequence of conditions $\langle p_i : i < \delta \rangle$ where $\delta < \kappa$. We form $q = \bigcup_{i<\delta} p_i$, so that q is a function with domain some limit ordinal $\gamma \in \kappa^+ \cap cof(< \kappa)$; let r be such that $\mathrm{dom}(r) = \gamma + 1$, $r \restriction \gamma = q$, and $r(\gamma) = 1$. Then it is easy to check that r is a condition and r extends each p_i for $i < \delta$.

For the rest, let $\dot{C}$ name a club subset of κ^+ and $\dot{f}$ name a κ-sequence of ordinals. Given a condition p, we build a decreasing sequence of conditions p_i for $i < \kappa$ so that

(1) $p_0 = p$.
(2) $p_{i+1} \leq p_i$.
(3) p_{i+1} determines $\dot{f}(i)$, and forces that some ordinal between $\max \mathrm{dom}(p_i)$ and $\max \mathrm{dom}(p_{i+1})$ is in $\dot{C}$.
(4) For each limit ordinal $i < \kappa$, $\max \mathrm{dom}(p_i) = \sup_{j<i} \max \mathrm{dom}(p_j)$, and $p_i(\max \mathrm{dom}(p_i)) = \{\max \mathrm{dom}(p_j) : j < i\}$.

Let $\eta = \sup_{i<\kappa} \max \mathrm{dom}(p_i)$. We may now find lower bounds q and q' for the sequence of p_i, each with domain $(\eta + 1) \cap cof(\kappa)$, by setting $q(\eta) = 1$ and $q'(\eta) = \{\max \mathrm{dom}(p_j) : j < \kappa\}$. Either one will determine the value of $\dot{f}$, and in addition $q \Vdash \eta \in \dot{C} \cap B$ and $q' \Vdash \eta \in \dot{C} \cap B'$. □

Now for the precise description of our forcing iteration: as usual each poset is to be defined in the generic extension by the iteration of the preceding posets. Round 0 is slightly different from the subsequent rounds, and round n will be divided into two steps for $n > 0$. We will write $\mathbb{R}_n$ for the forcing at round n, and for $n > 0$ we will break up $\mathbb{R}_n$ as $\mathbb{R}^1_n * \mathbb{R}^2_n$ where $\mathbb{R}^i_n$ is step i.

- Round 0: We force with $\mathbb{R}_0 =_{\mathrm{def}} Coll(\omega_1, < \kappa_0)$.
- Round n step 1 for $n > 0$: We force with the poset $\mathbb{R}^1_n =_{\mathrm{def}} \mathbb{P}(\omega_n)$ from Definition 3, which adds a set $B_{n-1} \subseteq S^{n+1}_n$, and also a partial square on the complement of B_{n-1} in S^{n+1}_n.
- Round n step 2 for $n > 0$: By the Lévy-Solovay theorem κ_n is still supercompact, and so there is a Laver function f_n. We perform an iteration $\mathbb{R}^2_n$ of length κ_n with ω_n-supports. At stage γ we do nothing unless one of the following conditions is satisfied:
  (1) There is $i \leq n - 1$ such that $f(\gamma)$ is an $\mathbb{R}^2_n \restriction \gamma$-name $\dot{S}$ for a stationary subset of S^{n+1}_i.

(2) $f(\gamma)$ is an $\mathbb{R}^2_n \restriction \gamma$-name $\dot{\mathbb{Q}}$ for an ω_{n+1}-directed closed forcing poset.

In this case we force with

$$CUB(\omega_{n+1}, S^{n+1}_{<n} \cup (S^{n+1}_n \setminus B_{n-1}) \cup (Tr(S) \cap B_{n-1}))$$

if $f(\gamma)$ names a stationary subset S of S^{n+1}_i for some $i \leq n-1$, and we force with $\mathbb{Q}$ if $f(\gamma)$ names an ω_{n+1}-directed closed forcing poset $\mathbb{Q}$.

Let $\mathbb{P}_0$ be the trivial forcing and $\mathbb{P}_{n+1} = \mathbb{P}_n * \mathbb{R}_n$, and let $\mathbb{P}_\omega$ be the inverse limit of the $\mathbb{P}_n$. It is easy to see that $\mathbb{R}_n$ is κ_n-c.c. and ω_n-directed closed, and also by Lemma 5 that it adds no ω_n-sequences of ordinals. Our final model is the generic extension of the universe by $\mathbb{P}_\omega$. We note that by the closure and distributivity properties of $\mathbb{R}_n$, every ω_n-sequence of ordinals in the final model is in the extension by $\mathbb{P}_n$. By Lemma 5, for $n > 0$ the set $S^{n+1}_n \setminus B_{n-1}$ is still stationary at the end of round n, and so will still be stationary in the final model.

Because round 0 was so simple, the analysis for round 1 is a little more straightforward than for subsequent rounds, and we will give it separate treatment. This gives us a chance to introduce some of the main ideas for later rounds. In everything that follows we assume that H_ω is $\mathbb{P}_\omega$-generic over V, and that G_n is the induced $\mathbb{R}_n$-generic filter. For $n > 0$ we let $G_n = G^1_n * G^2_n$ where G^i_n is $\mathbb{R}^i_n$-generic.

Lemma 7.

(1) *Forcing with $\mathbb{R}_1$ over $V[G_0]$ adds no ω_1-sequences of ordinals.*
(2) *In $V[G_0 * G_1]$, B_0 is stationary.*
(3) *In $V[G_0 * G_1]$ every stationary subset of S^2_0 reflects at almost every point of B_0.*
(4) *In $V[H_\omega]$, B_0 is stationary and every stationary subset of S^2_0 reflects at almost every point of B_0.*

Proof. It follows from Lemma 5 that $\mathbb{R}_1$ adds no ω_1-sequences of ordinals. It is easy to check that every stationary subset of S^2_0 in $V[G_0 * G_1]$ is named by $f_1(\alpha)$ for some α, and so we added a club to witness that the set reflects almost everywhere in B_0. Every ω_2-sequence of ordinals from $V[H_\omega]$ appears in $V[G_0 * G_1]$. It remains only to show that B_0 is stationary in $V[G_0 * G_1]$. This we will do by producing a generic elementary embedding j_0 with domain $V[G_0 * G_1]$ and critical point $\kappa_0 = \omega_2$, such that $\kappa_0 \in j_0(B_0)$.

Fix $j_0 : V \to M_0$ an embedding which witnesses that κ_0 is κ_1-supercompact in V. We may extend the identity embedding $id : \mathbb{R}_0 \to j_0(\mathbb{R}_0)$ to an embedding of $\mathbb{R}_0 * \mathbb{R}_1$ into $j_0(\mathbb{R}_0)$ with an ω_1-closed quotient. So then as usual we may lift j_0 to an embedding

$j_0 : V[G_0] \to M_0[G_0 * G_1 * H]$, where H is generic over $V[G_0 * G_1]$ for countably closed forcing. Working in $M_0[G_0 * G_1 * H]$, we define m_0 to be the condition in $j_0(\mathbb{R}^1_1)$ such that $\mathrm{dom}(m_0) = \kappa_0 + 1$, $m_0 \restriction \kappa_0 = \bigcup G^1_1$, $m_0(\kappa_0) = 1$. This is a legitimate condition: after forcing with $j_0(\mathbb{R}_0)$ the cofinality of κ_0 is ω_1, and the club sets appearing in m_0 are coherent.

It is easy to see that m_0 is a *strong master condition* for $j_0 : V[G_0] \to M_0[G_0 * G_1 * H]$ and G^1_1, in the sense that m_0 is a lower bound for j_0“G^1_1. We may now force over $V[G_0 * G_1 * H]$ to get a generic filter K for $j_0(\mathbb{R}^1_1)$ with j_0“$G^1_1 \subseteq K$, and then lift again to get $j_0 : V[G_0 * G^1_1] \to M_0[G_0 * G_1 * H * K]$. Note that by the choice of m_0, $\kappa_0 \in j_0(B_0)$.

Now we construct by induction conditions $m^*_\beta \in j_0(\mathbb{R}^2_1 \restriction \beta)$ for $\beta \leq \kappa_1$, such that $supp(m^*_\beta) = j_0$“β, m^*_β is a strong master condition for $j_0 : V[G_0 * G^1_1] \to M_0[G_0 * G_1 * H * K]$ and $G^2_1 \restriction \beta$, and $m^*_\beta \restriction j_0(\gamma) = m^*_\gamma$ for $\gamma < \beta$. There is no problem with the size of the supports because κ_1 has been collapsed to have cardinality ω_1 by $j_0(\mathbb{R}_0)$. The coordinates in j_0“κ_1 are the only ones that need attention because $\mathbb{R}^2_1$ is an iteration with ω_1-supports, and so $supp(j_0(r)) = j_0$“$supp(r)$ for all $r \in \mathbb{R}^2_1$.

We will need to make sure that each of the objects m^*_β is a member of $M_0[G_0 * G_1 * H * K]$. To do this we will first prove that $j_0 \restriction G^2_1 \in M_0[G_0 * G_1 * H * K]$, and then show that m^*_β can be constructed from $j_0 \restriction G^2_1$ by a uniform procedure which is definable in $M_0[G_0 * G_1 * H * K]$.

To show that $j_0 \restriction G^2_1 \in M_0[G_0 * G_1 * H * K]$, we will use the hypothesis that $j_0 : V \to M_0$ witnesses the κ_1-supercompactness of κ_0. In particular we know that $V \models {}^{\kappa_1} M_0 \subseteq M_0$. By a routine chain condition argument, working in V we may find a set of terms $Z \subseteq V_{\kappa_1}$ such that every condition in $\mathbb{R}^2_1$ is of the form $i_{G_o * G^1_1}(\dot{\tau})$ for some $\dot{\tau} \in Z$. By closure $j_0 \restriction Z \in M_0$. By definition if $p = i_{G_o * G^1_1}(\dot{\tau}) \in \mathbb{R}^2_1$ then $j(p) = i_{G_0 * G_1 * H * K}(j(\dot{\tau}))$. Since $j_0 \restriction Z \in M_0$, and both G^2_1 and $G_0 * G_1 * H * K$ are in $M_0[G_0 * G_1 * H * K]$, it follows that $j_0 \restriction G^2_1 \in M_0[G_0 * G_1 * H * K]$.

For each $\beta < \kappa_1$, there is in $V[G_0 * G^1_1 * (G^2_1 \restriction \beta)]$ a function LB_β such that

(1) $\mathrm{dom}(LB_\beta) = V_{\kappa_1}^{V[G_0 * G^1_1 * (G^2_1 \restriction \beta)]}$.
(2) For every pair $(a, b) \in \mathrm{dom}(LB_\beta)$ such that a is ω_2-directed closed and $b \subseteq a$ is a directed set of size less than ω_2, $LB_\beta(a, b)$ is a lower bound for b.

Working in $V[G_0 * G^1_1]$ we choose a sequence $\langle \dot{LB}_\beta : \beta < \kappa_1 \rangle$ such that $\dot{LB}_\beta$ is a $\mathbb{R}^2_1$-name for a suitable function LB_β.

After these preliminaries, we may finally describe the inductive construction of the conditions m^*_β. At limit stages λ we let m^*_λ be the unique sequence such that $supp(m^*_\lambda) = j_0``\lambda$ and $m^*_\lambda \restriction j_0(\beta) = m^*_\beta$ for $\beta < \lambda$. We will argue shortly that $m^*_\lambda \in M_0[G_0 * G_1 * H * K]$, and in particular it is a condition in $j_0(\mathbb{R}^2_1 \restriction \lambda)$. Before giving that argument we describe the successor step.

Suppose that we have defined m^*_β. We force below this condition to get L_β with $j_0``G^2_1 \restriction \beta \subseteq L_\beta$, and then lift to get $j_0 : V[G_0 * G^1_1 * G^2_1 \restriction \beta] \to M_0[G_0 * G_1 * H * K * L_\beta]$. There is nothing to do unless the iteration $\mathbb{R}^2_1$ is non-trivial at stage β, so suppose that it is. We distinguish the two cases: the Laver function f_1 guesses a stationary subset of S^2_0 or an ω_2-directed closed forcing poset.

Case 1: $f_1(\beta)$ is an $\mathbb{R}^2_1 \restriction \beta$ name for a stationary set $S \subseteq S^2_0$.

S is a stationary set of cofinality ω ordinals, and the forcing at stage β adds a club set C with $C \subseteq S^2_0 \cup (S^2_1 \setminus B_0) \cup (Tr(S) \cap B_0)$.

The rest of the iteration $\mathbb{R}^2_1$ is countably closed, so S is still stationary in $V[G_0 * G_1]$. Similarly $H * K * L_\beta$ is generic over $V[G_0 * G_1]$ for countably closed forcing, so S is still stationary in $V[G_0 * G_1 * H * K * L_\beta]$, and hence in $M_0[G_0 * G_1 * H * K * L_\beta]$.

Since $j_0(S) \cap \kappa_0 = S$ and S is stationary, we see that $\kappa_0 \in j_0(Tr(S) \cap B_0)$. It follows that $C \cup \{\kappa_0\}$ is a condition in $j_0(CUB(\omega_2, S^2_0 \cup (S^2_1 \setminus B_0) \cup (Tr(S) \cap B_0))$. It is now easy to see that if we prolong m^*_β by adding $C \cup \{\kappa_0\}$ at coordinate $j(\beta)$, we obtain a suitable master condition $m^*_{\beta+1}$.

Case 2: $f_1(\beta)$ is an $\mathbb{R}^2_1 \restriction \beta$ name for an ω_2-directed closed forcing poset $\mathbb{Q}$.

At stage β the forcing $\mathbb{R}^2_1$ adds a filter F which is generic for $\mathbb{Q}$. By the supercompactness assumption, an argument similar to the one we gave that $j_0 \restriction G^2_1 \in M_0[G_0 * G_1 * H * K]$ shows that $j_0``F \in M_0[G_0 * G_1 * H * K * L_\beta]$. Also $j_0``F$ is a directed set of size less than $j_0(\kappa_0)$ in the poset $j_0(\mathbb{Q})$, and if we let $LB_\beta = i_{G^2_1 \restriction \beta}(\dot{LB}_\beta)$ then $q =_{\text{def}} j_0(LB_\beta)(j_0``F)$ is a lower bound for $j_0``F$. We prolong m^*_β by adding a name for q at coordinate $j(\beta)$, to obtain $m^*_{\beta+1}$.

We have defined the objects m^*_β in a uniform way from $j_0 \restriction G^2_1$ and $j_0(\langle \dot{LB}_\beta : \beta < \kappa_1 \rangle)$, so by a straightforward induction they all lie in $M_0[G_0 * G_1 * H * K]$.

At the end of the construction we let $m = m_0 * m^*_{\kappa_1}$. Forcing below m we can lift to get $j_0 : V[G_0 * G^1_1 * G^2_1] \to M_0[G_0 * G_1 * H * K * L]$. $\square$

Now we prove a similar lemma for the subsequent rounds of the iteration. The general outline is similar, but we have to work harder at certain points. As before we will extend a certain embedding j_{n-1} witnessing the κ_n-supercompactness of κ_{n-1}, and embed $\mathbb{R}_{n-1} * \mathbb{R}_n$ into $j_{n-1}(\mathbb{R}_{n-1})$.

The main new technical issue is that the closure of the relevant posets is no longer sufficient to preserve the stationarity of the relevant stationary sets. For example we will need to know that stationary subsets of S^{n+1}_{n-1} are preserved by a final segment of $j_{n-1}(\mathbb{R}_{n-1})$, which is only ω_{n-1}-closed.

Lemma 8. *Let $n > 1$.*

(1) *Forcing with $\mathbb{R}_n$ over $V[G_0 * \ldots * G_{n-1}]$ adds no ω_n-sequences of ordinals.*
(2) *In $V[G_0 * \ldots * G_n]$, B_{n-1} is still stationary.*
(3) *In $V[G_0 * \ldots * G_n]$, every stationary subset of $S^{n+1}_{\leq n-1}$ reflects at almost every point of B_{n-1}.*
(4) *In $V[H_\omega]$, B_{n-1} is stationary and every stationary subset of $S^{n+1}_{\leq n-1}$ reflects at almost every point of B_{n-1}.*

Proof. It follows from Lemma 5 that $\mathbb{R}_n$ adds no ω_n-sequences of ordinals. Since every name for a stationary subset of $S^{n+1}_{\leq n-1}$ appears in the course of the iteration, it follows from the definition of the iteration that after forcing with $\mathbb{R}_n$ every stationary subset of $S^{n+1}_{\leq n-1}$ reflects at almost every point of B_{n-1}. We will finish by producing a generic embedding j_{n-1} with critical point ω_{n+1} and domain $V[G_0 * \ldots * G_n]$, such that $\omega_{n+1} \in j_{n-1}(B_{n-1})$.

Let $W = V[G_0 * \ldots G_{n-2}]$. Recall that in the course of the construction we observed that (by the Lévy-Solovay theorem) κ_{n-1} is still supercompact in W, and we used this to choose a Laver function f_{n-1}, which then dictated the course of the iteration $\mathbb{R}_{n-1}$.

As we have already observed, $\mathbb{R}_n$ is ω_n-directed closed in $W[G_{n-1}]$, in particular cardinals up to ω_n are preserved. We define $\mathbb{R}^*_n =_{\text{def}} \mathbb{R}_n * Coll(\omega_n, \omega_{n+1})$. Using the Laver property of f_{n-1} we then choose $j_{n-1} : W \to M_{n-1}$ such that j_{n-1} witnesses the κ_n-supercompactness of κ_{n-1}, and $j(f_{n-1})(\kappa_{n-1})$ is a name in $W^{\mathbb{R}_{n-1}}$ for the poset $\mathbb{R}^*_n$.

Since the poset $\mathbb{R}^*_n$ is ω_n-directed closed, it is the forcing which will be used at stage κ_{n-1} in the iteration $j_{n-1}(\mathbb{R}_{n-1})$. So then as usual we may lift j_{n-1} to an embedding $j_{n-1} : W[G_{n-1}] \to M_{n-1}[G_{n-1} * G^*_n * H]$, where $G^*_n = G_n * C$ for C which is $Coll(\omega_n, \omega_{n+1})$-generic over $W[G_{n-1} * G_n]$.

By the properties which we have already established for $\mathbb{R}_{n-1}$, forcing with $j_{n-1}(\mathbb{R}_{n-1})$ preserves the cardinals ω_k for $k \leq n$, and $j_{n-1}(\kappa_{n-1})$ becomes the new ω_{n+1}, while κ_{n-1} is collapsed and becomes an ordinal of cofinality ω_n. Working in $M_{n-1}[G_{n-1} * G^*_n * H]$, we define m_{n-1} to be the condition in $j_{n-1}(\mathbb{R}^1_n)$ such that $\mathrm{dom}(m_{n-1}) = \kappa_{n-1} + 1$, $m_{n-1} \restriction \kappa_{n-1} = \bigcup G^1_n$, and $m_{n-1}(\kappa_{n-1}) = 1$.

Easily m_{n-1} is a strong master condition for $j_{n-1} : W[G_{n-1}] \to M_{n-1}[G_{n-1} * G^*_n * H]$ and G^1_n. Using this master condition we force over $W[G_{n-1} * G^*_n * H]$ to get a suitable generic filter K for $j_{n-1}(\mathbb{R}^1_n)$, and lift to get $j_{n-1} : W[G_{n-1} * G^1_n] \to M_{n-1}[G_{n-1} * G^*_n * H * K]$. We have arranged that $\kappa_{n-1} \in j_{n-1}(B_{n-1})$.

Just as in the proof of Lemma 7 we build conditions $m^*_\beta \in j_{n-1}(\mathbb{R}^2_n \restriction \beta)$ such that $supp(m^*_\beta) = j_{n-1}``\beta$, m^*_β is a strong master condition for $j_{n-1} : W[G_{n-1} * G^1_n] \to M_{n-1}[G_{n-1} * G^*_n * H * K]$ and $G^2_n \restriction \beta$, and $m^*_\beta \restriction j_{n-1}(\gamma) = m^*_\gamma$ for $\gamma < \beta$. Arguments similar to those in the proof of Lemma 7 show that $j_{n-1} \restriction G^2_n \in M_{n-1}[G_{n-1} * G^*_n * H * K]$, and just as in that proof we will build the conditions m^*_β in a uniform way which is definable in $M_{n-1}[G_{n-1} * G^*_n * H * K]$.

The limit stage is exactly as in Lemma 7. The difference in the successor step will be that it takes more work to show that a stationary set $S \subseteq S^{n+1}_{\le n-1}$ arising in the course of the iteration $\mathbb{R}_n$ remains stationary out to the point where it can be used in the construction of the master condition.

Given m^*_β we force to get L_β with $j_{n-1}``G^2_n \restriction \beta \subseteq L_\beta$, and then lift to get $j_{n-1} : W[G_{n-1} * G^1_n * G^2_n \restriction \beta] \to M_{n-1}[G_{n-1} * G^*_n * H * K * L_\beta]$. The interesting cases arise when f_{n-1} guesses a stationary subset of $S^{n+1}_{\le n-1}$ or an ω_{n+1}-directed closed forcing poset.

Case 1a: $f_{n-1}(\beta)$ is an $\mathbb{R}^n_1 \restriction \beta$ name for a stationary set $S \subseteq S^{n+1}_j$ for $j < n - 1$.

S is stationary in $W[G_{n-1} * G^1_n * G^2_n \restriction \beta]$. By the resemblance between W and M_{n-1}, $S \in M_{n-1}[G_{n-1} * G^1_n * G^2_n \restriction \beta]$ and a fortiori it is stationary in that model.

We now observe that $M_{n-1}[G_{n-1} * G^*_n * H * K * L_\beta]$ is a generic extension of $M_{n-1}[G_{n-1} * G^1_n * G^2_n \restriction \beta]$ by ω_n-closed forcing, and so by Lemma 3 the set S is still stationary in $M_{n-1}[G_{n-1} * G^*_n * H * K * L_\beta]$.

Let C be the club set in κ_{n-1} which was added by the iteration $\mathbb{R}_n$ at stage β, and prolong m^*_β by adding $C \cup \{\kappa_{n-1}\}$ at coordinate $j(\beta)$. Exactly as in the proof of Lemma 7 we obtain a suitable master condition $m^*_{\beta+1}$.

Remark 1. *This is precisely the argument used by Jech and Shelah* [2].

Case 1b: $f_{n-1}(\beta)$ is an $\mathbb{R}^n_1 \restriction \beta$ name for a stationary set $S \subseteq S^{n+1}_{n-1}$.

As in case 1a, $S \in M_{n-1}[G_{n-1} * G_n^1 * G_n^2 \restriction \beta]$ and it is stationary in that model. The key point is again to show that S remains stationary in $M_{n-1}[G_{n-1} * G_n^* * H * K * L_\beta]$, but this time it is harder because we do not have enough closure for an appeal to Lemma 3.

The factor forcing to prolong $G_n^2 \restriction \beta$ to G_n^* is ω_n-closed and collapses ω_{n+1}, so we may fix $F \in M_{n-1}[G_{n-1} * G_n^*]$ with $F : \omega_n \to \omega_{n+1}$ increasing, continuous and cofinal. As we remarked earlier $S^n_{n-1} \setminus B_{n-2}$ is stationary in the universe $M_{n-1}[G_{n-1}]$, so by ω_n-closure it is still stationary in $M_{n-1}[G_{n-1} * G_n^*]$. It is easy to see that GCH holds in this universe, and so we may apply Lemma 4 to conclude that

$$U = \{\alpha \in S^n_{n-1} - B_{n-2} : F(\alpha) \in S\}$$

is stationary in $M_{n-1}[G_{n-1} * G_n^*]$. Notice that in any further extension in which U is a stationary subset of ω_n, S will also remain stationary.

We know that $\mathbb{R}_{n-1}$ adds no ω_{n-1}-sequences, and by elementarity the same is true of $j_{n-1}(\mathbb{R}_{n-1})$. So from the point of view of the universe $M_{n-1}[G_{n-1} * G_n^*]$, H is a generic object for an iteration $\mathbb{S}$ with ω_{n-1}-supports in which at every stage we either force with some ω_n-directed closed forcing poset or shoot a club set through a stationary subset of ω_n which contains $S^n_{<n-1} \cup B^c_{n-2}$. By Lemma 5, forcing with $\mathbb{S}$ preserves the stationarity of U.

Finally $\mathbb{R}_n$ is ω_n-closed, and so by elementarity $j_{n-1}(\mathbb{R}_n)$ is also ω_n-closed. So U is stationary in $M_{n-1}[G_{n-1} * G_n^* * H * K * L_\beta]$, and therefore S is stationary in this universe.

We finish as in Case 1a: let C be the club set in κ_{n-1} which was added by the iteration $\mathbb{R}_n$ at stage β, and prolong m^*_β by adding $C \cup \{\kappa_{n-1}\}$ at coordinate $j(\beta)$.

Case 2: $f_{n-1}(\beta)$ is an $\mathbb{R}^2_n \restriction \beta$ name for an ω_{n+1}-directed closed forcing poset $\mathbb{Q}$.

In this case the argument is just like that for Case 2 in Lemma 7.

Now we let $m = m_{n-1} * m^*_{\kappa_n}$. By Lemma 5 $\mathbb{R}_n$ adds no ω_n-sequences of ordinals. Forcing below m we lift to get $j_{n-1} : W[G_{n-1} * G_n] \to M_{n-1}[G_{n-1} * G_n^* * H * K * L]$. $\square$

This concludes the proof of Theorem 1.

Note: The authors would like to thank the anonymous referee for an exceptionally thorough and helpful report on the first version of this paper.

References

[1] J. Cummings and M. Foreman, The tree property, Advances in Mathematics 133 (1998) 1–32.

[2] T. Jech and S. Shelah, Full reflection of stationary sets below $\aleph_\omega$, Journal of Symbolic Logic 55 (1990), 822–830.

[3] T. Jech, Stationary subsets of inaccessible cardinals, Axiomatic set theory (Contemporary Mathematics 31), 115–142, American Mathematical Society, Providence, RI, 1984.

[4] R. B. Jensen, The fine structure of the constructible hierarchy, Annals of mathematical logic 4 (1972), 229–308.

[5] R. Laver, Making the supercompactness of κ indestructible under κ-directed closed forcing, Israel Journal of Mathematics 29 (1978), 385–388.

[6] A. Lévy and R. M. Solovay, Measurable cardinals and the continuum hypothesis, Israel Journal of Mathematics 5 (1967), 234–248.

[7] M. Magidor, Reflecting stationary sets, Journal of Symbolic Logic 47 (1982), 755–771.

[8] M. Magidor, Private communication.

[9] M. Magidor, Private communication.

[10] S. Shelah, Cardinal Arithmetic, Oxford University Press, Oxford, 1994.