A global version of a theorem of Ben-David and Magidor *†

Arthur W. Apter ‡
Department of Mathematics
Baruch College of CUNY
New York NY 10010 USA
awabb@cunyvm.cuny.edu
http://math.baruch.cuny.edu/∼apter

James Cummings §
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh PA 15213 USA
jcumming@andrew.cmu.edu
http://www.math.cmu.edu/users/jcumming

October 4, 1999

Abstract

We prove a consistency result about square principles and stationary reflection which generalises the result of Ben-David and Magidor [4].

*1991 Mathematics Subject Classifications: primary 03E35; secondary 03E05, 03E55

†Keywords: Square, weak square, stationary reflection, Radin forcing, large cardinals

‡Supported by the Volkswagen-Stiftung (RiP-program at Oberwolfach). In addition, this research was partially supported by PSC-CUNY Grants 665337 and 667379.

§Supported by the Volkswagen-Stiftung (RiP-program at Oberwolfach). In addition, this research was partially supported by NSF Grant DMS-9703945.

1 Introduction

In this paper we prove a consistency result about square principles and stationary reflection, which is a generalisation of a theorem from Ben-David and Magidor's paper [4]. We begin by giving some pertinent facts and definitions.

Definition 1.1 *Let κ be an uncountable regular cardinal, let $S \subseteq \kappa$ be stationary, and let $\alpha < \kappa$. Then S* reflects at α *iff* $\mathrm{cf}(\alpha) > \omega$ *and $S \cap \alpha$ is stationary in α. S* reflects *iff there exists $\alpha < \kappa$ such that S reflects at α. S is* non-reflecting *iff S does not reflect. A sequence $\vec{S} = \langle S_i : i < \beta \rangle$ of stationary subsets of κ* reflects simultaneously *iff there exists $\alpha < \kappa$ such that S_i reflects at α for every $i < \beta$; $\vec{S}$* reflects simultaneously to cofinality μ *iff there exists α such that* $\mathrm{cf}(\alpha) = \mu$ *and $S_i \cap \alpha$ is stationary for all $i < \beta$.*

Large cardinals can be used to get instances of simultaneous reflection. In particular we will use the following fact (due to Solovay). Since we could not find a reference, we sketch the proof.

Fact 1.2 *If κ is κ^+ supercompact and $\vec{S}$ is a sequence of stationary subsets of $\{ \alpha < \kappa^+ \mid \mathrm{cf}(\alpha) < \kappa \}$ such that $\mathrm{lh}(\vec{S}) < \kappa$, then there are unboundedly many $\mu < \kappa$ such that*

1. *μ is the successor of an inaccessible.*

2. *$\vec{S}$ reflects simultaneously to cofinality μ.*

Proof: Let $\vec{S} = \langle S_\alpha : \alpha < \nu \rangle$ where $\nu < \kappa$ and each S_α is a stationary subset of $\{ \alpha < \kappa^+ \mid \mathrm{cf}(\alpha) < \kappa \}$. Let $\rho < \kappa$. Let $j : V \longrightarrow M$ be an elementary embedding such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \kappa^+$ and ${}^{\kappa^+}M \subseteq M$. Let $\lambda = \sup(j``\kappa^+)$, where it follows from the closure of M that $\mathrm{cf}_M(\lambda) = \kappa^+ = \kappa^+_M$ and $\lambda < j(\kappa^+)$.

$j(\vec{S}) = \langle j(S_\alpha) : \alpha < \nu \rangle$ and $j(\rho) = \rho < \kappa$, and also κ is inaccessible in M. It will therefore suffice to show that

$$M \models \text{“}j(S_\alpha) \cap \lambda \text{ is stationary for every } \alpha < \nu\text{”}$$

and then appeal to the elementarity of j to see that

$$V \models \text{“there is an inaccessible } \zeta > \rho \text{ such that } \vec{S} \text{ reflects to cofinality } \zeta^{+}\text{”}$$

Clearly $j``S_\alpha \subseteq j(S_\alpha) \cap \lambda$. If $D \subseteq \lambda$ is club then it is easy to see that $j^{-1}``D$ is $< \kappa$-club in κ^+, so that there is some $\beta \in S_\alpha$ with $j(\beta) \in D$. Thus $j(S_\alpha) \cap \lambda$ is stationary (even in V!) and we are done.

♦

Definition 1.3 *Let κ be an infinite cardinal, and let $\lambda \leq \kappa$ be a cardinal. Then a $\square_{\kappa,\lambda}$-sequence is a sequence $\langle \mathcal{C}_\alpha : \lim(\alpha), \alpha < \kappa^+ \rangle$ such that*

1. $1 \leq |\mathcal{C}_\alpha| \leq \lambda$.
2. *For every $C \in \mathcal{C}_\alpha$*
   (a) *C is club in α.*

(b) *o.t.*$(C) \leq \kappa$.

(c) *For all* $\beta \in \lim(C)$, $C \cap \beta \in \mathcal{C}_\beta$.

We say that "$\Box_{\kappa,\lambda}$ *holds" iff there exists a* $\Box_{\kappa,\lambda}$*-sequence. A* $\Box_{\kappa,<\lambda}$-sequence *is defined as above with clause 1 replaced by "*$1 \leq |\mathcal{C}_\alpha| < \lambda$*".*

The principle $\Box_{\kappa,\lambda}$ was defined by Schimmerling [14] in his work on the core model for one Woodin cardinal. It is a common generalisation of two principles studied by Jensen [8], the principles $\Box_\kappa = \Box_{\kappa,1}$ and $\Box^*_\kappa = \Box_{\kappa,\kappa}$.

With these definitions in hand we are ready to state the main theorem of this paper.

Theorem 1 *Let GCH hold and let* κ *be a* κ^{+5}*-supercompact cardinal. Then there exists a forcing poset* $\mathbb{P}$ *such that in* $V^{\mathbb{P}}$

1. κ *is* κ^{+5}*-supercompact.*

2. *For every singular cardinal* $\lambda < \kappa$

   (a) *There exists* $S \subseteq \lambda^+$ *stationary such that if* $\vec{S}$ *is a sequence of stationary subsets of* S *and* $\mathrm{lh}(\vec{S}) < \mathrm{cf}(\lambda)$ *then* $\vec{S}$ *reflects to cofinality* μ *for unboundedly many* $\mu < \lambda$.

   (b) *The combinatorial principle* $\Box_{\lambda,\mathrm{cf}(\lambda)}$ *holds.*

Notice that truncating the generic extension by $\mathbb{P}$ at κ will give a set model of the theory "ZFC + there exists a proper class of cardinals δ which are δ^{+4}-supercompact + every singular cardinal λ has the above properties".

This theorem is (as we explain below) a generalisation of the following result of Ben-David and Magidor.

Fact 1.4 (Ben-David and Magidor [4]) *If κ is κ^+-supercompact there is a generic extension in which*

1. $\kappa = \aleph_\omega$, $\kappa^+ = \aleph_{\omega+1}$.
2. $\Box_{\aleph_\omega}$ *fails.*
3. $\Box^*_{\aleph_\omega}$ *holds.*

The model of [4] is built by using a modification of Magidor's "supercompact Prikry forcing with interleaved forcing" from [10]. Apter and Henle [2] showed that a somewhat similar proof can be made to work using κ which is only κ^+-strongly compact. Using the ideas of [2] and methods of Gitik for iterating Prikry-type forcing, Apter was able to show

Fact 1.5 (Apter [1]) *Con(ZFC + GCH + κ is an inaccessible limit of cardinals δ which are δ^+ strongly compact) $\Longrightarrow$ Con(ZFC + κ is inaccessible + For every cardinal $\lambda < \kappa$, there exists a stationary $S \subseteq \lambda^{+\omega+1}$ such that if $S' \subseteq S$ is stationary, then S' reflects at δ for unboundedly many $\delta < \lambda^{+\omega+1}$ (so $\neg\Box_{\lambda^{+\omega}}$) + For every cardinal $\lambda < \kappa$, $\Box_{\lambda^{+\omega},\omega}$).*

In the model of [4] there is a uniform ultrafilter on $\aleph_{\omega+1}$ which is λ-indecomposable for $\aleph_0 < \lambda < \aleph_\omega$. This implies [4, Lemma 2.2] that every stationary subset of $S = \{ \alpha < \aleph_{\omega+1} \mid \mathrm{cf}(\alpha) > \omega \}$ reflects, and from this stationary reflection principle it follows [4, Lemma 2.1] that $\Box_{\aleph_\omega}$ fails.

In the model of [4] the transfer principle $(\aleph_1, \aleph_0) \longrightarrow (\aleph_{\omega+1}, \aleph_\omega)$ holds. It follows from this transfer principle that there is a special $\aleph_{\omega+1}$-Aronszajn tree, and hence by work of Jensen [8] that $\Box^*_{\aleph_\omega}$ holds.

Our model for Theorem 1 is built using Foreman and Woodin's "supercompact Radin forcing with interleaved forcing" from their consistency proof [7] for "ZFC + GCH fails everywhere". Their work builds on ideas from [10] and Radin's paper [13].

It is worth remarking that the similarity between our model for Theorem 1 and the model of [4] is even more pronounced than has been shown so far. The methods of Section 4 of this paper can be used to show that in the model of [4] any finite sequence of stationary subsets of S reflects simultaneously, and it follows from Fact 1.8 that $\Box_{\aleph_\omega,\omega}$ holds in the model of [4].

The machinery that we use to get $\Box_{\lambda,\mathrm{cf}(\lambda)}$ to hold in the model of Theorem 1 is based on the following distinctive property of the forcing $\mathbb{P}$: every singular cardinal of $V^{\mathbb{P}}$ below κ is inaccessible in V. This idea originates in the proof of Fact 1.8. Before stating Fact 1.8 we make a technical definition.

Definition 1.6 *Let κ be regular and uncountable. $X \subseteq \kappa$ is a $> \omega$-*club *subset of κ if and only if X is unbounded in κ and X is closed under suprema of uncountable cofinality.*

Remark 1.7 *It is easy to see that the $> \omega$-club subsets of κ generate a* *normal filter on κ. We shall refer to this filter as the "$>\omega$-club filter".*

Fact 1.8 (Cummings and Schimmerling [6]) *Let $V \models$ "κ is inaccessible". Let $W \supseteq V$ be a model such that*

1. *κ and κ^+ are cardinals in W.*
2. *There is $E \in W$ such that*
   (a) *o.t.$(E) = \omega$.*
   (b) *For all $D \in V$, if $V \models$ "D is a $>\omega$-club subset of κ" then $E \cap D \neq \emptyset$.*

Then $W \models \square_{\kappa,\omega}$.

Remark 1.9 *In particular if κ is measurable in V, and W is a generic extension by Prikry forcing at κ, then $\square_{\kappa,\omega}$ holds in W.*

In the context of Theorem 1 we are changing cofinalities to values other than ω, and will have to prepare the ground model in order to get an analogue to Fact 1.8. This preparation uses ideas of Baumgartner.

We now quote some facts whose proofs can be found in [5]. Taken together they indicate that the result of Theorem 1 is close to being optimal.

Fact 1.10 (Cummings, Foreman and Magidor [5]) *Let λ be singular. If $\square_{\lambda,\mu}$ holds for $\mu < \lambda$ and $S \subseteq \lambda^+$ is stationary then there exists $\vec{T} = \langle T_i : i < \mathrm{cf}(\lambda) \rangle$ such that each T_i is a stationary subset of S and $\vec{T}$ does not reflect simultaneously to cofinality μ for any μ with $\mathrm{cf}(\lambda) < \mu < \lambda$.*

Fact 1.11 (Schimmerling) *Let λ be singular. If $\square_{\lambda,\mu}$ holds for $\mu < \mathrm{cf}(\lambda)$ and $S \subseteq \lambda^+$ is stationary, then there exists $T \subseteq \lambda$ stationary and $\delta < \lambda$ such that T does not reflect at any point of cofinality greater than δ.*

Our forcing notation is fairly standard. We write $p \leq q$ when p is a stronger condition than q. $\mathrm{Add}(\kappa, \lambda)$ is the poset to add λ Cohen subsets of κ, $\mathrm{Coll}(\kappa, <\lambda)$ is the Levy collapse to make every ordinal in $[\kappa, \lambda)$ have cardinality κ. A poset is κ-closed when every descending sequence of length less than κ has a lower bound, and is κ-directed closed when every directed subset of size less than κ has a lower bound. $\mathrm{RO}(\mathbb{P})$ is the complete Boolean algebra corresponding to the poset $\mathbb{P}$.

The second author would like to thank Matt Foreman for telling him Fact 2.7, and showing him a proof that if κ is κ^+-supercompact then any finite sequence of stationary subsets of κ^+ reflects after doing Prikry forcing at κ.

2 Building squares

In this section we discuss the machinery we use to get $\square_{\lambda,\mathrm{cf}(\lambda)}$-sequences in the model of Theorem 1. We will define a combinatorial principle $\square^B_\kappa$ (due to Baumgartner) and prove the following lemma, which is a generalisation of Fact 1.8. The point here is that when the cofinality of κ is changed to be uncountable we need some amount of square in the ground model to see that the desired form of square holds in the extension.

Lemma 2.1 *Let $V \models$ "κ is inaccessible and $\Box^B_\kappa$". Let $W \supseteq V$ be a model such that*

1. *κ and κ^+ are cardinals in W and λ is regular in W.*

2. *There is $E \in W$ such that*

   (a) *E is closed and unbounded in κ.*

   (b) *o.t.$(E) = \lambda$.*

   (c) *For all $D \in V$, if $V \models$ "D is a $> \omega$-club subset of κ" then $E \cap D \neq \emptyset$.*

Then $W \models \Box_{\kappa,\lambda}$.

Remark 2.2 *If κ is measurable and $A \subseteq \kappa$ is a $> \omega$-club subset of κ then $A \in U$ for every normal measure U on κ. It follows that if C is a generic club of order type λ added by any of the standard forcings for changing cofinality (Prikry forcing [12] [where the generic set C, having order type ω, is trivially considered as being club in κ], Magidor forcing [11] or Radin forcing [13]) then the technical condition 2 in the statement of Lemma 2.1 holds; in fact C will be eventually contained in every $> \omega$-club subset of κ from the ground model. This will also hold for the modified Radin forcing $\mathbb{P}_a$ which we define in the proof of Theorem 1.*

It is also easy to see that condition 2 holds if $W \models \mathrm{cf}(\kappa) = \lambda \geq \aleph_2$. We do not know whether it holds always when $W \models \mathrm{cf}(\kappa) = \lambda = \aleph_1$ or $W \models \mathrm{cf}(\kappa) = \lambda = \omega$.

Before giving the proof of this lemma we discuss some ideas that are used in the proof, define $\square_\kappa^B$ and show that this principle is consistent with κ being a large cardinal.

2.1 Good matrices

We will use the idea of a *good matrix*, which comes from the proof of Fact 1.8. In the interest of making this paper self-contained we have reproduced some results from [6].

Definition 2.3 *Let κ be inaccessible and let $S = \{ \alpha < \kappa^+ \mid \mathrm{cf}(\alpha) < \kappa \}$.*

A good matrix *is an array of sets*

$$\langle C_{\alpha i} : \alpha \in S, i \in X_\alpha \rangle$$

such that

1. *$C_{\alpha i}$ is club in α.*

2. *X_α contains a $> \omega$-club subset of κ.*

3. *o.t.$(C_{\alpha i}) < \kappa$.*

4. *If $i \in X_\alpha$ and $\beta \in \lim C_{\alpha i}$ then $i \in X_\beta$ and $C_{\alpha i} \cap \beta = C_{\beta i}$.*

5. *If $i, j \in X_\alpha$ and $i < j$ then $C_{\alpha i} \subseteq C_{\alpha j}$.*

6. *If $\alpha, \beta \in S$ with $\beta < \alpha$ then $\beta \in \lim C_{\alpha i}$ for some $i \in X_\alpha$ (and thus for all large $i \in X_\alpha$ by the preceding clause).*

Fact 2.4 (Cummings and Schimmerling [6]) *Let the cardinal κ be inaccessible. Then there exists a good matrix for κ.*

Remark 2.5 *When building a good matrix the hardest stages are those of the form $\beta + \omega$ where $\mathrm{cf}(\beta) = \kappa$, which are treated in Case 5 of the inductive definition below. To prepare for these stages, we need to make sure in Case 4 that if $\omega < \mathrm{cf}(\alpha) < \kappa$ then X_α is as large as possible.*

Accordingly when $\omega < \mathrm{cf}(\alpha) < \kappa$ we choose X_α as the "maximally fat" set of indices i on which it is possible to define $C_{\alpha i}$. X_α will be in the $> \omega$-club filter on κ but is not necessarily an actual $> \omega$-club subset of κ.

Proof:[Fact 2.4]

We construct a good matrix by induction on $\alpha \in S$.

Case 1: $\alpha = \omega$. We set $X_\omega = \kappa$ and $C_{\omega i} = \omega$ for all i.

Case 2: $\alpha = \beta + \omega$ for limit β with $\mathrm{cf}(\beta) < \kappa$ (that is to say $\beta \in S$). We set $X_\alpha = X_\beta$ and $C_{\alpha i} = C_{\beta i} \cup [\beta, \alpha)$ for all $i \in X_\alpha$.

Clearly $C_{\alpha i}$ is club in α. $X_\alpha = X_\beta$ and so X_α is in the $> \omega$-club filter, and $\mathrm{o.t.}(C_{\alpha i}) = \mathrm{o.t.}(C_{\beta i}) + \omega < \kappa$. If $i \in X_\alpha$ and $\gamma \in \lim C_{\alpha i}$ then either $\gamma \in \lim C_{\beta i}$ or $\gamma = \beta$. In the former case we have by induction that $i \in X_\gamma$ and $C_{\gamma i} = C_{\beta i} \cap \gamma$, in the latter that $i \in X_\beta = X_\gamma$ and $C_{\gamma i} = C_{\beta i}$: in either case $C_{\alpha i} \cap \gamma = C_{\gamma i}$.

If $i, j \in X_\alpha$ with $i < j$ then by induction $C_{\beta i} \subseteq C_{\beta j}$, so that $C_{\alpha i} \subseteq C_{\alpha j}$. Finally if $\gamma \in S \cap \alpha$ then either $\gamma \in S \cap \beta$ or $\gamma = \beta$: if $\gamma \in S \cap \beta$ then by induction $\gamma \in \lim C_{\beta i}$ for some i and then $\gamma \in \lim C_{\alpha i}$ for the same i, while if $\gamma = \beta$ then $\gamma \in \lim C_{\alpha i}$ for every $i \in X_\alpha$.

Case 3: $\mathrm{cf}(\alpha) = \omega$ and α is a limit of limit ordinals. We choose $\langle \alpha_m : m < \omega \rangle$ an increasing sequence of ordinals in S which is cofinal in α. We set

$$X_\alpha = \{ \ i < \kappa \mid \forall m < \omega \ \ i \in X_{\alpha_m} \text{ and } \forall m < n < \omega \ \ \alpha_m \in \lim C_{\alpha_n i} \ \}.$$

X_α is in the $> \omega$-club filter because it is a final segment of $\bigcap_j X_{\alpha_j}$.

We observe that if $i \in X_\alpha$ then $C_{\alpha_m i} = C_{\alpha_n i} \cap \alpha_j$ for all $m < n < \omega$. We now set $C_{\alpha i} = \bigcup_m C_{\alpha_m i}$ for all $i \in X_\alpha$.

$C_{\alpha i}$ is club in α because every initial segment is an initial segment of $C_{\alpha_m i}$ for some m. A similar argument shows that o.t.$(C_{\alpha i}) < \kappa$. If $\beta \in \lim C_{\alpha i}$ then $\beta \in \lim C_{\alpha_m i}$ for some m, and by induction $i \in X_\beta$ and $C_{\beta i} = C_{\alpha_m i} \cap \beta = C_{\alpha i} \cap \beta$.

If $i, j \in X_\alpha$ with $i < j$ then by induction $C_{\alpha_m i} \subseteq C_{\alpha_m j}$ for all $m < \omega$, so that $C_{\alpha i} \subseteq C_{\alpha j}$. Finally if $\beta \in S \cap \alpha$ then $\beta \in S \cap \alpha_m$ for some m, and so by induction $\beta \in \lim C_{\alpha_m i}$ for all large $i \in X_{\alpha_m}$; it follows that $\beta \in \lim C_{\alpha i}$ for any large enough $i \in X_\alpha$.

Case 4: $\omega < \mathrm{cf}(\alpha) < \kappa$. Let $\mathrm{cf}(\alpha) = \rho$ say. As in Case 3 we fix $\langle \alpha_m : m < \rho \rangle$ an increasing and continuous sequence of members of S which is cofinal in α. We define

$$Y_\alpha = \{ \ i < \kappa \mid \forall m < \rho \ \ i \in X_{\alpha_m} \text{ and } \forall m < n < \rho \ \ \alpha_m \in \lim C_{\alpha_n i} \ \}.$$

Note that the precise nature of Y_α depends on the sequence $\langle \alpha_m : m < \rho \rangle$ used in its definition. Exactly as in Case 3 Y_α is in the $> \omega$-club filter, and if $i \in Y_\alpha$ then $C_{\alpha_m i} = C_{\alpha_n i} \cap \alpha_m$ for all $m < n < \rho$.

We now let

$$X_\alpha = \{\ i < \kappa \mid \exists E \text{ club in } \alpha\ \forall \gamma \in \lim(E)\ (i \in X_\gamma \text{ and } E \cap \gamma = C_{\gamma i})\ \}.$$

If $i \in Y_\alpha$ and we let $E = \bigcup_m C_{\alpha_m i}$ then it is easy to check that E witnesses $i \in X_\alpha$, so that $Y_\alpha \subseteq X_\alpha$. We observe that

- X_α is independent of the choice of the sequence $\langle \alpha_m : m < \rho \rangle$.
- By its definition and clause 4 in the definition of a good matrix, X_α is the "maximally fat" set of indices i for which we can hope to define $C_{\alpha i}$.

Suppose that $i \in X_\alpha$ and E, E' are both clubs in α witnessing this. Then $E \cap E'$ is club in α and

$$E = \bigcup_{\gamma \in \lim(E \cap E')} C_{\gamma i} = E'.$$

For each $i \in X_\alpha$, we now define $C_{\alpha i}$ to be the unique E which is club in α and is such that $\forall \gamma \in \lim(E)\ E \cap \gamma = C_{\gamma i}$. Notice that if $i \in Y_\alpha$ then automatically $C_{\alpha i} = \bigcup_m C_{\alpha_m i}$.

Since every initial segment of $C_{\alpha i}$ is an initial segment of $C_{\gamma i}$ for some $\gamma < \alpha$, o.t.$(C_{\alpha i}) < \kappa$. If $\beta \in \lim C_{\alpha i}$ then $\beta \in \lim C_{\gamma i}$ for some $\gamma \in \lim C_{\alpha i}$, and we have by induction that $i \in X_\beta$ and $C_{\beta i} = C_{\gamma i} \cap \beta = C_{\alpha i} \cap \beta$.

Let $i, j \in X_\alpha$ with $i < j$. Let $C_{\alpha i} = E$ and $C_{\alpha j} = F$. Then

$$E = \bigcup_{\gamma \in \lim(E \cap F)} C_{\gamma i} \subseteq \bigcup_{\gamma \in \lim(E \cap F)} C_{\gamma j} = F,$$

that is to say that $C_{\alpha i} \subseteq C_{\alpha j}$. Finally we may argue as in Case 3 that $S \cap \alpha \subseteq \bigcup_{i \in Y_\alpha} \lim C_{\alpha i}$, which suffices since $Y_\alpha \subseteq X_\alpha$.

Case 5: $\alpha = \beta + \omega$ where $\mathrm{cf}(\beta) = \kappa$. We fix $\langle \beta_i : i < \kappa \rangle$ an increasing and continuous sequence of members of S which is cofinal in β. Let

$$Z = \{\ i < \kappa \mid \forall j < i \ \ i \in X_{\beta j} \text{ and } \forall j < k < i \ \ \beta_j \in \lim C_{\beta_k i}\ \}.$$

We claim that Z is in the $> \omega$-club filter. To see this first observe that if $D = \{\ i < \kappa \mid \forall j < i \ \ i \in X_{\beta_j}\ \}$ then D is a diagonal intersection of sets in the $> \omega$-club filter, and therefore is in the $> \omega$-club filter. Define $F : [\kappa]^2 \longrightarrow \kappa$ by setting $F(j, k)$ equal to the least $i \in X_{\beta_k}$ with $\beta_j \in \lim C_{\beta_k i}$, and let C be the club set of $i < \kappa$ such that $F``[i]^2 \subseteq i$. If $i \in D \cap C$ then

1. Since $i \in D$, $\forall j < i \ \ i \in X_{\beta_j}$.

2. If $j, k < i$ then since $i \in C$ we have $F(j, k) < i$, and by definition $F(j, k) \in X_{\beta_k}$ and $\beta_j \in \lim C_{\beta_k F(j,k)}$. Since $i \in D$ we also have $i \in X_{\beta_k}$, and so by the properties of a good matrix $C_{\beta_k F(j,k)} \subseteq C_{\beta_k i}$ and so $\beta_j \in \lim C_{\beta_k i}$.

It follows that $D \cap C \subseteq Z$, and $D \cap C$ is easily seen to be in the $> \omega$-club filter.

We let $X_\alpha = \{ \zeta \in D \cap C \mid \mathrm{cf}(\zeta) > \omega \}$. Let $i \in X_\alpha$ and consider the construction at level β_i; since $\mathrm{cf}(i) > \omega$ and the sequence $\langle \beta_n : n < \kappa \rangle$ is continuous, $\mathrm{cf}(\beta_i) = \mathrm{cf}(i) > \omega$ and the relevant clause of the definition is Case 4.

If we let $E = \bigcup_{j<i} C_{\beta_j i}$ then the fact that $i \in Z$ and the coherence properties of the good matrix imply that $\forall \gamma \in \lim(E)\ E \cap \gamma = C_{\gamma i}$, so that by the definition of X_{β_i} and $C_{\beta_i i}$ from Case 4 $i \in X_{\beta_i}$ and $C_{\beta_i i} = \bigcup_{j<i} C_{\beta_j i}$. Note that if $i \in Z$, the sequence $B = \langle \beta_j : j < i \rangle$ is very likely different from the sequence $A = \langle \alpha_m : m < \mathrm{cf}(i) \rangle$ used in the definition of Y_{β_i}. Even though A and B agree on a club of order type $\mathrm{cf}(i)$, this isn't necessarily enough to allow us to infer that $i \in Y_\alpha$ for every $\alpha \in A$, something that would be critical in allowing us to infer that $i \in Y_{\beta_i}$. The "maximal fatness" of X_{β_i}, however, ensures this isn't a problem and that $i \in X_{\beta_i}$.

We define

$$C_{\alpha i} = C_{\beta_i i} \cup \{\beta_i\} \cup [\beta, \alpha).$$

Clearly $C_{\alpha i}$ is club in α, and $\mathrm{o.t.}(C_{\alpha i}) = \mathrm{o.t.}(C_{\beta_i i}) + \omega < \kappa$. If $\gamma \in \lim C_{\alpha i}$ then either $\gamma \in \lim C_{\beta_i i}$ or $\gamma = \beta_i$, and in either case it is easy to see that $i \in X_\gamma$ and $C_{\gamma i} = C_{\beta_i i} \cap \gamma = C_{\alpha i} \cap \gamma$.

Let $i, j \in X_\alpha$ with $i < j$. By induction

$$C_{\beta_i i} = \bigcup_{k<i} C_{\beta_k i} \subseteq \bigcup_{k<i} C_{\beta_k j} \subseteq \bigcup_{k<j} C_{\beta_k j} = C_{\beta_j j}.$$

Since $C_{\beta_j j}$ is club in β_j and $C_{\beta_i i}$ is cofinal in β_i, it follows that $\beta_i \in C_{\beta_j j}$. Therefore by definition $C_{\alpha i} \subseteq C_{\alpha j}$.

Finally let $\gamma \in S \cap \alpha$, and observe that since $\beta \notin S$ we have $S \cap \alpha = S \cap \beta$. Find i such that $\gamma < \beta_i$, and then $j \in X_\alpha$ such that $i < j$ and $\gamma \in \lim C_{\beta_i j}$. Since $C_{\beta_j j} = \bigcup_{k<j} C_{\beta_k j}$, $\gamma \in \lim C_{\beta_j j}$.

This concludes the proof.

♦

2.2 The principle $\square^B_\kappa$

The following version of $\square_\kappa$ was studied by Baumgartner. Its main interest for us is that unlike the original $\square_\kappa$ principle it is consistent for κ supercompact.

Definition 2.6 *Let κ be regular. A $\square^B_\kappa$-sequence is a sequence $\langle C_\alpha : \alpha \in T \rangle$ where*

1. *T is a set of limit ordinals less than κ^+.*

2. *$\{ \alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \kappa \} \subseteq T$.*

3. *For all $\alpha \in T$, C_α is a club subset of α with o.t.$(C_\alpha) \leq \kappa$.*

4. *If $\alpha \in T$ and $\beta \in \lim(C_\alpha)$ then $\beta \in T$ and $C_\beta = C_\alpha \cap \beta$.*

As usual we say "$\square^B_\kappa$ holds" if there is a $\square^B_\kappa$-sequence.

Fact 2.7 (Baumgartner [3]) *Let κ be regular. Then there exists a forcing poset $\mathbb{P} = \mathbb{P}(\kappa)$ such that*

1. *$\mathbb{P}$ is κ-directed closed.*

2. *$\mathbb{P}$ is strategically closed for the game of length $\kappa + 1$.*

3. $\Vdash_{\mathbb{P}}$ *"$\square^B_\kappa$ holds".*

Proof: We force with the set of initial segments of successor length of such a sequence. More formally let $\mathbb{P}$ be the set of sequences $\langle C_\alpha : \alpha \in s \rangle$ where

1. s is a bounded set of limit ordinals less than κ^+, with a maximal element γ.

2. $\{ \alpha < \gamma \mid \mathrm{cf}(\alpha) = \kappa \} \subseteq s$.

3. For all $\alpha \in s$, C_α is a club subset of α with $\mathrm{o.t.}(C_\alpha) \leq \kappa$.

4. If $\alpha \in s$ and $\beta \in \lim(C_\alpha)$ then $\beta \in s$ and $C_\beta = C_\alpha \cap \beta$.

$\mathbb{P}$ is ordered as follows: if $p = \langle C_\alpha : \alpha \in s \rangle$ and $q = \langle D_\alpha : \alpha \in t \rangle$ then $p \leq q$ iff $t = s \cap (\max(t) + 1)$ and $C_\alpha = D_\alpha$ for all $\alpha \in t$.

Now we take the claims made in the theorem one by one. We adopt the convention "a sequence is a function is a set of ordered pairs", so that $\langle C_\alpha : \alpha \in s \rangle = \{ (\alpha, C_\alpha) \mid \alpha \in s \}$.

1. Since the partial ordering on $\mathbb{P}$ is treelike it suffices to show that $\mathbb{P}$ is κ-closed. Let $\langle p_i : i < \zeta \rangle$ be a decreasing ζ-sequence of conditions for some $\zeta < \kappa$, and let $s_i = \mathrm{dom}(p_i)$ and $\gamma_i = \max(s_i)$. Let $\beta^* = \sup_i \gamma_i$, $\beta = \beta^* + \omega$, $c = [\beta^*, \beta)$. Define $p = \bigcup_i p_i \cup \{(\beta, c)\}$. It is routine to check that p is a condition, the key point being that since $\mathrm{cf}(\beta^*) = \mathrm{cf}(\zeta) < \kappa$ we are under no obligation to put β^* into the domain of p.

2. We describe a winning strategy for player II in the game of length $\kappa+1$ played on $\mathbb{P}$. Let the move made at stage i be $p_i = \langle C_\alpha : \alpha \in s_i \rangle$, with $\max(s_i) = \gamma_i$. Player II will play to guarantee that if i, j are even with $i < j$ then $\gamma_i \in \lim(C_{\gamma_j})$ and $C_{\gamma_i} = C_{\gamma_j} \cap i$. Suppose we have reached an even stage α and it is the turn of Player II.

   Case I. α is a successor ordinal, say $\alpha = \beta + 2$. We set $\gamma_\alpha = \gamma_{\beta+1} + \omega$, $C_{\gamma_\alpha} = C_{\gamma_\beta} \cup \{\gamma_\beta\} \cup [\gamma_{\beta+1}, \gamma_\alpha)$, $s_\alpha = s_{\beta+1} \cup \{\gamma_\alpha\}$, and finally $p_\alpha = p_{\beta+1} \cup \{(\gamma_\alpha, C_{\gamma_\alpha})\}$.

   Case II. α is a limit ordinal with $\alpha \leq \kappa$. We set $\gamma_\alpha = \sup_{j<\alpha} \gamma_j$, $C_{\gamma_\alpha} = \{ \gamma_j \mid j < \alpha \}$, and finally $p_\alpha = \bigcup_{\gamma<\alpha} p_\gamma \bigcup \{(\gamma_\alpha, C_{\gamma_\alpha})\}$.

   It is routine to check that this is a successful strategy for player II. The key points are that II always plays a set of order type less than or equal to κ, and that if α is limit and $\beta \in \lim(C_{\gamma_\alpha})$ then $\beta = \gamma_\delta$ for limit $\delta < \alpha$ and so

$$C_{\gamma_\delta} = \{ \gamma_j \mid j < \delta \} = C_{\gamma_\alpha} \cap \gamma_\delta.$$

3. It is easy to see that for all $p \in \mathbb{P}$ and $\beta < \kappa^+$ there exists $q \leq p$ such that $\max(\mathrm{dom}(q)) > \beta$. The proof is a routine induction on β, using the same idea as in the proof of strategic closure to get through stages of cofinality κ. By strategic closure, $\mathbb{P}$ adds no κ-sequence of ordinals so preserves all cardinals and cofinalities up to κ^+. It follows that if G is $\mathbb{P}$-generic then $\bigcup G$ is a $\square^B_\kappa$-sequence in $V[G]$.

$\blacklozenge$

We will need to know later that $\Box^B_\kappa$ is consistent with κ being a large cardinal.

We will use the following basic facts, which originated in Silver's work on Reverse Easton forcing.

Fact 2.8 *Let M and N be models of ZFC, let $k : M \longrightarrow N$ be an elementary embedding. Let $\mathbb{P} \in M$ be a forcing poset and let G and H be such that*

1. *G is $\mathbb{P}$-generic over M.*
2. *H is $k(\mathbb{P})$-generic over N.*
3. *$\forall p \in G \; k(p) \in H$.*

Then there is a unique elementary embedding $k^+ : M[G] \longrightarrow N[H]$ such that k^+ extends k and $k^+(G) = H$.

Fact 2.9 *Let M and N be models of ZFC with $M \subseteq N$. Let λ be an N-cardinal. Let $\mathbb{P} \in M$ be a forcing poset such that*

1. *$N \models$ "$\mathbb{P}$ is λ-closed".*
2. *$\{ A \in M \mid A$ is a maximal antichain in $\mathbb{P} \}$ has cardinality λ in N.*

Then for every $p \in \mathbb{P}$ there exists $H \in N$ such that $p \in H$ and H is $\mathbb{P}$-generic over M.

Theorem 2 *Let κ be κ^{+n}-supercompact for some n with $0 < n < \omega$, and let GCH hold. Let $\mathbb{P}_{\kappa+1}$ be the Reverse Easton iteration of length $\kappa + 1$ where at every inaccessible $\alpha \leq \kappa$ we force with $\mathbb{P}(\alpha)_{V^{\mathbb{P}_\alpha}}$ (where $\mathbb{P}(\alpha)$ is the forcing to add $\square^B_\alpha$ defined in Fact 2.7).*

Then in $V^{\mathbb{P}_{\kappa+1}}$

1. *V-cardinals greater than or equal to κ are preserved.*

2. *GCH holds.*

3. *κ is κ^{+n}-supercompact.*

Proof: $\mathbb{P}_\kappa$ is a κ-c.c. forcing of size κ, so forcing with $\mathbb{P}_\kappa$ preserves cardinals greater than or equal to κ and GCH holds at and above κ in $V^{\mathbb{P}_\kappa}$.

We saw in the proof of Fact 2.7 that $\mathbb{P}(\kappa)$ adds no κ-sequences of ordinals, so it preserves all cardinals and cofinalities up to κ^+. $2^\kappa = \kappa^+$ in $V^{\mathbb{P}_\kappa}$ so $\mathbb{P}(\kappa)$ has cardinality κ^+, hence it is κ^{++}-c.c. and preserves all cardinals and cofinalities above κ^+. Standard arguments show that GCH still holds at and above κ in $V^{\mathbb{P}_{\kappa+1}}$, and a routine induction using the above analysis shows that GCH holds below κ in $V^{\mathbb{P}_{\kappa+1}}$.

To show the preservation of supercompactness, suppose that G is $\mathbb{P}_\kappa$-generic over V and g is $\mathbb{P}(\kappa)$-generic over $V[G]$. Fix $j : V \longrightarrow M$ which is the ultrapower map defined from some supercompactness measure on $\mathcal{P}_\kappa(\kappa^{+n})$. Notice that $\kappa^{+n+1} = \kappa^{+n+1}_M < j(\kappa)$. In particular if we let η be the least M-inaccessible greater than κ then $\eta > \kappa^{+n+1}$. $|\mathcal{P}_\kappa \kappa^{+n}| = \kappa^{+n}$, and it follows using GCH that $j(\kappa^{++}) < \kappa^{+n+2}$.

$j(\mathbb{P}_{\kappa+1})$ is an iterated forcing in M, and the resemblance between V and M implies that $\mathbb{P}_{\kappa+1}$ is an initial segment of $j(\mathbb{P}_{\kappa+1})$. By an easy chain condition argument $V[G][g] \models {}^{\kappa^{+n}}M[G][g] \subseteq M[G][g]$.

Let $\mathbb{R}$ be the forcing $j(\mathbb{P}_\kappa)/G * g$. Then in $M[G][g]$ the forcing $\mathbb{R}$ is η-closed and has $j(\kappa)$ maximal antichains. $|j(\kappa)| = \kappa^{+n+1}$, so working in $V[G][g]$ we may build a descending κ^{+n+1}-chain of conditions in $\mathbb{R}$ to decide each antichain in $M[G][g]$. Thus we may build $H \in V[G][g]$ which is $\mathbb{R}$-generic over $M[G][g]$.

$M[G][g][H] \subseteq V[G][g]$, and $V[G][g] \models {}^{\kappa^{+n}}M[G][g][H] \subseteq M[G][g][H]$. Since $j``G \subseteq G * g * H$, working in $V[G][g]$ we may lift j to get a new embedding (which we also denote by j, without risk of confusion) $j : V[G] \longrightarrow M[G][g][H]$.

Now we use Silver's "master condition" argument. Consider $p =_{\mathrm{def}} \bigcup j``g$. $p \in M[G][g][H]$ and p is a sequence $\langle D_\beta : \beta \in A\rangle$, where A is a bounded subset of $j(\kappa^+)$ because

$$M[G][g][H] \models \mathrm{cf}(\sup(j``\kappa^+)) = \kappa^+ < j(\kappa^+).$$

The only way in which p falls short of being a condition in $\mathbb{P}(j(\kappa))$ is that A does not have a largest element. Since $\sup(A) = \sup(j``\kappa^+)$ and $M \models \mathrm{cf}(\sup(j``\kappa^+)) = \kappa^+ < j(\kappa)$, we are not obliged to define a club at $\sup(A)$. Accordingly we let $\gamma = \sup(A) + \omega$, let $D_\gamma = \{ \sup(A) + n \mid n < \omega \}$ and $A^* = A \cup \{\gamma\}$. Now $p^* =_{\mathrm{def}} \langle D_\beta : \beta \in A^*\rangle$ is a condition and $\forall q \in g\ p^* \leq j(q)$.

In $M[G][g][H]$, the forcing $\mathbb{P}(j(\kappa))$ is κ^{+n+1}-closed and has $j(\kappa^{++})$ antichains. Since $|j(\kappa^{++})| = \kappa^{+n+1}$ and

$$V[G][g] \models {}^{\kappa^{+n}}M[G][g][H] \subseteq M[G][g][H],$$

we may build a chain of conditions in $\mathbb{P}(j(\kappa))$ to decide each antichain in $M[G][g][H]$, and so build $h \in V[G][g]$ which is $\mathbb{P}(j(\kappa))$-generic over the model $M[G][g][H]$. Taking the first element of that chain to be p^* we may assume that $p^* \in h$ and hence that $j``g \subseteq h$.

So we may lift j again to get $j : V[G][g] \longrightarrow M[G][g][H][h]$. This map is defined in $V[G][g]$ and witnesses that κ is κ^{+n}-supercompact in $V[G][g]$.

♦

2.3 Proof of Lemma 2.1

We are now ready to prove Lemma 2.1.

Proof:[Lemma 2.1] As before, let

$$S = \{ \alpha < \kappa^+ \mid \mathrm{cf}(\alpha) < \kappa \}.$$

Let

$$\langle C_{\alpha i} : \alpha \in S, i \in X_\alpha \rangle$$

be a good matrix and let $\langle D_\alpha : \alpha \in T \rangle$ witness the truth of $\square^B_\kappa$. We will define a $\square_{\kappa,\lambda}$-sequence $\langle \mathcal{E}_\alpha : \alpha < \kappa^+ \rangle$ in W. Let $E \in W$ be a club in κ which has order type λ and meets every $> \omega$-club from V, and let $E = \langle \kappa_i : i < \lambda \rangle$ with κ_i increasing.

Case 1: $V \models \mathrm{cf}(\alpha) < \kappa$, $\alpha \notin T$.

Let $\mathcal{E}_\alpha = \{ C_{\alpha\kappa_j} \mid \kappa_j \in X_\alpha \}$.

Case 2: $V \models \mathrm{cf}(\alpha) < \kappa$, $\alpha \in T$.

Let $\mathcal{E}_\alpha = \{ C_{\alpha\kappa_j} \mid \kappa_j \in X_\alpha \} \cup \{D_\alpha\}$.

Case 3: $V \models \mathrm{cf}(\alpha) = \kappa$ (which implies that $\alpha \in T$).

Let $\mathcal{E}_\alpha = \{D_\alpha\}$.

We verify that this is a $\Box_{\kappa,\lambda}$-sequence. Clearly each $\mathcal{E}_\alpha$ is a family of closed unbounded subsets of α which have order type at most κ, and the cardinality of $\mathcal{E}_\alpha$ is at most λ. Each $\mathcal{E}_\alpha$ is non-empty by our assumption on the club E.

Let $C \in \mathcal{E}_\alpha$ and let $\beta \in \lim C$. There are two possibilities.

1. $V \models \mathrm{cf}(\alpha) < \kappa$ and $C = C_{\alpha\kappa_j}$ for some $j < \lambda$ with $\kappa_j \in X_\alpha$. By the coherence properties of a good matrix, we have $\kappa_j \in X_\beta$ and $C \cap \beta = C_{\beta\kappa_j}$. The definition of $\mathcal{E}_\beta$ now tells us that $C \cap \beta \in \mathcal{E}_\beta$.

2. $C = D_\alpha$ for some $\alpha \in T$. By the coherence properties of a $\Box^B_\kappa$-sequence, $\beta \in T$ and $C \cap \beta = D_\beta$. Again the definition of $\mathcal{E}_\beta$ tells us that $C \cap \beta \in \mathcal{E}_\beta$.

♦

Remark 2.10 *If $\lambda = \omega$ (that is to say we are in the case of Fact 1.8) then we do not need the assumption that $\Box^B_\kappa$ holds in V. In this case we just set*

$$\mathcal{E}_\alpha = \{ C_{\alpha\kappa_j} \mid \kappa_j \in X_\alpha \}$$

for all α with $V \models \mathrm{cf}(\alpha) < \kappa$, and set $\mathcal{E}_\alpha = \{X_\alpha\}$ for some X_α cofinal in κ with order type ω when $V \models \mathrm{cf}(\alpha) = \kappa$.

3 Radin forcing

We assume in this section that GCH holds and that κ is a κ^{+5}-supercompact cardinal. We describe a partial ordering $\mathbb{P}$ which will have the properties that

1. κ is κ^{+5}-supercompact in $V^{\mathbb{P}}$.
2. If $\mu < \kappa$ is any limit cardinal in $V^{\mathbb{P}}$ then $V \models$ "μ is μ^+-supercompact".

The forcing $\mathbb{P}$ will be obtained by a minor modification in the construction of Foreman and Woodin's model where GCH fails everywhere [7].

In [7] Foreman and Woodin are working with a cardinal κ which is $\beth_\omega(\kappa)$-supercompact and is such that $\beth_n(\kappa)$ is weakly inaccessible for each n. They show (see Section 7 of their paper) that it is possible to construct some forcing $\mathbb{P}$ such that

1. $\mathbb{P}$ preserves cardinals.
2. κ is $\beth_3(\kappa)$-supercompact in $V^{\mathbb{P}}$.
3. The forcing $\mathbb{P}$ adds a club set $C \subseteq \kappa$ such that if C is enumerated in increasing order as $\langle \kappa_i : i < \kappa \rangle$ then
   (a) Each κ_i is a large cardinal in V.

(b) The forcing adds an $\mathrm{Add}(\beth_4(\kappa_i), \kappa_{i+1})$-generic object to V for each $i < \kappa$.

Truncating $V^{\mathbb{P}}$ at κ then gives a model in which GCH fails everywhere.

We will build $\mathbb{P}$ in a very similar way so that

1. κ is κ^{+5}-supercompact in $V^{\mathbb{P}}$.

2. The forcing $\mathbb{P}$ adds a club set $C \subseteq \kappa$ such that if C is enumerated in increasing order as $\langle \kappa_i : i < \kappa \rangle$ then

   (a) $\kappa_0 = \aleph_0$, κ_i is a large cardinal in V for $i > 0$.

   (b) The forcing adds a $\mathrm{Coll}(\kappa_i^{+6}, < \kappa_{i+1})$-generic object to V for each $i < \kappa$.

   (c) Cardinals not lying in the intervals $(\kappa_i^{+6}, \kappa_{i+1})$ are preserved.

We will assume that the reader has a copy of [7] to hand. Rather than reproducing the excellent exposition in that paper with minor changes, we will mostly just give references to that paper and leave the reader to fill in the details. We attempt to use notation which parallels that of [7]. In particular we define

$$\mathcal{P}_\gamma \delta = \{ X \subseteq \delta \mid \text{o.t.}(X) < \gamma, X \cap \gamma \in \gamma \}.$$

Given $x, y \in \mathcal{P}_\gamma \delta$ we define a relation "$x \prec y$" by

$$x \prec y \iff (x \subseteq y \wedge \text{o.t.}(x) < y \cap \gamma).$$

This definition really depends on the values of γ and δ but these should always be clear from the context.

We begin by setting $\sigma = \kappa^{+5}$, $\rho = \kappa^{+4}$, $\lambda^* = \kappa^{+3}$. For the rest of this paper we fix j such that

1. $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \sigma$, ${}^{\sigma}M \subseteq M$.

2. j arises as the ultrapower map associated with some normal fine ultrafilter U on the set $\mathcal{P}_\kappa \sigma$.

We will define a sequence $\vec{M} = \langle M_\alpha : \alpha < \lambda^* \rangle$ such that $M_0 = j``\lambda^*$ and for $\alpha > 0$ M_α is a measure on

$$Z = \mathcal{P}_\kappa \lambda^* \times V_\kappa \times V_\kappa.$$

The definition of $\vec{M}$ will be recursive, and will go as follows: for $0 < \alpha < \lambda^*$ we will define some function g_α such that $\mathrm{dom}(g_\alpha) = A_\alpha \in M_\alpha$ and $g_\alpha : A_\alpha \to V_\kappa$, and will then set

$$M_\alpha = \{ \, X \subseteq Z \mid (M(0), \langle M_\beta : 0 < \beta < \alpha \rangle, \langle g_\beta : 0 < \beta < \alpha \rangle) \in j(X) \, \}.$$

It is routine to check that $(M(0), \langle M_\beta : 0 < \beta < \alpha \rangle, \langle g_\beta : 0 < \beta < \alpha \rangle) \in j(Z)$, so M_α is a κ-complete measure on Z for all $\alpha < \lambda^*$. The argument of Lemma 3.1 from [7] shows that $\mathrm{Ult}(V, M_\alpha)$ is closed under λ^*-sequences.

We will need to know that M_α concentrates on triples whose second entries are constructed in a manner similar to that in which $\vec{M}$ is constructed; we prove this by a reflection argument. We define $j_0 : V \longrightarrow M_0^*$ to be the ultrapower map arising from the ultrafilter

$$U_0 = \{ \, x \in \mathcal{P}_\kappa \rho \mid j``\rho \in j(x) \, \}.$$

Defining $F : \mathcal{P}_\kappa \sigma \longrightarrow \mathcal{P}_\kappa \rho$ by $F(x) = x \cap \rho$ it is clear that $X \in U_0 \iff F^{-1}[X] \in U$. It follows that F induces a map from $M_0^* = \mathrm{Ult}(V, U_0)$ to $M = \mathrm{Ult}(V, U)$ given by $k : [f]_{U_0} \longmapsto [f \circ F]_U$, and it is routine to check that k is an elementary embedding and that $k \circ j_0 = j$.

Since $\rho + 1 \subseteq \mathrm{rge}(k)$, $\mathrm{crit}(k) > \rho$. Since $\rho^+_{M_0^*} = \rho^+_M = \sigma$ but $\rho^{++}_{M_0^*} < \rho^{++}_M$ it follows that $\mathrm{crit}(k) = \rho^{++}_{M_0^*}$. We also know that $H_\sigma^{M_0^*} = H_\sigma^M = H_\sigma$, and can argue in a standard way that $k \restriction H_\sigma = \mathrm{id}$. If we now define a sequence $\vec{N}$ in the same way as $\vec{M}$ is defined, save that j is replaced by j_0, then a routine inductive proof shows that $\vec{N} = \vec{M}$; the key point is that for every $\alpha \leq \lambda^*$ the sequence $\vec{M} \restriction \alpha$ lies in M_0^* and is fixed by k.

By GCH we see that $U_0 \in M$. Let $i_0 : M \longrightarrow N = \mathrm{Ult}(M, U_0)$ be the ultrapower map. $i_0 \restriction H_\sigma = j_0 \restriction H_\sigma$ and so if $\vec{N}_0$ is defined like $\vec{M}$ using i_0 in place of j then $\vec{N}_0 = \vec{N} = \vec{M}$. It follows that M_α will concentrate on triples whose second entries are sequences of measures constructed in the same manner as $\vec{M}$.

We defer for the moment the question of exactly how the function g_α is defined; we will return to this point after some general discussion of the construction of $\mathbb{P}$. The first step in the construction of $\mathbb{P}$ will be to build some forcing $\mathbb{R}$ which adds to the universe sequences $\langle x_i : i < \kappa \rangle$ and $\langle G_i : i < \kappa \rangle$ such that

1. $x_i \in \mathcal{P}_\kappa \lambda^*$, $i < j \Longrightarrow x_i \prec x_j$.

2. The sequence $\vec{x}$ is continuous in the sense that if $\alpha < \kappa$ is a limit ordinal then $x_\alpha = \bigcup_{i<\alpha} x_i$.

3. If we let $\kappa_i = x_i \cap \kappa$ then G_i is $\mathrm{Coll}(\kappa_i^{+6}, < \kappa_{i+1})$-generic over V.

4. $\lambda^* = \bigcup_{i<\kappa} x_i$.

The forcing $\mathbb{R}$ does too much damage to the cardinal structure of V, in particular it collapses κ_i^+ for many i; we will accordingly define $\mathbb{P}$ as a projection of $\mathbb{R}$ which adds $\langle (\kappa_i, G_i) : i < \kappa \rangle$ but preserves all those cardinals not explicitly collapsed by the G_i.

The forcing $\mathbb{R}$ is best considered as a generalised version of Prikry forcing. A condition will contain some information about the g's and x's along with some constraint on the g's and x's which can be added.

By its definition, M_α ($\alpha > 0$) will concentrate on the set of triples (u, v, w) such that

1. $u \in \mathcal{P}_\kappa \lambda^*$.

2. Defining $\kappa_u = u \cap \kappa$ and $\lambda_u = \mathrm{o.t.}(u)$,

   (a) $\mathrm{dom}(v) = \mathrm{dom}(w) = [1, \beta)$ for some $\beta < \lambda_u$.

   (b) $v(\alpha)$ is a measure on $Z_u = \mathcal{P}_{\kappa_u} \lambda_u \times V_{\kappa_u} \times V_{\kappa_u}$.

   (c) $w(\alpha)$ is a function such that $\mathrm{dom}(w(\alpha)) \in v(\alpha)$ and $\mathrm{rge}(w(\alpha)) \subseteq V_{\kappa_u}$.

The building blocks of the forcing $\mathbb{R}$ will be triples of this general form. A condition will determine a finite sequence of triples $\langle (u_i, v_i, w_i) : i < n \rangle$ such that $u_0 \prec u_1 \prec \ldots \prec u_{n-1}$. If $\mathrm{dom}(v_i) = [1, \beta_i)$ then the condition will associate with the triple (u_i, v_i, w_i) a sequence $\langle A_i(j) : 1 \leq j < \beta_i \rangle$ such that $A_i(j) \in v_i(j)$ for $j \in [1, \beta_i)$. The idea is that the sets $A_i(j)$ will constrain the triples (u, v, w) that can be interpolated between $(u_{i-1}, v_{i-1}, w_{i-1})$ and (u_i, v_i, w_i) when the condition is extended.

Unfortunately the measure $v_i(j)$ concentrates on the set Z_{u_i}, while a candidate for interpolation is a triple (u, v, w) as above where $u \in \mathcal{P}_\kappa \lambda^*$ and $u \prec u_i$. While v and w are of the right form (because $\lambda_u < \kappa_{u_i}$), u is not literally a member of $\mathcal{P}_{\kappa_{u_i}} \lambda_{u_i}$. We define some functions that will be used to deal with this problem.

If $u, u' \in \mathcal{P}_\kappa \lambda^*$ and $u \prec u'$ then (since $u \subseteq u'$) there is a natural map $i_{uu'} : \lambda_u \to \lambda_{u'}$ induced by the inclusion map from u to u'. Notice that $i_{uu'} \restriction \kappa_u = \mathrm{id}$. Abusing notation, we also denote by $i_{uu'}$ the map from Z_u to $Z_{u'}$ given by

$$i_{uu'}(a, b, c) = (i_{uu'}(a), b, c).$$

In the situation of the last paragraph, the appropriate question to ask will be whether $i_{uu_i}(u, v, w) \in A_i(j)$ where $j = i_{uu_i}(\mathrm{lh}(v))$.

Similarly if $u \in \mathcal{P}_\kappa \lambda^*$ the inclusion map induces a natural map $i_u : \lambda_u \to \lambda^*$ with $i_u \restriction \kappa_u = \mathrm{id}$. As in the preceding paragraph we abuse notation slightly and define $i_u : Z_u \longrightarrow Z$ by $i_u(a, b, c) = (i_u(a), b, c)$.

It is clear from the definitions that if $u \prec v$ then $i_u = i_v \circ i_{uv}$, and similarly if $u \prec v \prec w$ then $i_{uw} = i_{vw} \circ i_{uv}$.

Following [7] we will henceforth identify the triple (u, v, w) with the pair $(u \frown v, w)$, and will also regard $u \frown v$ as lying in $\mathcal{P}_\kappa \lambda^*$. Following the notation of [7] we usually write such a pair as $(\vec{u}, \vec{w})$, where with the conventions we are now using

1. $\vec{u}$ is a sequence whose domain is some ordinal β; we write $\mathrm{lh}(\vec{u}) = \beta$ and sometimes with a mild abuse of notation $\mathrm{lh}((\vec{u}, \vec{w})) = \beta$. $\mathrm{dom}(\vec{w}) = [1, \beta)$.

2. u_0 is an element of $\mathcal{P}_\kappa \lambda^*$. We will define $\kappa_{\vec{u}} = u_0 \cap \kappa$, $\lambda_{\vec{u}} = \mathrm{o.t.}(u_0)$. We also let $\kappa_{(\vec{u},\vec{w})} = \kappa_{\vec{u}}$ and $\lambda_{(\vec{u},\vec{w})} = \lambda_{\vec{u}}$.

3. For $0 < \alpha < \beta$, u_α is a measure on $\mathcal{P}_{\kappa_{\vec{u}}} \lambda_{\vec{u}} \times V_{\kappa_{\vec{u}}}$, and w_α is a function with domain lying in $u(\alpha)$ and with range a subset of $V_{\kappa_{\vec{u}}}$.

Definition 3.1 *$(\vec{u}, \vec{w})$ is a* good pair *if conditions 1, 2, and 3 above hold. If $(\vec{u}, \vec{w})$ is a good pair and $u_0 \prec x$ then $i_{u_0 x}(\vec{u}, \vec{w}) = (\vec{u}^*, \vec{w})$ where $\vec{u}^*$ is the sequence obtained from $\vec{u}$ by replacing u_0 with $i_{u_0 x}(u_0)$.*

The sequence of lemmas and definitions which follows runs exactly parallel to the corresponding discussion in [7]. We leave the proofs to the reader and just make a few motivating remarks. We will state some results about the sequence $\vec{M}$ before defining the functions g_α, but this is not problematic because these results do not depend on the exact definition of the g's.

Lemma 3.2 *Fix a well-ordering* $\lhd$ *of* $H_{\lambda^{+6}}$. *There exists a sequence of mutually disjoint sets* $\langle A_\alpha : 0 < \alpha < \lambda^* \rangle$ *such that if* $(\vec{u}, \vec{h}) \in A_\beta$ *then*

1. $(\vec{u}, \vec{h})$ *is a good pair.*
2. $\mathrm{lh}(\vec{u}) < \lambda_{\vec{u}}$, $i_{\vec{u}}(\mathrm{lh}(\vec{u})) = \beta$.
3. *The structure* $\langle H_{sup(u_0)^{+5}}, \in, (\vec{u}, \vec{h}), \lhd \rangle$ *is elementarily equivalent to the structure* $\langle H^M_{sup(j``\lambda^*)^{+5}}, \in, (\vec{M} \restriction \beta, \vec{g} \restriction \beta), \lhd \rangle$.

The point of this lemma is that M_α concentrates on sequences which resemble $(\vec{M} \restriction \alpha, \vec{g} \restriction \alpha)$. In particular if $(\vec{u}, \vec{h})$ is in A_α then since $U_0 \in H^M_{sup(j``\lambda^*)^{+5}}$ it will follow that there is some elementary embedding i such that

1. i witnesses that $\kappa_{\vec{u}}$ is $\kappa_{\vec{u}}^{+4}$-supercompact.
2. $\vec{u}$ and $\vec{h}$ are constructed from i in the same way that $\vec{M}$ and $\vec{g}$ are constructed from j.

As in [7] we may define a class U_∞ of pairs $(\vec{u}, \vec{h})$ such that if $(\vec{u}, \vec{h}) \in U_\infty$ then

1. $(\vec{u}, \vec{h})$ has the properties from clauses 1, 2 and 3 of Lemma 3.2.
2. Each measure in $\vec{u}$ concentrates on a subset of U_∞.

Henceforth we assume that all pairs $(\vec{u}, \vec{h})$ which we consider are drawn from U_∞.

At this point we are finally ready to be more precise about the definition of the functions g_α involved in the definition of $\vec{M}$. At the same time we will define a map π, which will eventually be used to define the forcing $\mathbb{P}$ as a projection of the forcing $\mathbb{R}$.

g_α will be chosen so that $\mathrm{dom}(g_\alpha) \in M_\alpha$ and $g_\alpha(\vec{u}, \vec{h}) \in \mathrm{Coll}(\kappa_{\vec{u}}^{+6}, < \kappa)$ for all $(\vec{u}, \vec{h}) \in \mathrm{dom}(g_\alpha)$.

We define a map π whose domain is U_∞. $\pi(\vec{u}, \vec{f})$ will be defined by induction on $\kappa_{\vec{u}}$ to be $(\vec{v}, \vec{\mathcal{G}})$ where

1. $\mathrm{lh}(\vec{v}) = \mathrm{lh}(\vec{u})$, $\mathrm{dom}(\vec{\mathcal{G}}) = \mathrm{dom}(\vec{f})$.

2. $v_0 = \kappa_{\vec{u}}$.

3. For $0 < \alpha < \mathrm{lh}(\vec{v})$

   (a) v_α is the measure on V_{v_0} defined by

   $$v_\alpha = \{ X \subseteq V_{v_0} \mid \pi^{-1}[X] \in u(\alpha) \}.$$

   (b) $\mathcal{G}_\alpha$ is a certain filter on the Boolean algebra given by the ultraproduct $Q(\vec{v}, \alpha) = \prod_a \mathrm{RO}(\mathrm{Coll}(\kappa_a^{+6}, < \kappa))/v_\alpha$. $\mathcal{G}_\alpha$ is generated by elements of the form $[b(\vec{u}, \vec{f}, \vec{A}, \alpha)]_{v_\alpha}$, where we define

   $$b(\vec{u}, \vec{f}, \vec{A}, \alpha)(c) = \bigvee \{ f_\alpha(a) \mid a \in A_\alpha, \pi(a) = c \}$$

   for each $c \in V_{v_0}$ and each sequence $\vec{A}$ such that $\mathrm{lh}(\vec{A}) = \mathrm{lh}(\vec{v})$ and $A_\alpha \in v_\alpha$ for all α.

We define $U^\pi_\infty = \{\ \pi(a) \mid a \in U_\infty\ \}$.

Exactly as in [7] we may now choose the sequence of functions g_α in such a way that $\pi(\vec{M}, \vec{g}) = (\vec{w}, \vec{\mathcal{F}})$ with each of the $\mathcal{F}_\alpha$ an ultrafilter on the appropriate Boolean algebra. To be more precise we use the closure of $\mathrm{Ult}(V, M_\alpha)$ to choose the g_α in such a way that for every $b \in Q(\vec{w}, \alpha)$ there is $\vec{A}$ such that $[b(\vec{M}, \vec{g}, \vec{A}, \alpha)]_{w_\alpha}$ either refines or is incompatible with b.

A key point in the construction that follows is this: if $\vec{k}$ is a sequence of functions with $B_\alpha = \mathrm{dom}(k_\alpha) \in w_\alpha$, $B_\alpha \subseteq \mathrm{dom}(g_\alpha)$ and $k_\alpha(a) \leq g_\alpha(a)$ for all a then $[b(\vec{M}, \vec{k}, \vec{b}, \alpha)]_{w_\alpha} \in \mathcal{F}_\alpha$ for all α. This will be crucial in the proof that $\mathbb{P}$ collapses only those cardinals which it ought to.

We are now ready to define the forcing $\mathbb{R}$. In fact we will define for each $a \in U_\infty$ a forcing $\mathbb{R}_a$. Our final $\mathbb{R}$ will be $\mathbb{R}_a$ where a is a certain initial segment of $(\vec{M}, \vec{g})$.

Following [7] we say that $(\vec{u}, \vec{f}, \vec{A}, \vec{k}, s)$ is a *suitable quintuple* iff

EITHER

1. $(\vec{u}, \vec{f}) \in U_\infty$.

2. $\vec{A}$ is a sequence such that $A_\alpha \in u_\alpha$ for $0 < \alpha < \mathrm{lh}(\vec{u})$.

3. $\vec{k}$ is a sequence such that for $0 < \alpha < \mathrm{lh}(\vec{u})$

    (a) $\mathrm{dom}(k_\alpha) = A_\alpha$.

    (b) For all $a \in A_\alpha$, $k_\alpha(a) \in \mathrm{Coll}(\kappa_a^{+6}, < \kappa_{\vec{u}})$ and $k_\alpha(a) \leq f_\alpha(a)$.

4. $s \in \mathrm{Coll}(\kappa_{\vec{u}}^{+6} < \kappa)$.

OR

1. $\vec{u} = \langle \aleph_0 \rangle$, where by convention we set $\kappa_{\vec{u}} = \aleph_0$.

2. $\vec{f} = \vec{A} = \vec{k} = 0$.

3. $s \in \mathrm{Coll}(\aleph_6, < \kappa)$.

When $u_0 \prec x$ we let $i_{u_0 x}(\vec{u}, \vec{f}, \vec{A}, \vec{k}, s) = (\vec{u}^*, \vec{f}, \vec{A}, \vec{k}, s)$, where u^* is the result of replacing u_0 by $i_{u_0 x}$ in u.

Definition 3.3 *Let $a = (\vec{u}, \vec{f}) \in U_\infty$. A condition in $\mathbb{R}_a$ is a sequence*

$$\langle t_1, \ldots, t_n, (\vec{u}, \vec{f}, \vec{A}, \vec{k}) \rangle$$

such that

1. *Each t_i is a suitable quintuple $t_i = (\vec{u}_i, \vec{f}_i, \vec{A}_i, \vec{k}_i, s_i)$.*

2. *$t_1 = (\langle \aleph_0 \rangle, 0, 0, 0, s_1)$.*

3. *$(\vec{u}_i)_0 \prec (\vec{u}_{i+1})_0$ for $1 \leq i < n$ and $(\vec{u}_i)_0 \in \mathcal{P}_{\kappa_{\vec{u}}} \lambda_{\vec{u}}$ for $1 \leq i \leq n$.*

4. *$\vec{A}$ is a sequence of measure one sets for $\vec{u}$. $\mathrm{dom}(k_\delta) = A_\delta$ and $k_\delta(a) \in \mathrm{Coll}(\kappa_a^{+6}, < \kappa_{\vec{u}})$ for all $a \in A_\delta$.*

5. *$s_i \in \mathrm{Coll}(\kappa_{\vec{u}_i}^{+6}, < \kappa_{\vec{u}_{i+1}})$ for $1 \leq i < n$ and $s_n \in \mathrm{Coll}(\kappa_{\vec{u}_n}^{+6}, < \kappa_{\vec{u}})$.*

Intuitively this condition is intended to give a certain amount of information about a pair of sequences $\vec{x}$ and $\vec{G}$ where

1. $\vec{x}$ is a continuous and $\prec$-increasing chain of sets from $\mathcal{P}_{\kappa_{\vec{u}}}\lambda_{\vec{u}}$ with union $\lambda_{\vec{u}}$.

2. G_i is $\mathrm{Coll}(\kappa_{x_i}^{+6}, < \kappa_{x_{i+1}})$-generic.

The ordering on $\mathbb{R}_a$ will be defined accordingly in Definition 3.5.

Definition 3.4 *Let* $q^0 = (\vec{u}^0, \vec{f}^0, \vec{A}^0, \vec{k}^0, s^0)$ *and* $q^1 = (\vec{u}^1, \vec{f}^1, \vec{A}^1, \vec{k}^1, s^1)$ *be suitable quintuples.*

1. q^1 shrinks q^0 *iff*

   (a) $(\vec{u}^1, \vec{f}^1) = (\vec{u}^0, \vec{f}^0)$.

   (b) $A^1_\alpha \subseteq A^0_\alpha$ *for all* α.

   (c) $k^1_\alpha(a) \leq k^0_\alpha(a)$ *for all* $a \in A^1_\alpha$.

   (d) $s^1 \leq s^0$.

2. q^1 *is* addable *to* q^0 *iff*

   (a) $(\vec{u}^1)_0 \prec (\vec{u}^0)_0$ *(so we may define a map* $i = i_{(\vec{u}^1)_0(\vec{u}^0)_0}$*).*

   (b) $i(\mathrm{lh}(\vec{u}^1)) < \mathrm{lh}(\vec{u}_0)$.

   (c) $i(\vec{u}^1, \vec{f}^1) \in A^0_{i(\mathrm{lh}(\vec{u}^1))}$.

   (d) $i``A^1_\delta \subseteq A^0_{i(\delta)}$ *for* $0 < \delta < \mathrm{lh}(\vec{u}^1)$.

   (e) $k^1_\delta(a) \leq k^0_{i(\delta)}(i(a))$ *for all* $a \in A^1_\delta$, $0 < \delta < \mathrm{lh}(\vec{u}^1)$.

   (f) $s^1 \leq k^0_{i(\mathrm{lh}(\vec{u}^1))}(i(\vec{u}^1, \vec{f}^1))$.

Notice that the definition of addability did not involve s^0, so that with a mild abuse of language we may say that "$(\vec{u}^1, \vec{f}^1, \vec{A}^1, \vec{k}^1, s^1)$ is addable to $(\vec{u}^0, \vec{f}^0, \vec{A}^0, \vec{k}^0)$".

Definition 3.5 *Let $a = (\vec{u}, \vec{f}) \in U_\infty$ and let*

$$\begin{aligned} p^0 &= \langle t_1^0, \ldots, t_n^0, (\vec{u}, \vec{f}, \vec{A}^0, \vec{k}^0) \rangle \\ p^1 &= \langle t_1^1, \ldots, t_m^1, (\vec{u}, \vec{f}, \vec{A}^1, \vec{k}^1) \rangle \end{aligned}$$

be two conditions in $\mathbb{R}_a$, where $t_e^d = (\vec{u}_e^d, \vec{f}_e^d, \vec{A}_e^d, \vec{k}_e^d, s_e^d)$.

Then $p^1 \leq p^0$ iff

1. *$n \leq m$.*

2. *$A_\alpha^1 \subseteq A_\alpha^0$ and $a \in A_\alpha^1 \Longrightarrow k_\alpha^1(a) \leq k_\alpha^0(a)$ for all $0 < \alpha < \mathrm{lh}(\vec{u})$.*

3. *For every i with $1 \leq i \leq n$ there is j with $1 \leq j \leq m$ such that t_j^1 shrinks t_i^0.*

4. *For each j with $1 \leq j \leq m$ one of the following statements holds:*

   (a) *t_j^1 is addable to t_1^0.*

   (b) *There exists i such that $1 \leq i \leq n$ and t_j^1 shrinks t_i^0.*

   (c) *There exists i such that $1 \leq i < n$, $(\vec{u}_i^0)_0 \prec (\vec{u}_j^1)_0 \prec (\vec{u}_{i+1}^0)_0$ and t_j^1 is addable to t_{i+1}^0.*

   (d) *$(\vec{u}_n^0)_0 \prec (\vec{u}_j^1)_0$ and $i_{(\vec{u}_j^1)_0}(t_j^1)$ is addable to $(\vec{u}, \vec{f}, \vec{A}^0, \vec{k}^0)$.*

We are now able to define the projected forcing $\mathbb{P}$. A *suitable quintuple for* $\mathbb{P}$ is $(\vec{v}, \vec{\mathcal{G}}, \vec{B}, \vec{b}, s)$ where

EITHER

1. $(\vec{v}, \vec{\mathcal{G}}) \in U^{\pi}_{\infty}$.

2. $\vec{B}$ is a sequence of measure one sets for $\vec{v}$.

3. b_γ is a function with domain B_γ such that $a \in B_\gamma \implies b_\gamma(a) \in \mathrm{RO}(\mathrm{Coll}(\kappa_a^{+6}, < \kappa_{\vec{v}}))$.

4. $[b_\gamma]_{v_\gamma} \in \mathcal{G}_\gamma$.

5. $s \in \mathrm{Coll}(\kappa_{\vec{v}}^{+6}, < \kappa)$.

OR

1. $\vec{v} = \langle \aleph_0 \rangle$, where by convention $\kappa_{\vec{v}} = \aleph_0$.

2. $\vec{\mathcal{G}} = \vec{B} = \vec{b} = 0$.

3. $s \in \mathrm{Coll}(\aleph_6, < \kappa)$.

In the natural way we may define a map π which takes suitable quintuples for $\mathbb{R}$ to suitable quintuples for $\mathbb{P}$. We let $\pi(\vec{u}, \vec{f}, \vec{A}, \vec{k}, s) = (\vec{v}, \vec{\mathcal{G}}, \vec{B}, \vec{b}, s)$ where

1. $\pi(\vec{u}, \vec{f}) = (\vec{v}, \vec{\mathcal{G}})$.

2. $B_\gamma = \pi``A_\gamma$.

3. $b_\gamma = b(\vec{u}, \vec{k}, \vec{A}, \gamma)$.

For the special case of a quintuple of form $q = (\langle \aleph_0 \rangle, 0, 0, 0, s)$ we set $\pi(q) = q$.

Now we define $\mathbb{P}_c$ for every $c \in U^\pi_\infty$.

Definition 3.6 *Let $c = (\vec{v}, \vec{G}) \in U^\pi_\infty$. A condition in $\mathbb{P}_c$ is a sequence*

$$\langle t_1, \ldots, t_n, (\vec{v}, \vec{\mathcal{G}}, \vec{B}, \vec{b}) \rangle$$

such that

1. *Each t_i is a suitable quintuple for $\mathbb{P}$, say $t_i = (\vec{v}_i, \vec{\mathcal{G}}_i, \vec{B}_i, \vec{b}_i, s_i)$.*

2. *$t_1 = (\langle \aleph_0 \rangle, 0, 0, 0, s_1)$.*

3. *$\kappa_{\vec{v}_i} < \kappa_{\vec{v}_{i+1}}$ for $1 \le i < n$ and $\kappa_{\vec{v}_n} < \kappa_{\vec{v}}$.*

4. *$\vec{B}$ is a sequence of measure one sets for $\vec{v}$. b_γ is a function such that $\mathrm{dom}(b_\gamma) = B_\gamma$, $b_\gamma(a) \in \mathrm{RO}(\mathrm{Coll}(\kappa_a^{+6}, < \kappa_{\vec{v}}))$ for all $a \in B_\gamma$, and $[b_\gamma]_{v_\gamma} \in \mathcal{G}_\gamma$.*

5. *$s_i \in \mathrm{Coll}(\kappa_{\vec{v}_i}^{+6}, < \kappa_{\vec{v}_{i+1}})$ for $1 \le i < n$ and $s_n \in \mathrm{Coll}(\kappa_{\vec{v}_n}^{+6}, < \kappa_{\vec{v}})$.*

If $p = \langle t_1, \ldots, t_n, (\vec{v}, \vec{\mathcal{G}}, \vec{B}, \vec{b}) \rangle \in \mathbb{P}_c$ then we will divide p into a "lower part" and an "upper part", to be more precise we let $\mathrm{lp}(p) = \langle t_1, \ldots, t_n \rangle$ and $\mathrm{up}(p) = (\vec{v}, \vec{\mathcal{G}}, \vec{B}, \vec{b})$.

Definition 3.7 *Let $q^0 = (\vec{v}^0, \vec{\mathcal{G}}^0, \vec{B}^0, \vec{b}^0, s^0)$ and $q^1 = (\vec{v}^1, \vec{\mathcal{G}}^1, \vec{B}^1, \vec{b}^1, s^1)$ be suitable quintuples for $\mathbb{P}$.*

1. q^1 shrinks q^0 *iff*

   (a) $(\vec{v}^1, \vec{\mathcal{G}}^1) = (\vec{v}^0, \vec{\mathcal{G}}^0)$.

   (b) $B^1_\alpha \subseteq B^0_\alpha$ *for all* α.

   (c) $b^1_\alpha(a) \leq b^0_\alpha(a)$ *for all* $a \in B^1_\alpha$.

   (d) $s^1 \leq s^0$.

2. q^1 *is* addable *to* q^0 *iff*

   (a) *There is* $\gamma < \mathrm{lh}(\vec{v}^0)$ *such that* $(\vec{v}^1, \vec{\mathcal{G}}^1) \in B_\gamma$ *and* $s^1 \leq b^0_\gamma(\vec{v}^1, \vec{\mathcal{G}}^1)$.

   (b) *There is an increasing* $e : \mathrm{lh}(\vec{v}^1) \longrightarrow \mathrm{lh}(\vec{v}^0)$ *such that* $B^1_\alpha \subseteq B^0_{e(\alpha)}$ *and* $b^1_\alpha(a) \leq b^0_{e(\alpha)}(a)$ *for all* $0 < \alpha < \mathrm{lh}(\vec{v}^1)$ *and all* $a \in B^1_\alpha$.

Definition 3.8 *Let* $c = (\vec{v}, \vec{\mathcal{G}}) \in U^\pi_\infty$ *and let*

$$\begin{aligned} p^0 &= \langle t^0_1, \ldots, t^0_n, (\vec{v}, \vec{\mathcal{G}}, \vec{B}^0, \vec{b}^0)\rangle \\ p^1 &= \langle t^1_1, \ldots, t^1_m, (\vec{v}, \vec{\mathcal{G}}, \vec{B}^1, \vec{b}^1)\rangle \end{aligned}$$

be two conditions in $\mathbb{P}_c$*, where* $t^d_e = (\vec{v}^d_e, \vec{\mathcal{G}}^d_e, \vec{B}^d_e, \vec{b}^d_e, s^d_e)$.

Then $p^1 \leq p^0$ *iff*

1. $n \leq m$.

2. $B^1_\alpha \subseteq B^0_\alpha$ *and* $a \in B^1_\alpha \Longrightarrow b^1_\alpha(a) \leq b^0_\alpha(a)$ *for all* $0 < \alpha < \mathrm{lh}(\vec{v})$.

3. *For every* i *with* $1 \leq i \leq n$ *there is* j *with* $1 \leq j \leq m$ *such that* t^1_j *shrinks* t^0_i.

4. For each j with $1 \leq j \leq m$ one of the following statements holds:

(a) t^1_j is addable to t^0_1.

(b) There exists i such that $1 \leq i \leq n$ and t^1_j shrinks t^0_i.

(c) There exists i such that $1 \leq i < n$, $(\vec{v}^0_i)_0 < (\vec{v}^1_j)_0 < (\vec{v}^0_{i+1})_0$ and t^1_j is addable to t^0_{i+1}.

(d) $(\vec{v}^0_n)_0 < (\vec{v}^1_j)_0$ and t^1_j is addable to $(\vec{v}, \vec{\mathcal{G}}, \vec{B}^0, \vec{b}^0)$.

At this point we are finally ready to define the forcings $\mathbb{P}$ and $\mathbb{R}$. To do this we observe that by GCH there are only $2^{2^\kappa} = \kappa^{++}$ many measures on V_κ, so that if $(\vec{w}, \vec{\mathcal{F}}) = \pi(\vec{M}, \vec{g})$ then the sequence $\vec{w}$ must contain a repetition. Accordingly we define $\lambda_1 < \lambda^*$ to be minimal such that for some $\lambda_0 < \lambda_1$ we have $w_{\lambda_0} = w_{\lambda_1}$. We then set $\mathbb{R} = \mathbb{R}_{(\vec{M}\restriction\lambda_1, \vec{g}\restriction\lambda_1)}$ and $\mathbb{P} = \mathbb{P}_{(\vec{w}\restriction\lambda_1, \vec{\mathcal{F}}\restriction\lambda_1)}$.

The following properties are then proved by easy adaptations of proofs in [7].

Lemma 3.9 *$\mathbb{P}_a$ has the κ_a^+-chain condition.*

Lemma 3.10 *$\mathbb{P}_a$ adds to the universe a sequence $\vec{a}$ of members of U^π_∞ such that if $\kappa_i = \kappa_{a_i}$ then $\kappa_0 = \aleph_0$ and $\vec{\kappa}$ is continuous, increasing and cofinal in κ_a. $\mathrm{lh}(a_i) > 1$ exactly when i is a limit ordinal. $\mathbb{P}_a$ also adds a sequence $\vec{G}$ such that G_i is $\mathrm{Coll}(\kappa_i^{+6}, < \kappa_{i+1})$-generic for all $i < \mathrm{lh}(\vec{\kappa}) = \mathrm{lh}(\vec{G})$.*

Lemma 3.11 *The generic club added by $\mathbb{P}_a$ is eventually contained in every set $X \subseteq \kappa_a$ such that $\{ v \mid \kappa_v \in X \}$ is measure one for all the measures* *appearing in a. In particular the generic club is eventually contained in every $> \omega$-club subset of κ_a.*

Lemma 3.12 *Let $a = (\vec{v}, \vec{\mathcal{G}}) \in U^{\pi}_{\infty}$, let*

$$p = \langle t_1, \ldots, t_n, (\vec{v}, \vec{\mathcal{G}}, \vec{B}, \vec{b}) \rangle \in \mathbb{P}_a$$

with $t_i = (\vec{v}_i, \vec{\mathcal{G}}_i, \vec{B}_i, \vec{b}_i, s_i)$.

1. *If $\mathrm{lh}(v_i) > 1$ then*

   $$\mathbb{P}_a/p \simeq \mathbb{P}_{(\vec{v}_i, \vec{\mathcal{G}}_i)}/q \times \mathbb{P}_a/r$$

   where

   $$q = \langle t_1, \ldots, t_{i-1}, (\vec{v}_i, \vec{\mathcal{G}}_i, \vec{B}_i, \vec{b}_i) \rangle$$

   and

   $$r = \langle (\langle \kappa_{\vec{v}_i} \rangle, \langle\rangle, \langle\rangle, \langle\rangle, s_i), t_{i+1}, \ldots, t_n, (\vec{v}, \vec{\mathcal{G}}, \vec{B}, \vec{b}) \rangle.$$

2. *If also $\mathrm{lh}(\vec{v}_{i+1}) = 1$ then $\mathbb{P}_a/p \simeq \mathbb{P}_{(v_i, \mathcal{G}_i)}/q \times \mathrm{Coll}(\kappa^{+6}_{\vec{v}_i}, < \kappa_{\vec{v}_{i+1}})/s_i \times \mathbb{P}_a/r^*$ where*

   $$r^* = \langle t_{i+1}, \ldots, t_n, (\vec{v}, \vec{\mathcal{G}}, \vec{B}, \vec{b}) \rangle.$$

Lemma 3.13 *Let F be $\mathbb{P}_a$-generic with $p \in F$. Let $\vec{a}$ be the sequence from U^{π}_{∞} added by F, and let $\vec{G}$ be the corresponding sequence of generic collapses.*

If $\mathrm{lh}(a_i) > 1$ then $V[F] = V[F_1 \times F_2]$ where

1. *F_1 is $\mathbb{P}_{a_i}$-generic.*

2. F_2 *is* $\mathbb{P}_a$*-generic.*

3. F_1 *adds* $\vec{a} \restriction i$ *and* $\vec{G} \restriction i$*.*

4. F_2 *adds* $\vec{a}^*$, $\vec{G}^*$ *where* $a^*_j = a_{i+j}$ *and* $G^*_j = G_{i+j}$*.*

Abusing notation, we will denote F_1 by "$F \restriction i$".

If p, q are conditions in $\mathbb{P}_a$ we say that p is a *direct extension* of q and write $p \leq^* q$ when p and q are sequences of the same length. To put it another way, p is obtained from q by merely shrinking the quintuples which are present in p and the relevant parts of the final quadruple. We write $p \leq^* q$ when p is a direct extension of q.

Lemma 3.14 *With the same hypotheses and notation as Lemma 3.12, if* $b \in \mathrm{RO}(\mathbb{P}_a)$ *is a boolean value then there exist a maximal antichain* $A \subseteq \mathbb{P}_{(\vec{v}_i, \vec{\mathcal{G}}_i)}/q$ *and a condition* $r^* \leq^* r$ *such that* (s, r^*) *decides* b *for all* $s \in A$*.*

Lemma 3.15 *Let* F *be* $\mathbb{P}_a$*-generic and let* $\vec{\kappa}$, $\vec{G}$ *be as in Lemma 3.10. Let* $\lambda < \kappa_a$ *be a cardinal and let* i *be the largest limit ordinal such that* $\kappa_i \leq \lambda$*. Let* $n < \omega$ *be minimal such that* $\lambda < \kappa^{+6}_{i+n}$*.*

Let $G^* = F \restriction i \times \prod_{j<n} G_{i+j}$*. Then* G^* *is* $\mathbb{P}_{a_i} \times \prod_{j<n} \mathrm{Coll}(\kappa^{+6}_{i+j}, < \kappa_{i+j+1})$*-generic and* $(\mathcal{P}\lambda)_{V[F]} = (\mathcal{P}\lambda)_{V[G^*]}$*. In particular, if* $\lambda < \kappa$ *and does not lie in an interval* $(\kappa^{+6}_i, \kappa_{i+1})$ *then* λ *is preserved.*

Lemma 3.16 *Let* G *be* $\mathbb{P}$*-generic. Then* κ *is* κ^{+5}*-supercompact in* $V[G]$*.*

One more remark is in order before we begin to exploit the forcing $\mathbb{P}$. If $A \subseteq \kappa$ and $\kappa \in j(A)$ then every measure w_α will concentrate on the set of a with $\kappa_a \in A$. Accordingly by forcing below an appropriate condition we may ensure that the generic sequence $\vec{\kappa}$ added by $\mathbb{P}$ consists of points from A.

4 Proof of the main theorem

We are now in a position to prove Theorem 1.

Proof:[Theorem 1] We begin with a model (V_0 say) in which GCH holds and κ is κ^{+5}-supercompact. We do a Reverse Easton iteration of the sort described in Theorem 2 and obtain a model V in which

1. GCH holds.

2. κ is κ^{+5}-supercompact.

3. $\Box^B_\kappa$ holds.

Let

$$A = \{ \alpha < \kappa \mid \alpha \text{ is } \alpha^+\text{-supercompact and } \Box^B_\alpha \text{ holds} \}.$$

It is easy to see that $\kappa \in j(A)$.

Working in V, we now use j to build a forcing notion $\mathbb{P}$ as in the preceding section. Let $G_{\mathbb{P}}$ be $\mathbb{P}$-generic over V and let $\vec{a}$, $\vec{\kappa}$, $\vec{G}$ be as in Lemma 3.10. Forcing below a suitable condition we may assume that $\kappa_i \in A$ for all $i > 0$.

We know by the work of the last section that

1. κ is κ^{+5}-supercompact in $V[G_{\mathbb{P}}]$.

2. If a cardinal λ is in $(\kappa_i^{+6}, \kappa_{i+1})$ for some i then λ is collapsed to have cardinality κ_i^{+6} in $V[G_{\mathbb{P}}]$. Otherwise λ is preserved.

The following claim will establish Theorem 1.

Claim 4.1 *Let $\lambda < \kappa$ be a singular cardinal of $V[G_{\mathbb{P}}]$. Then in $V[G_{\mathbb{P}}]$*

1. *$\lambda = \kappa_i$ for some limit $i < \kappa$.*

2. *The combinatorial principle $\square_{\lambda,\mathrm{cf}(\lambda)}$ holds.*

3. *If $S = \{ \alpha < \lambda^+ \mid \mathrm{cf}(\alpha) < \lambda \}_V$ then S is stationary. Moreover if $\vec{S}$ is a sequence of stationary subsets of S and $\mathrm{lh}(\vec{S}) < \mathrm{cf}(\lambda)$ then $\vec{S}$ reflects to cofinality μ for unboundedly many $\mu < \lambda$.*

Proof:

We take the various claims in turn.

1. Since λ is singular, λ is a limit cardinal. We have collapsed all but finitely many cardinals in each interval $[\kappa_j, \kappa_{j+1})$ so the only possibility is that $\lambda = \kappa_i$ for some limit i. Notice that we must have $\mathrm{lh}(a_i) > 1$ because sequences of length 1 correspond to successor points on the sequence $\vec{\kappa}$.

2. Since $\lambda = \kappa_i$, $\lambda \in A$ and so $V \models \square^B_\lambda$. $\lambda^+_V = \lambda^+_{V[G_{\mathbb{P}}]}$ by Lemma 3.15, so $V[G_{\mathbb{P}}] \models \square_{\lambda,\mathrm{cf}(\lambda)}$ by Lemma 3.11 and Lemma 2.1.

3. We know $\lambda = \kappa_i$ where i is limit and $\mathrm{lh}(a_i) > 1$. By Lemma 3.15 we know $(\mathcal{P}\lambda^+)_{V[G_{\mathbb{P}}]} = (\mathcal{P}\lambda^+)_{V[G_{\mathbb{P}} \restriction \lambda]}$. Since $\mathbb{P}_{a_i}$ is λ^+-c.c. the set S is still stationary in $V[G_{\mathbb{P}} \restriction i]$, and so is stationary in $V[G_{\mathbb{P}}]$.

   It suffices to prove the desired reflection statement in $V[G_{\mathbb{P}} \restriction \lambda]$. We will do this using the fact that $\lambda \in A$, so that λ is λ^+-supercompact in V. Let $G^*_j = G_{\mathbb{P}} \restriction j$ for $j \leq \mathrm{lh}(\vec{a})$.

   Let $\mu = \mathrm{cf}_{V[G_{\mathbb{P}}]}(\lambda) = \mathrm{cf}_{V[G^*_i]}(\lambda)$. Until further notice we work in $V[G^*_i]$. Fix a sequence $\langle \tau_j : j < \mu \rangle$ which is increasing and cofinal in i. Then $\langle \kappa_{\tau_j} : j < \mu \rangle$ is increasing and cofinal in $\kappa_i = \lambda$.

   If the reflection claim fails we can find $p \in G^*_i$, $\langle \dot{S}_k : k < \nu \rangle$ for some $\nu < \mu$, and $\sigma < \lambda$ such that $p \Vdash \dot{S}_k \subseteq \hat{S}$ and

   $$p \Vdash^V_{\mathbb{P}_{a_i}} \text{“}\langle \dot{S}_k : k < \nu \rangle \text{ reflects simultaneously at no cofinality} > \sigma\text{”}.$$

   Let S_k be the realisation of the term $\dot{S}_k$ by the generic G^*_i. Then $S_k = \bigcup_{j<\mu} S^j_k$ where

   $$S^j_k = \{\ \alpha \mid \exists q \in G^*_i\ \mathrm{lp}(q) \in G^*_{\tau_j}, q \Vdash \alpha \in \dot{S}_k\ \}.$$

   Notice that S^j_k increases with j.

   For each $k < \nu$ there exists $j < \mu$ such that S^j_k is stationary. Increasing j further we may also arrange that $S^j_k \cap X_j$ is stationary where

   $$X_j = \{\ \alpha < \kappa^+ \mid \mathrm{cf}(\alpha) < \kappa_{\tau_j}\ \}_V.$$

   Since $\nu < \mu = \mathrm{cf}(\mu)$ we may find j so large that

(a) $S^j_k \cap X_j$ is stationary for all k.

(b) $\mathrm{lp}(p) \in G^*_{\tau_j}$.

We will now work in $V[G^*_{\tau_j}]$ until further notice. Let

$$T_k = \{\ \alpha \mid \exists q \in \mathbb{P}_{a_i}\ \mathrm{lp}(q) \in G^*_{\tau_j}, q\ \Vdash \alpha \in \dot{S}_k \cap X_j\ \}.$$

$T_k \in V[G^*_{\tau_j}] \subseteq V[G^*_i]$, $S^j_k \cap X_j \subseteq T_k$ and $S^j_k \cap X_j$ is stationary in $V[G^*_i]$, so clearly T_k is stationary in $V[G^*_{\tau_j}]$.

Now λ is still λ^+-supercompact in $V[G^*_{\tau_j}]$ because $|\mathbb{P}_{a_{\tau_j}}| < \lambda$. $T_k = \bigcup_{p \in G^*_{\tau_j}} \{\ \alpha \mid p \Vdash \alpha \in \dot{T}_k\ \}$, and so there is $p_k \in G^*_{\tau_j}$ such that $Y_k = \{\ \alpha \mid p_k \Vdash \alpha \in \dot{T}_k\ \}$ is stationary.

That is to say, for every k we may find $Y_k \subseteq T_k$ such that Y_k is stationary and $Y_k \in V$. $\langle Y_k : k < \nu \rangle$ is not necessarily in V because p_k depends on k and $G^*_{\tau_j}$, but since $\mathbb{P}_{a_{\tau_j}}$ is λ-c.c. we may find a family $X \in V$ of stationary subsets of X_j such that $|X| < \lambda$ and $\{\ Y_k \mid k < \nu\ \} \subseteq X$.

We now work in V. Appealing to Fact 1.2 we may find γ such that all the Y_k reflect at γ, and $\mathrm{cf}(\gamma)$ is the successor of an inaccessible with $\max\{\sigma, \kappa^{+6}_{\tau_j}\} < \mathrm{cf}(\gamma) < \lambda$.

Choose $D \subseteq \gamma$ club such that $\mathrm{o.t.}(D) = \mathrm{cf}(\gamma)$. Using the "strong factorisation" property from Lemma 3.14, we work in $V[G^*_{\tau_j}]$ and choose for each $k < \nu$ and each $\beta \in Y_k \cap D$ an upper part $q(k, \beta)$ such that

$$\exists r \in G^*_{\tau_j}\ r \frown q(k, \beta) \Vdash \beta \in \dot{S}_k.$$

Since $\max\{|\mathbb{P}_{a_{\tau_j}}|, \nu, \mathrm{cf}(\gamma)\} < \lambda$ we may find a set Y of upper parts for $\mathbb{P}_{a_i}$ such that $Y \in V$, $|Y| < \lambda$ and $\{ q(k, \beta) \mid k < \nu, \beta \in Y_k \cap D \} \subseteq Y$. The filters which appear in a_i are all λ-complete so we may find a single upper part q such that q shrinks all the $q(k, \beta)$ as well as the upper part of p.

Shrinking further if necessary we can guarantee that the condition $\langle q \rangle$ forces that the generic club added by $\mathbb{P}_{a_i}$ has minimal entry greater than $\mathrm{cf}(\gamma)$. We force below $\langle q \rangle$ to get an H which is $\mathbb{P}_{a_i}$-generic over $V[G^*_{\tau_j}]$. Let μ^* be the least entry on the generic club added by H. We force h which is $\mathrm{Coll}(\kappa^{+6}_{\tau_j}, < \mu^*)_V$-generic over $V[G^*_{\tau_j}][H]$.

By the second part of Lemma 3.12, $G^*_{\tau_j} \times h \times H$ can be rearranged as a generic object $G^\dagger$ for $\mathbb{P}_{a_i}$. What is more we have arranged that $p \in G^\dagger$ and that $Y_k \cap D \subseteq \dot{S}_k^{G^\dagger}$ for all k. We will reach a contradiction by showing that $Y_k \cap D$ is stationary in $V[G^\dagger]$.

Since $\mathrm{o.t.}(D) = \mathrm{cf}_V(\gamma)$ we may collapse $Y_k \cap D$ to get Y'_k a stationary subset of $\mathrm{cf}_V(\gamma)$. Since $\mu^* > \mathrm{cf}_V(\gamma)$ it will suffice to show that Y'_k is stationary in $V[G^*_{\tau_j} \times h]$.

Y'_k consists of ordinals of cofinality less than κ_{τ_j}. $\mathrm{cf}(\gamma)$ is the successor of an inaccessible (θ say), so $\Box^*_\theta$ holds. So $\mathrm{Coll}(\kappa^{+6}_{\tau_j}, < \mu^*)$ will preserve (see [9]) the stationarity of Y'_k. $|\mathbb{P}_{\tau_j}| = \kappa^+_{\tau_j}$ so Y'_k (and thus $Y_k \cap D$) will be stationary in $V[G^*_{\tau_j} \times h]$ as required.

We now have a contradiction, so the simultaneous reflection property holds.

♦

This concludes the proof of Theorem 1.

♦

References

[1] A. Apter, "Strong compactness and a global version of a theorem of Ben-David and Magidor", to appear in the Mathematical Logic Quarterly.

[2] A. Apter and J. Henle, "On box, weak box and strong compactness", Bulletin of the London Mathematical Society **24** (1992), 513–518.

[3] J. Baumgartner, unpublished.

[4] S. Ben-David and M. Magidor, "The weak $\square^*$ is really weaker than the full $\square$", Journal of Symbolic Logic **51** (1986), 1029-1033.

[5] J. Cummings, M. Foreman and M. Magidor, "Scales, squares and stationary reflection", submitted to the Journal of Mathematical Logic.

[6] J. Cummings and E. Schimmerling, "Indexed squares", submitted to the Israel Journal of Mathematics.

[7] M. Foreman and W. H. Woodin, "The generalized continuum hypothesis can fail everywhere", Annals of Mathematics **133** (1991), 1–35.

[8] R. Jensen, "The fine structure of the constructible hierarchy", Annals of Mathematical Logic **4** (1972), 229–308.

[9] P. Komjath, "Stationary reflection for uncountable cofinality", Journal of Symbolic Logic **51** (1986), 147–151.

[10] M. Magidor, "On the singular cardinals problem I", Israel Journal of Mathematics **28** (1977), 1–31.

[11] M. Magidor, "Changing cofinality of cardinals", Fundamenta Mathematicae **99** (1978), 61–71.

[12] K. Prikry, "Changing measurable into accessible cardinals", Dissertationes Mathematicae (Rozprawy Matematyczne) **68** (1970), 5–52.

[13] L. Radin, "Adding closed cofinal sequences to large cardinals", Annals of Mathematical Logic **22** (1982), 243–261.

[14] E. Schimmerling, "Combinatorial principles in the core model for one Woodin cardinal", Annals of Pure and Applied Logic **74** (1995), 153–201.