DOWKER AND SUPER-DOWKER FILTERS

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Abstract. Our main results show that a very simple forcing construction can be used to add Dowker and super-Dowker filters:

- Let κ be uncountable with $\kappa^{<\kappa} = \kappa$. Let G be generic over V for $Add(\kappa, \kappa^{++})$. Then in $V[G]$ there is a Dowker filter on κ^+.
- Let V be Laver's model in which κ is supercompact and the supercompactness of κ is indestructible under κ-directed closed forcing, and let G be generic for $Add(\kappa, \kappa^{++})$. Then in $V[G]$ there is a super-Dowker filter on κ^+.

1. Introduction

Dowker [3] raised the question of the existence of a certain type of filter. Such filters are now known as *Dowker filters.* The consistency of the existence of Dowker filters was shown by Balogh and Gruenhage [1] using an ingenious iterated forcing argument. Freiling and Payne [4] made a detailed study of Dowker filters and related objects; in particular they introduced the stronger notion of a *super-Dowker filter* and showed that if every set of reals has the Baire property (an assumption which contradicts AC, but holds for example in the Solovay model) then there is a super-Dowker filter.

Our main results show that a very simple forcing construction can be used to add Dowker and super-Dowker filters:

- Let κ be uncountable with $\kappa^{<\kappa} = \kappa$. Let G be generic over V for $Add(\kappa, \kappa^{++})$. Then in $V[G]$ there is a Dowker filter on κ^+.
- Let V be Laver's model in which κ is supercompact and the supercompactness of κ is indestructible under κ-directed closed forcing, and let G be generic for $Add(\kappa, \kappa^{++})$. Then in $V[G]$ there is a super-Dowker filter on κ^+.

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2. Preliminaries

We begin with the definitions and basic properties of Dowker and super-Dowker filters.

Definition 2.1. Let $\mathcal{F}$ be a filter on a set X. Then $A \subseteq X$ is *$\mathcal{F}$-small* if $X \setminus A \in \mathcal{F}$; $A \subseteq X$ is *$\mathcal{F}$-positive*, and we write $A \in \mathcal{F}^+$, if it is not $\mathcal{F}$-small.

Definition 2.2. Let $\mathcal{F}$ be a filter on a set X and $Y \subseteq X$. If $f : Y \longrightarrow \mathcal{F}$ and x and y are elements of Y let $\psi(f, x, y)$ denote the formula

$$\text{“}x \neq y \ \text{ and } \ y \in f(x) \ \text{ and } \ x \in f(y).\text{”}$$

We can view f as defining a directed graph in which there is a directed edge from x to y when $y \in f(x)$. In this language the formula ψ asserts that there are directed edges in both directions between the vertices x and y.

We list some properties that a filter $\mathcal{F}$ on a set X may have:

I $\quad \forall f : X \longrightarrow \mathcal{F}\ \exists x\ \exists y\ \ \psi(f, x, y)$.

II $\quad \forall e : X \longrightarrow 2\ \exists f_e : X \longrightarrow \mathcal{F}\ \forall x\ \forall y\ \psi(f_e, x, y) \Longrightarrow e(x) = e(y)$.

II$^+$ $\ \forall e : X \longrightarrow 2\ \exists f_e : X \longrightarrow \mathcal{F}\ \exists i\ \forall x\ \forall y\ \psi(f_e, x, y) \Longrightarrow e(x) = e(y) = i$.

Definition 2.3. Let $\mathcal{F}$ be a filter on X.

(1) $\mathcal{F}$ is a *Dowker filter* if it enjoys properties I and II.

(2) $\mathcal{F}$ is a *super-Dowker filter* if it enjoys properties I and II$^+$.

Dowker proved some fundamental results about Dowker filters.

Theorem 2.4. [3] *If $\mathcal{F}$ is a Dowker filter on X then:*

(1) X is not the union of countably many $\mathcal{F}$-small sets.

(2) The least cardinality of an $\mathcal{F}$-positive set is strictly less than the least cardinality of an $\mathcal{F}$-large set.

(3) All countable subsets of X are $\mathcal{F}$-small.

It follows immediately that if X carries a Dowker filter then $\overline{\overline{X}} \geq \omega_2$.

Definition 2.5. If $\mathcal{F}$ is a super-Dowker filter on X, then let $\mathcal{F}^*$ be the family of sets $E \subseteq X$ with the following property: there is some $f : X \longrightarrow \mathcal{F}$ such that for every $\{x, y\} \in [X]^2$ if $\psi(f, x, y)$ then x and y are in E.

It is easy to see that $\mathcal{F}^*$ is an ultrafilter, and that $\mathcal{F} \subseteq \mathcal{F}^*$.

Before Dowker filters were known to be consistent in ZFC, Freiling and Payne introduced super-Dowker filters and studied their properties, primarily in the context of ZF. They observed in particular that under ZFC the ultrafilter $\mathcal{F}^*$ associated with a super-Dowker filter $\mathcal{F}$ is countably complete and non-principal, so that super-Dowker filters can only exist at or above the first measurable cardinal.

Balogh and Gruenhage gave a forcing construction for Dowker filters. Starting with a model where GCH holds and κ is either inaccessible or the successor of a regular cardinal, they built a forcing extension by κ-closed and κ^+-c.c. forcing in which $2^\kappa = \kappa^{++}$ and κ^+ carries a Dowker filter. Their filter has the property that the intersection of fewer than κ many large sets is non-empty, which implies that it is not obtained from a Dowker filter $\mathcal{F}_0$ on a cardinal $\lambda \leq \kappa$ by the "trivial" construction of forming $\{ X \subseteq \kappa^+ : X \cap \lambda \in \mathcal{F}_0 \}$. Our arguments owe much to theirs.

The most important question about Dowker filters remains open.

Question 2.6. Does ZFC prove the existence of a Dowker filter?

Two other important questions are also still open. The first is a weakening of the previous question.

Question 2.7. Is the existence of a Dowker filter on a cardinal κ^+ consistent with $2^\kappa = \kappa^+$?

The second asks about Dowker filters at successors of singulars.

Question 2.8. Is the existence of a Dowker filter on the successor of a singular cardinal consistent relative to ZFC?

3. Combinatorial facts about filters

We turn to some general technical facts about filters which will be useful in the ensuing sections. These were proved by Balogh and Gruenhage for a filter analogous to our filters $\mathcal{F}^j(\Gamma)$ defined below (see Definition (4.12)), but are quite general. We have included the proofs in order to make this paper self-contained.

Lemma 3.1. *Let λ be regular and uncountable. Let $\mathcal{G}$ be a λ-complete filter on some set W, let $C \in \mathcal{G}^+$ and let $h : W \to [Z]^{<\omega}$ for some set Z.*

Then:

(1) *There is $R \in [Z]^{<\lambda}$ such that for all $R' \in [Z]^{<\lambda}$ with $R' \supseteq R$*
$$\{w \in C : h(w) \cap R' \subseteq R\} \in \mathcal{G}^+.$$

(2) *There exist a set $a \in [Z]^{<\omega}$ and a set $D \subseteq C$ with $D \in \mathcal{G}^+$ such that*
$$\forall z \in Z \setminus a \ \{w \in D : z \in h(w)\} \notin \mathcal{G}^+.$$

Proof. We take the two claims in turn.

(1) Suppose not. Define by recursion an ω-sequence of sets $R_n \in [Z]^{<\lambda}$ such that:

(a) $R_0 = \emptyset$,

(b) $R_n \subseteq R_{n+1}$ and $\{w \in C : h(w) \cap R_{n+1} \subseteq R_n\}$ is $\mathcal{G}$-small.

Since λ is uncountable and C is positive for the λ-complete filter $\mathcal{G}$, there is $w \in C$ such that $h(w) \cap R_{n+1} \nsubseteq R_n$ for all n, contradicting the assumption that $h(w)$ is finite.

(2) Suppose not. Appealing to the facts that $\lambda > \omega$ and $\mathcal{G}$ is λ-complete, we may find a $\mathcal{G}$-positive set $D_0 \subseteq C$ and a natural number n such that $\overline{\overline{h(w)}} = n$ for all $w \in D_0$. We now choose $D_i \in \mathcal{G}^+$, $a_i \in [Z]^{<\omega}$ and $z_i \in Z$ for $i \leq n+1$ such that

(a) $a_i = \{z_j : j < i\}$.

(b) $z_i \in Z \setminus a_i$.

(c) $D_{i+1} = \{w \in D_i : z_i \in h(w)\}$.

This is possible at each stage because (by hypothesis) a_i and D_i do not satisfy the conclusion.

Let $w \in D_{n+1}$. Then $w \in D_{i+1}$ for all $i \leq n$, so that $z_i \in h(w)$ for $i \leq n$. This is a contradiction because the points z_i for $i \leq n$ are $n+1$ distinct elements of $h(w)$, yet $\overline{\overline{h(w)}} = n$ since $w \in D_0$.

4. Generic set constructions

In this section we describe some constructions of certain subsets of κ^+ in the generic extension by $Add(\kappa, \kappa^{++})$, where κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. In later sections we will use these sets to generate Dowker and super-Dowker filters. We note that under our cardinal arithmetic assumption $Add(\kappa, \kappa^{++})$ is κ-closed and κ^+-cc, so that all cardinals are preserved.

4.1. **Functions and sets.** Let $g : \kappa^+ \times \kappa^+ \longrightarrow 2$ be a function. We associate to g the sequence of sets $\langle A_\beta : \beta < \kappa^+ \rangle$, where for each $\beta < \kappa^+$ we set

$$A_\beta = \{\alpha : g(\alpha, \beta) = 1\}.$$

Let $e : \kappa^+ \longrightarrow 2$ be a function. Such functions correspond in the obvious way to partitions of κ^+ into two disjoint pieces. We define two ways of altering g using e:

- Let $g^{D,e} : \kappa^+ \times \kappa^+ \longrightarrow 2$ be the function defined by:

$$g^{D,e}(\alpha,\beta) = \begin{cases} 0 & \text{if } \alpha < \beta,\ g(\alpha,\beta) = g(\beta,\alpha) = 1 \\ & \text{ and } e(\alpha) \neq e(\beta) \\ g(\alpha,\beta) & \text{otherwise.} \end{cases}$$

- Let $g^{SD,e} : \kappa^+ \times \kappa^+ \longrightarrow 2$ be the function defined by:

$$g^{SD,e}(\alpha,\beta) = \begin{cases} 0 & \text{if } \alpha < \beta,\ g(\alpha,\beta) = g(\beta,\alpha) = 1 \\ & \text{ and } e(\alpha)e(\beta) = 0 \\ g(\alpha,\beta) & \text{otherwise.} \end{cases}$$

4.2. **Generic sets.**

Definition 4.1. Let $\mathbb{P}$ be the poset whose conditions are partial functions p from $\kappa^{++} \times \kappa^+ \times \kappa^+$ to 2 with $\overline{\overline{p}} < \kappa$, ordered by extension. Clearly $\mathbb{P}$ is isomorphic to $Add(\kappa, \kappa^{++})$. Let G be a $\mathbb{P}$-generic filter and let $f = \bigcup G$, so that $f : \kappa^{++} \times \kappa^+ \times \kappa^+ \longrightarrow 2$ and $V[G] = V[f]$. For each $i < \kappa^{++}$ let $f^i = f(i, ., .)$, that is $f^i : \kappa^+ \times \kappa^+ \longrightarrow 2$ and f^i is defined by $f^i(\alpha,\beta) = f(i,\alpha,\beta)$.

Notation 4.2. For $p \in \mathbb{P}$ let

$$\mathrm{dom}^1(p) = \{\alpha \,:\, \exists i\, \exists \beta\ (i,\alpha,\beta) \in \mathrm{dom}(p) \text{ or } (i,\beta,\alpha) \in \mathrm{dom}(p)\},$$

and

$$\mathrm{dom}^0(p) = \{i \,:\, \exists \alpha\, \exists \beta\ (i,\alpha,\beta) \in dom(p)\}.$$

We note that $\mathrm{dom}(p) \subseteq \mathrm{dom}^0(p) \times \mathrm{dom}^1(p) \times \mathrm{dom}^1(p)$.

By standard counting arguments we may produce a sequence $\langle \dot{e}_i \,:\, i < \kappa^{++} \rangle$ of canonical $\mathbb{P}$-names for functions from κ^+ to 2, such that every $\mathbb{P}$-name for such a function is forced to equal $\dot{e}_i$ for some i.

Working in $V[G]$, we define various sets.

Definition 4.3. Let $e_i = \dot{e}_i^G$ and $E_i = \{\gamma < \kappa^+ \,:\, e_i(\gamma) = 1\}$ for each $i < \kappa^{++}$.

Definition 4.4. For $x = (i,\beta) \in \kappa^{++} \times \kappa^+$, in $V[G]$ we set

$$\begin{aligned} A_x =& \{\alpha \,:\, (f^i)(\alpha,\beta) = 1\}, \\ A^D_x =& \{\alpha \,:\, (f^i)^{D,e_i}(\alpha,\beta) = 1\}, \text{ and} \\ A^{SD}_x =& \{\alpha \,:\, (f^i)^{SD,e_i}(\alpha,\beta) = 1\}. \end{aligned}$$

Definition 4.5. For $a \subseteq \kappa^{++} \times \kappa^+$ we set

$$A_a = \bigcap_{x\in a} A_x,\ A^D_a = \bigcap_{x\in a} A^D_x \text{ and } A^{SD}_a = \bigcap_{x \in a} A^{SD}_x.$$

We will use sets of the form A_x, A_x^D and A_x^{SD} to generate various filters. In order to see that these filters are proper we need a technical fact. Before stating it we make three easy remarks:

Remark 4.6.

(1) Since $\mathbb{P}$ is κ-closed, $([X]^{<\kappa})^V = ([X]^{<\kappa})^{V[G]}$ for any set $X \in V$.

(2) $A_x^{SD} \subseteq A_x^D \subseteq A_x$ for all $x \in \kappa^{++} \times \kappa^+$.

(3) When $\alpha < \beta$ with $\beta \in A_{(i,\alpha)}$, then $\beta \in A_{(i,\alpha)}^{SD}$. In the more vivid digraph language, the digraph determined by $f^{i,SD}$ is obtained from the digraph determined by f^i by deleting only "downwards" directed edges.

Lemma 4.7. *Let $B \in [\kappa^+]^\kappa \cap V$ and $a \in [\kappa^{++} \times \kappa^+]^{<\kappa}$. There is some $\alpha < \kappa^+$ such that*

$$V[G] \models \text{“ } \alpha \in A_a^{SD} \cap B. \text{ ”}$$

Proof. Let $p \in \mathbb{P}$, and find $\alpha \in B$ such that $\alpha \notin \pi_1[a] \cup \mathrm{dom}^1(p)$. Then choose $q \leq p$ such that $q(i, \alpha, \beta) = 1$ and $q(i, \beta, \alpha) = 0$ for every pair $(i, \beta) \in a$.

Forcing below q, we obtain a model $V[G]$ in which for each pair $(i, \beta) \in a$ we have $f^i(\alpha, \beta) = 1$ and $f^i(\beta, \alpha) = 0$, and hence $\alpha \in A_{(i,\beta)}$ while $\beta \notin A_{(i,\alpha)}$.

It follows from the definitions that we have $(f^i)^{SD,e}(\alpha, \beta) = 1$ for all $(i, \beta) \in a$, regardless of the values of $e_i(\alpha)$ and $e_i(\beta)$, so that $\alpha \in A_a^{SD}$. ▲

Corollary 4.8. Let $B \in [\kappa^+]^\kappa \cap V$ and $a \in [\kappa^{++} \times \kappa^+]^{<\kappa}$. Then

$$V[G] \models A_a^{SD} \cap B \neq \emptyset.$$

Corollary 4.9. Let $B \in [\kappa^+]^{\kappa^+} \cap V$ and $a \in [\kappa^{++} \times \kappa^+]^{<\kappa}$. Then

$$V[G] \models A_a^{SD} \cap B \text{ is unbounded in } \kappa^+$$

4.3. **Filters.** Working in $V[G]$, for each $\Gamma \in [\kappa^{++}]^{\kappa^{++}}$ we let $\mathcal{F}^j(\Gamma)$ be the filter generated by all sets of the form $A_{(i,\beta)}^j$ for $i \in \Gamma$, where j is one of 'no superscript', D or SD. To be a little more explicit,

Definition 4.10. Let $\Gamma \in [\kappa^{++}]^{\kappa^{++}}$ and $j \in \{\text{'no superscript'}, D, SD\}$. Then

$$\mathcal{F}^j(\Gamma) = \{B \subseteq \kappa^+ \,:\, \exists a \in [\Gamma \times \kappa^+]^{<\omega} \; A_a^j \subseteq B\}.$$

Remark 4.11. It follows easily from Lemma (4.7) that for $\Gamma \in [\kappa^{++}]^{\kappa^{++}}$ the $\mathcal{F}^j(\Gamma)$-large sets are unbounded in κ^+, but there are bounded $\mathcal{F}^j(\Gamma)$-positive sets.

Following Balogh and Gruenhage we introduce auxiliary filters $\mathcal{F}^j_\kappa(\Gamma)$, the least κ-complete filters extending the filters $\mathcal{F}^j(\Gamma)$ and containing all sets of the form $\kappa^+ \setminus \gamma$ for $\gamma < \kappa^+$; more explicitly

Definition 4.12. For $\Gamma \in [\kappa^{++}]^{\kappa^{++}}$ and j one of 'no superscript', D or SD

$$\mathcal{F}^j_\kappa(\Gamma) = \{\, B \subseteq \kappa^+ \,:\, \exists a \in [\Gamma \times \kappa^+]^{<\kappa}\ \exists \gamma < \kappa^+\ (A^j_a \cap (\kappa^+ \setminus \gamma) \subseteq B)\,\}.$$

Remark 4.13. In contrast to what pertains for the filters $\mathcal{F}^j(\Gamma)$, every $\mathcal{F}^j_\kappa(\Gamma)$-positive set is unbounded in κ^+.

5. An ultrafilter after forcing over a Laver prepared model

In this section we suppose that $V = V_0[H]$ where κ is supercompact in V_0 and H is generic over V_0 for a Laver preparation iteration $\mathbb{L}$, and we refer to [2] for well-known facts about lifting elementary embeddings.

Proposition 5.1. In $V[G]$ there is a κ-complete uniform weakly normal ultrafilter U on κ^+ such that $A^{SD}_{(i,\delta)} \in U$ for all $i < \kappa^{++}$ and $\delta < \kappa^+$.

Proof. The argument is a variation on Laver's argument that κ is supercompact in $V[G]$. Let $j : V_0 \longrightarrow M$ be an embedding witnessing that κ is κ^{+++}-supercompact in V_0, with the additional properties that

- $j(\mathbb{L}) \restriction \kappa + 1 \simeq \mathbb{L} * Add(\kappa, \kappa^{++})$
- The "tail forcing" $j(\mathbb{L})/H * G$ is κ^{+++}-closed in $M[H * G]$.

Note that by standard arguments $V_0[H * G] \models {}^{\kappa^{+++}}M[H * G] \subseteq M[H * G]$, so that $j(\mathbb{L})/H * G$ is κ^{+++}-closed in $V_0[H * G] = V[G]$. Note also that $j \restriction \kappa^{++} \in M$.

Since $\mathbb{L}$ is an iteration with bounded supports, if I is $j(\mathbb{L})/H * G$-generic over $V[G]$ then $j``H \subseteq H * G * I$, and it follows that we may lift $j : V_0 \longrightarrow M$ to obtain $j : V_0[H] = V \longrightarrow M[H * G * I]$. Now $j``G \in M[H * G * I]$, $M[H * G * I] \models \overline{\overline{j``G}} < j(\kappa)$, and $j``G$ is directed, so (by the elementarity of j), if m_0 is the union of $j``G$ then $m_0 \in j(Add(\kappa, \kappa^{++}))$ and $m_0 \leq j(p)$ for all $p \in G$.

It is easy to see that:

- $m_0^\eta = m_0(\eta, ., .) \neq \emptyset$ if and only if $\eta \in j``\kappa^{++}$.
- If $\eta = j(i)$ for some $i < \kappa^{++}$ then $\mathrm{supp}(m_0^\eta) = j``\kappa^+ \times j``\kappa^+$.
- For γ, $\delta < \kappa^+$ and $\eta = j(i)$, $m_0(\eta, j(\gamma), j(\delta)) = 1$ if $\gamma \in A_{(i,\delta)}$ and $m_0(\eta, j(\gamma), j(\delta)) = 0$ if $\gamma \notin A_{(i,\delta)}$.

For our purposes we need a strengthening m_1 of m_0. This is defined as follows, where we set $\mu = \sup j``\kappa^+$:

- $m_1^\eta = m_1(\eta, ., .) \neq \emptyset$ if and only if $\eta \in j``\kappa^{++}$.
- If $\eta = j(i)$ for some $i < \kappa^{++}$ then:
    - $\mathrm{supp}(m_1^\eta) = (j``\kappa^+ \cup \{\mu\}) \times j``\kappa^+$
    - $m_1^\eta \restriction j``\kappa^+ \times j``\kappa^+ = m_0^\eta$
    - $m_1(\eta, \mu, j(\delta)) = 1$ for all $\delta < \kappa^+$

It is clear that $m_1 \in j(Add(\kappa, \kappa^{++}))$.

Since $m_1 \leq m_0 \leq j(p)$ for all $p \in G$, if we force over $V[G * I]$ with $j(Add(\kappa, \kappa^{++}))$ below m_1 we obtain a generic object J such that $j``G \subseteq J$, and so by standard arguments we may lift the embedding j to obtain a generic embedding $j : V[G] \longrightarrow M[H * G * I * J]$ defined in $V[G * I * J]$. We note that by our choice of m_1 we have arranged that $\mu \in j(A_{(i,\delta)})$ for all $i < \kappa^{++}$ and $\delta < \kappa^+$; since $\mu > j(\delta)$ it follows by Remark 4.6.3 that $\mu \in j(A^{SD}_{(i,\delta)})$.

Let $U = \{B \in V[G] : \mu \in j(B)\}$. Since $2^{\kappa^+} = \kappa^{++}$ in $V[G]$ and the forcing $(j(\mathbb{L})/H * G) * j(Add(\kappa, \kappa^{++}))$ is κ^{+++}-closed in $V[G]$, we see that $U \in V[G]$. Standard arguments now show that U is a κ-complete uniform weakly normal ultrafilter on κ^+ and that $A^{SD}_{(i,\delta)} \in U$ for all $i < \kappa^{++}$ and $\delta < \kappa^+$ ▲

6. A Dowker filter and a super-Dowker filter

Theorem 6.1. *Let κ be uncountable with $\kappa^{<\kappa} = \kappa$. Let G be generic over V for $Add(\kappa, \kappa^{++})$. Then in $V[G]$ there is a Dowker filter on κ^+.*

Theorem 6.2. *Let V be Laver's model in which κ is supercompact and the supercompactness of κ is indestructible under κ-directed closed forcing, and let G be generic for $Add(\kappa, \kappa^{++})$. Then in $V[G]$ there is a super-Dowker filter on κ^+.*

We run the proofs of our main theorems simultaneously, so we must synchronize notation. We work in $V[G]$ in either case.

Notation 6.3. We will work in two settings, which we call the *arbitrary setting* and the *prepared supercompact setting*.

- In the arbitrary setting, κ is an arbitrary uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$. In this setting we work in $V[G]$ and set $j = D$, $\Gamma = \kappa^{++}$, $\mathcal{F} = \mathcal{F}^D(\kappa^{++})$ and $\mathfrak{F} = \mathcal{F}^D_\kappa(\kappa^{++})$.
- In the prepared supercompact setting, κ is supercompact, V is a Laver prepared extension and as in the previous section U is κ-complete uniform ultrafilter on κ^+ defined in $V[G]$ with the property

that $A^{SD}_{i,\beta} \in U$ for all $i < \kappa^{++}$ and $\beta < \kappa^+$. Recall from Definition 4.3 that we enumerated all partitions of κ^+ as (E_i, E_i^c) for $i < \kappa^{++}$. In this setting we work in $V[G]$ and set $j = SD$, $\Gamma = \{i < \kappa^{++} : E_i \in U\}$, $\mathcal{F} = \mathcal{F}^{SD}(\Gamma)$ and $\mathfrak{F} = U$.

We note that in both settings $\mathfrak{F}$ is a κ-complete uniform filter on κ^+ which extends $\mathcal{F}$.

It is clear that, in $V[G]$, in the arbitrary setting the filter $\mathcal{F}$ enjoys property II, and in the prepared supercompact setting $\mathcal{F}$ enjoys property II$^+$. So to finish the proofs of the theorems, that we have a Dowker or a super-Dowker filter, we just need to prove that $\mathcal{F}$ has property I.

Proposition 6.4. In $V[G]$, the filter $\mathcal{F}$ has property I.

Proof. In $V[G]$ let $\langle B_\alpha : \alpha < \kappa^+\rangle$ be a sequence of sets such that $B_\alpha \in \mathcal{F}$ for all $\alpha < \kappa^+$. Let $\phi : \kappa^+ \longrightarrow [\Gamma \times \kappa^+]^{<\omega}$ be such that $A^j_{\phi(\alpha)} \subseteq B_\alpha$ for all $\alpha < \kappa^+$. Then clearly it will suffice to establish property I for the sequence $\langle A^j_{\phi(\alpha)} : \alpha < \kappa^+\rangle$.

We will now make appeals to both halves of Lemma (3.1) with $\lambda = \kappa$ and $\mathcal{G} = \mathfrak{F}$.

Firstly, we set $C = \kappa^+$ and let $h : \kappa^+ \longrightarrow [\Gamma]^{<\omega}$ be given by $h(\alpha) = \pi_0[\phi(\alpha)]$. By Lemma (3.1.2) there is a $D \in \mathfrak{F}^+$ and an $a \in [\Gamma]^{<\omega}$ such that:

- $\bullet_1$ If $i \notin a$ then $\{\alpha \in D : i \in \pi_0[\phi(\alpha)]\}$ is $\mathfrak{F}$-small.

For each $\alpha < \kappa^+$ define $\varepsilon_\alpha : a \longrightarrow 2$ by $\varepsilon_\alpha(i) = e_i(\alpha)$.

Shrinking D if necessary we may also find a function $\varepsilon : a \to 2$ such that:

- $\bullet_2$ $\varepsilon_\alpha = \varepsilon$ for all $\alpha \in D$.

We note that in the prepared supercompact setting, $E_i \in U = \mathfrak{F} = \mathfrak{F}^+$ for all $i \in a$, and so necessarily $D \subseteq E_i$ and $\varepsilon(i) = 1$ for all $i \in a$.

Then we set $C = D$ and $h = \phi$. Applying Lemma (3.1.1) we find a set $R \in [\Gamma \times \kappa^+]^{<\kappa}$ such that:

- $\bullet_3$ For all $R' \in [\Gamma \times \kappa^+]^{<\kappa}$ with $R' \supseteq R$
$$\{\alpha \in D : \phi(\alpha) \cap R' \subseteq R\} \in \mathfrak{F}^+.$$

We set $A^j = A^j_R$, and note that in each setting $A^j \in \mathfrak{F}$.

Now we work in V. Let $\dot{\phi}$ be a name for ϕ, so that it is forced that properties $\bullet_1$, $\bullet_2$ and $\bullet_3$ above hold. Let $\dot{a}$, $\dot{D}$, $\dot{\varepsilon}$, $\dot{R}$, $\dot{A}^j$, $\langle \dot{\varepsilon}_\alpha : \alpha < \kappa^+\rangle$, $\dot{\Gamma}$, $\dot{\mathcal{F}}$, $\dot{\mathfrak{F}}$ name the relevant objects.

Find $p \in \mathbb{P}$, $a \in [\kappa^{++}]^{<\omega}$, $\varepsilon : a \to 2$, and $R \in [\kappa^{++} \times \kappa^+]^{<\kappa}$ such that:

- $\bullet_4$ $p \Vdash \dot{a} = \check{a}$, $\dot{\varepsilon} = \check{\varepsilon}$, $\dot{R} = \check{R}$, and $\dot{D} \in \dot{\mathfrak{F}}^+$.

We note that:

- $\bullet_5$ $p \Vdash \check{a} \subseteq \dot{\Gamma}$ and $p \Vdash \check{R} \subseteq \dot{\Gamma} \times \kappa^+$.
- $\bullet_6$ $p \Vdash \forall \alpha \in \dot{D}\ \dot{\varepsilon}_\alpha = \check{\varepsilon}$.

We have that p forces that the set $\dot{D} \cap \dot{A}^j$ is unbounded in κ^+. So there are unboundedly many $\alpha < \kappa^+$ such that some extension p'_α of p forces that $\alpha \in \dot{A}^j \cap \dot{D}$. Hence we can find an unbounded set $H_0 \subseteq \kappa^+$ and $\langle (p_\alpha, x_\alpha) : \alpha \in H_0 \rangle$ such that:

- $\bullet_7$ $p_\alpha \leq p$ and $p_\alpha \Vdash \alpha \in \dot{D} \cap \dot{A}^j$.
- $\bullet_8$ $x_\alpha \in [\kappa^{++} \times \kappa^+]^{<\omega}$ and $p_\alpha \Vdash \dot{\phi}(\alpha) = x_\alpha$, in particular $p_\alpha \Vdash \check{x}_\alpha \subseteq \dot{\Gamma} \times \kappa^+$.
- $\bullet_9$ $\mathrm{dom}(p_\alpha) = \mathrm{dom}^0(p_\alpha) \times \mathrm{dom}^1(p_\alpha) \times \mathrm{dom}^1(p_\alpha)$.
- $\bullet_{10}$ $a \cup \pi_0[R] \cup \pi_0[x_\alpha] \subseteq \mathrm{dom}^0(p_\alpha)$ and $\{\alpha\} \cup \pi_1[R] \cup \pi_1[x_\alpha] \subseteq \mathrm{dom}^1(p_\alpha)$.

By repeated applications of the Δ-system lemma we find an unbounded set $H_1 \subseteq H_0$ together with $\Delta^0 \in [\kappa^{++}]^{<\kappa}$, $\Delta^1 \in [\kappa^+]^{<\kappa}$, $r : \Delta^0 \times \Delta^1 \times \Delta^1 \to 2$ such that:

- $\bullet_{11}$ $\langle \mathrm{dom}^0(p_\alpha) : \alpha \in H_1 \rangle$ forms a Δ-system with root Δ^0, $\langle \mathrm{dom}^1(p_\alpha) : \alpha \in H_1 \rangle$ forms a Δ-system with root Δ^1, and $p_\alpha \restriction \Delta^0 \times \Delta^1 \times \Delta^1 = r$ for all $\alpha \in H_1$.

It is easy to see that:

- $\bullet_{12}$ $R \subseteq \Delta^0 \times \Delta^1$ and $a \subseteq \Delta^0$.
- $\bullet_{13}$ $r \leq p$.

Replacing H_1 by $H_1 \setminus \Delta_1$, we may also assume that:

- $\bullet_{14}$ For all $\alpha \in H_1$, $\alpha \in \mathrm{dom}^1(p_\alpha) \setminus \Delta^1$.

Force below r to obtain a generic extension $V[G]$.

Working in $V[G]$, we will choose an ordinal $\beta \in D$ satisfying various conditions: this will be possible since the first four conditions hold for $\mathfrak{F}$-almost every $\beta \in D$ and the fifth one holds on an $\mathfrak{F}$-positive set of elements of D.

- $\bullet_{15}$ $\beta \notin \Delta^1$ ($\mathfrak{F}$-almost every $\beta \in D$ satisfies this property because $\mathfrak{F}$ concentrates on final segments of κ^+).
- $\bullet_{16}$ $\pi_0[\phi(\beta)] \cap \Delta^0 \subseteq a$ ($\mathfrak{F}$-almost every $\beta \in D$ satisfies this property since $\overline{\overline{\Delta^0 \setminus a}} < \kappa$, $\{\beta \in D : i \in \pi_0[\phi(\beta)]\}$ is $\mathfrak{F}$-small for each $i \in \Delta^0 \setminus a$ by ($\bullet_1$) above, and $\mathfrak{F}$ is κ-complete).

- $\bullet_{17}$ $\beta \in A^j_{\Delta^0\times\Delta^1}$ ($\mathfrak{F}$-almost every $\beta \in D$ satisfies this property because $\overline{\overline{\Delta^0 \times \Delta^1}} < \kappa$, so that $A^j_{\Delta_0\times\Delta_1} \in \mathfrak{F}$).
- $\bullet_{18}$ $\beta \in A^j$ ($\mathfrak{F}$-almost every $\beta \in D$ satisfies this property since $A^j \in \mathfrak{F}$)
- $\bullet_{19}$ $\phi(\beta) \cap (\Delta^0 \times \Delta^1) \subseteq R$ (an $\mathfrak{F}$-positive set of $\beta \in D$ satisfies this property by ($\bullet_3$) above, setting $R' = \Delta^0 \times \Delta^1$).

We now work in V, and choose $\overline{p} \le r$ which forces that $\beta \in \dot{D}$ and has properties ($\bullet_{15}$)-($\bullet_{19}$), and also forces that $\dot{\phi}(\beta) = y$ for some $y \in [\kappa^{++} \times \kappa^+]^{<\omega}$, so that $\overline{p} \Vdash y \subseteq \dot{\Gamma} \times \kappa^+$. Clearly:

- $\bullet_{20}$ $\pi_0[y] \cap \Delta^0 \subseteq a$.
- $\bullet_{21}$ $y \cap (\Delta^0 \times \Delta^1) \subseteq R$.

Extending $\overline{p}$ if necessary, we may also assume that:

- $\bullet_{22}$ $\{\beta\} \cup \pi_1[y] \subseteq \mathrm{dom}^1(\overline{p})$ and $\pi_0[y] \subseteq \mathrm{dom}^0(\overline{p})$.

We now appeal to ($\bullet_{11}$) to choose $\alpha \in H_1$ such that:

- $\bullet_{23}$ $(\mathrm{dom}^1(p_\alpha) \setminus \Delta^1) \cap \mathrm{dom}^1(\overline{p}) = \emptyset$.
- $\bullet_{24}$ $(\mathrm{dom}^0(p_\alpha) \setminus \Delta^0) \cap \mathrm{dom}^0(\overline{p}) = \emptyset$.

We claim that $\overline{p}$ is compatible with p_α. This holds because

$\mathrm{dom}^1(\overline{p}) \cap \mathrm{dom}^1(p_\alpha) \subseteq \Delta^1$ by ($\bullet_{23}$),

$\mathrm{dom}^0(\overline{p}) \cap \mathrm{dom}^0(p_\alpha) \subseteq \Delta^0$ by ($\bullet_{24}$),

$p_\alpha \restriction \Delta^0 \times \Delta^1 \times \Delta^1 = r$ by ($\bullet_{11}$), and

$\overline{p} \restriction \Delta^0 \times \Delta^1 \times \Delta^1 = r$ by the choice of $\overline{p}$.

The strategy to complete the proof will be to extend $p_\alpha \cup \overline{p}$ to a condition p^* forcing that $\alpha \in A^j_{\phi(\beta)}$ and $\beta \in A^j_{\phi(\alpha)}$. Before defining p^* we will give some motivation for its definition.

Since $p_\alpha \cup \overline{p}$ forces that $\phi(\beta) = y$ and $\phi(\alpha) = x_\alpha$, we need to force that $\alpha \in A^j_y$ and $\beta \in A^j_{x_\alpha}$. Since p_α forces that $\alpha \in A^j_R$, we need only arrange that α is (ultimately) forced into $A^j_{y\setminus R}$. Since $\overline{p}$ forces that $\beta \in A^j_{\Delta^0\times\Delta^1}$, we need only arrange that β is (ultimately) forced into $A^j_{x_\alpha\setminus\Delta^0\times\Delta^1}$.

In order to accomplish this by extending $p_\alpha \cup \overline{p}$, we will need to control the values of the generic function f at tuples of the form (i, α, γ) and (i, γ, α) for $(i, \gamma) \in y \setminus R$, and also at tuples of the form (j, β, δ) and (j, δ, β) for $(j, \delta) \in x_\alpha \setminus R$. The key point is that (as we will see) none of these tuples appear in $\mathrm{dom}(p_\alpha) \cup \mathrm{dom}(\overline{p})$, and that these four types of tuples are mostly distinct. We note that:

- $\bullet_{25}$ By ($\bullet_{21}$) we have $y \cap \Delta^0 \times \Delta^1 \subseteq R$, so that $y \setminus R = y \setminus \Delta^0 \times \Delta^1$.
- $\bullet_{26}$ By ($\bullet_{14}$) we have $\alpha \in \mathrm{dom}^1(p_\alpha) \setminus \Delta^1$, so that by ($\bullet_{23}$) we have $\alpha \notin \mathrm{dom}^1(\bar{p})$.
- $\bullet_{27}$ By ($\bullet_{22}$) and ($\bullet_{15}$) we have $\beta \in \mathrm{dom}^1(\bar{p}) \setminus \Delta^1$, so that by ($\bullet_{23}$) we have $\beta \notin \mathrm{dom}^1(p_\alpha)$.

Claim 6.5. For all $(i,\gamma) \in y \setminus \Delta^0 \times \Delta^1$, (i,α,γ) and (i,γ,α) are not elements of $\mathrm{dom}(p_\alpha) \cup \mathrm{dom}(\bar{p})$.

Proof. By ($\bullet_{26}$) it is clear that these tuples are not elements of $\mathrm{dom}(\bar{p})$. Now we note that by our hypothesis on (i,γ) and ($\bullet_{22}$), $i \in \pi^0[y] \subseteq \mathrm{dom}^0(\bar{p})$ and $\gamma \in \pi^1[y] \subseteq \mathrm{dom}^1(\bar{p})$. If $(i,\alpha,\gamma) \in \mathrm{dom}(p_\alpha)$ then also $i \in \mathrm{dom}^0(p_\alpha)$ and $\gamma \in \mathrm{dom}^1(p_\alpha)$, so that by ($\bullet_{23}$) and ($\bullet_{24}$) we have $i \in \Delta^0$ and $\gamma \in \Delta^1$. This contradicts our assumption on (i,γ). ▲

Claim 6.6. For all $(j,\delta) \in x_\alpha \setminus \Delta^0 \times \Delta^1$, (j,β,δ) and (j,δ,β) are not elements of $\mathrm{dom}(p_\alpha) \cup \mathrm{dom}(\bar{p})$.

Proof. By ($\bullet_{27}$), it is clear that these tuples are not elements of $\mathrm{dom}(p_\alpha)$. By ($\bullet_{10}$) $j \in \mathrm{dom}^0(p_\alpha)$ and $\delta \in \mathrm{dom}^1(p_\alpha)$, so if either tuple lies in $\mathrm{dom}(\bar{p})$ then arguing as in the last claim we have $j \in \Delta^0$ and $\delta \in \Delta^1$, contradicting our assumption on (j,δ). ▲

Note that:

- $\bullet_{28}$ By ($\bullet_{26}$) and ($\bullet_{22}$),
$$\alpha \notin \mathrm{dom}^1(\bar{p}) \text{ and } \{\beta\} \cup \pi_1[y] \subseteq \mathrm{dom}(\bar{p}), \text{ so } \alpha \neq \beta \text{ and } \alpha \notin \pi_1[y].$$
- $\bullet_{29}$ By ($\bullet_{27}$) and ($\bullet_{10}$),
$$\beta \notin \mathrm{dom}^1(p_\alpha) \text{ and } \pi_1[x_\alpha] \subseteq \mathrm{dom}^1(p_\alpha), \text{ so } \beta \notin \pi_1[x_\alpha].$$

Now let us define:

- $B_1 = \{(i,\alpha,\gamma) : (i,\gamma) \in y \setminus \Delta^0 \times \Delta^1\}$.
- $B_2 = \{(i,\gamma,\alpha) : (i,\gamma) \in y \setminus \Delta^0 \times \Delta^1\}$.
- $C_1 = \{(k,\beta,\delta) : (k,\delta) \in x_\alpha \setminus \Delta^0 \times \Delta^1\}$.
- $C_2 = \{(k,\delta,\beta) : (k,\delta) \in y \setminus \Delta^0 \times \Delta^1\}$.

Claim 6.7. $B_1 \cap C_1$, $B_1 \cap B_2$, $B_2 \cap C_2$ and $C_1 \cap C_2$ are all empty.

Proof. Immediate from ($\bullet_{28}$) and ($\bullet_{29}$). ▲

Claim 6.8. For all $(i,\alpha,\beta) \in B_1 \cap C_2$, $i \in a$.

Proof. From the definitions, $(i, \beta) \in y \setminus \Delta^0 \times \Delta^1$ and $(i, \alpha) \in x_\alpha \setminus \Delta^0 \times \Delta^1$. We note that by $(\bullet_{22})$ and $(\bullet_{10})$ we have $i \in \pi_0[y] \subseteq \mathrm{dom}^0(\overline{p})$, and also $i \in \pi_0[x_\alpha] \subseteq \mathrm{dom}^0(p_\alpha)$, so that by $(\bullet_{24})$ we have $i \in \Delta^0$. By $(\bullet_{20})$ we have $\pi_0[y] \cap \Delta_0 \subseteq a$, and we conclude that $i \in a$. ▲

We also note that $(i, \alpha, \beta) \in B_1 \cap C_2 \iff (i, \beta, \alpha) \in B_2 \cap C_1$.

We now define a condition $p^* \leq p_\alpha \cup \overline{p}$ with $\mathrm{dom}(p^*) = \mathrm{dom}(p_\alpha) \cup \mathrm{dom}(\overline{p}) \cup (B_1 \cup C_1 \cup B_2 \cup C_2)$. We set $p^*(t) = 1$ for $t \in B_1$ and $t \in C_1$, and $p^*(t) = 0$ for $t \in B_2 \setminus C_1$ and $t \in C_2 \setminus B_1$.

Claim 6.9. p^* forces that $\alpha \in A^j_{\phi(\beta)}$.

Proof. As we saw already, it is enough to verify that p^* forces $\alpha \in A^j_{(i,\gamma)}$ for all $(i, \gamma) \in y \setminus (\Delta^0 \times \Delta^1)$. We fix such an (i, γ) and note that we need to verify that p^* forces $(f^i)^{j,e_i}(\alpha, \gamma) = 1$.

By definition $(i, \alpha, \gamma) \in B_1$, so that p^* forces $f^i(\alpha, \gamma) = 1$, and $(i, \gamma, \alpha) \in B_2$. If $(i, \gamma, \alpha) \notin C_1$ then p^* forces $f^i(\gamma, \alpha) = 0$ and so easily p^* forces $(f^i)^{j,e_i}(\alpha, \gamma) = 1$.

If $(i, \gamma, \alpha) \in C_1$ then $(i, \alpha, \gamma) \in C_2$, so that $\gamma = \beta$ and by Claim 6.8 we have $i \in a$. Since $p^* \leq p_\alpha \cup \overline{p}$, $p^* \Vdash \alpha, \beta \in \dot{D} \cap \dot{A}^j$ and so by $(\bullet_2)$, $(\bullet_4)$ and $(\bullet_6)$, $p^* \Vdash \dot{\varepsilon}_\alpha(i) = \varepsilon(i) = \dot{\varepsilon}_\beta(i)$. Recalling that in $V[G]$ we defined $\epsilon_\zeta(j) = e_j(\zeta)$ for all relevant j and ζ, we see that $p^* \Vdash \dot{e}_i(\alpha) = \dot{e}_i(\beta)$. Moreover, as we noted after $(\bullet_2)$, in the prepared supercompact setting $\varepsilon(i) = 1$ for all $i \in a$, so that $p^* \Vdash \dot{e}_i(\alpha) = \dot{e}_i(\beta) = 1$. It follows that again, in both settings, p^* forces $(f^i)^{j,e_i}(\alpha, \gamma) = 1$. ▲

Claim 6.10. p^* forces that $\beta \in A^j_{\phi(\alpha)}$.

Proof. It is enough to verify that p^* forces $\beta \in A^j_{(k,\delta)}$ for all $(k, \delta) \in x_\alpha \setminus (\Delta^0 \times \Delta^1)$. The argument is exactly parallel to that for the previous claim. ▲

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