DIAMOND AND ANTICHAINS

JAMES CUMMINGS AND ERNEST SCHIMMERLING

Abstract. It is obvious that \( \diamond \) implies the existence of an antichain of stationary sets of cardinality \( 2^{\omega_1} \), which is the largest possible cardinality. We show that the obvious antichain is not maximal and find a less obvious extension of it by \( \aleph_2 \) more stationary sets.

Let \( A = \langle A_\alpha \mid \alpha < \omega_1 \rangle \) be a \( \diamond \)-sequence. For \( X \subseteq \omega_1 \), let

\[
S_X = \{ \alpha < \omega_1 \mid X \cap \alpha = A_\alpha \}.
\]

Then each \( S_X \) is stationary. Moreover, if \( X \neq Y \), then \( S_X \cap S_Y \) is bounded in \( \omega_1 \). Thus \( \langle S_X \mid X \subseteq \omega_1 \rangle \) is an antichain of stationary sets of cardinality \( 2^{\omega_1} \). Part of what we will show is that \( \langle S_X \mid X \subseteq \omega_1 \rangle \) is not a maximal antichain.

First we record some easy observations. In the Boolean completion of \( P(\omega_1)/NS \), \( S_X \) is the Boolean value of the sentence

\[
\text{If } j_G : V \longrightarrow M \text{ is the generic ultrapower embedding (where the wellfounded part of } M \text{ is identified with its Mostowski collapse), then}
\]

\[
j_G(A)_{\omega_1^V} = X.
\]

Moreover, the Boolean value of the sentence

\[
j_G(A)_{\omega_1^V} \in V
\]

is precisely the join of the antichain \( \langle S_X \mid X \subseteq \omega_1 \rangle \).

If \( X \) is a set of countable ordinals, then we write \( \text{acc}(X) \) for the set of accumulation points of \( X \), that is \( \{ \alpha < \omega_1 \mid \sup(X \cap \alpha) = \alpha \} \).

Definition 1. Let \( T = \{ \alpha < \omega_1 \mid S_{A_\alpha} \cap \text{acc}(A_\alpha) = \{ \alpha \} \} \).

In other words, \( T \) is the collection of \( \beta < \omega_1 \) such that \( \beta = \sup(A_\beta) \) but there are no accumulation points \( \alpha < \beta \) of \( A_\beta \) such that \( A_\alpha = A_\beta \cap \alpha \).

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Lemma 2. $T$ is stationary and $|T \cap S_X| \leq 1$ for all $X \subseteq \omega_1$. In particular, $(S_X \mid X \subseteq \omega_1)$ is not a maximal antichain in NS.

Proof. Consider an arbitrary club $C \subseteq \omega_1$. Then $\text{acc}(C)$ is a club subset of $C$ and $S_C \cap \text{acc}(C)$ is stationary. Let $\alpha$ be the least element of $S_C \cap \text{acc}(C)$. Then $\alpha$ is the unique element of $T \cap S_C \cap \text{acc}(C)$. In particular, $\alpha \in T \cap C$. This shows that $T$ is stationary.

Now let $X \subseteq \omega_1$ and $\alpha < \beta$ be elements of $S_X$. Then $A_\alpha = A_\beta \cap \alpha$. It is enough to show that if $\beta \in T$, then $\alpha \not\in T$. Assume $\beta \in T$. Because $\alpha \in \beta \cap S_{A_\beta}$ and $\beta \in T$, we may conclude that $\alpha \not\in \text{acc}(A_\beta)$. So $\alpha \not\in \text{acc}(A_\alpha)$. Hence $\alpha \not\in T$. □

Let us record the easy observation that $T$ is the Boolean value of the sentence

$$j_G(A)_\omega^Y \text{ is unbounded in } \omega_1^Y \text{ but if } \alpha < \omega_1^Y \text{ is an accumulation point of } j_G(A)_\omega^Y, \text{ then } A_\alpha \neq j_G(A)_\omega^Y \cap \alpha.$$ 

We have shown that our $\diamond$-sequence is also a $\diamond\langle T^\prime \rangle$ sequence where $T^\prime = \omega_1 - T$. Many useful examples are obtained by the standard trick of redefining $A_\alpha$ for $\alpha \in T$. For example, if $\alpha \in T$, then let $\alpha' \subseteq \alpha$ be an unbounded subset of $\alpha$ of order type $\omega$, while if $\alpha \in T^\prime$, then keep $\alpha' = \alpha$. Observe that $T$ forces

$$j_G(A')_\omega^Y \text{ is unbounded in } \omega_1^Y \text{ and has order type } \omega.$$ 

Note that in this example, if $T'$ is computed from $A'$ the same way that $T$ was computed from $A$, then $T \subseteq T'$.

Our next goal is to define sets $T_{\delta, \rho}$ so that

$$(T_{\delta, \rho} \mid \delta < \omega_2 \& \rho < \omega_1) \cap \langle S_X \mid X \subseteq \omega_1 \rangle$$

is an antichain of stationary sets. For $\omega_1 \leq \delta < \omega_2$, let $e_\delta$ map $\omega_1$ onto $\delta$ with $e_\delta(0) = 0$. This choice will remain fixed for the rest of our note.

Definition 3. By recursion on $\delta < \omega_2$, define $\partial^\delta X$ for $X \subseteq \omega_1$ as follows. Let

$$\partial^0 X = \text{acc}(X).$$

If $\delta = \gamma + 1$, then let

$$\partial^\delta X = \{ \alpha < \omega_1 \mid \text{ot}(\alpha \cap S_X \cap \partial^\gamma X) = \alpha \}.$$ 

If $\delta$ is a countable limit ordinal, then let

$$\partial^\delta X = \bigcap_{\gamma < \delta} \partial^\gamma X.$$ 

If $\delta$ is an uncountable limit ordinal, then let

$$\partial^\delta X = \bigcup_{i < \omega_1} \partial^{\delta^{\langle i \rangle}} X.$$
We remark that there is an alternative approach in which the countable limit stages in the definition of $\partial^\delta X$ are handled using diagonal intersections instead of intersections. There is very little difference between the two approaches, and so we simply picked one randomly here.

**Lemma 4.** Let $\delta < \omega_2$.

1. For all $X \subseteq \omega_1$ and $\alpha < \omega_1$,
   \[ \alpha \cap \partial^\delta X = \alpha \cap \partial^\delta (\alpha \cap X). \]
   If, in addition, $\alpha \in \text{acc}(X)$ and $A_\alpha = X \cap \alpha$, then
   \[ \alpha \in \partial^\delta X \iff \alpha \in \partial^\delta A_\alpha. \]

2. For all unbounded $X \subseteq \omega_1$, $\partial^\delta X$ is club in $\omega_1$.

3. For all $\varepsilon$ such that $\delta \leq \varepsilon < \omega_2$, there exists $\iota < \omega_1$ such that
   \[ (\partial^\delta X) - \varepsilon \subseteq \partial^\delta X \]
   whenever $X \subseteq \omega_1$.

4. If $C$ is a club subset of $\omega_1$, then
   \[ \partial^\delta C \subseteq C. \]

The proof of Lemma 4 is standard. Let us write $\iota_{\delta, \varepsilon}$ for the least $\iota$ as in Part 3 of Lemma 4. Our requirement that $e_\delta(0) = 0$ was to ensure that $\iota_{0, \delta} = 0$.

**Definition 5.** For $\delta < \omega_2$ and $\rho < \omega_1$, let

\[ T_{\delta, \rho} = \{ \alpha < \omega_1 \mid \alpha \in \partial^\delta A_\alpha \& \text{ot} (\alpha \cap S_{A_\alpha} \cap \partial^\delta A_\alpha) = \rho \}. \]

In other words, $T_{\delta, \rho}$ is the set of $\alpha < \omega_1$ such that $S_{A_\alpha} \cap \partial^\delta A_\alpha$ has order type $\rho + 1$ and greatest element $\alpha$. Note that $T_{0, 0} = T$.

**Theorem 6.** $(T_{\delta, \rho} \mid \delta < \omega_2 \& \rho < \omega_1) \cap (S_X \mid X \subseteq \omega_1)$ is an antichain of stationary sets. Moreover, if $P$ and $Q$ are distinct elements of this antichain, then $P \cap Q$ is bounded in $\omega_1$.

Theorem 6 follows immediately from Lemmas 7 – 12 below.

**Lemma 7.** For $\delta < \omega_2$ and $\rho < \omega_1$, $T_{\delta, \rho}$ is stationary in $\omega_1$.

**Proof.** Let $C$ be an arbitrary club subset of $\omega_1$. Then $\partial^\delta C$ is a club subset of $C$ and $S_C \cap \partial^\delta C$ is stationary. Let $\alpha$ be the $\rho$-th element in the increasing enumeration of $S_C \cap \partial^\delta C$. Then $\alpha$ is the unique element of $T_{\delta, \rho} \cap S_C \cap \partial^\delta C$. In particular, $\alpha \in T_{\delta, \rho} \cap C$. \(\square\)

**Lemma 8.** Let $\delta < \omega_2$ and $\rho < \sigma < \omega_1$. Then
\[ T_{\delta, \rho} \cap T_{\delta, \sigma} = \emptyset. \]
Proof. Otherwise, there would be an ordinal \( \alpha \) such that \( \alpha \) is both the \( \rho \)-th and the \( \sigma \)-th element of \( S_{A_\alpha} \cap \partial^\delta A_\alpha \). This is absurd. \( \square \)

**Lemma 9.** Let \( \delta < \omega_2 \) and \( \rho < \omega_1 \). Then

\[
|T_{\delta,\rho} \cap S_X| \leq 1
\]

whenever \( X \subseteq \omega_1 \).

**Proof.** Let \( \alpha < \beta \) be elements of \( S_X \). Then \( A_\alpha = A_\beta \cap \alpha \). For contradiction, suppose that both \( \alpha \) and \( \beta \) are elements of \( T_{\delta,\rho} \). Then, for some \( \sigma < \rho \), \( \alpha \) is the \( \sigma \)-th element of \( \beta \cap S_{A_\beta} \cap \partial^\delta A_\beta \). By Part 1 of Lemma 4 and the fact that \( A_\alpha = A_\beta \cap \alpha \), we have that that \( \alpha \in T_{\delta,\rho} \). This contradicts Lemma 8. \( \square \)

**Lemma 10.** Let \( \delta < \varepsilon < \omega_2 \) and \( \rho, \sigma < \omega_1 \). Suppose that \( \varepsilon \) is a limit ordinal. If \( \varepsilon \) is countable, then

\[
T_{\delta,\rho} \cap T_{\varepsilon,\sigma} \subseteq (\delta + 1) \cup \{\rho\}.
\]

If, on the other hand, \( \varepsilon \) is uncountable and \( \epsilon_\varepsilon (\varepsilon) = \delta + 1 \), then

\[
T_{\delta,\rho} \cap T_{\varepsilon,\sigma} \subseteq \varepsilon \cup \{\rho\}.
\]

**Proof.** First suppose that \( \varepsilon \) is uncountable. Let

\[
\alpha \in T_{\delta,\rho} \cap T_{\varepsilon,\sigma}
\]

such that \( \alpha > \nu \). Because \( \alpha \in T_{\varepsilon,\sigma} \),

\[
\alpha \in \partial^{\delta+1} A_\alpha.
\]

So

\[
\text{ot}(\alpha \cap S_{A_\alpha} \cap \partial^\delta A_\alpha) = \alpha.
\]

Because \( \alpha \in T_{\delta,\rho} \),

\[
\text{ot}(\alpha \cap S_{A_\alpha} \cap \partial^\delta A_\alpha) = \rho.
\]

Thus \( \alpha = \rho \).

The proof when \( \varepsilon \) is countable is essentially the same, so we omit it. \( \square \)

**Lemma 11.** Let \( \gamma < \omega_2 \) and \( \rho, \sigma < \omega_1 \). Then

\[
T_{\gamma,\rho} \cap T_{\gamma+1,\sigma} \subseteq \{\rho\}.
\]

**Proof.** Let \( \alpha \in T_{\gamma,\rho} \cap T_{\gamma+1,\sigma} \). Then \( \alpha \) is both the \( \rho \)-th and the \( \alpha \)-th element of \( S_{A_\alpha} \cap \partial^\delta A_\alpha \). So \( \alpha = \rho \). \( \square \)

**Lemma 12.** Suppose that \( \delta < \varepsilon \) and \( \rho, \sigma < \omega_1 \). Then \( T_{\delta,\rho} \cap T_{\varepsilon+1,\sigma} \) is bounded in \( \omega_1 \).
Proof. Assume for contradiction that $T_{\delta,\rho} \cap T_{\varepsilon+1,\sigma}$ is unbounded in $\omega_1$. Consider an arbitrary $\alpha \in T_{\delta,\rho} \cap T_{\varepsilon+1,\sigma}$. Then $\alpha$ is the $\rho$-th element of

$$S_{A_{\alpha}} \cap \partial^\varepsilon A_{\alpha}$$

and the $\sigma$-th element of

$$S_{A_{\alpha}} \cap \partial^{\varepsilon+1} A_{\alpha}.$$ So $\alpha$ is the $\alpha$-th element of

$$S_{A_{\alpha}} \cap \partial A_{\alpha}$$

by the definition of $\partial^{\varepsilon+1} A_{\alpha}$.

Recall that by Part 3 of Lemma 4, there is an ordinal $\iota = \iota_{\delta,\varepsilon}$ such that for all $\alpha > \iota$,

$$\partial A_{\alpha} - \iota \subseteq \partial^\varepsilon A_{\alpha}.$$ Let us assume that $\alpha$ is large enough that $\alpha > \iota$ and $\iota + \alpha = \alpha$. Then $\alpha$ is the $\alpha$-th element of

$$(S_{A_{\alpha}} \cap \partial^\varepsilon A_{\alpha}) - \iota.$$ Hence $\alpha$ is the $\alpha$-th element of

$$S_{A_{\alpha}} \cap \partial^\varepsilon A_{\alpha}.$$ Therefore $\alpha = \rho$. \qed

That completes the proof of Theorem 6. We close with some remarks and a conjecture.

It is natural to consider the sets $T_{\delta,\rho}$ as being ordered lexicographically according to the index $(\delta, \rho)$. The reader may wonder why we have been listing sets of the form $S_X$ after all those of the form $T_{\delta,\rho}$. The reason is as follows. Let $X$ be unbounded in $\omega_1$. Recall that forcing below $S_X$ produces a generic ultrapower in which

$$X = j_G(A)_{\omega_1^V}.$$ Observe that $\omega_1$ is the $\omega_1$-st element of $\partial^\varepsilon X$ for all $\delta < \omega_2$. Thus, in the obvious sense,

$$\omega_1 \in \partial^{\omega_2} j_G(A)_{\omega_1^V},$$

which goes beyond what happens if we force below any of the sets $T_{\delta,\rho}$.

We conjecture that the join of the antichain

$$\langle T_{\delta,\rho} \mid \delta < \omega_2 \& \rho < \omega_1 \rangle$$

is the Boolean value of the sentence

$$j(A)_{\omega_1^V} \not\in V \text{ and } j(A)_{\omega_1^V} \text{ is unbounded in } \omega_1^V,$$
perhaps under one of the familiar hypotheses on the canonical functions.

It is easy to see that $(\mathcal{T}_{\delta,\rho} \mid \delta < \omega_2 \land \rho < \omega_1)^{V} \cap \langle S_X \mid X \subseteq \omega_1 \rangle$ is not necessarily a maximal antichain. The example we have in mind involves redefining $A$ on $T$ again as follows. Let $A''_\alpha$ be a subset of $\omega$ that codes a wellorder of type $\alpha$ whenever $\alpha \in T$, while if $\alpha \in T^c$, then let $A''_\alpha = A_\alpha$. Then $T$ forces

$$j(A'')_{\omega_1} \notin V \text{ and } j(A''_\alpha) \subseteq \omega.$$ 

Let $R_\beta = \{ \alpha < \omega_1 \mid \text{sup}(A_\alpha) = \beta \}$ for $\beta < \omega_1$. It is obvious that $\langle R_\beta \mid \beta < \omega_1 \rangle$ is an antichain the join of which is the the Boolean value of

$$j(A)_{\omega_1} \text{ is bounded in } \omega^V_1.$$ 

Thus our conjecture above says exactly that

$$\langle T_{\delta,\rho} \mid \delta < \omega_2 \land \rho < \omega_1 \rangle \cap \langle S_X \mid X \subseteq \omega_1 \land \text{sup}(X) = \omega_1 \rangle \cap \langle R_\beta \mid \beta < \omega_1 \rangle$$

is a maximal antichain. Even if the conjecture turns out to be true, we still would not know how to describe the Boolean value of

$$j(A)_{\omega_1} \notin V \text{ and } j(A)_{\omega_1} \text{ is bounded in } \omega^V_1.$$ 

as the join of a specific antichain.

E-mail address: jcumming@andrew.cmu.edu
E-mail address: eschimme@andrew.cmu.edu

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213-3890, USA