DIAMOND AND ANTICHAINS

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Abstract. It is obvious that $\diamondsuit$ implies the existence of an antichain of stationary sets of cardinality $2^{\aleph_1}$, which is the largest possible cardinality. We show that the obvious antichain is not maximal and find a less obvious extension of it by $\aleph_2$ more stationary sets.

Let $A = \langle A_\alpha \mid \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence. For $X \subseteq \omega_1$, let

$$S_X = \{\alpha < \omega_1 \mid X \cap \alpha = A_\alpha\}.$$

Then each S_X is stationary. Moreover, if $X \neq Y$, then $S_X \cap S_Y$ is bounded in ω_1. Thus $\langle S_X \mid X \subseteq \omega_1 \rangle$ is an antichain of stationary sets of cardinality $2^{\aleph_1}$. Part of what we will show is that $\langle S_X \mid X \subseteq \omega_1 \rangle$ is not a maximal antichain.

First we record some easy observations. In the Boolean completion of $\mathcal{P}(\omega_1)/\mathrm{NS}$, S_X is the Boolean value of the sentence

If $j_G : V \longrightarrow M$ is the generic ultrapower embedding (where the wellfounded part of M is is identified with its Mostowski collapse), then

$$j_G(A)_{\omega_1^V} = X.$$

Moreover, the Boolean value of the sentence

$$j_G(A)_{\omega_1^V} \in V$$

is precisely the join of the antichain $\langle S_X \mid X \subseteq \omega_1 \rangle$.

If X is a set of countable ordinals, then we write $\mathrm{acc}(X)$ for the set of accumulation points of X, that is $\{\alpha < \omega_1 \mid \sup(X \cap \alpha) = \alpha\}$.

Definition 1. *Let $T = \{\alpha < \omega_1 \mid S_{A_\alpha} \cap \mathrm{acc}(A_\alpha) = \{\alpha\}\}$.*

In other words, T is the collection of $\beta < \omega_1$ such that $\beta = \sup(A_\beta)$ but there are no accumulation points $\alpha < \beta$ of A_β such that $A_\alpha = A_\beta \cap \alpha$.

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Lemma 2. *T is stationary and $|T \cap S_X| \leq 1$ for all $X \subseteq \omega_1$. In particular, $\langle S_X \mid X \subseteq \omega_1 \rangle$ is not a maximal antichain in* NS.

Proof. Consider an arbitrary club $C \subseteq \omega_1$. Then $\operatorname{acc}(C)$ is a club subset of C and $S_C \cap \operatorname{acc}(C)$ is stationary. Let α be the least element of $S_C \cap \operatorname{acc}(C)$. Then α is the unique element of $T \cap S_C \cap \operatorname{acc}(C)$. In particular, $\alpha \in T \cap C$. This shows that T is stationary.

Now let $X \subseteq \omega_1$ and $\alpha < \beta$ be elements of S_X. Then $A_\alpha = A_\beta \cap \alpha$. It is enough to show that if $\beta \in T$, then $\alpha \notin T$. Assume $\beta \in T$. Because $\alpha \in \beta \cap S_{A_\beta}$ and $\beta \in T$, we may conclude that $\alpha \notin \operatorname{acc}(A_\beta)$. So $\alpha \notin \operatorname{acc}(A_\alpha)$. Hence $\alpha \notin T$. $\square$

Let us record the easy observation that T is the Boolean value of the sentence

> $j_G(A)_{\omega_1^V}$ *is unbounded in* ω_1^V *but if* $\alpha < \omega_1^V$ *is an accumulation point of* $j_G(A)_{\omega_1^V}$*, then* $A_\alpha \neq j_G(A)_{\omega_1^V} \cap \alpha$.

We have shown that our $\diamondsuit$-sequence is also a $\diamondsuit(T^c)$ sequence where $T^c = \omega_1 - T$. Many useful examples are obtained by the standard trick of redefining A_α for $\alpha \in T$. For example, if $\alpha \in T$, then let A'_α be an unbounded subset of α of order type ω, while if $\alpha \in T^c$, then keep $A'_\alpha = A_\alpha$. Observe that T forces

> $j_G(A')_{\omega_1^V}$ *is unbounded in* ω_1^V *and has order type* ω.

Note that in this example, if T' is computed from A' the same way that T was computed from A, then $T \subseteq T'$.

Our next goal is to define sets $T_{\delta,\rho}$ so that

$$\langle T_{\delta,\rho} \mid \delta < \omega_2 \ \&\ \rho < \omega_1 \rangle^\frown \langle S_X \mid X \subseteq \omega_1 \rangle$$

is an antichain of stationary sets. For $\omega_1 \leq \delta < \omega_2$, let e_δ map ω_1 onto δ with $e_\delta(0) = 0$. This choice will remain fixed for the rest of our note.

Definition 3. *By recursion on $\delta < \omega_2$, define $\partial^\delta X$ for $X \subseteq \omega_1$ as follows. Let*

$$\partial^0 X = \operatorname{acc}(X).$$

If $\delta = \gamma + 1$, then let

$$\partial^\delta X = \{\alpha < \omega_1 \mid \operatorname{ot}(\alpha \cap S_X \cap \partial^\gamma X) = \alpha\}.$$

If δ is a countable limit ordinal, then let

$$\partial^\delta X = \bigcap_{\gamma < \delta} \partial^\gamma X.$$

If δ is an uncountable limit ordinal, then let

$$\partial^\delta X = \mathop{\triangledown}_{\iota < \omega_1} \partial^{e_\delta(\iota)} X.$$

We remark that there is an alternative approach in which the countable limit stages in the definition of $\partial^\delta X$ are handled using diagonal intersections instead of intersections. There is very little difference between the two approaches, and so we simply picked one randomly here.

Lemma 4. *Let $\delta < \omega_2$.*

(1) *For all $X \subseteq \omega_1$ and $\alpha < \omega_1$,*
$$\alpha \cap \partial^\delta X = \alpha \cap \partial^\delta(\alpha \cap X).$$
If, in addition, $\alpha \in \operatorname{acc}(X)$ and $A_\alpha = X \cap \alpha$, then
$$\alpha \in \partial^\delta X \iff \alpha \in \partial^\delta A_\alpha.$$
(2) *For all unbounded $X \subseteq \omega_1$, $\partial^\delta X$ is club in ω_1.*
(3) *For all ε such that $\delta \leq \varepsilon < \omega_2$, there exists $\iota < \omega_1$ such that*
$$(\partial^\varepsilon X) - \iota \subseteq \partial^\delta X$$
whenever $X \subseteq \omega_1$.
(4) *If C is a club subset of ω_1, then*
$$\partial^\delta C \subseteq C.$$

The proof of Lemma 4 is standard. Let us write $\iota_{\delta,\varepsilon}$ for the least ι as in Part 3 of Lemma 4. Our requirement that $e_\delta(0) = 0$ was to ensure that $\iota_{0,\delta} = 0$.

Definition 5. *For $\delta < \omega_2$ and $\rho < \omega_1$, let*
$$T_{\delta,\rho} = \left\{\alpha < \omega_1 \mid \alpha \in \partial^\delta A_\alpha \ \&\ \operatorname{ot}\left(\alpha \cap S_{A_\alpha} \cap \partial^\delta A_\alpha\right) = \rho\right\}.$$

In other words, $T_{\delta,\rho}$ is the set of $\alpha < \omega_1$ such that $S_{A_\alpha} \cap \partial^\delta A_\alpha$ has order type $\rho + 1$ and greatest element α. Note that $T_{0,0} = T$.

Theorem 6. *$\langle T_{\delta,\rho} \mid \delta < \omega_2 \ \&\ \rho < \omega_1\rangle^\frown\langle S_X \mid X \subseteq \omega_1\rangle$ is an antichain of stationary sets. Moreover, if P and Q are distinct elements of this antichain, then $P \cap Q$ is bounded in ω_1.*

Theorem 6 follows immediately from Lemmas 7 – 12 below.

Lemma 7. *For $\delta < \omega_2$ and $\rho < \omega_1$, $T_{\delta,\rho}$ is stationary in ω_1.*

Proof. Let C be an arbitrary club subset of ω_1. Then $\partial^\delta C$ is a club subset of C and $S_C \cap \partial^\delta C$ is stationary. Let α be the ρ-th element in the increasing enumeration of $S_C \cap \partial^\delta C$. Then α is the unique element of $T_{\delta,\rho} \cap S_C \cap \partial^\delta C$. In particular, $\alpha \in T_{\delta,\rho} \cap C$. $\square$

Lemma 8. *Let $\delta < \omega_2$ and $\rho < \sigma < \omega_1$. Then*
$$T_{\delta,\rho} \cap T_{\delta,\sigma} = \emptyset.$$

Proof. Otherwise, there would be an ordinal α such that α is both the ρ-th and the σ-th element of $S_{A_\alpha} \cap \partial^\delta A_\alpha$. This is absurd. □

Lemma 9. *Let $\delta < \omega_2$ and $\rho < \omega_1$. Then*

$$|T_{\delta,\rho} \cap S_X| \leq 1$$

whenever $X \subseteq \omega_1$.

Proof. Let $\alpha < \beta$ be elements of S_X. Then $A_\alpha = A_\beta \cap \alpha$. For contradiction, suppose that both α and β are elements of $T_{\delta,\rho}$. Then, for some $\sigma < \rho$, α is the σ-th element of $\beta \cap S_{A_\beta} \cap \partial^\delta A_\beta$. By Part 1 of Lemma 4 and the fact that $A_\alpha = A_\beta \cap \alpha$, we have that that $\alpha \in T_{\delta,\sigma}$. This contradicts Lemma 8. □

Lemma 10. *Let $\delta < \varepsilon < \omega_2$ and $\rho, \sigma < \omega_1$. Suppose that ε is a limit ordinal. If ε is countable, then*

$$T_{\delta,\rho} \cap T_{\varepsilon,\sigma} \subseteq (\delta + 1) \cup \{\rho\}.$$

If, on the other hand, ε is uncountable and $e_\varepsilon(\iota) = \delta + 1$, then

$$T_{\delta,\rho} \cap T_{\varepsilon,\sigma} \subseteq \iota \cup \{\rho\}.$$

Proof. First suppose that ε is uncountable. Let

$$\alpha \in T_{\delta,\rho} \cap T_{\varepsilon,\sigma}$$

such that $\alpha > \iota$. Because $\alpha \in T_{\varepsilon,\sigma}$,

$$\alpha \in \partial^{\delta+1} A_\alpha.$$

So

$$\mathrm{ot}(\alpha \cap S_{A_\alpha} \cap \partial^\delta A_\alpha) = \alpha.$$

Because $\alpha \in T_{\delta,\rho}$,

$$\mathrm{ot}(\alpha \cap S_{A_\alpha} \cap \partial^\delta A_\alpha) = \rho.$$

Thus $\alpha = \rho$.

The proof when ε is countable is essentially the same, so we omit it. □

Lemma 11. *Let $\gamma < \omega_2$ and $\rho, \sigma < \omega_1$. Then*

$$T_{\gamma,\rho} \cap T_{\gamma+1,\sigma} \subseteq \{\rho\}.$$

Proof. Let $\alpha \in T_{\gamma,\rho} \cap T_{\gamma+1,\sigma}$. Then α is both the ρ-th and the α-th element of $S_{A_\alpha} \cap \partial^\gamma A_\alpha$. So $\alpha = \rho$. □

Lemma 12. *Suppose that $\delta < \varepsilon$ and $\rho, \sigma < \omega_1$. Then $T_{\delta,\rho} \cap T_{\varepsilon+1,\sigma}$ is bounded in ω_1.*

Proof. Assume for contradiction that $T_{\delta,\rho} \cap T_{\varepsilon+1,\sigma}$ is unbounded in ω_1. Consider an arbitrary $\alpha \in T_{\delta,\rho} \cap T_{\varepsilon+1,\sigma}$. Then α is the ρ-th element of

$$S_{A_\alpha} \cap \partial^\delta A_\alpha$$

and the σ-th element of

$$S_{A_\alpha} \cap \partial^{\varepsilon+1} A_\alpha.$$

So α is the α-th element of

$$S_{A_\alpha} \cap \partial^\varepsilon A_\alpha$$

by the definition of $\partial^{\varepsilon+1} A_\alpha$.

Recall that by Part 3 of Lemma 4, there is an ordinal $\iota = \iota_{\delta,\varepsilon}$ such that for all $\alpha > \iota$,

$$\partial^\varepsilon A_\alpha - \iota \subseteq \partial^\delta A_\alpha.$$

Let us assume that α is large enough that $\alpha > \iota$ and $\iota + \alpha = \alpha$. Then α is the α-th element of

$$(S_{A_\alpha} \cap \partial^\varepsilon A_\alpha) - \iota.$$

Hence α is the α-th element of

$$S_{A_\alpha} \cap \partial^\delta A_\alpha.$$

Therefore $\alpha = \rho$. □

That completes the proof of Theorem 6. We close with some remarks and a conjecture.

It is natural to consider the sets $T_{\delta,\rho}$ as being ordered lexicographically according to the index (δ, ρ). The reader may wonder why we have been listing sets of the form S_X after all those of the form $T_{\delta,\rho}$. The reason is as follows. Let X be unbounded in ω_1. Recall that forcing below S_X produces a generic ultrapower in which

$$X = j_G(A)_{\omega_1^V}.$$

Observe that ω_1 is the ω_1-st element of $\partial^\delta X$ for all $\delta < \omega_2$. Thus, in the obvious sense,

$$\omega_1 \in \partial^{\omega_2} j_G(A)_{\omega_1^V} ,$$

which goes beyond what happens if we force below any of the sets $T_{\delta,\rho}$.

We conjecture that the join of the antichain

$$\langle T_{\delta,\rho} \mid \delta < \omega_2 \ \&\ \rho < \omega_1 \rangle$$

is the Boolean value of the sentence

$$j(A)_{\omega_1^V} \notin V \textit{ and } j(A)_{\omega_1^V} \textit{ is unbounded in } \omega_1^V,$$

perhaps under one of the familiar hypotheses on the canonical functions.

It is easy to see that $\langle T_{\delta,\rho} \mid \delta < \omega_2 \ \& \ \rho < \omega_1\rangle^\frown\langle S_X \mid X \subseteq \omega_1\rangle$ is not necessarily a maximal antichain. The example we have in mind involves redefining A on T again as follows. Let A''_α be a subset of ω that codes a wellorder of type α whenever $\alpha \in T$, while if $\alpha \in T^c$, then let $A''_\alpha = A_\alpha$. Then T forces

$$j(A'')_{\omega_1^V} \notin V \ \textit{and}\ j(A'')_{\omega_1^V} \subseteq \omega.$$

Let $R_\beta = \{\alpha < \omega_1 \mid \sup(A_\alpha) = \beta\}$ for $\beta < \omega_1$. It is obvious that $\langle R_\beta \mid \beta < \omega_1\rangle$ is an antichain the join of which is the the Boolean value of

$$j(A)_{\omega_1^V} \ \textit{is bounded in}\ \omega_1^V.$$

Thus our conjecture above says exactly that

$$\langle T_{\delta,\rho} \mid \delta < \omega_2 \ \& \ \rho < \omega_1\rangle^\frown\langle S_X \mid X \subseteq \omega_1 \ \& \ \sup(X) = \omega_1\rangle^\frown\langle R_\beta \mid \beta < \omega_1\rangle$$

is a maximal antichain. Even if the conjecture turns out to be true, we still would not know how to describe the Boolean value of

$$j(A)_{\omega_1^V} \notin V \ \textit{and}\ j(A)_{\omega_1^V} \ \textit{is bounded in}\ \omega_1^V.$$

as the join of a specific antichain.

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