CANONICAL STRUCTURE IN THE UNIVERSE OF SET THEORY: PART TWO

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ABSTRACT. We prove a number of consistency results complementary to the ZFC results from our paper [4]. We produce examples of non-tightly stationary mutually stationary sequences, sequences of cardinals on which every sequence of sets is mutually stationary, and mutually stationary sequences not concentrating on a fixed cofinality. We also give an alternative proof for the consistency of the existence of stationarily many non-good points, show that diagonal Prikry forcing preserves certain stationary reflection properties, and study the relationship between some simultaneous reflection principles. Finally we show that the least cardinal where square fails can be the least inaccessible, and show that weak square is incompatible in a strong sense with generic supercompactness.

1. INTRODUCTION

In our paper [4] we prove a number of ZFC results concerning PCF theory, mutual stationarity, square principles and stationary reflection. In that paper we discussed the informal notion of *canonical structure*. This notion is supposed to capture the idea of structure that is not arbitrarily determined by non-constructive existence assumption. For example, structure that requires the axiom of choice to prove its existence may still be independent of any choices made in proving it exists. Cardinals of uncountable cofinality fall into this category. Other examples might include fine structure models of large cardinals. Large cardinal axioms are non-constructive assumptions (as opposed to e.g. the pairing axiom, where we know exactly what the intended object is). However, as a consequence of their existence there is various canonical structure, such as $U \cap L[U]$ for U a normal ultrafilter on a measurable cardinal κ.

The notion of canonical structure is different from the notion of absoluteness. We illustrate this with an example. Assuming the Axiom of Choice, the collection of real numbers has some well-ordered cardinality $\mathfrak{c}$ and this cardinality is independent of the choices made to show it exists. Similarly, one needs the Axiom of Choice to prove that the least regular uncountable ordinal ($\aleph_1$) exists. Both of these objects are "canonical" in our sense, but it is independent of ZFC whether they are in fact identical. We would like to say that these distinct examples of structure that may or may not determine identical objects.

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In this paper we continue to explore canonical structure by proving consistency results complementary to the ZFC results in [4].

After some preliminaries in Section 2, we show the following results.

- In Section 3, we give a forcing construction for a sequence of stationary sets which is mutually stationary but not tightly stationary. The proof involves a combinatorial principle which we dub Coherent Squares.
- In Section 4, we give another forcing construction for a sequence of stationary sets which is mutually stationary but not tightly stationary. The proof involves some lemmas about uniform structures and mutual stationarity which are of independent interest. We also show the consistency of a splitting property for mutually stationary sequences.
- In Section 5 we show that on an increasing ω-sequence of measurable cardinals, any sequence of stationary sets is mutually stationary. We also show that for any Prikry-generic sequence, a tail of the sequence has this property.
- In Section 6 we give an alternative proof of a theorem by Shelah, that there can exist sequences of stationary sets on the $\aleph_n$ for n finite which are mutually stationary and do not concentrate on a fixed cofinality.
- In Section 7 we give an alternative construction for a model in which the set of non-good points of cofinality $\aleph_1$ in $\aleph_{\omega+1}$ is non-stationary. We also show that if we are given an increasing ω-sequence of measurable cardinals such that the successor of their supremum exhibits a certain stationary reflection property, then the reflection property is preserved by diagonal Prikry forcing.
- In Section 8 we show that the principle saying that for all λ any family of fewer than η many stationary subsets of $[\lambda]^{\aleph_0}$ reflect does not imply simultaneous reflection of η many sets of ω-cofinal ordinals. The proof uses Martin's Maximum.
- In Section 9 we show that it is consistent that the least λ for which $\Box_\lambda$ fails is inaccessible.
- In Section 10 we show that if $\Box^*_\mu$ holds for a singular cardinal μ of cofinality ω, then a cardinal-preserving countably closed forcing poset can not create any instances of supercompactness below μ. This shows that there is an essential problem in a result by Ben-David and Shelah [2].

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2. Preliminaries

In this section we give some background material on mutual and tight stationarity and PCF theory. For more details we refer the reader to [4].

The idea of *mutual stationarity* was introduced by Foreman and Magidor [12] in their work on the non-saturation of the non-stationary ideal on $P_\kappa\lambda$.

Definition 2.1. Let $\langle S_\kappa : \kappa \in K\rangle$ be such that $S_\kappa \subseteq \kappa$ for all $\kappa \in K$, where K is a set of regular uncountable cardinals.

(1) If N is a set, then N *meets* $\langle S_\kappa : \kappa \in K\rangle$ if and only if $\sup(N \cap \kappa) \in S_\kappa$ for all $\kappa \in N \cap K$.

(2) $\langle S_\kappa : \kappa \in K\rangle$ is *mutually stationary* if and only if for every algebra $\mathcal{A}$ on $\sup(K)$ there exists $N \prec \mathcal{A}$ such that N meets $\langle S_\kappa : \kappa \in K\rangle$.

If $S \subseteq \mathrm{P}(X)$ then S is a *stationary subset of* $\mathrm{P}(X)$ if and only if for every algebra $\mathcal{A}$ on X there is $B \in S$ such that $B \prec \mathcal{A}$. The sequence $\langle S_\kappa : \kappa \in K\rangle$ is mutually stationary if and only if the set of subsets of $\sup(K)$ which meet $\langle S_\kappa : \kappa \in K\rangle$ is a stationary subset of $\mathrm{P}(\sup(K))$. By standard facts [13, Lemma 0] about generalised stationarity, if X is any set with $\sup(K) \subseteq X$ then $\langle S_\kappa : \kappa \in K\rangle$ is mutually stationary if and only if the set of subsets of X which meet $\langle S_\kappa : \kappa \in K\rangle$ is a stationary subset of $\mathrm{P}(X)$.

It is easy to see that if $\langle S_\kappa : \kappa \in K\rangle$ is mutually stationary then S_κ is stationary for each κ. Foreman and Magidor showed that the converse is false in general, but is true if $S_\kappa \subseteq \kappa \cap \mathrm{cof}(\omega)$ for all κ. In order to get versions of Solovay's splitting theorem and Fodor's theorem Foreman and Magidor introduced the notion of *tight structure* and *tightly stationary sequence*.

Definition 2.2. Let K be a set of regular cardinals, let $\theta = \mathrm{cf}(\theta) > \sup(K)$, and let $\mathcal{A} = (H_\theta, \in, <_\theta)$. Let $M \prec \mathcal{A}$.

Then M is *tight for* K if and only if

(1) $K \in M$.
(2) For all $g \in \prod_{\kappa \in M \cap K}(M \cap \kappa)$ there exists $h \in M \cap \prod K$ such that $g(\kappa) < h(\kappa)$ for all $\kappa \in M \cap K$.

If $|K| \subseteq M$ then $K \subseteq M$, and in this case tightness has a simpler formulation. When $K \subseteq M$, M is tight for K exactly when $M \cap \prod K$ is cofinal in $\prod_{\kappa \in K} M \cap \kappa$.

Definition 2.3. Let K be a set of regular cardinals and let M be a set. The characteristic function of M (on K) is the function χ^K_M with domain K given by $\chi^K_M : \kappa \longmapsto \sup(M \cap \kappa)$.

If a structure M is such that $K \subseteq M$, then tightness of M amounts to saying that every function in $\prod K$ which is pointwise dominated by χ_M is pointwise dominated by some function in $M \cap \prod K$, that is to say $M \cap \prod K$ is cofinal in $\prod K$ below χ^K_M.

Definition 2.4. Let K be a set of regular cardinals and let $\langle S_\kappa : \kappa \in K\rangle$ be such that $S_\kappa \subseteq \kappa$ for all $\kappa \in K$. Let $\theta = \sup(K)^+$. The sequence $\langle S_\kappa : \kappa \in K\rangle$ is *tightly stationary* if and only if for every algebra $\mathcal{A}$ on H_θ there is $N \prec \mathcal{A}$ such that N is tight for K and N meets $\langle S_\kappa : \kappa \in K\rangle$.

PCF theory gives a very general technique for analysing singular cardinals, but for our purposes in this paper we will restrict ourselves to the special case when the singular cardinal is $\aleph_\omega$. Shelah has shown that

- There is an infinite set $A \subseteq \omega$ and a sequence of functions $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ which is a *scale* (that is to say an increasing and cofinal sequence) in $\prod_{n \in A} \aleph_n$ under the eventual domination ordering.
- Modulo finite sets there is a unique maximal choice for the set A.

For the rest of this discussion we fix A to be the maximal set as above, and also fix $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ a scale in $\prod_{n \in A} \aleph_n$. A function g from A to the ordinals is said to be an *exact upper bound* for $\langle f_\alpha : \alpha < \beta\rangle$ iff $f_\alpha <^* g$ for all $\alpha < \beta$, and for every $h < g$ there is $\alpha < \beta$ such that $h <^* f_\alpha$. For example the function $n \mapsto \aleph_n$ is an exact upper bound for $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$.

Without loss of generality we assume that the scale $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ is *continuous*, which means that whenever an exact upper bound for $\langle f_\alpha : \alpha < \beta \rangle$ exists then f_β is such an upper bound. It is easy to see that modulo finite sets exact upper bounds are unique, so that if h is any exact upper bound for $\langle f_\alpha : \alpha < \beta \rangle$ then $h =^* f_\beta$.

We will be especially interested in the *good points* of this kind of scale. An ordinal $\beta < \aleph_{\omega+1}$ of uncountable cofinality is *good* if there exists an exact upper bound h for $\langle f_\alpha : \alpha < \beta \rangle$ such that $\mathrm{cf}(h(n)) = \mathrm{cf}(\beta)$ for all n with $cf(\beta) < \omega_n$. The set of good points is stationary in every uncountable cofinality and is an important invariant of the universe of set theory; see for example [11], [6] and [18].

There is a useful alternative characterization of good points. The point β is good if and only if it has uncountable cofinality and for every A unbounded in β there exist $B \subseteq A$ unbounded in β and $k < \omega$ such that $\langle f_\alpha(n) : \alpha \in B \rangle$ is strictly increasing for all $n > k$.

One reason for us to be interested in good ordinals is that they give a characterization of tight structures. We showed [12] that if $M \prec H_\theta$, $0 < m < \omega$ and PCF is trivial (that is to say that $A = \omega$, so there is a scale of length $\aleph_{\omega+1}$ in $\prod_n \aleph_n$ modulo the ideal of finite sets.) then the following are equivalent:

1. (1) The structure M is tight for $\{\aleph_n : n < \omega\}$ and $\mathrm{cf}(M \cap \aleph_n) = \aleph_m$ for all large $n < \omega$.
2. (2) If $\gamma = \sup(M \cap \aleph_{\omega+1})$ then γ is a good point of cofinality $\aleph_m$ and $f_\gamma(n) = \chi_M(\aleph_n)$ for all large $n < \omega$.

The kind of uniform cofinality assumption which appears in the result we just quoted is ubiquitous enough to deserve a name. We will say that a structure M is $\aleph_m$*-uniform* if $\mathrm{cf}(M \cap \aleph_n) = \aleph_m$ for $m < n < \omega$.

If $0 < m < \omega$ then every internally approachable structure of length and cardinality $\aleph_m$ is tight for $\{\aleph_n : n < \omega\}$, so there are stationarily many $\aleph_m$-uniform tight structures. Zapletal [12] showed that there are stationarily many $\aleph_m$-uniform non-tight structures.

Without the assumption that PCF is trivial, we can give a more complicated description of the uniform tight structures. We refer the reader to [4, Theorem 5.6] for the missing details. Let $K = \{\aleph_n : n < \omega\}$, let $\vec{B} = \langle B_\lambda : \lambda \in \mathrm{pcf}(K) \rangle$ be a sequence of PCF generators for K and let $\vec{f} = \langle f^\lambda_\alpha : \alpha < \lambda, \lambda \in \mathrm{pcf}(K) \rangle$ be such that $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ is an ω-club minimal scale in $\prod B_\lambda / J_{<\lambda}$ for each λ. We showed [4] that if $0 < m < \omega$ and $M \prec (H_\theta, \vec{B}, \vec{f})$ is $\aleph_m$-uniform and tight for K then χ^K_M can be written as the pointwise supremum of finitely many functions of the form $f^\lambda_{\sup(M \cap \lambda)}$.

If M is an $\aleph_m$-uniform substructure of some expansion $\mathcal{A}$ of $(H_\theta, \in, <_\theta)$, and M^* is the Skolem hull in $\mathcal{A}$ of $M \cup \aleph_m$, then as we see later in Lemma 6.3 $\sup(M \cap \aleph_n) = \sup(M^* \cap \aleph_n)$ for $m < n < \omega$. So M^* is $\aleph_m$-uniform and contains $\aleph_m$. We showed [4] that for $0 < m < \omega$, if N is $\aleph_m$-uniform and contains $\aleph_m$ then the set $N \cap \aleph_\omega$ is closed under bounded suprema of length less than $\aleph_m$; in particular for $m < n < \omega$ there is a club subset of $N \cap \aleph_n$ which has order type $\aleph_n$ and is contained in $N \cap \aleph_n$.

3. A non-tight mutually stationary sequence

Foreman and Magidor [12] raised the question as to whether every mutually stationary sequence is tightly stationary. In this section we give a forcing construction showing that a negative answer is consistent; we do not know whether a negative answer follows from the axioms of ZFC. Given $k > 0$ we will construct by forcing a sequence $\langle T_n : k < n < \omega \rangle$ with $T_n \subseteq \aleph_n \cap \mathrm{cof}(\aleph_k)$, which is mutually stationary but not tightly stationary.

We start by defining a combinatorial principle Coherent Squares (CS). The principle asserts the existence of $\square_{\aleph_n}$-sequences for $0 < n \leq \omega$, together with a scale of length $\aleph_{\omega+1}$ in $\prod_{n<\omega} \aleph_n$ which relates the $\square_{\aleph_n}$-sequences for $n < \omega$ to the $\square_{\aleph_\omega}$-sequence. We note that the scale involved in the principle CS is a "Very Good Scale" in the sense of our paper [6]. This principle is closely related to some combinatorial principles of Donder, Jensen and Stanley [8] and Donder, Jensen and Koppelberg [7].

Definition 3.1. For each $n \leq \omega$ let $I_n = \{ \alpha : \aleph_n < \alpha < \aleph_{n+1} \}$. The principle CS asserts that there exist sequences

$$\langle C^n_\alpha : \alpha \in I_n \cap LIM, 0 < n \leq \omega \rangle, \langle f_\alpha : \alpha \in I_\omega \cap LIM \rangle,$$

such that

(1) For all n and all α in $S_n \cap LIM$
  (a) The set C^n_α is club in α, and $C^n_\alpha \subseteq I_n$.
  If the cofinality of α is less than $\aleph_n$ then the order type of C^n_α is less than $\aleph_n$.
  (b) For every limit point β of C^n_α, $C^n_\beta = C^n_\alpha \cap \beta$.

(2) For all α in $S_\omega \cap LIM$, f_α is a function such that
  (a) There exists $k < \omega$ such that $\mathrm{dom}(f_\alpha) = \{ n : k < n \leq \omega \}$ and $\mathrm{ot}(C^\omega_\alpha) \leq \aleph_k$.
  (b) For all n in $\mathrm{dom}(f_\alpha)$, $f_\alpha(n) \in I_n \cap LIM$.
  (c) For every limit point β of C^ω_α, $\mathrm{dom}(f_\alpha) \subseteq \mathrm{dom}(f_\beta)$.
  (d) For all n in $\mathrm{dom}(f_\alpha)$,
$$\lim(C^n_{f_\alpha(n)}) = \{ f_\beta(n) : \beta \in \lim(C^\omega_\alpha) \}.$$

(3) The sequence $\langle f_\alpha : \alpha \in I_\omega \cap LIM \rangle$ forms a scale in $\prod_n \aleph_{n+1}$, that is to say it is increasing and cofinal in the eventual domination ordering.

Remark 3.2. Notice that $\langle C^n_\alpha : \alpha \in I_n \cap LIM \rangle$ is essentially a $\square_{\aleph_n}$-sequence, with the (purely cosmetic) difference that the underlying set is I_n rather than $\aleph_{n+1}$.

Remark 3.3. If $\langle f_\alpha : \alpha \in I_\omega \rangle$ is the scale in $\prod \aleph_n$ modulo the ideal of finite sets given by the principle CS, then $\langle f_\alpha \rangle$ is continuous and ω-club minimal. Moreover, it is a very good scale in the sense of [6].

All this follows from the observation that if α has uncountable cofinality then for large n the sequence $\langle f_\gamma(n) : \gamma \in \lim(C^\omega_\alpha) \rangle$ is continuous and increasing with supremum $f_\alpha(n)$.

To motivate the principle CS we show that it can be used to generate a sequence of stationary sets which is not tightly stationary. We suppose that $\langle f_\alpha \rangle$ and $\langle C^n_\alpha \rangle$ are as in Definition 3.1. Given $k < \omega$ and a sequence of limit ordinals $\langle \gamma_n : k \leq n < \omega \rangle$ such that $\gamma_n < \aleph_n$ for all n, we define a sequence of sets by

$$T_n = \{ \alpha \in I_n \cap LIM : \mathrm{cf}(\alpha) = \aleph_k, \mathrm{ot}(C^n_\alpha) \geq \gamma_n \}.$$

The stationarity of the sets T_n follows from a general fact about $\square_\kappa$-sequences for κ regular.

Lemma 3.4. *If κ is regular and $\langle C_\alpha : \alpha < \kappa^+ \rangle$ is a $\square_\kappa$-sequence, then for every $\eta < \kappa$ the set $\{\delta < \kappa^+ : \mathrm{ot}(C_\delta) > \eta\}$ is stationary.*

Proof. Let C be club in κ^+ and let ζ be a limit point of C with cofinality κ. $C_\zeta \cap C$ is club in ζ with order type κ, and so we may find δ a limit point of $C_\zeta \cap C$ such that $\mathrm{ot}(C_\zeta \cap \delta) > \eta$. Clearly $\delta \in C$, and by the coherence property of the square sequence $C_\zeta \cap \delta = C_\delta$ and so $\mathrm{ot}(C_\delta) > \eta$. We showed $\{\delta < \kappa^+ : \mathrm{ot}(C_\delta) > \eta\}$ meets every club subset of κ^+ and so it is stationary. $\square$

Lemma 3.5. *If $\langle \gamma_n : k \leq n < \omega \rangle$ is unbounded in $\aleph_\omega$ $\gamma_n < \aleph_n$ for every n, then $\langle T_n : k \leq n < \omega \rangle$ is not tightly stationary.*

Proof. Let N be a tight structure and for each $i \leq \omega$ let $\alpha_{i+1} = \sup(N \cap \aleph_{i+1})$. Suppose for a contradiction that $\alpha_i \in T_i$ for all i.

The sequence $\langle f_\alpha : \alpha \in I_\omega \cap LIM \rangle$ forms a continuous scale, so by the characterization of uniform tight structures in terms of PCF theory which we discussed in Section 2 there exists $m < \omega$ such that $f_{\alpha_\omega}(n) = \alpha_n$ for all $n \geq m$. If $n \geq m$ then

$$\lim(C^n_{\alpha_n}) = \lim(C^n_{f_{\alpha_\omega}(n)}) = \{\ f_\beta(n)\ :\ \beta \in \lim(C^\omega_{\alpha_\omega})\ \}.$$

Notice also that if $\beta, \gamma \in \lim(C^\omega_{\alpha_\omega})$ and $\beta < \gamma$ then $\beta \in \lim(C^\omega_\gamma)$, and so $f_\beta(n) \in \lim(C^n_{f_\gamma(n)})$ and in particular $f_\beta(n) < f_\gamma(n)$. It follows that $\mathrm{ot}(C^n_{\alpha_n}) = \mathrm{ot}(C^\omega_{\alpha_\omega})$ for all $n \geq m$. Since $\langle \gamma_n : k \leq n < \omega \rangle$ is unbounded in $\aleph_\omega$, we may find $n \geq m$ such that $\gamma_n > \mathrm{ot}(C^\omega_{\alpha_\omega})$. It follows that $\alpha_n \notin T_n$, which is a contradiction. $\square$

We define a forcing iteration of length $\omega + 1$ which forces CS to hold. At stage n for $n < \omega$ we force with a version of Jensen's poset for adding a square sequence $\vec{C}^n$, where conditions prescribe an initial segment of $\vec{C}^n$. At stage ω we force with conditions which prescribe initial segments of $\vec{C}^\omega$ and $\vec{f}$.

Definition 3.6. For $n < \omega$, $\mathbb{Q}_n$ is the set of sequences

$$q = \langle C^{q,n}_\alpha : \alpha \in I_n \cap LIM \cap (\beta + 1) \rangle$$

where

(1) The ordinal β is a limit ordinal in S_n. We refer to β as the *length of* q and write $\beta = \mathrm{lh}(q)$.
(2) For all α in $S_n \cap LIM \cap (\beta + 1)$
  (a) The set $C^{q,n}_\alpha$ is club in α and $C^{q,n}_\alpha \subseteq I_n$.
  (b) The order type of $C^{q,n}_\alpha$ is less than $\aleph_n$ if the cofinality of α is less than $\aleph_n$.
  (c) For every limit point γ of $C^{q,n}_\alpha$, $C^{q,n}_\gamma = C^{q,n}_\alpha \cap \gamma$.

If $q, r \in \mathbb{Q}_n$ then $q \leq r$ if and only if

(1) The length of q is greater than or equal to the length of r.
(2) For all $\alpha \in I_n \cap LIM \cap (\mathrm{lh}(r) + 1)$, $C^{q,n}_\alpha = C^{r,n}_\alpha$.

Before stating the main facts about $\mathbb{Q}_n$ we recall the concept of strategic closure. Let λ be an infinite cardinal. A poset $\mathbb{Q}$ is $(\lambda + 1)$-strategically closed if and only if player Even has a winning strategy for the game in which two players (Even and Odd) build a decreasing $\lambda + 1$-sequence in $\mathbb{Q}$, where Odd plays at all odd stages and Even plays at all non-zero even stages including limit stages, and Even wins if she can move at stage λ. We refer the reader to Foreman's paper on games [9] for more about strategic closure, noting here only that a $(\lambda + 1)$-strategically closed poset adds no λ-sequences and hence preserves all cardinals less than or equal to λ^+.

The following facts are standard, see for example [6].

Fact 3.7. *Let $n < \omega$.*

(1) *The poset $\mathbb{Q}_n$ is countably closed.*
(2) *The poset $\mathbb{Q}_n$ is $(\aleph_n + 1)$-strategically closed.*
(3) *If $2^{\aleph_n} = \aleph_{n+1}$ then $|\mathbb{Q}_n| = \aleph_{n+1}$, so in particular $\mathbb{Q}_n$ has the $\aleph_{n+2}$-c.c.*
(4) *If $p \in \mathbb{Q}_n$, γ is a limit ordinal in S_n and $\mathrm{lh}(p) < \gamma$, then there is $q \leq p$ such that $\mathrm{lh}(q) = \gamma$.*
(5) *If $\vec{C}^n = \langle C^n_\alpha : \alpha \in I_n \cap LIM \rangle$ is $\mathbb{Q}_n$-generic and $\delta \in (\aleph_n + 1) \cap LIM$, then*
$$V[\vec{C}^n] \models \text{“} \{ \alpha \in I_n \cap LIM \ : \ \mathrm{ot}(C^n_\alpha) = \delta \} \text{ is stationary in } \aleph_{n+1}\text{”}.$$

We now assume that V satisfies GCH. We define $\mathbb{P}_\omega$ as an iteration with full support, where at stage n we force with $\mathbb{Q}_n$ as defined in $V^{\mathbb{P}_n}$. As usual we let $\dot{\mathbb{Q}}_n$ be a $\mathbb{P}_n$-name for $\mathbb{Q}_n$. In $V^{\mathbb{P}_\omega}$ let $\langle C^n_\alpha : \alpha \in I_n \cap LIM \rangle$ be the sequence added by $\mathbb{Q}_n$, and define $\mathbb{Q}_\omega$ as follows.

Definition 3.8. The poset $\mathbb{Q}_\omega$ is the set of pairs of sequences
$$q = (\langle C^{q,\omega}_\alpha : \alpha \in I_\omega \cap LIM \cap (\beta + 1) \rangle, \langle f^q_\alpha : \alpha \in I_\omega \cap LIM \cap (\beta+1) \rangle)$$
where

(1) The ordinal β is a limit ordinal in I_ω. We call β the *length* of q and write $\beta = \mathrm{lh}(q)$.
(2) For all $\alpha \in I_\omega \cap LIM \cap (\beta + 1)$
   (a) The set $C^{q,\omega}_\alpha$ is club in α and $C^{q,\omega}_\alpha \subseteq I_\omega$.
   (b) The order type of $C^{q,\omega}_\alpha$ is less than $\aleph_\omega$.
   (c) For every limit point γ of $C^{q,\omega}_\alpha$, $C^{q,\omega}_\gamma = C^{q,\omega}_\alpha \cap \gamma$.
(3) For all α in $I_\omega \cap LIM \cap (\beta + 1)$, f_α is a function such that
   (a) There exists $k < \omega$ such that $\mathrm{dom}(f_\alpha) = \{ n : k < n \leq \omega \}$ and $\mathrm{ot}(C^{q,\omega}_\alpha) \leq \aleph_k$.
   (b) For all $n \in \mathrm{dom}(f_\alpha)$, $f_\alpha(n) \in I_n \cap LIM$.
   (c) For all limit points β of $C^{q,\omega}_\alpha$, $\mathrm{dom}(f_\alpha) \subseteq \mathrm{dom}(f_\beta)$.
   (d) For all $n \in \mathrm{dom}(f_\alpha)$,
$$\lim(C^n_{f_\alpha(n)}) = \{ f_\beta(n) \ : \ \beta \in \lim(C^{q,\omega}_\alpha) \}.$$
(4) If $\alpha_1 < \alpha_2 \leq \beta$, then $f_{\alpha_1} <^* f_{\alpha_2}$.

If $q, r \in \mathbb{Q}_\omega$ then $q \leq r$ if and only if

(1) The length of q is greater than or equal to the length of r.
(2) For all α in $S_n \cap LIM \cap (\mathrm{lh}(r) + 1)$, $C^{q,\omega}_\alpha = C^{r,\omega}_\alpha$ and $f^q_\alpha = f^r_\alpha$.

When we construct members of $\mathbb{Q}_\omega$ we will generally only verify that the “coherence” clause 3d holds.

Lemma 3.9. *The forcing poset $\mathbb{Q}_\omega$ is countably closed in $V^{\mathbb{P}_\omega}$.*

Proof. Let $\langle q_i : i < \omega \rangle$ be a strictly decreasing ω-sequence of conditions, and define $\beta_i = \mathrm{lh}(q_i)$ and $\beta = \sup_{i<\omega} \beta_i$. We define a condition q as follows.

(1) The length of q is β.
(2) For all $\alpha \in I_n \cap LIM \cap (\mathrm{lh}(q_i) + 1)$, $C^{q,\omega}_\alpha = C^{q_i,\omega}_\alpha$ and $f^q_\alpha = f^{q_i}_\alpha$.
(3) The set $C^{q,\omega}_\beta$ is cofinal in β with order type ω.

(4) For all $n < \omega$, $f^q_\beta(n) \geq \sup \{ f^q_{\beta_i}(n) \ : \ i \in \omega \}$ and $\mathrm{ot}(C^n_{f^q_\beta(n)}) = \omega$.

The choice of $f^q_\beta(n)$ is possible because $\mathrm{ot}(C^n_\delta) = \omega$ for a cofinal set of $\delta < \aleph_{n+1}$. The clause 3d of the definition of $\mathbb{Q}_n$ is satisfied trivially because neither $C^{q,\omega}_\beta$ nor any of the $C^n_{f^q_\beta(n)}$ has any limit points. □

We now define $\mathbb{P}_{\omega+1} = \mathbb{P}_\omega * \mathbb{Q}_\omega$. A standard argument shows that

$$\mathbb{P}_{\omega+1} \simeq \mathbb{P}_n * \mathbb{R}_n,$$

where $\mathbb{R}_n$ is the full support iteration of length $\omega + 1$ with factors $\langle \mathbb{Q}_i : n \leq i \leq \omega \rangle$. For notational convenience we will index the steps of $\mathbb{R}_n$ by the set $\{ i \ : \ n \leq i \leq \omega \}$ rather than $\omega + 1$.

For the next few lemmas we work in $V^{\mathbb{P}_n}$.

Lemma 3.10. *The set of p in $\mathbb{R}_n$ such that $p \restriction i \setminus n$ decides $\mathrm{lh}(p(i))$ for all i with $n \leq i \leq \omega$ is dense.*

Proof. This is easy, as each of the $\mathbb{Q}_i$ is countably closed. □

From now on we will assume that all $p \in \mathbb{R}_n$ have this property. Accordingly we will write $\mathrm{lh}_i(p)$ for the unique ordinal γ such that $p \restriction i \Vdash \check{\gamma} = \mathrm{lh}(p(i))$.

Notation: If $p \in \mathbb{R}_n$ then we write

$$p(i) = \langle \dot{C}^{p,i}_\alpha : \alpha \in S_i \cap LIM \cap (\mathrm{lh}_i(p) + 1) \rangle$$

for $i < \omega$, and let $\langle \dot{C}^{p,\omega}_\alpha : \alpha \in I_\omega \cap LIM \cap (\mathrm{lh}_\omega(p) + 1) \rangle$ be the first component of $p(\omega)$ and $\langle \dot{f}^p_\alpha : \alpha \in I_\omega \cap LIM \cap (\mathrm{lh}_\omega(p) + 1) \rangle$ the second component.

Definition 3.11. A condition $p \in \mathbb{R}_n$ is *flat* if and only if $p \restriction \omega \setminus n$ forces that

(1) $\mathrm{dom}(\dot{f}^p_{\mathrm{lh}_\omega(p)}) = \{ i \ : \ n \leq i < \omega \}$.
(2) For all $i \in \mathrm{dom}(\dot{f}^p_{\mathrm{lh}_\omega(p)})$, $\dot{f}^p_{\mathrm{lh}_\omega(p)}(i) = \mathrm{lh}_i(p)$.

Lemma 3.12. *The set of flat conditions in $\mathbb{R}_n$ is dense.*

Proof. Given p we first find $q \leq p$ such that

- $\mathrm{lh}_\omega(q) = \mathrm{lh}_\omega(p)$.
- $q \restriction \omega \setminus n$ decides $f^p_{\mathrm{lh}_\omega(p)}$.
- $\mathrm{lh}_i(q) > f^p_{\mathrm{lh}_\omega(p)}(i)$ and $\mathrm{ot}(C^{q,i}_{\mathrm{lh}_i(q)}) = \omega$ for all i with $n \leq i < \omega$.

Then we find $r \leq q$ such that

(1) $\mathrm{lh}_i(r) = \mathrm{lh}_i(q)$ for all $i < \omega$, and $\mathrm{lh}_\omega(r) = \mathrm{lh}_\omega(q) + \omega$.
(2) $\mathrm{ot}(C^{r,\omega}_{\mathrm{lh}_\omega(r)}) = \omega$.
(3) $\mathrm{dom}(f^r_{\mathrm{lh}_\omega(r)}) = \{ i \ : \ n \leq i < \omega \}$ and $f^r_{\mathrm{lh}_\omega(r)}(i) = \mathrm{lh}_i(r)$ for all i.

Clearly r is a condition, $r \leq p$ and r is flat. □

Lemma 3.13. *For all $n < \omega$, $\mathbb{R}_n$ is $(\aleph_n + 1)$-strategically closed.*

Proof. We describe a strategy for player Even in the game, where, without loss of generality, we may assume that Odd plays a flat condition at each odd stage. Even's moves will also be flat conditions. Suppose that γ is even and that so far the sequence $\langle p_i : i < \gamma \rangle$ has been played.

γ successor: If $\gamma = \delta + 1$, then Even defines p_γ as follows.

(1) $\mathrm{lh}_i(p_\gamma) = \mathrm{lh}_i(p_\delta) + \omega$ and $C^{p_\gamma,i}_{\mathrm{lh}_i(p_\gamma)} = \{\mathrm{lh}_i(p_\delta) + j : j < \omega\}$ for $n \leq i \leq \omega$.
(2) $\mathrm{dom}(f^{p_\gamma}_{\mathrm{lh}_\omega(p_\gamma)}) = \{ \, i \, : \, n \leq i < \omega \, \}$.
(3) $f^{p_\gamma}_{\mathrm{lh}_\omega(p_\gamma)}(i) = \mathrm{lh}_i(p_\gamma)$ for $n \leq i < \omega$.

γ **limit:** Even defines p_γ as follows.
(1) $\mathrm{lh}_i(p_\gamma) = \bigcup_{\beta<\gamma} \mathrm{lh}_i(p_\beta)$ and $C^{p_\gamma,i}_{\mathrm{lh}_i(p_\gamma)} = \{ \, \mathrm{lh}_i(p_\beta) \, : \, \beta < \gamma \, \}$ for $n \leq i \leq \omega$.
(2) $\mathrm{dom}(f^{p_\gamma}_{\mathrm{lh}_\omega(p_\gamma)}) = \{ \, i \, : \, n \leq i < \omega \, \}$.
(3) $f^{p_\gamma}_{\mathrm{lh}_\omega(p_\gamma)}(i) = \mathrm{lh}_i(p_\gamma)$ for $n \leq i < \omega$.

As usual we only check that $p_\gamma(\omega)$ satisfies clause 3d from the definition of $\mathbb{Q}_\omega$. We observe first that if $\delta \in \lim(C^{p_\gamma,i}_{\mathrm{lh}_i(p_\gamma)})$ then $\delta = \mathrm{lh}_i(p_\beta)$ for some limit $\beta < \gamma$. At stage β player Even defined $C^{p_\beta,i}_{\mathrm{lh}_i(p_\beta)} = \{ \, \mathrm{lh}_i(p_\alpha) \, : \, \alpha < \beta \, \}$ for $n \leq i \leq \omega$, and $f^{p_\beta}_{\mathrm{lh}_\omega(p_\beta)}(i) = \mathrm{lh}_i(p_\beta)$ for $n \leq i < \omega$.

It follows that for $n \leq i < \omega$

$$\begin{aligned} \lim(C^{p_\gamma,i}_{\mathrm{lh}_i(p_\gamma)}) &= \{ \, \mathrm{lh}_i(p_\beta) \, : \, \beta \in \gamma \cap LIM \, \} \\ &= \{ \, f^{p_\beta}_{\mathrm{lh}_\omega(p_\beta)}(i) \, : \, \beta \in \gamma \cap LIM \, \} \\ &= \{ \, f^{p_\gamma}_\delta(i) \, : \, \delta \in \lim(C^{p_\gamma,\omega}_{\mathrm{lh}_\omega(p_\gamma)}) \, \} \end{aligned}$$

□

Lemma 3.14. *Let $G_{\omega+1}$ be $\mathbb{P}_{\omega+1}$-generic and let G_ω be the induced $\mathbb{P}_\omega$-generic filter. Then*
(1) *The models V and $V[G_{\omega+1}]$ have the same cardinals and cofinalities up to $\aleph_{\omega+1}$.*
(2) *The principle CS holds in $V[G_{\omega+1}]$.*
(3) *Every $\aleph_\omega$-sequence of ordinals from $V[G_{\omega+1}]$ is in $V[G_\omega]$.*

Proof. This is fairly routine. We only check that $\langle f_\alpha : \alpha \in I_\omega \cap LIM \rangle$ forms a scale. To see this let $g \in (\prod_n \aleph_{n+1})^{V[G_{\omega+1}]}$, and observe that $g \in V$ because $\mathbb{P}_{\omega+1}$ is countably closed. Now let p be an arbitrary condition. Find $q \leq p$ such that $\mathrm{lh}_\omega(q) = \mathrm{lh}_\omega(p)$ and $\mathrm{lh}_i(q) \geq g(i)$ for all i, and then find $r \leq q$ such that r is flat. By construction $f^r_{\mathrm{lh}_\omega(r)}(i) = \mathrm{lh}_i(r) \geq lh_i(q) \geq g(i)$ for all $i < \omega$, and we are done. □

We now work in $V[G_{\omega+1}]$. We recall that given $k < \omega$ and a sequence of limit ordinals $\langle \gamma_n : k \leq n < \omega \rangle$ such that $\gamma_n < \aleph_n$ for all n, we defined a sequence of sets by

$$T_n = \{ \, \alpha \in I_n \cap LIM \, : \, \mathrm{cf}(\alpha) = \aleph_k, \mathrm{ot}(C^n_\alpha) \geq \gamma_n \, \}.$$

We showed in Lemma 3.5 that a suitable choice of $\langle \gamma_n : k \leq n < \omega \rangle$ will generate a sequence which is not tightly stationary.

Lemma 3.15. *Let $G_{\omega+1} \subset \mathbb{Q}_\omega$ be generic. Then in $V[G_{\omega+1}]$, for all sequences $\langle \gamma_n : k \leq n < \omega \rangle$ the sequence $\langle T_n : k \leq n < \omega \rangle$ is mutually stationary (where T_n is defined as above.)*

Proof. Let $\langle \gamma_n : k \leq n < \omega \rangle$ be a sequence of ordinals in $V[G_{\omega+1}]$. Then, by the closure of $\mathbb{Q}_\omega$, $\langle \gamma_n : k \leq n < \omega \rangle$ lies in V. We showed in Lemma 3.14 that the models $V[G_\omega]$ and $V[G_{\omega+1}]$ have the same $\aleph_\omega$-sequences of ordinals, so it is enough to check that this is so in $V[G_\omega]$. We use the fact that

$$\mathbb{P}_\omega \simeq \mathbb{P}_k * \mathbb{S}_k$$

where $\mathbb{S}_k$ is the iteration of length ω with full support and factors $\mathbb{Q}_n$ for $k \le n < \omega$. We will do a density argument in $\mathbb{S}_k$ similar to the proof given above that $\mathbb{R}_k$ is strategically closed.

Let $\dot{H}$ be a name for a function from ${}^{<\omega}\aleph_\omega$ to $\aleph_\omega$. Let $p_0 \in \mathbb{S}_n$ be arbitrary. Extending p_0 if necessary, we may assume without loss of generality that $\mathrm{ot}(C^{p_0,i}_{\mathrm{lh}_i(p_0)}) = \gamma_i$.

We will argue in the model $V^{\mathbb{P}_k}$, using the closure of $\mathbb{S}_k$. In particular we will now understand $\dot{H}$ as an $\mathbb{S}_k$-name appropriate for forcing over $V^{\mathbb{P}_k}$. We describe an inductive construction of a decreasing chain of conditions $\langle p_j : j \le \aleph_k \rangle$ in $\mathbb{S}_k$, and an increasing chain of sets $\langle A_j : j \le \aleph_k \rangle$ such that $|A_j| \le \aleph_k$.

Stage zero: p_0 has already been determined, and we set $A_0 = \emptyset$.
Successor stages: Suppose that p_j, A_j have been defined. We start by choosing $B_{j+1} \subseteq \aleph_\omega$ such that $|B_{j+1}| \le \aleph_k$, $A_j \subseteq B_{j+1}$, and $\mathrm{lh}_i(p_j) \le \sup(B_{j+1} \cap \aleph_{i+1})$ for $i \ge k$. We then choose $q_j \le p_j$ and A_{j+1} so that q_j forces the $\dot{H}$-closure of B_{j+1} to be A_{j+1}, and $\mathrm{lh}_i(q_j) \ge \sup(A_{j+1} \cap \aleph_{i+1})$ for $i \ge k$. We define p_{j+1} as follows:

(1) $p_{j+1} \le q_j$.
(2) $\mathrm{lh}_i(p_{j+1}) = \mathrm{lh}_i(q_j) + \omega$.
(3) $C^{p_{j+1},i}_{\mathrm{lh}_i(p_{j+1})} = \{ \mathrm{lh}_i(q_j) + l \ : \ l < \omega \}$.

Limit stages: Suppose that j is limit and we have defined $\langle p_k : k < j \rangle$ and $\langle A_k : k < j \rangle$. Define p_j by

(1) $p_j \le p_k$ for all $k < j$.
(2) $\mathrm{lh}_i(p_j) = \sup_{k<j} \mathrm{lh}_i(p_k)$.
(3) $C^{p_j,i}_{\mathrm{lh}_i(p_j)} = C^{p_0,i}_{\mathrm{lh}_i(p_0)} \cup \{ \mathrm{lh}_i(p_k) \ : \ k < j \}$.

Let $A_j = \bigcup_{k<j} A_k$.

If j is limit it is routine to check that p_j is a condition, $\sup(A_j \cap \aleph_{i+1}) = \mathrm{lh}_i(p_j)$, and that p_j forces that A_j is closed under $\dot{H}$.

Let $A = A_{\aleph_k}$, $p = p_{\aleph_k}$. Then p forces that $\sup(A \cap \aleph_{i+1}) = \mathrm{lh}_i(p)$ for all i, and p also forces that $\mathrm{ot}(C^i_{\mathrm{lh}_i(p)}) = \gamma_i + \aleph_k$ for all i. It follows that $p \Vdash \forall i \ \sup(A \cap \aleph_{i+1}) \in T_i$. □

We summarise the main result of this section in a theorem.

Theorem 3.16. *It is consistent that for every integer $k > 0$ there exists a sequence $\langle T_n : k < n < \omega \rangle$ such that $T_n \subseteq \aleph_n \cap \mathrm{cof}(\aleph_k)$, and the sequence is mutually stationary but not tightly stationary.*

4. Another non-tight mutually stationary sequence

Steprans and Foreman found another consistency proof for the existence of a sequence $\langle S_n : k < n < \omega \rangle$ such that $S_n \subseteq \aleph_n \cap \mathrm{cof}(\aleph_k)$, and the sequence is mutually stationary but not tightly stationary. The model is easily described: fixing an integer $k > 0$, we force with the Cohen poset $\mathrm{Add}(\aleph_0, \aleph_\omega)$ for adding a subset S of $\aleph_\omega$ with finite conditions and define $S_n = \aleph_n \cap \mathrm{cof}(\aleph_k) \cap S$ for each $n > k$. We claim that in $V[S]$ the sequence $\langle S_n : k < n < \omega \rangle$ is as required.

We start by showing that $\langle S_n : k < n < \omega \rangle$ is not tightly stationary. This part of the argument is due to Steprans (under the assumption that $2^{\aleph_\omega} = \aleph_{\omega+1}$).

We observe that the poset $\mathrm{Add}(\aleph_0, \aleph_\omega)$ has the countable chain condition. Working in V we fix a sequence of PCF generators $\langle B_\lambda : \lambda \in \mathrm{pcf}(K)\rangle$ and a family $\langle f^\lambda_\alpha : \lambda \in \mathrm{pcf}(K), \alpha < \lambda\rangle$ of ω-club minimal scales, where $K = \{\aleph_n : n < \omega\}$. By the countable chain condition it is still the case in $V[S]$ that $\langle B_\lambda\rangle$ is a sequence of generators and $\langle f^\lambda_\alpha\rangle$ is a matrix of ω-club minimal scales.

If N is a tight $\aleph_k$-uniform structure in $V[S]$ then as we discussed in Section 2, it follows from [4, Theorem 5.6] that χ_N can be computed in an absolute way from finitely many of the functions f^λ_α, and so $\chi_N \in V$. An easy density argument shows that $V \cap \prod_n S_n = \emptyset$, so that the tight structure N can not meet the sequence $\langle S_n : n < \omega\rangle$.

It remains to be seen that the sequence $\langle S_n : n < \omega\rangle$ is mutually stationary in $V[S]$. Let $F \in V[S]$ be a function from ${}^{<\omega}\aleph_\omega$ to $\aleph_\omega$. We start by showing that it is enough to consider structures which lie in the ground model.

Lemma 4.1. *Let $\mathbb{P}$ be a c.c.c. forcing poset, let λ be a cardinal and let $\dot F$ be a $\mathbb{P}$-name for a function from ${}^{<\omega}\lambda$ to λ. There is a function $f \in V$ from ${}^{<\omega}\lambda$ to λ such that if G is $\mathbb{P}$-generic and $X \in V[G]$ is a subset of λ closed under f, then X is closed under $\dot F^G$.*

Proof. It follows from the c.c.c. that if $x \in {}^{<\omega}\lambda$ then there are only countably many possibilities for $\dot F(x)$. Fix an enumeration of these possibilities as $\langle g(x, n) : n < \omega\rangle$ and then define f as follows: if $y \in {}^{<\omega}\lambda$ and $\mathrm{lh}(y) = 2^m(2n + 1)$ then $f(y) = g(y \restriction m, n)$. □

For the rest of this section we mean by "structure" an elementary substructure of $(H_{\aleph_{\omega+1}}, \in, <, F)$. Let $\mathcal{F}$ be the set of characteristic functions of $\aleph_k$-uniform structures with respect to the set $\{\aleph_n : k < n < \omega\}$; for notational simplicity we consider the domain of an element of $\mathcal{F}$ to be $\{n : k < n < \omega\}$. Let T be the tree consisting of all proper initial segments of all elements of $\mathcal{F}$. We prove two lemmas about T, which may have some independent interest.

Lemma 4.2. *Every infinite branch of T is a member of $\mathcal{F}$.*

Proof. Let χ be a branch of T, and find structures $\langle M_j : k < j < \omega\rangle$ such that $\sup(M_j \cap \aleph_n) = \chi(n)$ for all n and j with $k < n \leq j$. As we noted in Section 2, we may as well assume that $\aleph_k \subseteq M_j$ and then may find $C^j_n \subseteq M_j \cap \chi(n)$ which is club in $\chi(n)$, for all n and j with $k < n \leq j$. For all $n > k$ let $D_n = \bigcap_{j \geq n} C^j_n$, so that D_n is club in $\chi(n)$. We note that if $\gamma \in D_n$ then $\gamma \in M_j$ for all large j.

Let M be the Skolem hull of $\bigcup_{n>k} D_n$. We claim that M is a structure with characteristic function χ. It is clear that $\chi_M(n) \geq \chi(n)$ for all $n > k$. To see that the reverse inequality holds, let $\alpha \in M \cap \aleph_n$ and fix s a finite subset of $\bigcup_{n>k} D_n$ such that α is in the hull of s. Since s is finite we may find j so large that $j \geq n$ and $s \subseteq M_j$, so that $\alpha \in M_j \cap \aleph_n$ and therefore $\alpha < \chi_{M_j}(n)$ Since $\chi_{M_j}(n) = \chi(n)$, we are done. □

Remark 4.3. Notice that in any cardinal-preserving extension of V, the argument works to show that every infinite branch of T is the characteristic function of some $\aleph_k$-uniform structure. In particular this is true in $V[S]$.

We now show the tree T has a stationary branching subtree U.

Lemma 4.4. *There is a tree $U \subseteq T$ such that for all $j > k$ and $t \in U$ with $\mathrm{dom}(t) = \{n : k < n < j\}$, $\{\alpha < \aleph_j : t^\frown\alpha \in U\}$ is stationary in $\aleph_j$.*

Proof. We will use the Gale-Stewart theorem [19] on the determinacy of games with open payoff sets. We denote by $\langle\langle \beta_0, \dots \beta_{j-1} \rangle\rangle$ the function f with domain $\{n : k < n < k + j + 1\}$ given by $f(n) = \beta_{n-k-1}$.

Consider the following two-player game of perfect information between two players I and II. Player I's i^{th} move is a set of ordinals A_i, player II's i^{th} move is an ordinal α_i. We suppose that

(1) $\langle\langle \alpha_i : i < j \rangle\rangle \in T$ for all j.
(2) A_j is a subset of $\{\beta < \aleph_{k+j+1} : \langle\langle \alpha_0, \dots \alpha_{j-1}, \beta \rangle\rangle \in T\}$, and A_j is non-stationary in $\aleph_{k+j+1}$.
(3) $\alpha_j \notin A_j$.

The first player to violate these conditions loses, and if play continues for ω moves then II wins.

Intuitively the idea is that II is trying to build an infinite branch of T, and that player I is allowed to block a non-stationary set of potential successors at each stage. Similar games appear in the game analysis of Namba forcing by Shelah [21].

We claim that II has a winning strategy. Since the game is open for player I, it follows from the Gale-Stewart theorem that it suffices to show I has no winning strategy. Suppose for a contradiction that I has a winning strategy τ, and find an $\aleph_k$-uniform structure M with $\tau \in M$.

We will construct a run of the game where I plays according to τ but the wrong player (player II) wins. At her j^{th} move player II will play $\alpha_j = \sup(M \cap \aleph_{k+j+1})$. We check that this gives a win for player II.

Suppose that I has played $\langle A_i : i \leq j \rangle$, II has played $\langle \alpha_i : i < j \rangle$, and $\alpha_i \notin A_i$ for $i < j$. In general A_j will not be in M. However if we define B to be the union of the set of all A such that I plays A at stage j in some run of the game where I plays according to τ, then $B \in M$ because $\tau \in M$. Since B is the union of at most $\aleph_{k+j}$ non-stationary subsets of $\aleph_{k+j+1}$, B is non-stationary.

Let $C \in M$ be a club subset of $\aleph_{k+j+1}$ which is disjoint from B. Since C is unbounded in α_j by elementarity, $\alpha_j \in C$ and thus $\alpha_j \notin B$. By construction $A_j \subseteq B$, thus $\alpha_j \notin A_j$. It follows that II wins the game, contradiction!

We now fix a winning strategy σ for player II. We define U to be the set of all $\langle\langle \alpha_0, \dots \alpha_{j-1} \rangle\rangle$ such that $\alpha_0, \dots \alpha_{j-1}$ is an initial segment of II's sequence of plays in some run of the game where II plays according to σ. To finish the proof, we show that U has stationary branching.

Let $\langle\langle \alpha_0, \dots \alpha_{j-1} \rangle\rangle \in U$ and suppose that it represents II's response to I's playing $A_0, A_1, \dots A_{j-1}$. Let

$$B = \{\beta < \aleph_{k+j+1} : \langle\langle \alpha_0, \dots \alpha_{j-1}, \beta \rangle\rangle \in U\},$$

and suppose for a contradiction that B is non-stationary. Let I play B as his j^{th} move and let β be the response dictated by σ. Then by the definition of U, $\langle\langle \alpha_0, \dots \alpha_{j-1}, \beta \rangle\rangle \in U$ and so $\beta \in B$. This means that player II loses immediately, contradicting the assumption that σ was a winning strategy.

□

It is easy to check that for every $n > k$, S_n meets every stationary subset of $\aleph_n \cap \operatorname{cof}(\aleph_k)$ from the ground model. Since U has stationary branching, we may build by induction a branch b of U which is in $\prod_{n>k} S_n$. By Lemma 4.2 we may construct a structure M such that $\chi_M = b$. This shows that the sequence $\langle S_n \rangle$ is mutually stationary.

We summarise the main result of this section in a theorem.

Theorem 4.5. *Let $S \subseteq \aleph_\omega$ be V-generic for* $\mathrm{Add}(\aleph_0, \aleph_\omega)$*, and define $S_n = \aleph_n \cap \mathrm{cof}(\aleph_k) \cap S$ for each $n > k$. In $V[S]$ the sequence $\langle S_n : k < n < \omega \rangle$ is mutually stationary but not tightly stationary.*

Remark 4.6. It is clear from the proof that a large class of forcing posets could be used in place of $\mathrm{Add}(\aleph_0, \aleph_\omega)$. To be more precise, essentially the same proof will work for any forcing poset $\mathbb{P}$ such that

(1) The poset $\mathbb{P}$ is $\aleph_\omega$-c.c. and cardinal-preserving.
(2) Forcing with $\mathbb{P}$ adds a sequence $\langle S_n : k < n < \omega \rangle$ with $S_n \subseteq \aleph_n \cap \mathrm{cof}(\aleph_k)$ such that
  (a) In the extension by $\mathbb{P}$, $V \cap \prod_n S_n = \emptyset$.
  (b) For all $S \in V$ such that $S \subseteq \aleph_n \cap \mathrm{cof}(\aleph_k)$ and $V \models$ "S is stationary", $S_n \cap S \neq \emptyset$.

Similar ideas can be used to show that adding enough Cohen reals gives a model in which every mutually stationary sequence can be split.

Lemma 4.7. *Let $0 < k < \omega$ and let $\langle U_n : k < n < \omega \rangle$ be a mutually stationary sequence of sets with $U_n \subseteq \aleph_n \cap \mathrm{cof}(\aleph_k)$. Let T^* be the tree of functions f such that*

- $\mathrm{dom}(f) = \{n : k < n \leq j\}$ *for some $j > k$.*
- *There is a structure M such that M meets $\langle U_n : k < n < \omega \rangle$ and $f(n) = \sup(M \cap \aleph_n)$ for $k < n \leq j$.*

Then there is a subtree $U^* \subseteq T^*$ such that for all $j > k$ and $t \in U^*$ with $\mathrm{dom}(t) = \{n : k < n < j\}$, $\{\alpha < \aleph_j : t^\frown\alpha \in U^*\}$ is stationary in $\aleph_j$.

Proof. The proof is very similar to that of Lemma 4.4. Two players I and II collaborate to build a branch of T^*, with player I blocking out a non-stationary set of possible successors of the current position and player II choosing a successor which was not blocked by player I.

We need to check that I does not win, and so we suppose that τ is a strategy for player II. Since $\langle U_n : k < n < \omega \rangle$ is mutually stationary, we may find M such that M meets $\langle U_n : k < n < \omega \rangle$ and $\tau \in M$. As before, we may check that II can win against τ by playing $\sup(M \cap \aleph_{k+j+1})$ at move j of the game.

By the Gale-Stewart theorem there is a winning strategy σ for player II. As in Lemma 4.4 we may use σ to construct a suitable tree U^*, consisting of finite initial segments of runs of the game in which II plays according to σ. □

Lemma 4.8. *Let $0 < k < \omega$ and let $\langle U_n : k < n < \omega \rangle$ be a mutually stationary sequence of sets with $U_n \subseteq \aleph_n \cap \mathrm{cof}(\aleph_k)$. Let G be $\mathrm{Add}(\aleph_0, \aleph_\omega)$-generic over V. Then in $V[G]$ the following is true: there are partitions $\langle U_n^i : i < \omega \rangle$ of each U_n into ω disjoint stationary pieces, such that for all $f : \omega \longrightarrow \omega$ the sequence $\langle U_n^{f(n)} : k < n < \omega \rangle$ is mutually stationary.*

Proof. We can regard $\mathrm{Add}(\aleph_0, \aleph_\omega)$ as the finite support product of posets $\mathbb{P}_n$ for $k < n < \omega$, where $\mathbb{P}_n$ is the poset of finite partial functions from $\aleph_n$ to $\aleph_0$. We may then identify G with $\langle g_n : k < n < \omega \rangle$ where g_n is a map from $\aleph_n$ to $\aleph_0$, and we set $U_n^i = \{\alpha \in U_n : g_n(\alpha) = i\}$.

It is routine to check that each U_n^i is stationary. We now use Lemma 4.7 to construct a suitable tree U^*, and then given f we build a branch of U^* which lies in

$\prod_{n>k} U_n^{f(n)}$. We may now finish the argument exactly as in the proof of Theorem 4.5. □

Theorem 4.9. *If G is generic for $\mathrm{Add}(\aleph_0, \aleph_{\omega+1})$ then the following statement holds in $V[G]$: for all $k > 0$ and all mutually stationary $\langle U_n : k < n < \omega \rangle$ with $U_n \subseteq \aleph_n \cap \mathrm{cof}(\aleph_k)$, there are partitions $\langle U_n^i : i < \omega \rangle$ of each U_n into ω disjoint stationary pieces, such that for all $f : \omega \longrightarrow \omega$ the sequence $\langle U_n^{f(n)} : k < n < \omega \rangle$ is mutually stationary.*

Proof. By a chain condition argument the sequence $\langle U_n : k < n < \omega \rangle$ lies in the generic extension of V by some proper initial segment $\mathrm{Add}(\aleph_0, \lambda)$ of $\mathrm{Add}(\aleph_0, \aleph_{\omega+1})$, where $\lambda < \aleph_{\omega+1}$. Since $\mathrm{Add}(\aleph_0, \aleph_{\omega+1}) \simeq \mathrm{Add}(\aleph_0, \lambda) \times \mathrm{Add}(\aleph_0, \aleph_{\omega+1})$ we may as well assume that $\langle U_n : k < n < \omega \rangle \in V$. The theorem is now immediate from Lemma 4.8. □

5. Models in which every sequence is mutually stationary

Foreman and Magidor [12] pointed out that in general the question of which sequences $\langle S_n : n < \omega \rangle$ with $S_n \subseteq \aleph_n$ are mutually stationary is connected with the open question whether $\aleph_\omega$ can be a Jonsson cardinal. It is known that rather large singular cardinals of cofinality ω can be Jonsson: in particular Prikry proved that a singular limit of measurable cardinals is Jonsson and that a measurable cardinal remains Jonsson after doing Prikry forcing [20].

In this section we mildly strengthen these classical results by relating them to mutual stationarity. See the introduction to our previous paper [4] for more on the connection between mutual stationarity, Jonsson cardinals and Chang's conjecture.

Remark 5.1. Baumgartner [1] proved Theorem 5.2 in the special case where $S_n = \kappa_n \cap \mathrm{cof}(\aleph_{f(n)})$ for $f : \omega \longrightarrow 2$.

Theorem 5.2. *Let $\langle \kappa_i : i < \lambda \rangle$ be an increasing sequence of measurable cardinals where $\lambda = \mathrm{cf}(\lambda) < \kappa_0$. Let $S_i \subseteq \kappa_i$ be stationary for each $i < \lambda$, then $\langle S_i : i < \lambda \rangle$ is mutually stationary.*

Note that an immediate corollary is the well-known fact that $\sup \langle \kappa_i : i < \lambda \rangle$ is Jonsson.

Proof. Note that the hypothesis imply that for all $i < \lambda$, $\kappa_i > \sup \langle \kappa_j : j < i \rangle$. To simplify the bookkeeping we assume $\lambda = \omega$. Let $\theta = \sup_i \kappa_i$, and fix $\mathcal{M}$ a structure on H_θ. For each i let U_i be a normal measure on κ_i.

We will construct sets $J_i \in U_i$ such that $J_{i+1} \subseteq (\kappa_i, \kappa_{i+1})$ and the following indiscernibility property holds: for any positive integer n and any sequence $\langle k_j : j < n \rangle$ of positive integers, if $t_i, u_i \in [J_i]^{k_i}$ for $i < n$ and ϕ is any formula in the language of $\mathcal{M}$ then

$$\mathcal{M} \models \phi(t_0, \ldots t_{n-1}) \iff \mathcal{M} \models \phi(u_0, \ldots u_{n-1}).$$

To build the J_i, we define for each $j < \omega$ a sequence $\langle I_n^j : n < \omega \rangle$ such that

(1) $I_n^j \in U_n$.
(2) $I_{n+1}^j \subseteq (\kappa_n, \kappa_{n+1})$.
(3) $I_n^{j+1} \subseteq I_n^j$.
(4) For all $s \in [\kappa_{n-1}]^{<\omega}$ and all $\langle t_i : i \leq j \rangle$ with $t_i \in [I_{n+i}^j]^{<\omega}$, the $\mathcal{M}$-type of $(s, t_0, \ldots t_j)$ is determined by $(s, |t_0|, \ldots |t_j|)$.

Base case $j = 0$: We choose $I^0_n \in U_n$ as a set of order-indiscernibles for the structure obtained from $\mathcal{M}$ by adding a constant symbol for each element of κ_{n-1}. This is possible by Rowbottom's theorem.

Successor step: suppose we have constructed I^j_n. Let $s \in [\kappa_{n-1}]^{<\omega}$, $t \in [I^j_n]^{<\omega}$ and $u_i \in I^j_{n+i}$ for $1 \le i \le j+1$. By induction the $\mathcal{M}$-type of $(s, t, u_1, \ldots u_{j+1})$ is determined by $(s, t, |u_1|, \ldots |u_{j+1}|)$. Using Rowbottom's theorem and the completeness of U_n we may find $I^{j+1}_n \subseteq I^j_n$ such that if $t \in [I^{j+1}_n]^{<\omega}$ then for all $s, u_1, \ldots u_{j+1}$ the $\mathcal{M}$-type of $(s, t, u_1, \ldots u_{j+1})$ is determined by $(s, |t|, |u_1|, \ldots |u_{j+1}|)$.

We now set $J_n = \bigcap_j I^j_n$. To finish the proof of the theorem, we choose for each n a set $z_n \subseteq J_n$ with limit order-type such that $\sup(z_n) \in S_n$. Let N be the Skolem hull in $\mathcal{M}$ of the union of the sets z_n. We claim that $\sup(N \cap \kappa_n) = \sup(z_n)$ for each n.

Suppose that t is a Skolem term and that $t(a_0, \ldots a_j) < \kappa_i$ where $a_n \in [z_n]^{<\omega}$ and without loss of generality $j \ge i$. Let β be the least element of z_i with $\beta > \max(a_i)$. It must be that $t(a_0, \ldots a_j) < \beta$, for if not an application of indiscernibility shows that every element of J_i which is greater than $\max(a_i)$ is bounded by $t(a_0, \ldots a_j)$, and this is impossible since J_i is unbounded in κ_i. This shows that $t(a_0, \ldots a_j) < \sup(z_i)$, so $\sup(N \cap \kappa_i) = \sup(z_i)$ and we are done. □

We now turn to the situation in which $\langle \kappa_n : n < \omega \rangle$ is a Prikry-generic sequence in a measurable cardinal κ.

Remark 5.3. It is too much to ask that every Prikry-generic ω-sequence should have the property that every sequence of stationary sets is mutually stationary. For example if the sequence begins with $\aleph_1$ and $\aleph_2$ and Chang's conjecture is false then we can not meet the sets $S_0 = \aleph_1$, $S_2 = \aleph_2 \cap \mathrm{cof}(\aleph_1)$.

Theorem 5.4. *Let κ be measurable and let U be a normal measure on κ. Let $\mathbb{P}$ be the Prikry forcing defined from U. Then there is a condition $(\emptyset, A) \in \mathbb{P}$ which forces that if $\langle \kappa_n : n < \omega \rangle$ is the generic cofinal ω-sequence added by $\mathbb{P}$, then every sequence $\langle S_n : n < \omega \rangle$ with S_n stationary in κ_n for all n is mutually stationary.*

Proof. Suppose not. By the direct extension property for Prikry forcing, there is a condition $(\emptyset, A)$ and names $\dot{S}_n$ and $\dot{\mathcal{A}}$ such that $(\emptyset, A)$ forces that

- $\dot{\mathcal{A}}$ is an algebra on κ.
- $\dot{S}_n$ is stationary in κ_n.
- No substructure of $\dot{\mathcal{A}}$ meets $\langle \dot{S}_n : n < \omega \rangle$.

Let $\dot{F}$ be a name for a function $F : [\kappa]^{<\omega} \longrightarrow \kappa$ which is a Skolem function for $\mathcal{A}$. That is to say, $X \subseteq F``[X]^{<\omega}$ and $F``[X]^{<\omega} \prec \mathcal{A}$ for all infinite $X \subseteq \kappa$. Define a function $F^* : [\kappa]^{<\omega} \times [\kappa]^{<\omega} \longrightarrow \kappa$ as follows: $F^*(s, t)$ is equal to the unique β such that there is $E \in U$ with the property $(s, E) \Vdash \dot{F}(t) = \beta$ if such an E exists, and 0 otherwise.

By a standard application of Rowbottom's theorem and a diagonal intersection argument, we may find $B \in U$ such that for all $\delta < \kappa$ the set $B \setminus (\delta + 1)$ is a set of order-indiscernibles for $(\kappa, <, F^*, \{\gamma : \gamma \le \delta\})$. Now let $D = \{\gamma \in A \cap B : \sup(B \cap \gamma) = \gamma\}$. It is easy to check that $D \in U$, since U is normal.

We now force below the condition $(\emptyset, D)$ to get a generic increasing ω-sequence $G = \langle \kappa_n : n < \omega \rangle$. We use this to realise the names $\dot{S}_n$, $\dot{A}$ and $\dot{F}$ to get stationary sets S_n , an algebra $\mathcal{A}$ on κ and a Skolem function F for $\mathcal{A}$. Since $(\emptyset, D)$ refines $(\emptyset, A)$ there can be no substructure of $\mathcal{A}$ which meets $\langle S_n : n < \omega \rangle$.

Working in $V[G]$ we choose for each n a point $\gamma_n \in S_n$ with $\sup(B \cap \gamma_n) = \gamma_n$. Let $P = \bigcup_n B \cap [\kappa_{n-1}, \gamma_n)$ and let N be the closure of P under F. We claim that $\sup(N \cap \kappa_n) = \gamma_n$ for all n.

To see this, suppose for a contradiction that $F(w) = \beta$ for some $w \in [P]^{<\omega}$ and β such that $\gamma_n \leq \beta < \kappa_n$.

Find a condition $(s, E) \in G$ such that $(s, E) \Vdash \dot{F}(w) = \beta$, and notice that s must be a finite initial segment of $\langle \kappa_n : n < \omega \rangle$. Extending if necessary we may assume that $\mathrm{lh}(s) = m > n$. It is convenient to break up w and s as follows;

- $t = w \cap \kappa_{n-1}$, $u = w \cap [\kappa_{n-1}, \kappa_n)$, $v = w \cap (\kappa \setminus \kappa_n)$.
- $s_L = (\kappa_0, \ldots \kappa_{n-1})$, $s_H = (\kappa_n, \ldots \kappa_{m-1})$.

By the definition of the function F^*, we have

$$F^*(s_L^\frown s_H, t^\frown u^\frown v) = \beta < \kappa_n.$$

All points of B above $\sup(u)$ are chosen from a set of indiscernibles for a structure which has symbols for F^* and all ordinals below $\sup(u)$. Fix $\delta, \zeta \in B \cap \kappa_n$ with $\beta < \delta$ and $\sup(w \cap \kappa_n) < \zeta < \gamma_n$. We may choose a suitable δ because $\kappa_n \in D$ and so $B \cap \kappa_n$ is unbounded in κ_n.

The key points are that

- The sequences s_L, t and u consist of ordinals below $\sup(u)$.
- The sequences s_H and v consist of ordinals in B above κ_n.
- The ordinals δ and ζ lie in B and are between $\sup(u)$ and κ_n.

Since $F^*(s, w) < \delta$, it follows by indiscernibility that $F^*(s, w) < \zeta$. This is a contradiction, so $\sup(N \cap \kappa_n) = \gamma_n$ as required. It follows that N meets the sequence $\langle S_n : n < \omega \rangle$, contradiction! □

Corollary 5.5. *If $\langle \kappa_n : n < \omega \rangle$ is any Prikry generic sequence, then there exists m such that all sequences $\langle S_m : m \leq n < \omega \rangle$ with S_m stationary in κ_m for all $m \geq n$ are mutually stationary.*

Proof. Let A be as in the conclusion of Theorem 5.4, and find m such that $\kappa_n \in A$ for all $n \geq m$. □

6. Mutually stationary sequences not concentrating on a fixed cofinality

Theorems 5.2 and 5.4 show that if $\langle \kappa_n : n < \omega \rangle$ is an increasing sequence of reasonably large cardinals then every sequence of stationary sets can be mutually stationary. We now return to the problem of mutual stationarity for small cardinals.

Let $0 < l < \omega$, let $f : \omega \longrightarrow \{0, l\}$ be any function, and define $S^f_n = \{\alpha < \aleph_n : \mathrm{cf}(\alpha) = \aleph_{f(n)}\}$ for $n > l$. We will construct a model in which for every function f the sequence $\langle S^f_n : l < n < \omega \rangle$ is mutually stationary, starting from the assumption that there are infinitely many supercompact cardinals. This was originally done by Shelah, the simpler proof given here is due to Foreman and Magidor.

We will use some facts about IA structures. The first fact appears in in section 2 of [10].

Lemma 6.1. *Let $N \prec \mathcal{A}$ be a structure of some regular uncountable cardinality μ. Then the following are equivalent:*

(1) *N is IA of length and cardinality μ.*
(2) *For every μ-closed poset $\mathbb{P} \in N$ there is a sequence of elements $\langle p_\alpha : \alpha < \mu \rangle$ of $N \cap \mathbb{P}$, such that for every $D \in N$ a dense open subset of $\mathbb{P}$ there is $\alpha < \mu$ with $p_\alpha \in D$.*

The next fact is implicit in Foreman, Magidor and Shelah's paper [13] on Martin's Maximum.

Lemma 6.2. *Let $N \prec \mathcal{A}$ be internally approachable of length and cardinality μ, where μ is an uncountable regular cardinal. Let β be an ordinal such that $\beta < \sup(N \cap ON)$ and let $M = \mathrm{Sk}^{\mathcal{A}}(N \cup \{\beta\})$. Then M is internally approachable of length and cardinality μ.*

Proof. Let $\langle N_\alpha : \alpha < \mu \rangle$ be an internally approaching chain of models with union M. We may assume without loss of generality that $\beta < \sup(N_0 \cap ON)$. Define $M_\alpha = \mathrm{Sk}^{\mathcal{A}}(N_\alpha \cup \{\beta\})$, so that clearly the M_α form an increasing continuous chain of models of size less than μ whose union is M.

We claim that $M_\alpha = \{f(\beta) : f \in N_\alpha, \beta \in \mathrm{dom}(f)\}$. Clearly if $f \in N_\alpha$ then $f(\beta) \in M_\alpha$. Conversely if $y \in M_\alpha$ then $y = t(x, \beta)$ for some Skolem term t and parameter $x \in N_\alpha$. If $\gamma \in N_0 \cap ON$ with $\beta < \gamma$ then the (partial) function f with domain γ which maps α to $t(x, \alpha)$ is definable in H_θ from the parameters y, γ so $f \in N_\alpha$.

Fix an ordinal $\zeta < \mu$. β and $\langle N_\alpha : \alpha \leq \zeta \rangle$ are members of $M_{\zeta+1}$, so by the work of the last paragraph $\langle M_\alpha : \alpha \leq \zeta \rangle \in M_{\zeta+1}$. So M is internally approachable of length and cardinality μ, as claimed. □

The construction will proceed by starting with a structure which meets each $\aleph_n$ for $n > l$ in a set of cofinality $\aleph_l$, and judiciously adding ω many ordinals. The following well-known lemma [1] shows that adding an ordinal below $\aleph_m$ does no damage above $\aleph_m$.

Lemma 6.3. *Let $\mathcal{A} = (H_\theta, \in, <_\theta)$ for some large regular θ. Let $N \prec \mathcal{A}$, where $|N| = \aleph_n \subseteq N$ for some $n < \omega$. Let $n < m < \omega$, let β be an ordinal with $\sup(N \cap \aleph_m) < \beta < \aleph_m$, and let $N^* = \mathrm{Sk}^{\mathcal{A}}(N \cap \{\beta\})$. Then $\sup(N^* \cap \aleph_j) = \sup(N \cap \aleph_j)$ for $m < j < \omega$.*

Proof. Let t be a Skolem term. For each $x \in N$, N can compute the supremum of the set $\{t(x, \delta) : \delta < \aleph_m\} \cap \aleph_j$. □

For the rest of this section we will make the following assumption:
Assumption: there exists a sequence of ideals $\langle \mathcal{I}_n : l + 2 \leq n < \omega \rangle$ such that

(1) $\mathcal{I}_n$ is a uniform, $\aleph_n$-complete, normal ideal on $\aleph_n$.
(2) $\mathrm{P}(\aleph_n)/\mathcal{I}_n$ has an $\aleph_{l+1}$-closed dense subset.

This assumption is known to be consistent relative to the existence of infinitely many supercompact cardinals.

We now fix some large regular cardinal θ and a structure $\mathcal{A}$ which is an expansion of $(H_\theta, \in, <_\theta, \langle \mathcal{I}_n : l + 2 \leq n < \omega \rangle)$. If $N \prec \mathcal{A}$ has cardinality $\aleph_l$, and $\sup(N \cap \aleph_n) < \beta < \aleph_n$ for some n, we will say that *β is $\mathcal{I}_n$-generic for N* if and only if the following two conditions are satisfied

(1) For every $C \in N \cap \mathcal{I}_n$, $\beta \notin C$.
(2) The set $\{A \in N \cap \mathrm{P}(\aleph_n) : \beta \in A\}$ induces an N-generic filter on $N \cap \mathrm{P}(\aleph_n)/\mathcal{I}_n$.

Notice that by the first of these two conditions, if A and A' are subsets of $\aleph_n$ which both lie in N and are equivalent modulo $\mathcal{I}_n$, then $\beta \in A \iff \beta \in A'$.

Lemma 6.4. *If N is internally approachable of length and cofinality $\aleph_l$, then the set of $\beta < \aleph_n$ which are $\mathcal{I}_n$-generic for N is $\mathcal{I}_n$-positive.*

Proof. Let D be a dense $\aleph_{l+1}$-closed subset of $\mathrm{P}(\aleph_n)/\mathcal{I}_n$. By Lemma 6.1 we may find a decreasing sequence $\langle [A_\alpha] : \alpha < \aleph_l \rangle$ of elements of $N \cap D$, which meets every dense open subset of $\mathrm{P}(\aleph_n)/\mathcal{I}_n$ lying in N.

Let $[B] \in D$ be a lower bound for the sequence $\langle [A_\alpha] : \alpha < \aleph_l \rangle$. Since the ideal $\mathcal{I}_n$ is $\aleph_{l+1}$-complete, the set $C = \bigcap \{ \aleph_n \setminus X \ : \ X \in N \cap \mathcal{I}_n \}$ is in the dual of $\mathcal{I}_n$. For all $A \in N \cap P(\aleph_n), \alpha \in B \cap C$ we have $\alpha \in A$ iff A is in the filter generated by the sequence $\langle [A_\alpha] : \alpha < \aleph_l \rangle$. In particular, all $\alpha \in B \cap C$ are generic over N. □

The following lemma is the crucial one motivating our use of $\mathcal{I}_n$-generic ordinals. It indicates that when we add a suitable $\mathcal{I}_n$-generic ordinal we do not undo our work at cardinals below $\aleph_n$.

Lemma 6.5. *Let β be such that $\sup(N \cap \aleph_n) < \beta < \aleph_n$ and β is $\mathcal{I}_n$-generic for N. Let $N^* = \mathrm{Sk}^{\mathcal{A}}(N \cup \{\beta\})$. Then $N^* \cap \aleph_{n-1} = N \cap \aleph_{n-1}$.*

Proof. By the same argument that we used in Lemma 6.2, $N^* = \{f(\beta) : f \in N\}$. Let $\gamma \in N^* \cap \aleph_{n-1}$ and fix $f \in N$ such that $f : \aleph_n \longrightarrow \aleph_{n-1}$ and $\gamma = f(\beta)$.

The set of equivalence classes $[A]$ such that f is constant on A lies in N, and by normality it is dense in $\mathrm{P}(\aleph_n)/\mathcal{I}_n$. Since β is a generic ordinal there is $A \in N$ such that $\beta \in A$ and f is constant on A. It follows that $\gamma \in N$. □

Remark 6.6. We may also give an essentially equivalent proof of Lemma 6.5 phrased in the language of ultrafilters and elementary embeddings. Let M be the transitive collapse of N, let M^* be the collapse of N^*, and let $j : M \longrightarrow M^*$ be the elementary embedding from M to M^* corresponding to the inclusion map from N to N^*. Let U be the M-ultrafilter on the collapse of κ_n which is induced by β.

It is routine to check that $M^* = Ult(M, U)$ and j is the associated elementary embedding j^M_U. j has critical point equal to the collapse of κ_n, so in particular j fixes the collapse of κ_{n-1}. It follows that $N \cap \kappa_{n-1} = N^* \cap \kappa_{n-1}$.

Theorem 6.7. *Let $f : \omega \longrightarrow \{0, l\}$ be any function and let $T_n = \{\alpha < \aleph_n : \mathrm{cf}(\alpha) = \aleph_{f(n)}\}$ for $n > l$. The sequence $\langle T_n : l < n < \omega \rangle$ is mutually stationary.*

Proof. It will suffice to build a structure $M \prec \mathcal{A}$ such that $\mathrm{cf}(M \cap \aleph_{l+1}) = \aleph_l$ and $\mathrm{cf}(M \cap \aleph_n) = f(n)$ for $n > l + 1$. If necessary we may then use Lemma 6.3 to add in ω ordinals below $\aleph_{l+1}$ and adjust $\mathrm{cf}(M \cap \aleph_{l+1})$.

Let $\mathcal{A}$ be some expansion of $(H_\theta, \in, <_\theta)$. Let $N \prec \mathcal{A}$ be an internally approachable structure of length and cardinality $\aleph_l$. In particular, $\sup(N \cap \aleph_n)$ has cofinality $\aleph_l$ for every $n > l$.

If f is constant with value l there is nothing to do, so we assume that f takes the value 0 at least once. Let $\langle n_k : k < \omega \rangle$ be a sequence of integers such that $n_k > 2$, $f(n_k) = 0$ for all k, and for all $n > l + 1$ such that $f(n) = 0$ there are infinitely many k such that $n_k = n$.

We construct sequences $\langle N_k : k < \omega \rangle$ of structures and $\langle \beta_k : k < \omega \rangle$ of ordinals by recursion on k.

- $N_0 = N$.

- If $n_k = n$ then β_k is some ordinal such that $\sup(N_k \cap \aleph_n) < \beta_k < \aleph_n$ and β_k is $\mathcal{I}_n$-generic for N_k.
- $N_{k+1} = \mathrm{Sk}^{\mathcal{A}}(N_k \cup \{\beta_k\})$.

The construction can proceed, because by Lemma 6.2 the structure N_k is internally approachable of length and cofinality $\aleph_l$ for every $k < \omega$. Lemmas 6.3 and 6.5 imply that $\sup(N_{k+1} \cap \aleph_j) = \sup(N_k \cap \aleph_j)$ for $j \neq k$, so if we set $N_\omega = \bigcup_k N_k$ then we see that

- $N_\omega \prec \mathcal{A}$.
- $\mathrm{cf}(N_\omega \cap \aleph_{l+1}) = \aleph_l$.
- $\mathrm{cf}(N_\omega \cap \aleph_j) = \aleph_l$ if $f(j) = l$, $j > l + 1$.
- $\mathrm{cf}(N_\omega \cap \aleph_j) = \aleph_0$ if $f(l) = 0$, $j > l + 1$.

This shows that $\langle T_n : l < n < \omega \rangle$ is mutually stationary. □

If we are willing to leave gaps between the cardinals where we want cofinality ω, then we can reduce the hypothesis of Theorem 6.7 to infinitely many measurable cardinals. Explicitly: If there are infinitely many measurable cardinals and $A \subset \omega \setminus 2$ is such that for all $n \in A, n+1 \notin A$, then there is a forcing extension where $\aleph_n$ carries a normal $\aleph_n$-complete ideal on $\aleph_n$ with a dense set that is closed under decreasing sequences of length $\aleph_{n-2}$. In the resulting model, it can be shown that if $f : \omega \to \{0, l\}$ is such that $f^{-1}(0) \subset A$, then the sequence of sets $\langle T_n : l < n < \omega \rangle$ is mutually stationary. The proof is exactly as above.

7. Good points and diagonal Prikry forcing

In this section we record two forcing constructions involving large cardinals, PCF and reflection. The first construction gives a simple proof that it is consistent for there to be stationarily many non-good points in $\aleph_{\omega+1}$.

7.1. **Good points.** As we mentioned in the introduction to this paper, various models are known in which the set of non-good points of cofinality $\aleph_1$ is stationary in $\aleph_{\omega+1}$.

- Levinski, Magidor and Shelah [16] have shown that the Chang's conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is consistent, and Foreman and Magidor [11] have shown that if $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ then the set of non-good points of cofinality $\aleph_1$ is stationary.
- In unpublished work Magidor [17] has shown that the same conclusion follows from Martin's Maximum.

In this section we record the remark that Shelah's construction [14] for making the set of non-approachable points of cofinality $\aleph_1$ stationary also makes the set of non-good points stationary.

We start by assuming that κ is supercompact and that the GCH holds. It follows from GCH that there exists a scale $\langle f_\alpha : \alpha < \kappa^{+\omega+1} \rangle$ in $\prod_{n<\omega} \kappa^{+n}$ under the eventual domination ordering; to see this enumerate $\prod_{n<\omega} \kappa^{+n}$ as $\langle g_\eta : \eta < \kappa^{+\omega+1} \rangle$, write each $\alpha < \aleph_{\omega+1}$ as an increasing union $\bigcup_n X^\alpha_n$ with $|X^\alpha_n| < \kappa^{+n}$, and inductively choose f_α so that $f_\alpha(n)$ is greater than $f_\eta(n)$ and $g_\eta(n)$ for all $\eta \in X^\alpha_n$.

The basic idea is that this scale contains many non-good points of cofinality less than κ, and that we will "miniaturise" this situation by some judicious cardinal collapsing. Fix $j : V \longrightarrow M$ witnessing that κ is $\kappa^{+\omega+1}$-supercompact, and note that j is discontinuous at κ^{+n} for $n < \omega$ and also at $\kappa^{+\omega+1}$. Let $\gamma = \sup(j``\kappa^{+\omega+1})$

and let $H \in \prod j(\kappa^{+n})$ be given by $H(n) = \sup(j``\kappa^{+n})$. Let $j(\langle f_\alpha : \alpha < \kappa^{+\omega+1}\rangle) = \langle g_\beta : \beta < j(\kappa^{+\omega+1})\rangle$, and observe that by the elementarity of j and the closure of M the sequence $\langle g_\beta : \beta < j(\kappa^{+\omega+1})\rangle$ is a scale in $\prod_n j(\kappa^{+n})$ under eventual domination. It is easy to see that H is an exact upper bound for $\langle g_\beta : \beta < \gamma\rangle$.

We claim that there is an inaccessible $\delta < \kappa$ such that for stationarily many $\eta \in \kappa^{+\omega+1} \cap \mathrm{cof}(\delta^{+\omega+1})$ there an exact upper bound h for $\langle f_\alpha : \alpha < \eta\rangle$, with $\mathrm{cf}(h(n)) = \delta^{+n}$ for all n. If the claim fails then fix for each δ a club C_δ witnessing the non-stationarity of the relevant set, and let $C = \bigcap_\delta C_\delta$. Since C is club we see that $\gamma \in j(C)$, and since $\mathrm{cf}(\gamma) = \kappa^{+\omega+1}$ and $\langle g_\beta : \beta < \gamma\rangle$ has an exact upper bound H with $\mathrm{cf}(H(n)) = \kappa^{+n}$ for all n we get a contradiction by elementarity.

We now fix a suitable inaccessible $\delta < \kappa$ and let S be the stationary set of $\eta \in \kappa^{+\omega+1} \cap \mathrm{cof}(\delta^{+\omega+1})$ such that there is an exact upper bound h for $\langle f_\alpha : \alpha < \eta\rangle$, with $\mathrm{cf}(h(n)) = \delta^{+n}$ for all n. We force with $\mathbb{P} \times \mathbb{Q}$ where $\mathbb{P} = Col(\omega, \delta^{+\omega})$ and $\mathbb{Q} = Col(\delta^{+\omega+2}, < \kappa)$. Let $G \times H$ be $\mathbb{P} \times \mathbb{Q}$-generic.

The usual chain condition and closure arguments tell us that $\delta_V^{+\omega+1}$ is the new $\aleph_1$, $\kappa_V^{+n} = \aleph_{n+3}$ and $\kappa_V^{+\omega+1} = \aleph_{\omega+1}$. By Easton's lemma all $\delta_V^{+\omega+1}$-sequences of ordinals from $V[G][H]$ lie in $V[G]$. Since $\mathbb{P} \times \mathbb{Q}$ is κ-c.c. it is also routine to check that S is still stationary in $V[G][H]$ and that $\langle f_\alpha : \alpha < \kappa^{+\omega+1}\rangle$ is a scale in $\prod_{n<\omega} \kappa_V^{+n}$.

To finish we claim that if $\eta \in S$ then η is not a good point in $V[G][H]$. Suppose for a contradiction that such an η is good, and fix an unbounded set $A \subseteq \eta$ of order type $\delta_V^{+\omega+1}$ and $k < \omega$ such that $\langle f_\alpha(n) : \alpha \in A\rangle$ is strictly increasing for $n > k$. As we pointed out above, $A \in V[G]$. Since $\mathbb{P}$ has cardinality $\delta^{+\omega}$ it follows that there is $B \subseteq A$ with $B \in V$ and B unbounded in η.

The set B will serve as a witness that in V the point η is good of cofinality $\delta^{+\omega+1}$. This implies that an exact upper bound g for $\langle f_\alpha : \alpha < \eta\rangle$ exists with $\mathrm{cf}(g(n)) = \delta^{+\omega+1}$ for all n, contradicting the fact that $\eta \in S$ and that exact upper bounds are unique modulo finite alteration.

To summarise we have proved the following result.

Theorem 7.1. *If κ is $\kappa^{+\omega+1}$-supercompact then in some generic extension the set of non-good points of cofinality $\aleph_1$ in $\aleph_{\omega+1}$ is stationary.*

If we could make $\delta^{+\omega+1}$ into $\aleph_2$ by some small forcing we could get the consistency of the set of non-good points of cofinality $\aleph_2$ being stationary. Unfortunately this kind of cardinal collapse is provably very difficult and conjectured to be impossible [3].

7.2. **Diagonal Prikry forcing.** We showed in a previous paper [6] that Prikry forcing at a measurable cardinal κ preserves some of the stationary reflection properties of κ^+. Here we prove a similar result for diagonal Prikry forcing, using a rather similar argument.

We start by fixing some notation that we will use through this section. Suppose that we are given an increasing ω-sequence of measurable cardinals κ_n together with a normal measure U_n on each κ_n. A condition in the diagonal Prikry forcing determined by these data is a sequence $(\alpha_0, \ldots \alpha_{m-1}, B_m, B_{m+1}, \ldots)$ where $\kappa_{i-1} < \alpha_i < \kappa_i$ and $B_i \in U_i$. Given conditions $p = (\alpha_0, \ldots \alpha_{m-1}, B_m, B_{m+1}, \ldots)$ and $q = (\beta_0, \ldots \beta_{n-1}, C_n, C_{n+1}, \ldots)$, q extends p when $n \geq m$, $\beta_i = \alpha_i$ for $i < m$ and $\beta_i \in B_i$ for $m \leq i < n$.

We refer to the finite sequence $(\alpha_0, \dots \alpha_{m-1})$ as the lower part of the condition $(\alpha_0, \dots \alpha_{m-1}, B_m, B_{m+1}, \dots)$. It is well-known that diagonal Prikry forcing has the Prikry property, in the sense that any question about the forcing extension can be decided by shrinking the measure one sets in a condition, or to put it another way without changing the lower part.

We now let $\kappa = \bigcup_n \kappa_n$ and suppose that κ^+ has the following reflection property: for all n, any stationary susbet of $\kappa^+ \cap \mathrm{cof}(< \kappa_n)$ reflects at some point in $\kappa^+ \cap \mathrm{cof}(< \kappa_n)$. This will be the case for example if all of the κ_n are strongly compact. We claim that this reflection property is preserved by the diagonal Prikry forcing.

To see this fix n, a condition p and a name $\dot{T}$ for a stationary subset of $\kappa^+ \cap \mathrm{cof}(< \kappa_n)$. By extending p if necessary we may assume that the lower part of p has length at least n. For each lower part x which extends the lower part of p we let T_x be the set of α such that some extension of p with lower part x forces α into $\dot{T}$; since there are only κ possibilities for x, we may find x such that T_x is stationary.

By hypothesis there is $\gamma < \kappa^+$ with $\mathrm{cf}(\gamma) < \kappa_n$ such that $T_x \cap \gamma$ is stationary. We now fix $C \subseteq \gamma$ with order type $\mathrm{cf}(\gamma)$, and then use the completeness of the measures U_j for $j \geq n$ to find a single condition q with lower part x such that q forces that $C \cap T_x \subseteq \dot{T}$. Then q forces that $\dot{T}$ reflects at γ and we are done.

We summarise the results of this discussion in a theorem.

Theorem 7.2. *Let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of measurable cardinals with supremum κ, and suppose that for every n every stationary susbet of $\kappa^+ \cap \mathrm{cof}(< \kappa_n)$ reflects at some point in $\kappa^+ \cap \mathrm{cof}(< \kappa_n)$. Then this reflection property still holds in the generic extension by any diagonal Prikry forcing defined from some sequence of normal measures on the κ_n.*

Gitik and Magidor have devised several forcing posets for adding many diagonal Prikry sequences simultaneously. It would be interesting to combine their methods with those of Theorem 7.2.

8. Reflection and Martin's maximum revisited

Foreman, Magidor and Shelah [13] showed that Martin's Maximum implies that for all $\lambda \geq \aleph_2$, every stationary subset of $[H_\lambda]^{\aleph_0}$ reflects to a structure of size and uniform cofinality $\aleph_1$. We showed in the last section of [6] that forcing over a model of MM^+ we can get the consistency of this kind of reflection with the existence of two stationary subsets of a regular cardinal $\kappa \geq \aleph_2$ which do not reflect simultaneously, and Larson [15] independently obtained similar results. The following result generalises and sharpens these theorems: note in particular that we are reflecting to an IA structure and that we are only using MM (rather than MM^+) in the ground model.

Theorem 8.1. *Assume Martin's Maximum. Let κ be a regular cardinal with $\kappa > \aleph_1$ and let η be a (possibly finite) cardinal with $\eta \leq \aleph_1$. Then there is a forcing poset $\mathbb{P}$ which adds no bounded subsets of κ and such that in $V^{\mathbb{P}}$*

(1) *There are η stationary subsets of $\kappa \cap \mathrm{cof}(\omega)$ which do not reflect simultaneously.*
(2) *For every $\lambda > \aleph_1$, every set of fewer than η stationary subsets of $[H_\lambda]^{\aleph_0}$ simultaneously reflects to an internally approachable set in $[H_\lambda]^{\aleph_1}$.*

In particular, every collection of less than η stationary subsets of $\kappa \cap cof(\omega)$ simultaneously reflects.

Proof. We let $\mathbb{P}$ be the natural poset for partitioning $\kappa \cap \mathrm{cof}(\omega)$ into η many stationary sets which do not reflect simultaneously. Conditions in $\mathbb{P}$ are functions p such that $p : \eta \times (\alpha \cap \mathrm{cof}(\omega)) \longrightarrow 2$ for some $\alpha < \kappa$, with the properties that for each i we have $p(\nu, i) = 1$ for exactly one $\nu < \eta$, and that for every $\beta \in (\alpha + 1) \cap \mathrm{cof}(\aleph_1)$ there are $\nu < \eta$ and C club in β such that $f \restriction \{\nu\} \times C$ is identically zero.

It is easy to see that $\mathbb{P}$ is countably closed and is κ-strategically closed (a winning strategy for player II is to pick a coordinate $\nu < \eta$ and to write zero at that coordinate whenever it is her turn to play). It is also easy to see that $\mathbb{P}$ adds a partition of $\kappa \cap \mathrm{cof}(\omega)$ into η stationary sets which do not reflect simultaneously.

By Martin's Maximum the non-stationary ideal on $\aleph_1$ is $\aleph_2$-saturated in V. We claim that this is also the case in $V^{\mathbb{P}}$. To see this let $\langle \dot{A}_i : i < \aleph_2 \rangle$ be a $\mathbb{P}$-name for a counterexample to saturation, and note that V and $V^{\mathbb{P}}$ have the same subsets of $\aleph_1$; in particular they agree on the question of whether a subset of $\aleph_1$ is club, stationary or non-stationary. Since $\kappa \geq \aleph_2$ we may use the strategic closure of $\mathbb{P}$ to build a decreasing chain $\langle p_i : i < \aleph_2 \rangle$ of conditions in $\mathbb{P}$ such that $p_i \Vdash \dot{A}_i = \check{B}_i$ for some $B_i \in V$. Then $\langle B_i : i < \aleph_2 \rangle$ is a counterexample to saturation in V, which is a contradiction.

We let $\langle T_j : j < \eta \rangle$ be the sequence of stationary subsets of κ added by $\mathbb{P}$. Let $\zeta < \eta$ be a cardinal, let λ be a cardinal in $V^{\mathbb{P}}$ and let $\langle \dot{S}_i : i < \zeta \rangle$ be a sequence of $\mathbb{P}$-names for stationary subsets of $[H_\lambda]^{\aleph_0}$. (So there are at most countably many sets S_i.) Let μ be the maximum of λ and κ. We work towards showing that $\mathbb{P}$ forces that the sets S_i reflect simultaneously.

We now work in $V^{\mathbb{P}}$. For each $i < \zeta$ we will say that S_i is *social* if there exists $j < \eta$ such that for stationarily many $N \in [H_\mu]^{\aleph_0}$, $N \cap H_\lambda \in S_i$ and $\sup(N \cap \kappa) \in T_j$. In this case we let $j(i)$ be the least j with this property. If S_i is not social we say that S_i is *antisocial.* If $\eta < \aleph_1$ then all S_i are social, but if $\eta = \aleph_1$ this is not necessarily the case.

Let $j^* < \eta$ be least such that $j^* \neq j(i)$ for any social S_i. Since there are only countably many sets S_i we may fix a club set C_{bad} in $[H_\mu]^{\aleph_0}$ such that if S_i is antisocial, then $\sup(N \cap \kappa) \notin T_{j^*}$ for every $N \in C_{bad}$ such that $N \cap H_\lambda \in S_i$.

We now use a fact from [13]:

Claim. *Suppose that the non-stationary ideal on ω_1 is $\aleph_2$-saturated and $S \subset [H_\mu]^{\aleph_0}$ is stationary. Then there is a closed unbounded set $C \subset [H_\mu]^{\aleph_0}$ such that for all stationary $T \subset \{ N \cap \omega_1 : N \in C \cap S \}$ there are stationarily many $N \in C \cap S$ such that $N \cap \omega_1 \in T$.*

Proof. (Sketch) First note that if C, D are club sets in $[H_\mu]^{\aleph_0}$ and there is a $\xi \in \omega_1$ such that for all $N \in C$ if $\xi \in N$ then $N \in D$, then $\{N \cap \omega_1 : N \in C\} \subset \{N \cap \omega_1 : N \in D\}$ modulo the non-stationary ideal on ω_1.

Now build a sequence of closed unbounded sets $\langle C_\xi : \xi < \xi^* \rangle$ for some $\xi^* \leq \omega_2$ by induction. Let $C_0 = [H_\mu]^{\aleph_0}$ and given C_ξ choose $C_{\xi+1} \subset C_\xi$ if possible so that $\{ N \cap \omega_1 : N \in C_{\xi+1} \} \subsetneq \{ N \cap \omega_1 : N \in C_\xi \}$ modulo the non-stationary ideal. If this is not possible, then we set $\xi^* = \xi + 1$. At limit stages we take diagonal intersections.

Since the non-stationary ideal on ω_1 is $\aleph_2$-saturated, there is a $\xi^* < \omega_2$ where this sequence stops. If $\xi^* = \xi + 1$, then $C = C_\xi$ satisfies the conclusion of the claim. □

By the claim we can fix for each $i < \zeta$ a stationary set $U_i \subseteq \aleph_1$ such that

(1) If S_i is social then for every stationary $T \subseteq U_i$ there are stationarily many $N \in [H_\mu]^{\aleph_0}$ such that $N \cap H_\lambda \in S_i$, $\sup(N \cap \kappa) \in T_{j(i)}$ and $N \cap \aleph_1 \in T$.
(2) If S_i is antisocial then for every stationary $T \subseteq U_i$ there are stationarily many $N \in [H_\mu]^{\aleph_0}$ such that $N \cap H_\lambda \in S_i$ and $N \cap \aleph_1 \in T$.

Thinning out if necessary we arrange that the U_i are pairwise disjoint. By the closure of $\mathbb{P}$ we see that $\langle U_i : i < \zeta \rangle \in V$, and so working below a suitable condition in $\mathbb{P}$ we may assume that we have a fixed sequence $\langle U_i : i < \zeta \rangle$ which is in V.

Still working in $V^{\mathbb{P}}$ we define $\mathbb{Q}$ to be $Col(\aleph_1, H_\mu^V)$. If $F : \aleph_1 \simeq H_\mu^V$ is the bijection added by $\mathbb{Q}$ then in $V^{\mathbb{P}*\mathbb{Q}}$ we let $s_i = \{\delta < \aleph_1 : F``\delta \cap H_\lambda \in S_i\}$ and $t_j = \{\delta < \aleph_1 : \sup(F``\delta \cap \kappa) \in T_j\}$. Working in $V^{\mathbb{P}*\mathbb{Q}}$ we define a poset $\mathbb{R}$; conditions in $\mathbb{R}$ are closed bounded subsets of $\aleph_1$ consisting of ordinals δ such that $\delta \notin t_{j^*}$, and such that $\delta \in U_i$ implies $F``\delta \cap H_\lambda \in S_i$ for each i. The ordering on $\mathbb{R}$ is end-extension.

With a view to applying Martin's Maximum, we claim that $\mathbb{P}*\mathbb{Q}*\mathbb{R}$ is stationary set preserving. Let S be a stationary subset of $\aleph_1$. It is clear by the strategic closure of $\mathbb{P}$ that S is still stationary in $V^{\mathbb{P}}$, and we will work in $V^{\mathbb{P}}$ to argue that $\mathbb{Q} * \mathbb{R}$ preserves the stationarity of S. Let $\dot{C}$ be a $\mathbb{Q} * \mathbb{R}$-name for a club subset of $\aleph_1$ and let $(q_0, c_0) \in \mathbb{Q} * \mathbb{R}$. As usual when we are proving the preservation of stationarity, our goal is to find $(q, c) \leq (q_0, c_0)$ forcing that $\dot{C}$ meets S.

Shrinking S if necessary, and using the fact that there are only countably many sets U_i, we may assume that either S is disjoint from every U_i or $S \subseteq U_i$ for some i. We will treat these cases separately, and will also break up the second case according to the sociality or otherwise of S_i. We start by fixing some large regular cardinal θ.

Case 1: S is disjoint from every U_i. In this case we will choose a countable $M \prec H_\theta$ containing everything relevant such that $\delta =_{\mathrm{def}} M \cap \aleph_1 \in S$ and $\sup(M \cap \kappa) \notin T_{j^*}$. We then build a chain $\langle (q_n, c_n) : n < \omega \rangle$ of conditions in $M \cap \mathbb{Q} * \mathbb{R}$ which meets every dense subset of $\mathbb{Q} * \mathbb{R}$ lying in M, and let $q = \bigcup q_n$ and $c = \bigcup c_n$.

It is clear that $q \in \mathbb{Q}$ and q forces that $F``\delta = M \cap H_\mu^V$. Since $\sup(M \cap \kappa) \notin T_{j^*}$, q forces that $\delta \notin t_{j^*}$ and so $(q, c \cup \{\delta\})$ is a condition in $\mathbb{Q}*\mathbb{R}$. This condition forces that $\delta \in \dot{C}$ and we are done.

Case 2a: $S \subseteq U_i$ for a social S_i. In this case we choose a countable $M \prec H_\theta$ such that $\delta =_{\mathrm{def}} M \cap \aleph_1 \in S$, $M \cap H_\lambda \in S_i$ and $\sup(M \cap \kappa) \in T_{j(i)}$; this is possible by the choice of $j(i)$ and U_i. We define q and c as in case 1, and again q forces that $F``\delta = M \cap H_\mu^V$.

Since $j(i) \neq j^*$, q forces that $\delta \notin t_{j^*}$. By the choice of M we also see that q forces $\delta \in s_i$. Thus $(q, c \cup \{\delta\})$ is a condition in $\mathbb{Q} * \mathbb{R}$ and we are done.

Case 2b: $S \subseteq U_i$ for an antisocial S_i. In this case we choose $M \prec H_\theta$ such that $\delta =_{\mathrm{def}} M \cap \aleph_1 \in S$, $M \cap H_\lambda \in S_i$ and $M \cap H_\mu \in C_{bad}$. It follows from the choice of C_{bad} that $\sup(M \cap \kappa) \notin T_{j^*}$, and we may now proceed as in Case 2a.

We note that in the course of proving the claim, we also showed that if C is the club set added by $\mathbb{R}$ then $C \cap U_i$ is stationary for every i.

To finish the argument we will now apply Martin's Maximum to $\mathbb{P} * \mathbb{Q} * \mathbb{R}$ as in the last section of [6]. Meeting suitable dense sets we produce p, F and C together with $j^* < \eta$ and disjoint stationary $U_i \subseteq \aleph_1$ such that

(1) $\mathrm{dom}(p) = \eta \times \alpha$ for some $\alpha < \kappa$ of cofinality $\aleph_1$, with $p : \eta \times \alpha \longrightarrow 2$.

(2) $F : \aleph_1 \longrightarrow H_\mu$ and $F``\aleph_1$ is internally approachable (this approachability is easy to arrange, observing that for each $\gamma < \aleph_1$ the set of (p, q, r) such that $q \restriction \gamma \in \text{range}(q)$ is dense)
(3) $\sup F``\aleph_1 \cap \kappa = \alpha$.
(4) C is club in $\aleph_1$ and $C \cap U_i$ is stationary for every i.
(5) $p(j^*, \sup(F``\delta \cap \kappa)) = 0$ for every $\delta \in C$.
(6) For every $\bar{\alpha} < \alpha$, $p \restriction \eta \times \bar{\alpha} \in \mathbb{P}$.
(7) For every $\delta \in C \cap U_i$, there is some $\bar{\alpha} < \alpha$ such that $p \restriction \eta \times \bar{\alpha}$ forces that $F``\delta \cap H_\lambda \in S_i$.

Since we have arranged that $p(j^*, \nu) = 0$ for a club set of $\nu < \alpha$, p is itself a condition in $\mathbb{P}$. For each i we have arranged that $C \cap U_i$ is stationary and that p forces $\{F``\delta \cap H_\lambda : \delta \in C \cap U_i\} \subseteq S_i$, so that $p$ forces $S_i$ to reflect to $F``\aleph_1$.

□

9. The least cardinal where square fails

We showed [4] that if $\square_{\aleph_n}$ holds for every $n < \omega$ and CH holds then a certain weakening of $\square_{\aleph_\omega}$ holds. We then showed [5] that it is consistent for the full $\square_{\aleph_\omega}$ to fail under these circumstances. In this section we show that the least cardinal where square fails can be the least inaccessible.

Theorem 9.1. *It is consistent from large cardinals that the least λ where $\square_\lambda$ fails is the least inaccessible cardinal.*

Zeman pointed out that consistency that square first fails at the first inaccessible is a Mahlo cardinal. The model he constructs is the "usual" model $V^{Col(\kappa,\lambda)}$, where κ is the first inaccesible cardinal. The "usual" arguments show that square fails at κ in this model, and moreover, that if square held below κ in the ground model, it holds below κ in this model. Nonetheless we give the proof below as it seems that it may be useful in some other context.

Proof. (Sketch) Let GCH hold, let κ be supercompact and let λ be the least inaccessible cardinal greater than κ. Force that $\square_\eta$ holds for every η with $\eta < \lambda$ by a Reverse Easton iteration $\mathbb{P}$ of length λ. Note that $\mathbb{P}$ preserves cardinals, preserves the inaccessibility of λ and has cardinality λ. Now let $\mathbb{Q}$ be the Cohen forcing $\text{Add}(\aleph_0, \kappa)$, so that in $V^{\mathbb{P}*\mathbb{Q}}$ the cardinal λ is the least inaccessible cardinal.

We show that $\square_\lambda$ fails in $V^{\mathbb{P}*\mathbb{Q}}$ by showing that every stationary subset of λ^+ reflects. Let T be a stationary subset of λ^+ in $V^{\mathbb{P}*\mathbb{Q}}$, and use the fact that $\mathbb{P} * \mathbb{Q}$ has size λ to find a set $U \subseteq T$ such that $U \in V$ and U is stationary in V. Since κ is supercompact U reflects to some point η of cofinality δ^+, for some inaccessible δ with $\delta < \kappa$. We finish by showing that $\mathbb{P} * \mathbb{Q}$ preserves the stationarity of stationary subsets of δ^+, from which it follows that $T \cap \eta$ is stationary in $V^{\mathbb{P}*\mathbb{Q}}$.

We factor $\mathbb{P}$ as $\mathbb{P}_\delta * \mathbb{P}^\delta$, where $\mathbb{P}_\delta$ adds the $\square_\zeta$ sequences for $\zeta < \delta$ and $\mathbb{P}^\delta$ adds them for $\zeta \geq \delta$. Since $\mathbb{P}_\delta$ is δ^+-c.c. and $\mathbb{P}^\delta$ is $< \delta^+$-strategically closed in $V^{\mathbb{P}_\delta}$, forcing with $\mathbb{P}$ preserves stationary subsets of δ^+; since $\mathbb{Q}$ is is c.c.c the same is true of $\mathbb{P} * \mathbb{Q}$. □

10. A limiting result

In this last section of the paper we prove a result which limits the possibilities for creating a supercompact cardinal by forcing in the presence of weak squares. This result was motivated by the question "to what extent are weak squares compatible with stationary reflection?" A natural scenario for making a model with weak square at μ and some reflection is to make a model of $\square^*_\mu$ where some λ with $\lambda < \mu$ can be made supercompact by "mild" forcing. Ben-David and Shelah [2] attempted to give a proof of the consistency of weak square with reflection in which a generic supercompact embedding is resurrected by countably closed forcing, but the theorem that follows shows that their approach to the problem cannot work. See our paper on squares and reflection [6] for a consistency proof that uses the technique of resurrecting supercompact cardinals, but where the forcing which resurrects supercompactness is stationary set preserving for more delicate reasons.

A cardinal λ is *generically η- supercompact by countably closed forcing* iff there is a countable closed forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is an elementary embedding $j : V \to M$ with M a transitive class and $j``\eta \in M$.

Theorem 10.1. *Let $\lambda < \mu$ be cardinals with λ regular, μ strong limit and* $\mathrm{cf}(\mu) = \omega$. *If $\square^*_\mu$ holds then λ is not generically μ^+-supercompact by a countably closed forcing which preserves μ and μ^+.*

Proof. We wish to fix a sequence which witnesses $\square^*_\mu$ and has some additional properties. Starting with an arbitrary $\square^*_\mu$-sequence, we first replace each set $\mathcal{C}_\alpha$ by its closure under the power set operation; since μ is strong limit and the elements of C have order type less than μ, the resulting set still has size at most μ. We have produced a sequence $\langle \mathcal{C}_\alpha : \alpha < \mu^+ \rangle$ such that for all α

(1) $\mathcal{C}_\alpha$ is a set of subsets of α and $|\mathcal{C}_\alpha| \leq \mu$.
(2) If C is in $\mathcal{C}_\alpha$ then
  (a) $\mathrm{P}(C) \subseteq \mathcal{C}_\alpha$.
  (b) For every $\beta < \alpha$ with $\sup(C \cap \beta) = \beta$, $C \cap \beta \in \mathcal{C}_\beta$.
(3) $\mathcal{C}_\alpha$ contains at least one set which is club in α and has order-type $\mathrm{cf}(\alpha)$.

Now let $\mathbb{P}$ be some countably closed forcing poset which preserves μ and μ^+, let V_1 be some generic extension by $\mathbb{P}$. Suppose that in V_1, the generic μ^+-supercompactness of λ is witnessed by $j : V \longrightarrow M$.

Let $\gamma = \sup j``\mu^+$, so that $\mathrm{cf}(\gamma) = \mathrm{cf}_M(\gamma) = \mu^+$ and $\gamma < j(\mu^+)$. Let the image of our original $\square^*_\mu$-sequence under j be $\langle \mathcal{C}^j_\alpha : \alpha < j(\mu^+) \rangle$, and fix $C \in \mathcal{C}^j_\gamma$ which is club in γ and has order-type μ^+. The embedding j is continuous at points of cofinality ω, so that $j``\mu^+ \cap C$ is ω-club in γ. Let $\dot{C}$ name C.

Claim. *There do not exist $p \in \mathbb{P}$ and an unbounded subset D of μ^+ such that $p \Vdash j``D \subseteq \dot{C}$.*

Proof. If such p and D exist, let α be an accumulation point of D such that $\mathrm{cf}(\alpha) = \omega$ and $\mathrm{ot}(D \cap \alpha) \geq \mu$. The embedding j must be continuous at α, so p forces that $j(\alpha)$ is an accumulation point of $j(\dot{C})$, and so by coherence that $j(\dot{C}) \cap j(\alpha) \in \mathcal{C}^j_{j(\alpha)}$. If x is any countable subset of $D \cap \alpha$ then p forces that $j(x) = j``x$ and that $j``x \subseteq j(\dot{C}) \cap j(\alpha)$, so p forces that $j(x) \in \mathcal{C}^j_{j(\alpha)}$. By elementarity $x \in \mathcal{C}_\alpha$. This is impossible because there are μ^ω possibilities for x and $\mu^\omega \geq \mu^+ > |C_\alpha|$. □

Given $\alpha < \mu^+$, let α^* be a term for the least $\beta > \alpha$ with $j(\beta) \in \dot{C}$; we say that p *bounds* α^* if and only if there is $\gamma < \mu^+$ such that $p \Vdash \alpha^* \leq \gamma$. Not that if p does not bound α^* and $\beta > \alpha$ then p does not bound β^*.

Claim. *For every p there is an $\alpha < \mu^+$ such that p does not bound α^*.*

Proof. Suppose that p bounds α^* for every α, and define D to be the ω-club set of points β such that $\mathrm{cf}(\beta) = \omega$ and $p \Vdash \alpha^* < \beta$ for every $\alpha < \beta$. If $\beta \in D$ then it is forced that j is continuous at β, so that p forces $j(\beta)$ to be a limit point of $\dot{C}$ and hence $p \Vdash j(\beta) \in \dot{C}$. This contradicts the previous claim. □

Claim. *There exist a tree of conditions $\langle p_s : s \in {}^{<\omega}\mu \rangle$ and an increasing sequence $\langle \alpha_i : i < \omega \rangle$ of ordinals from μ^+ such that*

1. *If t extends s then $p_t \leq p_s$.*
2. *The condition p_s does not bound $\alpha^*_{\mathrm{lh}(s)}$.*
3. *For each $i < \mu$, $p_{s^\frown i}$ decides $\alpha^*_{\mathrm{lh}(s)}$ as some ordinal $\beta(s^\frown i)$ with $\beta(s^\frown i) < \alpha_{\mathrm{lh}(s)+1}$.*
4. *If $i \neq j$ then $\beta(s^\frown i) \neq \beta(s^\frown j)$.*

Proof. We observe that if p does not bound α^* and $\beta > \alpha$ then p does not bound β^*. We start by setting $p_0 = 1_{\mathbb{P}}$ and choosing α_0 such that p_0 does not bound α^*_0. Having defined α_n and p_s for $\mathrm{lh}(s) = n$, we use the fact that no p_s bounds α^*_n to choose the $p_{s^\frown i}$ and $\beta(s^\frown i)$ appropriately; we then choose α_{n+1} above all the $\beta(t)$ for $\mathrm{lh}(t) = n + 1$, with the property that α^*_{n+1} is not bounded by any p_t with $\mathrm{lh}(t) = n + 1$. □

Let $\alpha_\omega = \sup_{i<\omega} \alpha_i$. For each $f \in {}^\omega\mu$ let p_f be a lower bound for $\langle p_{f \restriction n} : n < \omega \rangle$ and let $x_f = \{ \beta(f \restriction n) : n < \omega \}$. By construction each p_f forces that $j(\alpha_\omega)$ is a limit point of C, so that arguing as in the proof of our first claim $p_f \Vdash j(x_f) \in \mathcal{C}^j_{j(\alpha_\omega)}$ and hence $x_f \in \mathcal{C}_{\alpha_\omega}$.

By construction the x_f are all distinct, and there are μ^ω possibilities for f. Therefore $|\mathcal{C}_{\alpha_\omega}| > \mu$, contradiction! It follows that j can not be a generic μ^+-supercompact embedding in $V^{\mathbb{P}}$. □

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