# CANONICAL STRUCTURE IN THE UNIVERSE OF SET THEORY: PART ONE

JAMES CUMMINGS, MATTHEW FOREMAN, AND MENACHEM MAGIDOR

ABSTRACT. We start by studying the relationship between two invariants isolated by Shelah, the sets of *good* and *approachable* points. As part of our study of these invariants, we prove a form of "singular cardinal compactness" for Jensen's square principle. We then study the relationship between *internally approachable* and *tight* structures, which parallels to a certain extent the relationship between good and approachable points. In particular we characterize the tight structures in terms of PCF theory and use our characterisation to prove some covering results for tight structures, along with some results on tightness and stationary reflection. Finally we prove some absoluteness theorems in PCF theory, deduce a covering theorem, and apply that theorem to the study of precipitous ideals.

## 1. INTRODUCTION

It is a distinguishing feature of modern set theory that many of the most interesting questions are not decided by ZFC, the theory in which we profess to work; to put it another way, ZFC admits a large variety of models. A natural response to this is to identify *invariants* which may take different values in different models, and which codify a large amount of information about a model.

Of particular interest are invariants which are *canonical*, in the sense that the Axiom of Choice is needed to show that they exist, but once shown to exist they are independent of the choices made. For example the uncountable regular cardinals are canonical in this sense.

Shelah discovered a large class of canonical invariants, the study of which he labeled PCF theory. These invariants include two which are central in this paper; Shelah [24, 26] (under some mild cardinal arithmetic assumptions on the singular cardinal $\mu$) defined two stationary subsets of $\mu^+$, the sets of *good* and *approachable* points. The definitions of these sets appear to depend on certain arbitrary choices, but (modulo the club filter) are in fact independent of these choices. Other canonical structures we study in this paper include the stationary sets of *tight* and *internally approachable* structures, and the collection of *good points on a scale*.

It is known that every approachable point is good and that weak forms of square, for example Jensen's weak square principle $\Box^*_\mu$, imply that every point is approachable. Foreman and Magidor [16] showed that their principle "Very weak square",

1991 *Mathematics Subject Classification.* Primary 03E35, 03E55; Secondary 03E05.

*Key words and phrases.* PCF theory, good ordinal, approachable ordinal, the ideal $I[\lambda]$, internally approachable structure, tight structure, square sequence, covering properties, precipitous ideal, mutual stationarity, stationary reflection.

The first author was partially supported by NSF Grants DMS-9703945 and DMS-0070549. The second author was partially supported by DMS-9803126 and DMS-0101155. The second and third authors were partially supported by the US-Israel Binational Science Foundation.

which captures some of the approachability implied by $\square^*_\mu$, implies many of the interesting consequences of $\square^*_\mu$. Cummings [5] showed that the assertion that every point is good severely constrains all $\mu^+$-preserving extensions of $V$.

Our first motivation for the work in this paper is the problem of the relationship between the sets of good and approachable points. This problem is trivial when weak squares exist, but non-trivial in general. For example it is consistent relative to large cardinals that not every point of cofinality $\aleph_1$ in $\aleph_{\omega+1}$ is good. We have speculated that perhaps the sets of good and approachable points coincide, and in Section 3 we prove that under some strong structural hypotheses this is the case.

The concept of an approachable ordinal is closely linked to that of an *internally approachable (IA)* structure. To be more precise, the set of approachable ordinals of cofinality $\eta$ can be characterized [16] as the set of ordinals which have the form $\sup(N \cap \mu^+)$ for some internally approachable $N$ of length and cardinality $\eta$.

Foreman and Magidor [17] isolated the concept of *tight structure* in their work on mutual stationarity and the non-saturation of the non-stationary ideal on $P_\kappa\lambda$, and tightness turns out to be closely related to the issues of goodness and approachability. In particular internally approachable structures are tight, and if $N$ is tight then $\sup(N \cap \mu^+)$ is good. Our second motivation for the work in this paper is the analogy

$$\frac{\text{Tight structures}}{\text{IA structures}} = \frac{\text{Good ordinals}}{\text{Approachable ordinals}}$$

Here is an outline of the paper. Section 2 contains some background material.

- In section 3 we prove a technical result about square-like sequences using the machinery of PCF theory. We use this to show that under some structural hypotheses all good points in $\aleph_{\omega+1}$ of cofinality greater than $\aleph_1$ are approachable, and also to show a kind of "singular cardinal compactness" for square sequences. For example we show that if CH holds and $\square_{\aleph_n}$ holds for all $n < \omega$, then there is a sequence $\langle C_\gamma : \gamma \in \aleph_{\omega+1} \cap \mathrm{cof}(\aleph_2) \rangle$ where $C_\gamma$ is a club subset of $\gamma$ with order type $\aleph_2$ and the $C_\gamma$ cohere at common limit points of uncountable cofinality.
- In section 4 we study the important property of *uniformity* for a structure, and show that sufficiently uniform structures can be reconstructed from their characteristic functions.
- In section 5 we characterize tight structures in terms of PCF theory. We also show that the properties of uniformity and tightness can sometimes be propagated from a set of regular cardinals $K$ to the set $\mathrm{pcf}(K)$.
- In section 6 we explore the relationship between tightness, covering properties and internal approachability. We prove theorems showing that under some circumstances tightness and internal approachability are equivalent. We also record a remarks on the connection between tightness and stationary reflection.
- In section 7 we prove some absoluteness results in PCF theory. We deduce a covering theorem, and use it to show that if $I$ is an ideal on $\aleph_1$ such that forcing with $P\aleph_1/I$ is sufficiently mild then $I$ is precipitous.

This paper contains only ZFC results. In the sequel [7] we prove a series of complementary consistency results. We would like to thank John Krueger for comments and corrections on an earlier draft of this paper.

## 2. Preliminaries

After a brief review of notation, we discuss the "canonical" concepts which are central in this paper: *internally approachable* and *tight* structures, and *approachable* and *good* ordinals.

We write $ON$ for the class of ordinals, $LIM$ for the limit ordinals and $SUCC$ for the successor ordinals. We write $CARD$ for the class of infinite cardinals, $REG$ for the regular cardinals and $SING$ for the singular cardinals. We denote by $\mathrm{cof}(\kappa)$ the class of ordinals of cofinality $\kappa$, and if $A$ is a set of ordinals we write $\mathrm{cf}(A)$ for the cofinality of $A$ considered as an ordered set. An *interval of regular cardinals* is a set of the form $REG \cap [\kappa, \lambda)$ for cardinals $\kappa$ and $\lambda$. We denote by $\tau^M$ the interpretation of a term $\tau$ in a model $M$, and by $\phi^M$ the relativisation of a formula $\phi$ to $M$. An *algebra* on a set $X$ is a structure for some countable first-order language which has $X$ as its underlying set.

### 2.1. Internally approachable and tight structures.

When $\theta$ is an uncountable regular cardinal, we will denote by $H_\theta$ the transitive set of those $X$ such that the transitive closure of $X$ has size less than $\theta$. We denote by $<_\theta$ some fixed well-ordering of $H_\theta$. By convention when $\theta < \chi$ we will assume that $<_\theta$ is the restriction of $<_\chi$ to $H_\theta$.

We will frequently be interested in structures of the form $\mathcal{A} = (H_\theta, \in, <_\theta)$; the advantage of building in a well-ordering is that if $X \subseteq H_\theta$ and $\mathrm{Sk}^{\mathcal{A}}(X)$ is the set of elements of $H_\theta$ definable in $\mathcal{A}$ with parameters from $X$, then $\mathrm{Sk}^{\mathcal{A}}(X) \prec \mathcal{A}$ and $\mathrm{Sk}^{\mathcal{A}}(X)$ is the smallest substructure of $\mathcal{A}$ containing $X$.

The definition of *internally approachable structure* appears in Foreman, Magidor and Shelah's paper [18] on Martin's Maximum. These structures are ubiquitous in modern set theory; see Lemma 2.3 and the remarks which follow it for some motivation.

**Definition 2.1.** Let $\theta$ be regular and let $\mathcal{A}$ be some algebra expanding $(H_\theta, \in, <_\theta)$. Let $N \prec \mathcal{A}$. $N$ is *internally approachable (IA)* if and only if there exist a limit ordinal $\delta$ and a sequence $\langle N_\alpha : \alpha < \delta \rangle$ such that

(1) $N = \bigcup_{\alpha<\delta} N_\alpha$.
(2) For all $\beta < \delta$, $\langle N_\alpha : \alpha < \beta \rangle \in N$.

In this case we will say that $N$ is *IA of length* $\delta$ and that $\langle N_\alpha : \alpha < \delta \rangle$ is an *approaching sequence for* $N$. We note that if we can always take an approaching sequence to be *continuous* in the sense that if $\beta$ is a limit ordinal, then $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$.

The length $\delta$ of the approaching sequence is not uniquely determined, but it is easy to see that $\mathrm{cf}(\delta)$ is uniquely determined; by a mild abuse of language we refer to this cofinality as the *cofinality of* $N$. In their paper on definable counterexamples to the continuum hypothesis Foreman and Magidor [15] give a detailed discussion of the lengths and cofinalities of approaching sequences and the sizes of the structures that appear in them.

All countable $N$ are IA, and if $N$ is approached by a sequence of length $\delta$ then $\delta \subseteq N$. We are often concerned with IA structures $N$ which have length and cardinality $\kappa$ for some regular uncountable cardinal $\kappa$. In this case we can cast the definition in a slightly different form, using the following easy lemma.

**Lemma 2.2.** *Let $\kappa$ be regular and uncountable, and let $N$ be an IA substructure of $\mathcal{A}$ with length and cardinality $\kappa$. Then there is a continuous approaching sequence $\langle M_\alpha : \alpha < \kappa \rangle$ for $M$, such that $M_\beta$ is an elementary substructure of $\mathcal{A}$ with cardinality less than $\kappa$ and $\langle M_\alpha : \alpha \leq \beta \rangle \in M_{\beta+1}$ for all $\beta < \kappa$.*

We will refer to increasing and continuous sequences $\langle M_\alpha : \alpha < \beta \rangle$ of elementary submodels of $\mathcal{A}$ such that $\langle M_\alpha : \alpha \leq \gamma \rangle \in M_{\gamma+1}$ for $\gamma + 1 < \beta$ as "continuous internally approaching chains of submodels".

The following lemma encapsulates some of the key properties of IA structures. For simplicity we only consider IA substructures with approaching sequences consisting of continuous internally approaching chains of submodels.

**Lemma 2.3.** *Let $\kappa$ be a regular cardinal. Let $\langle M_\alpha : \alpha < \kappa \rangle$ be a continuous increasing chain of elementary submodels of $\mathcal{A}$ such that $\langle M_\alpha : \alpha \leq \gamma \rangle \in M_{\gamma+1}$ for $\gamma < \kappa$. Let $M = \bigcup_{\alpha<\kappa} M_\alpha$ and let $\lambda = |M|$. Then*

(1) *$\kappa \subseteq M$, so in particular $\kappa \leq \lambda$.*
(2) *For all ordinals $\gamma \in M$ with $\mathrm{cf}(\gamma) > \lambda$*
   (a) *$\mathrm{cf}(M \cap \gamma) = \kappa$.*
   (b) *There is a closed unbounded set $C$ in $\sup(M \cap \gamma)$ with $C \subseteq M$.*
(3) *For every set $X \subseteq M$ with $|X| < \kappa$, there is $Y \in M$ with $|Y| < \lambda$ and $X \subseteq Y$.*
(4) *Let $K \in M$ be a set of regular cardinals such that $\lambda < \min(K)$ and $|K| < \kappa$. Then every function in $\prod_{\gamma \in K} M \cap \gamma$ is dominated pointwise by a function in $M \cap \prod K$.*
(5) *Let $\mathbb{P} \in M$ be a $\kappa$-closed and $(\lambda, \infty)$-distributive forcing poset. Then there is a decreasing sequence $\langle p_i : i < \kappa \rangle$ of conditions in $M \cap \mathbb{P}$ which meets every dense open subset of $\mathbb{P}$ lying in $M$.*

*Proof.* We sketch the proof.

(1) If $\beta < \kappa$ then $\langle M_\alpha : \alpha < \beta \rangle$ in $M$, and so by elementarity the length $\beta$ of this sequence is in $M$. We note that as a consequence $M_\beta \in M$ also.
(2) Since $\mathrm{cf}(\gamma) > \lambda = |M|$, $M \cap \gamma$ is bounded in $\gamma$. Since $\gamma \in M$ and each $M_\alpha$ is in $M$, the sequence $\langle \sup(M_\alpha \cap \gamma) : \alpha < \kappa \rangle$ is easily seen to be a continuous increasing sequence which is cofinal in $\sup(M \cap \gamma)$ and consists of ordinals in $M$.
(3) Since $|X| < \kappa$, there exists $\alpha < \kappa$ such that $X \subseteq M_\alpha$.
(4) Let $f$ be a function in $\prod_{\kappa \in K} M \cap \kappa$. We note that since $|K| < \kappa \subseteq M$ and $K \in M$, we have $K \subseteq M$. Since $|K| < \kappa$, we may find $\alpha < \kappa$ such that $f$ is in $\prod_{\kappa \in K} M_\alpha \cap \kappa$. We define $g$ with domain $K$ by $g(\kappa) = \sup(M_\alpha \cap \kappa)$. Since $|M_\alpha| \leq \lambda < \min(K)$ we see that $g \in \prod K$, and since $K$ and $M_\alpha$ both lie in $M$ we also have $g \in M$. Clearly $f(\kappa) < g(\kappa)$ for all $\kappa \in K$.
(5) By induction we choose $p_i$ to be the $<_\theta$-least condition in $\mathbb{P}$ which is a lower bound for $\langle p_j : j < i \rangle$ and lies in all the dense open subsets of $\mathbb{P}$ which are in $M_i$. Since $\langle M_j : j < i \rangle$ lies in $M$ it is easy to see that $\langle p_j : j < i \rangle$ lies in $M$, and hence $p_i$ lies in $M$.

□

Property 5 is not especially relevant in this paper, but it is highly significant in the context of [18] and [15]. Properties 2, 3 and 4 will all be of interest to us in what follows. In Section 4 we will make a detailed study of structures which have uniform cofinality as in Property 2. Property 3 is a kind of *internal covering* which we discuss at length in Section 6. Property 4 we call *tightness* and we axiomatise it in the following definition: actually we axiomatise something a little more general.

**Definition 2.4.** Let $K$ be a set of regular cardinals, let $\theta = \mathrm{cf}(\theta) > \sup(K)$, and let $\mathcal{A} = (H_\theta, \in, <_\theta)$. Let $M \prec \mathcal{A}$.

Then $M$ is *tight for* $K$ if and only if

(1) $K \in M$.
(2) For all $g \in \prod_{\kappa \in M\cap K}(M\cap\kappa)$ there exists $h \in M \cap \prod K$ such that $g(\kappa) < h(\kappa)$ for all $\kappa \in M \cap K$.

We note that if $|K| \subseteq M$ then $K \subseteq M$, and in this case tightness has the simpler form appearing in Property 4 above: when $K \subseteq M$, $M$ is tight for $K$ exactly when $M \cap \prod K$ is cofinal in $\prod_{\kappa\in K} M \cap \kappa$. It is natural to phrase this definition in terms of a standard idea, the *characteristic function of a structure.*

**Definition 2.5.** Let $K$ be a set of regular cardinals and let $M$ be a set. The *characteristic function* of $M$ (on $K$) is the function $\chi_M^K$ with domain $K$ given by $\chi_M^K : \kappa \longmapsto \sup(M \cap \kappa)$.

We will usually drop the superscript $K$ and write $\chi_M$ when the set $K$ is clear from the context. Typically we will be in a situation where $|M| < \min(K)$ and so $\chi_M \in \prod K$. If a structure $M$ is such that $K \subseteq M$, then tightness of $M$ amounts to saying that every function in $\prod K$ which is pointwise dominated by $\chi_M$ is pointwise dominated by some function in $M \cap \prod K$.

There are several reasons why it seems worthwhile to isolate the property of tightness. One reason is that there are many arguments in PCF theory which employ IA structures, and on closer inspection these arguments are typically just using the tightness guaranteed by Lemma 2.3. Another reason is that tight structures arise naturally in Foreman and Magidor's theory of *mutual stationarity.* We only give a cursory description of the theory of mutual stationarity, for more information see [17] and the sequel to this paper [7].

**Definition 2.6.** Let $K$ be a set of regular uncountable cardinals. Let $\langle S_\kappa : \kappa \in K\rangle$ be such that $S_\kappa \subseteq \kappa$ for all $\kappa \in K$.

(1) If $N$ is a set, then $N$ *meets* $\langle S_\kappa : \kappa \in K\rangle$ if and only if $\sup(N \cap \kappa) \in S_\kappa$ for all $\kappa \in N \cap K$ (equivalently $\chi_N \restriction N \cap K \in \prod_{\kappa\in N\cap K} S_\kappa$).
(2) $\langle S_\kappa : \kappa \in K\rangle$ is *mutually stationary* if and only if for every algebra $\mathcal{A}$ on $\sup(K)$ there exists $N \prec \mathcal{A}$ such that $N$ meets $\langle S_\kappa : \kappa \in K\rangle$.

It is easy to see that if $\langle S_\kappa : \kappa \in K\rangle$ is mutually stationary then $S_\kappa$ is stationary for each $\kappa$. Foreman and Magidor showed [17] that the converse is false in general, but is true if $S_\kappa \subseteq \kappa \cap \mathrm{cof}(\omega)$ for all $\kappa$.

Mutual stationarity can be seen as intermediate between the classical concept of stationarity for subsets of a regular uncountable cardinal, and the very general concept of stationarity introduced in [18]. We recall that if $S \subseteq \mathrm{P}(X)$ then $S$ is a *stationary subset of* $\mathrm{P}(X)$ if and only if for every algebra $\mathcal{A}$ on $X$ there is $B \in S$ such that $B \prec \mathcal{A}$. This is easily seen to be equivalent to demanding that for every $F : {}^{<\omega}X \longrightarrow X$ there is a non-empty $B \in S$ which is closed under $F$. When we need to distinguish between different flavors of stationarity we will refer to this last concept as *general stationarity.*

The sequence $\langle S_\kappa : \kappa \in K\rangle$ is mutually stationary if and only if the set of subsets of $\sup(K)$ which meet $\langle S_\kappa : \kappa \in K\rangle$ is a stationary subset of $\mathrm{P}(\sup(K))$. By standard facts [18, Lemma 0] about general stationarity, if $X$ is any set with $\sup(K) \subseteq X$ then $\langle S_\kappa : \kappa \in K\rangle$ is mutually stationary if and only if the set of subsets of $X$ which meet $\langle S_\kappa : \kappa \in K\rangle$ is a stationary subset of $\mathrm{P}(X)$. Subsequently we will often let $X = H_\theta$ for $\theta$ some regular cardinal greater than $\sup(K)$.

Two of the most useful facts about stationary subsets of a regular uncountable cardinal $\kappa$ are Fodor's lemma [13] and Solovay's splitting theorem [27]. It is open to what extent these results may be generalised to arbitrary mutually stationary sequences; Foreman and Magidor [17] identified a class of mutually stationary sequences, the *tightly stationary sequences,* for which versions of Fodor's lemma and Solovay's splitting theorem are available. As one might expect, a *tightly stationary sequence* is a sequence whose mutual stationarity is witnessed by tight structures.

**Definition 2.7.** Let $K$ be a set of regular cardinals and let $\langle S_\kappa : \kappa \in K\rangle$ be such that $S_\kappa \subseteq \kappa$ for all $\kappa \in K$. Let $\theta = \sup(K)^+$. The sequence $\langle S_\kappa : \kappa \in K\rangle$ is *tightly stationary* if and only if for every algebra $\mathcal{A}$ on $H_\theta$ there is $N \prec \mathcal{A}$ such that $N$ is tight for $K$ and $N$ meets $\langle S_\kappa : \kappa \in K\rangle$.

See Foreman and Magidor's paper [17] for the statements and proofs of Fodor's lemma and Solovay's theorem in the context of tight stationarity.

2.2. **Approachable and good points.** There is a close connection between internally approachable structures and the normal ideal $I[\lambda]$ defined by Shelah [24, 25, 26].

**Definition 2.8.** Let $\lambda$ be a regular uncountable cardinal. $S \in I[\lambda]$ if and only if there exists a club subset $E$ of $\lambda$ and a sequence $\langle a_\alpha : \alpha < \lambda\rangle$ of bounded subsets of $\lambda$, such that for all $\delta \in E \cap S$ there is $A \subseteq \delta$ unbounded in $\delta$ such that $\mathrm{ot}(A) = \mathrm{cf}(\delta) < \delta$ and for every $\beta < \delta$ there is $\gamma < \delta$ such that $A \cap \beta = a_\gamma$.

We discuss some alternative characterisations of $I[\lambda]$ in Section 7. The following result appears in [16] as part of the proof of Claim 4.4 in that paper, and gives one direction of the connection between $I[\lambda]$ and approachable structures.

**Lemma 2.9.** *Let $\lambda$ be regular and uncountable. Let $\theta > \lambda$ and let $\mathcal{A}$ be an algebra expanding $(H_\theta, \in, <_\theta)$. Let $S \in I[\lambda]$. Then there is a club subset $F$ of $\lambda$ such that for every $\delta \in F \cap S$ there is $M \prec \mathcal{A}$ such that $M$ is IA of length and cardinality* $\mathrm{cf}(\delta)$ *and* $\sup(M \cap \lambda) = \delta$.

To get a reasonable converse we fix a regular cardinal $\kappa$ less than $\lambda$ and assume that $\lambda^{<\kappa} = \lambda$. We let $I[\lambda, \kappa]$ be the restriction of $I[\lambda]$ to cofinality $\kappa$, that is the ideal of those $X \subseteq \lambda$ such that $X \cap \mathrm{cof}(\kappa) \in I[\lambda]$. We enumerate $[\lambda]^{<\kappa}$ as $\langle a_\alpha : \alpha < \lambda\rangle$, and let $S$ be the set of $\delta \in \lambda \cap \mathrm{cof}(\kappa)$ such that there is $A \subseteq \delta$ unbounded in $\delta$ with $\mathrm{ot}(A) = \kappa$ and every proper initial segment of $A$ enumerated as $a_\gamma$ for some $\gamma < \delta$. If we choose a different enumeration $\langle b_\alpha : \alpha < \lambda\rangle$ of $[\lambda]^{<\kappa}$ then $\{a_\alpha : \alpha < \beta\} = \{b_\alpha : \alpha < \beta\}$ for a club set of $\beta < \lambda$, so modulo the club filter $S$ is independent of the choice of the enumeration $\langle a_\alpha : \alpha < \lambda\rangle$.

It is not difficult to see that $S$ generates $I[\lambda, \kappa]$ modulo the club filter on $\lambda$, or to put it another way $S$ is (modulo club sets) the largest subset of $\lambda \cap \mathrm{cof}(\kappa)$ which is in $I[\lambda]$. We will refer to $S$ as "the set of approachable points of cofinality $\kappa$ in $\lambda$" with the understanding that this set is well-defined modulo the club filter. In this situation there is an easy converse to Lemma 2.9.

**Lemma 2.10.** *Let $\kappa$, $\lambda$ and $\theta$ be regular with $\kappa < \lambda < \theta$ and $\lambda^{<\kappa} = \lambda$. Let $\langle a_\alpha : \alpha < \lambda \rangle$ be an enumeration of $[\lambda]^{<\kappa}$ in order type $\lambda$, and let $S$ be defined as above. Let $N \prec (H_\theta, \in, <_\theta, \{\langle a_\alpha : \alpha < \lambda \rangle\})$ be IA of length and cardinality $\kappa$. Then* $\sup(N \cap \lambda) \in S$.

*Proof.* Let $N$ be approached by $\langle N_i : i < \kappa \rangle$, and let $\gamma_i = \sup(N_i \cap \lambda)$. Then every proper initial segment of $\langle \gamma_i : i < \kappa \rangle$ lies in $N$, so is enumerated as $a_\alpha$ for some $\alpha \in N \cap \lambda$. It follows that $\sup(N \cap \lambda) \in S$. □

Summarising, if $\lambda^{<\kappa} = \lambda$ then (modulo club sets) $S$ is the set of ordinals of the form $\sup(N \cap \lambda)$ where $N$ is IA of length and cardinality $\kappa$. If $C$ is any club subset of $\lambda$ we can add $C$ to the structure in the proof of Lemma 2.10, and find $N$ such that $\sup(N \cap \lambda) \in C \cap S$; it follows that $S$ is stationary. It is interesting to note that there is an attractive characterisation of $S$ in terms of forcing: results of Shelah [24] imply that every $\kappa^+$-closed forcing poset preserves the stationarity of subsets of $S$, but there is a $\kappa^+$-closed poset which destroys the stationarity of $\lambda \cap \mathrm{cof}(\kappa) \setminus S$.

The set $S$ is an example of the sort of "canonical invariant" discussed in the introduction. We will compare the set of approachable points with the set of *good points*, but before we can define the set of good points we need some PCF-theoretic preliminaries. For more information about PCF theory we refer the reader to Shelah's book [26], or the survey papers [3] and [1].

Given a set $X$ and an ideal $I$ on $X$, we refer to the sets in $I$ as *$I$-small sets*. We say that a set $Y \subseteq X$ is *$I$-large* if $X \setminus Y \in I$ and is *$I$-positive* if $Y \notin I$. Given ordinal valued functions $f$ and $g$ with domain $X$, we say that *$g$ dominates $f$ modulo $I$* (and write $f <_I g$) if $f(x) < g(x)$ for an $I$-large set of values of $x$; the relation $<_I$ is a strict partial ordering. Similarly we say *$g$ dominates $f$ pointwise* (and write $f < g$) if $f(x) < g(x)$ for all $x$. Given an ordinal valued function $f$ a *scale of length $\beta$ in* $(\prod_x f(x), <_I)$ is an increasing and cofinal sequence of length $\beta$ in $(\prod_x f(x), <_I)$.

For $Y$ a subset of $X$, a sequence of functions in ${}^X ON$ is *pointwise increasing on $Y$* if it is strictly increasing on every coordinate in $Y$. We note that a sequence which is pointwise increasing on an $I$-large set is $<_I$-increasing, but that in general the converse is not true. Two sequences $\langle f_\alpha : \alpha < \gamma \rangle$ and $\langle g_\alpha : \alpha < \gamma \rangle$ which are increasing with respect to $<_I$ are *cofinally interleaved (modulo $I$)* if for all $\alpha < \gamma$ there is $\beta < \gamma$ such that $f_\alpha <_I g_\beta$ and $g_\alpha <_I f_\beta$.

The function $g$ is a *an exact upper bound (eub)* for the $<_I$-increasing sequence $\langle f_\alpha : \alpha < \beta \rangle$ if and only if $f_\alpha <_I g$ for all $\alpha < \beta$, and for every $h$ with $h <_I g$ there exists $\alpha < \beta$ with $h <_I g_\alpha$. This is equivalent to $\langle f_\alpha : \alpha < \beta \rangle$ being a scale in $(\prod_x g(x), <_I)$ (with the caveat that the functions $f_\alpha$ need only be dominated by $g$ modulo $I$, rather than literally being members of $\prod_x g(x)$). It is easy to see that an exact upper bound, if one exists, is well-defined modulo $I$.

A point $\beta$ is *good* for a $<_I$-increasing sequence $\vec{f}$ (of length at least $\beta$) if and only if $\mathrm{cf}(\beta) > |X|$ and there exists an exact upper bound $h$ for $\langle f_\alpha : \alpha < \beta \rangle$ with the property that $\mathrm{cf}(h(x)) = \mathrm{cf}(\beta)$ for all $x$. The following lemma gives some useful equivalent characterisations of goodness: the implication from 3) to 1) gives an important construction principle for exact upper bounds.

**Lemma 2.11.** *The following are equivalent for $\langle f_\alpha : \alpha < \beta \rangle$ a $<_I$-increasing sequence with* $\mathrm{cf}(\beta) > |X|$.

(1) *$\beta$ is good.*

(2) *There is a sequence of functions of length* $\mathrm{cf}(\beta)$ *which is pointwise increasing and is cofinally interleaved with* $\langle f_\alpha : \alpha < \beta \rangle$ *modulo* $I$.
(3) *There is a sequence of functions of length* $\mathrm{cf}(\beta)$ *which is pointwise increasing on an* $I$*-large subset of* $X$ *and is cofinally interleaved with* $\langle f_\alpha : \alpha < \beta \rangle$ *modulo* $I$.

*Proof.* For 1) implies 2), let $h$ be an eub such that $\mathrm{cf}(h(x)) = \mathrm{cf}(\beta)$ for all $x$ and fix for each $x$ a sequence $\langle \alpha^x_i : i < \mathrm{cf}(\beta) \rangle$ increasing and cofinal in $h(x)$; now define $g_i(x) = \alpha^x_i$ and check that $\langle g_i : i < \mathrm{cf}(\beta) \rangle$ is pointwise increasing and cofinally interleaved with $\langle f_\alpha : \alpha < \beta \rangle$. The implication from 2) to 3) is immediate. For 3) implies 1) fix $\langle g_i : i < \mathrm{cf}(\beta) \rangle$ pointwise increasing on an $I$-large set and cofinally interleaved with $\langle f_\alpha : \alpha < \beta \rangle$; check that the pointwise supremum of $\langle g_i : i < \mathrm{cf}(\beta) \rangle$ is an exact upper bound for $\langle f_\alpha : \alpha < \beta \rangle$ which has cofinality $\mathrm{cf}(\beta)$ on an $I$-large set, and then alter it to get an exact upper bound which has cofinality $\mathrm{cf}(\beta)$ on all $x$. □

As an immediate corollary of Lemma 2.11, if $\beta$ is a good point then there is $C$ club in $\beta$ such that every point of $C$ with cofinality greater than $|X|$ is good. So the set of ungood points of cofinality greater than $|X|$ is quite thin, in the sense that if it is stationary then its stationarity can only reflect at points of itself.

**Example 2.12.** *If* $\langle f_\alpha : \alpha < \lambda \rangle$ *and* $\langle f'_\alpha : \alpha < \lambda \rangle$ *are two* $<_I$*-increasing sequences with the same exact upper bound* $g$ *then it is easy to see that they are cofinally interleaved. It follows that if* $\lambda$ *is a regular uncountable cardinal there is a club set of* $\beta < \lambda$ *such that* $\langle f_\alpha : \alpha < \beta \rangle$ *and* $\langle f'_\alpha : \alpha < \beta \rangle$ *are cofinally interleaved. Therefore the sets of good points for* $\langle f_\alpha : \alpha < \lambda \rangle$ *and* $\langle f'_\alpha : \alpha < \lambda \rangle$ *are equivalent modulo the club filter and so give an example of "canonical" structure.*

The following result by Shelah is central in PCF theory.

**Fact 2.13.** *If* $\mu$ *is a singular cardinal then there is a set* $K \subseteq \mu$ *of regular cardinals such that* $\mathrm{ot}(K) = \mathrm{cf}(\mu)$ *and there is a scale of length* $\mu^+$ *in* $\prod K$ *under the eventual domination ordering.*

As we just pointed out, the set of good points in such a scale is essentially independent of the choice of the scale so has some claim to be considered a canonical invariant. Of course there is still a dependence on $K$ but for small values of $\mu$ we can also make a canonical choice for $K$. The case of most interest to us is $\mu = \aleph_\omega$, and in this case work of Shelah shows that modulo finite sets there is a largest $K \subseteq \{\aleph_n : n < \omega\}$ such that $\prod K$ has a scale of length $\aleph_{\omega+1}$ in the eventual domination ordering. In this situation we refer to the set of good points in such a scale as the *good points in* $\aleph_{\omega+1}$.

The next lemma makes the connection between scales, good points and IA structures. It should be compared with Lemma 2.10.

**Lemma 2.14.** *Let* $|X| < \kappa < \lambda < \theta$ *with* $\kappa$, $\lambda$ *and* $\theta$ *regular. Let* $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ *be a* $<_I$*-increasing sequence in* ${}^X ON/I$*, and suppose there is an exact upper bound* $g$ *for* $\vec{f}$ *such that* $\mathrm{cf}(g(x)) > \kappa$ *for all* $x \in X$*. Let* $N \prec (H_\theta, \in, <_\theta, \{\vec{f}, g\})$ *be an IA structure of length and cardinality* $\kappa$*. Then* $\sup(N \cap \lambda)$ *is a good point for* $\vec{f}$*.*

*Proof.* We fix an internally approaching chain $\langle N_i : i < \kappa \rangle$ such that $X \in N_0$, $X \subseteq N_0$, $|N_i| < \kappa$ for all $i$ and the union of the chain is is $N$. We let $g_i(x) = \sup(N_i \cap g(x))$ for $i < \kappa$, and claim that $\langle g_i : i < \kappa \rangle$ will serve as a witness that $\sup(N \cap \lambda)$ is good.

Since $X \subseteq N_0$ and $\mathrm{cf}(g(x)) > \kappa$ for all $x \in X$, we see that $g_i < g$; moreover if $i < j$ then $N_i \in N_j$, so $g_i \in N_j$ and hence $g_i < g_j$.

Since $g_i \in N$ and $g_i$ is dominated by the exact upper bound $g$, it follows by elementarity that $g_i <_I f_\alpha$ for some $\alpha \in N \cap \lambda$. On the other hand if $\alpha < \sup(N \cap \lambda)$ then there exist $i < \kappa$ and $\beta \in N_i \cap \lambda$ with $\alpha < \beta$, so that $\mathrm{range}(f_\beta) \subseteq N_i$ and $f_\beta < g_i$. □

If $C$ is club in $\lambda$ we may add a predicate for $C$ to $\mathcal{A}$ and produce a good point of cofinality $\kappa$ lying in $C$. It follows that the set of good points of cofinality $\kappa$ is stationary.

We can now prove that, as we mentioned in the introduction, approachable points are good.

**Corollary 2.15.** *Let $\lambda$ be the successor of a singular cardinal $\mu$, let $K \subseteq \mu$ be an unbounded set of regular cardinals with $\mathrm{ot}(K) = \mathrm{cf}(\mu) < \min(K)$. Let $\vec{f}$ be a scale of length $\lambda$ in $\prod K$ under eventual domination. If $S \in I[\lambda]$ then almost all points of $S$ with cofinality greater than $\mathrm{cf}(\mu)$ are good for $\vec{f}$.*

*Proof.* Immediate from Lemmas 2.9 and 2.14 □

Recalling that the set of approachable points (when it can be defined) is the maximal set in $I[\lambda]$, we see that modulo the club filter every approachable point of cofinality greater than $\mathrm{cf}(\mu)$ is good.

## 3. Goodness, approachability and compactness for squares

One theme of this paper is the relationship between the concepts of *goodness* and *approachability*. As we showed in Corollary 2.15, approachable points are good. In this section we show that under certain circumstances the implication from approachability to goodness can be reversed. It is notable that the result presented here shows that scales can be used to derive squarelike principles at $\kappa$ from squares below $\kappa$.

Since approachable points are good, the problem "which good points are approachable" becomes trivial if almost every point is approachable. We digress briefly to review what is known about the extent of $I[\lambda]$.

It it is known that if Jensen's weak square principle $\Box^*_\mu$ holds then $\mu^+ \in I[\mu^+]$, so that large cardinals will be required to make models in which $I[\mu^+]$ is non-trivial. If $\mu^{<\mu} = \mu$ then $\Box^*_\mu$ holds, so GCH trivialises $I[\lambda]$ for $\lambda$ the successor of a regular cardinal. Shelah has shown that if $\lambda$ is regular then $\lambda^+ \cap \mathrm{cof}(< \lambda)$ is in $I[\lambda^+]$, and that if $\lambda$ is singular then for all regular $\kappa < \lambda$ there is a stationary subset of $\lambda^+ \cap \mathrm{cof}(\kappa)$ lying in $I[\lambda^+]$.

If $\kappa$ is supercompact and $\mathrm{cf}(\lambda) < \kappa < \lambda$, then $\lambda^+ \cap \mathrm{cof}(< \kappa) \notin I[\lambda^+]$. By doing some suitable Levy collapses (see [24] or [20] for details) this can be used to produce a model in which $\aleph_{\omega+1} \cap \mathrm{cof}(\aleph_1) \notin I[\aleph_{\omega+1}]$. As for successors of regular cardinals, Mitchell's model [23] in which there is no $\aleph_2$-Aronszajn tree also has the property that $\aleph_2 \cap \mathrm{cof}(\aleph_1) \notin I[\aleph_2]$. In recent work [22] Mitchell has produced a model in which $I[\aleph_2]$ is generated modulo the club filter by the set $\aleph_2 \cap \mathrm{cof}(\omega)$.

We now narrow our focus to the cardinal $\aleph_{\omega+1}$, where the points of cofinality $\aleph_1$ seem to play a special role. It is known (see [16] or [7]) to be consistent that stationarily many points of cofinality $\aleph_1$ are not good, and it is open whether $\aleph_{\omega+1} \cap \mathrm{cof}(\neq \aleph_1)$ is always in $I[\aleph_{\omega+1}]$. As we see shortly all points of cofinality greater than $2^{\aleph_0}$ are good. In particular under CH all points of cofinality greater than $\aleph_1$ are good.

Before we prove the main result of this section we need some technical preliminaries. We start with the concept of a *continuous sequence*. Let $\vec{f}$ be a $<_I$-increasing sequence. The sequence $\vec{f}$ is *continuous at* $\beta$ if and only either there is no exact upper bound for $\langle f_\alpha : \alpha < \beta \rangle$ or $f_\beta$ is such a bound. $\vec{f}$ is *continuous* if and only if it is continuous at every limit $\beta$. Given an arbitrary $<_I$-increasing sequence $\vec{f}$ of limit length $\lambda$, we may replace $f_\beta$ for $\beta < \lambda$ limit by an exact upper bound for $\langle f_\alpha : \alpha < \beta \rangle$ whenever such a bound exists, and get a continuous sequence which is cofinally interleaved with the original one.

If $\beta$ is a good point for $\vec{f}$ and $\langle g_i : i < \mathrm{cf}(\beta) \rangle$ is increasing on an $I$-large set and cofinally interleaved with $\langle f_\alpha : \alpha < \beta \rangle$ then let $g$ be the pointwise supremum of the sequence $\langle g_i : i < \mathrm{cf}(\beta) \rangle$. As we saw in Lemma 2.11 $g$ is an exact upper bound for $\langle f_\alpha : \alpha < \beta \rangle$ and so by continuity $f_\beta$ is also an exact upper bound: since exact upper bounds are unique modulo $I$, the functions $f_\beta$ and $g$ must agree on an $I$-large set.

Next we need an alternative characterisation of good points in scales of a special kind. When $X$ is an ordered set with no last element, and $I$ is the ideal of bounded subsets of $X$, we usually write $<^*$ for $<_I$ and $=^*$ for $<_I$. We refer to $<^*$ as the *eventual domination ordering*. This is the context of the "good scales" and "very good scales" studied in our paper [8] on scales, squares and reflection.

In a scale under eventual domination the definition of good point can be simplified. To be more precise the following statements are equivalent:

- The ordinal $\beta$ is good for $\vec{f}$.
- The cofinality of $\beta$ is greater than $|X|$, and for every unbounded $A \subseteq \beta$ there exists an unbounded $B \subseteq A$ and $x \in X$ such that $\langle f_\alpha : \alpha \in B \rangle$ is pointwise increasing on $\{y : x < y\}$.

The proof of the forward implication uses an "interleaving" argument of a type which is ubiquitous in PCF theory, so we give it in detail.

Since $\beta$ is good, we may fix $x_0 \in X$ and $\langle g_\alpha : \alpha < \mathrm{cf}(\beta) \rangle$ which is pointwise increasing on $\{y : x_0 < y\}$ and is cofinally interleaved with $\langle f_\alpha : \alpha < \beta \rangle$. Thinning out the sequence of $g_\alpha$ we may also assume that for every $\alpha$ there is $\gamma_\alpha \in A$ with $g_\alpha <^* f_{\gamma_\alpha} <^* g_{\alpha+1}$. Since $\mathrm{cf}(\beta) > |X|$ we may find $x \geq x_0$ and an unbounded set $B_0 \subseteq \mathrm{cf}(\beta)$ such that $g_\alpha(y) < f_{\gamma_\alpha}(y) < g_{\alpha+1}(y)$ for all $\alpha \in B_0$ and $y \geq x$. Let $B = \{\gamma_\alpha : \alpha \in B_0\}$, and observe that if $\alpha$ and $\alpha'$ are in $B_0$ with $\alpha < \alpha'$ and $y \geq x$ then $f_{\gamma_\alpha}(y) < g_{\alpha+1}(y) \leq g_{\alpha'}(y) < f_{\gamma'_\alpha}(y)$.

We will also need Shelah's important Trichotomy theorem [26].

**Fact 3.1.** *Let $I$ be an ideal on a set $X$ of cardinality $\kappa$, and let $\lambda$ be regular with $\kappa^+ < \lambda$. Let $\langle f_\alpha : \alpha < \lambda \rangle$ be a $<_I$-increasing sequence. Then one of the following must occur:*

(1) *There is an eub $g$ for $\langle f_\alpha : \alpha < \lambda \rangle$ such that $\mathrm{cf}(g(x)) > \kappa$ for all $x$.*
(2) *There exist an ultrafilter $U$ on $X$ disjoint from $I$ and sets $\langle S_x : x \in X \rangle$ with $|S_x| \leq \kappa$, such that for all $\alpha < \lambda$ there is $g \in \prod_{x \in X} S_x$ and $\beta < \lambda$ such that $f_\alpha <_U g <_U f_\beta$.*
(3) *There exists a function $h$ such that if $D_\alpha = \{x : f_\alpha(x) < h(x)\}$ then the sequence $\langle D_\alpha : \alpha < \lambda \rangle$ is not eventually constant modulo $I$.*

If $2^\kappa < \lambda$ then it is not hard to see that the second and third cases are impossible, so that there must exist an eub $g$ for $\langle f_\alpha : \alpha < \lambda \rangle$ such that $\mathrm{cf}(g(x)) > \kappa$ for all $\kappa$.

For the rest of this section we fix some infinite $A \subseteq \omega$ such that there is a scale of length $\aleph_{\omega+1}$ in $\prod_{n \in A} \aleph_n$ under eventual domination. We also fix $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ which is such a scale. Altering the scale at limits if necessary, we may assume that it has the following strengthened form of the continuity property: if $0 < k < \omega$ and $\alpha$ is a good point of cofinality $\aleph_k$, then $f_\alpha$ is an exact upper bound for $\langle f_\zeta : \zeta < \alpha \rangle$ (this is just continuity) and in addition $\mathrm{cf}(f_\alpha(n)) = \aleph_k$ for all $n > k$.

We already know that there are stationarily many good points for this scale in any uncountable cofinality but with some cardinal arithmetic assumptions we can say more.

If $2^{\aleph_0} < \aleph_k$ and $\beta \in \aleph_{\omega+1}$ is a point of cofinality $\aleph_k$ then it follows from Trichotomy that there is an eub $g$ for $\langle f_\alpha : \alpha < \beta \rangle$ with $\mathrm{cf}(g(n)) > \aleph_0$ for all $n \in A$. A little analysis (see [21] or [4] for the details) shows that $\mathrm{cf}(g(n)) = \aleph_k$ for cofinitely many $n$, so that $\beta$ is a good point: it follows from our assumptions on the scale that $f_\beta$ is an eub for $\langle f_\alpha : \alpha < \beta \rangle$ and $\mathrm{cf}(f_\beta(n)) = \aleph_k$ for all $n \in A$ with $n > k$.

The structural hypothesis which we need for our main result is a weakening of square which only refers to ordinals below $\aleph_\omega$ with a fixed cofinality.

**Definition 3.2.** Let $k$ be a natural number with $k \geq 1$. A $\square(\aleph_\omega, \mathrm{cof}(\omega_k))$-*sequence* is a sequence $\langle C_\alpha : \alpha \in \aleph_\omega \cap \mathrm{cof}(\omega_k) \rangle$ such that

1. For all $\alpha$, $C_\alpha$ is club in $\alpha$ and $\mathrm{ot}(C_\alpha) = \omega_k$.
2. For all $\alpha$, $\beta$ and $\gamma$, if $\gamma \in \lim(C_\alpha) \cap \lim(C_\beta)$ then $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

The next lemma is similar in spirit to the results on "improving squares" in our paper on scales, squares and reflection [8].

**Lemma 3.3.** *If $\square_{\aleph_n}$ holds for all $n$ with $k \leq n < \omega$ then there is a $\square(\aleph_\omega, \mathrm{cof}(\omega_k))$-sequence.*

*Proof.* Fix $\langle D^n_\alpha : \alpha < \aleph_{n+1} \rangle$ witnessing $\square_{\aleph_n}$, where we assume without loss of generality that $D^n_\alpha \subseteq \alpha \setminus (\aleph_n + 1)$ for $\alpha \in \aleph_{n+1} \setminus (\aleph_n + 1)$. We define $C_\alpha$ inductively.

**Base case:** If $\alpha \in \aleph_{k+1} \cap \mathrm{cof}(\omega_k)$, let $C_\alpha = D^k_\alpha$.

**Successor case:** Let $\alpha \in (\aleph_{n+1} \setminus (\aleph_n + 1)) \cap \mathrm{cof}(\omega_k)$ for $n > k$ and let $\alpha^* = \mathrm{ot}(D^n_\alpha)$. Since $\mathrm{cf}(\alpha) = \omega_k$ and $k \neq n$, we see that $\alpha^* < \aleph_n$. Since $D^n_\alpha$ is club in $\alpha$, $\mathrm{cf}(\alpha^*) = \mathrm{cf}(\alpha) = \omega_k$. By induction $C_{\alpha^*}$ has already been defined, and we define $C_\alpha$ by copying $C_{\alpha^*}$ into $D^n_\alpha$. To be more precise we set $C_\alpha = \{\gamma \in D^n_\alpha : \mathrm{ot}(D^n_\alpha \cap \gamma) \in C_{\alpha^*}\}$.

We now check that this definition succeeds. Let $\eta \in \lim(C_\gamma) \cap \lim(C_\delta)$. By our assumption on the sets $D^n_\alpha$, either $\gamma$ and $\delta$ are both less than $\aleph_{k+1}$ or they both lie in $\aleph_{n+1} \setminus (\aleph_n + 1)$ for some $n > k$.

**Case 1:** $\gamma, \delta < \aleph_{k+1}$. In this case $C_\gamma = D^k_\gamma$ and $C_\delta = D^k_\delta$, and it follows that $C_\gamma \cap \eta = D^k_\eta = C_\delta \cap \eta$ by the defining property of a $\square_{\aleph_k}$-sequence.

**Case 2:** $\aleph_n < \gamma, \delta < \aleph_{n+1}$ for $n > k$. In this case $\eta \in \lim(D^n_\gamma) \cap \lim(D^n_\delta)$ and so $D^n_\gamma \cap \eta = D^n_\delta \cap \eta$. Moreover if $\zeta = \mathrm{ot}(D^n_\gamma \cap \eta)$ then by the definition of $C_\gamma$ we have that $\zeta \in \lim(C_{\gamma^*})$, and similarly $\zeta \in \lim(C_{\delta^*})$. By induction $C_{\gamma^*} \cap \zeta = C_{\delta^*} \cap \zeta$, and so by definition $C_\gamma \cap \eta = C_\delta \cap \eta$. $\square$

The main theorem of this section shows that starting with a $\square(\aleph_\omega, \mathrm{cof}(\omega_k))$-sequence, we may lift it via PCF theory to a square-like sequence defined on the set of good points of cofinality $\omega_k$ in some scale of length $\aleph_{\omega+1}$. To be more precise we define the following square-like principle, which is obtained by allowing the club sets to be defined only at points of some set $S$ and weakening the coherence requirement so that it only applies at common limit points of uncountable cofinality.

**Definition 3.4.** Let $S \subseteq \aleph_{\omega+1} \cap \mathrm{cof}(\omega_k)$ for some $k > 1$. A $\square^{-\omega}_{\aleph_\omega}(S)$*-sequence* is a sequence $\langle E_\alpha : \alpha \in S\rangle$ such that

1. For every $\alpha \in S$, $E_\alpha$ is club in $\alpha$ and $\mathrm{ot}(E_\alpha) = \omega_k$.
2. For all $\gamma, \delta \in S$ and every $\eta$ of uncountable cofinality which is a common limit point of $E_\gamma$ and $E_\delta$, $E_\gamma \cap \eta = E_\delta \cap \eta$.

**Theorem 3.5.** *Let $k$ be an integer with $k \geq 2$, and assume that there is a $\square(\aleph_\omega, \mathrm{cof}(\omega_k))$-sequence. If $G$ is the set of good points of cofinality $\omega_k$ for the scale $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$, then there exists a $\square^{-\omega}_{\aleph_\omega}(G)$-sequence.*

*Proof.* Dropping finitely many points from the set $A$ if necessary, we may assume that all points in $A$ are greater than $k$. Fix a $\square(\aleph_\omega, \mathrm{cof}(\omega_k))$-sequence $\langle C_\alpha : \alpha \in \aleph_\omega \cap \mathrm{cof}(\omega_k)\rangle$. By our assumptions on the scale and the set $A$, for all $\gamma \in G$ the function $f_\gamma$ is an exact upper bound for $\langle f_\alpha : \alpha < \gamma\rangle$ and $\mathrm{cf}(f_\gamma(n)) = \aleph_k$ for all $n \in A$.

Given $\gamma \in G$, we define functions $g^\gamma_i$ for $i < \omega_k$ by setting $g^\gamma_i(n)$ equal to the $i^{\mathrm{th}}$ member of $C_{f_\gamma(n)}$. By construction the sequence of functions $\langle g^\gamma_i : i < \omega_k\rangle$ is pointwise increasing and is cofinally interleaved with $\langle f_\alpha : \alpha < \gamma\rangle$.

For almost every $\eta < \gamma$ of uncountable cofinality, there is $j < \omega_k$ such that $\langle g^\gamma_i : i < j\rangle$ is cofinally interleaved with $\langle f_\alpha : \alpha < \eta\rangle$; the sequence $\langle g^\gamma_i : i < j\rangle$ witnesses that $\eta$ is good. Fix such $\eta$ and $j$. By the argument of Theorem 2.11 and the uniqueness of exact upper bounds $f_\eta =^* \sup_{i<j} g^\gamma_i$, and since $C_{f_\gamma(n)}$ is closed we also see that $\sup_{i<j} g^\gamma_i(n) = g^\gamma_j(n)$. We conclude that $f_\eta =^* g^\gamma_j$.

For every $\gamma \in G$, we now define $D_\gamma = \{\, \eta < \gamma \,:\, \exists j \; f_\eta =^* g^\gamma_j \,\}$. The set $D_\gamma$ need not be club in $\gamma$, and so we let $E_\gamma$ be the closure of $D_\gamma$ in $\gamma$. We will show that $\langle E_\gamma : \gamma \in G\rangle$ is a $\square^{-\omega}_{\aleph_\omega}(G)$-sequence.

If $\eta$ is an accumulation point of $D_\gamma$ with uncountable cofinality, then there is a unique $k$ such that the functions $\{g^\gamma_j : j < k\}$ are cofinally interleaved with $\{f_\alpha : \alpha < \eta\}$. It follows that $\eta$ is good, and so by continuity we have $f_\eta =^* g^\gamma_k$ and thus $\eta \in D_\gamma$.

It is easy to see that $\mathrm{ot}(D_\gamma) = \omega_k$ and we just showed that $D_\gamma$ is closed under suprema of uncountable cofinality. Now let $\gamma, \delta$ be members of $G$. We claim that if $\eta$ is a common accumulation point of $D_\gamma$ and $D_\delta$ with uncountable cofinality, then $D_\gamma \cap \eta = D_\delta \cap \eta$.

Since $\eta$ is in $D_\gamma$, there is $j$ with $\mathrm{cf}(\eta) = \mathrm{cf}(j)$ such that $f_\eta(n) = g^\gamma_j(n)$ for all large $n$, so that $f_\eta(n) \in \lim(C_{f_\gamma(n)})$ for all large $n$. Similarly there is $k$ such that $f_\eta(n) = g^\delta_k(n)$ and $f_\eta(n) \in \lim(C_{f_\delta(n)})$ for all large $n$.

It follows that $C_{f_\gamma(n)} \cap f_\eta(n) = C_{f_\delta(n)} \cap f_\eta(n)$ for all large $n$, and hence that $j = k$ and that $g^\gamma_i = g^\delta_i$ for $i < j$. If $\zeta \in D_\gamma \cap \eta$ then $f_\zeta =^* g^\gamma_i$ for some $i < j$, and so $f_\zeta =^* g^\delta_i$ and $\zeta \in D_\delta \cap \eta$; similarly $D_\delta \cap \eta \subseteq D_\gamma \cap \eta$, so $D_\gamma \cap \eta = D_\delta \cap \eta$.

It is now routine to verify that $\langle E_\gamma : \gamma \in S\rangle$ is a $\square^{-\omega}_{\aleph_\omega}(G)$-sequence. $\square$

Theorem 3.5 has the following striking corollary.

**Corollary 3.6.** *Let CH hold and let $\square_{\aleph_n}$ hold for all $n < \omega$. Then for every integer $m \geq 2$ there exists a $\square^{-\omega}_{\aleph_\omega}(\operatorname{cof}\omega_m)$-sequence.*

**Remark 3.7.** With a little work we can put Corollary 3.6 in a more pleasing way. Extending our notation, let $\square^{-\omega}_{\aleph_n}(\operatorname{cof}(\aleph_k))$ be the statement that there exists $\langle C_\alpha : \alpha \in \aleph_{n+1} \cap \operatorname{cof}(\aleph_k)\rangle$ with $C_\alpha$ club in $\alpha$, and the $C_\alpha$ cohering at common limit points of uncountable cofinality. Let CH hold, let $k \geq 2$; then if $\square^{-\omega}_{\aleph_n}(\operatorname{cof}(\aleph_k))$ holds for all sufficiently large finite $n$, it holds for $n = \omega$. The proof is an easy variation on the one given above.

Theorem 3.5 also supplies a partial answer to the problem which motivates this paper, the relationship between goodness and approachability. It is easy to see that if there is a $\square^{-\omega}_{\aleph_\omega}(S)$-sequence then $S \in I[\aleph_{\omega+1}]$, and so Theorem 3.5 has the following corollary.

**Corollary 3.8.** *If $\square_{\aleph_n}$ holds for all finite $n$ then in $\aleph_{\omega+1}$ all good points of cofinality greater than $\aleph_1$ are approachable.*

We can also deduce some results about stationary reflection.

**Corollary 3.9.** *Let CH hold and let $\square_{\aleph_n}$ hold for all $n < \omega$. For all integers $m$ and $n$ such that $0 < m < n$ there is a stationary subset of $\aleph_{\omega+1} \cap \operatorname{cof}(\aleph_m)$ which does not reflect at any point of $\aleph_{\omega+1} \cap \operatorname{cof}(\aleph_n)$.*

*Proof.* Let $\langle C_\alpha : \alpha \in \aleph_{\omega+1} \cap \operatorname{cof}(\aleph_n)\rangle$ be a $\square^{-\omega}_{\aleph_\omega}(\operatorname{cof}\omega_n)$-sequence. Let $S$ be the set of points of cofinality $\aleph_m$ which occur as limit points of some $C_\alpha$, and given $\beta \in S$ let $D_\beta$ be the unique set such that $D_\beta = C_\alpha \cap \beta$ when $\beta \in \lim(C_\alpha)$.

For any club subset $D$ of $\aleph_{\omega+1}$, we can find $\alpha \in \lim(D)$ with $\operatorname{cf}(\alpha) = \aleph_n$. Then $D \cap C_\alpha$ is club in $\alpha$ and so we can find $\beta \in \lim(C_\alpha \cap D)$ with $\operatorname{cf}(\beta) = \aleph_m$. Clearly $\beta \in D \cap S$ so that $S$ is stationary.

Applying Fodor's lemma we may find $T \subseteq S$ stationary and $\gamma$ such that $\operatorname{ot}(D_\beta) = \gamma$ for all $\beta \in T$. For each $\alpha$ of cofinality $\aleph_n$ the club set $\lim(C_\alpha)$ meets $T$ exactly once, so that the stationarity of $T$ does not reflect at any point of cofinality $\aleph_n$. □

In our paper [9] it is shown that it is consistent that $\square_{\aleph_n}$ holds for $0 < n < \omega$ but $\square_{\aleph_\omega}$ fails. This is achieved by arranging that every stationary subset of $\aleph_{\omega+1} \cap \operatorname{cof}(\omega)$ reflects at some point of $\aleph_{\omega+1} \cap \operatorname{cof}(\aleph_1)$. In the sequel [7] to this paper we show that it is consistent that the least $\lambda$ for which $\square_\lambda$ fails should be the first inaccessible cardinal.

We can also combine the $\square^{-\omega}_{\aleph_\omega}(\operatorname{cof}\omega_m)$-sequences of Corollary 3.6 for different values of $m$ and get a weakening of the principle $\square_{\aleph_\omega,\omega}$ [8].

**Corollary 3.10.** *Let CH hold and let $\square_{\aleph_n}$ hold for all $n < \omega$. Then there exists $\langle \mathcal{C}_\gamma : \gamma < \aleph_{\omega+1}\rangle$ such that for every limit ordinal $\gamma$*

1. *$\mathcal{C}_\gamma$ is a countable family of subsets of $\gamma$, each with order type less than $\aleph_\omega$.*
2. *For every $C \in \mathcal{C}_\gamma$ and every $\beta \in \lim(C)$ with uncountable cofinality, $C \cap \beta \in \mathcal{C}_\beta$.*

Similar arguments can be used with some other square like principles. The following "Strong Non-Reflection" principle was introduced by Dzamonja and Shelah [11], and has been used by Cummings, Dzamonja and Shelah [6, 10] in the investigation of stationary reflection.

**Definition 3.11.** Let $\lambda$ and $\kappa$ be regular and uncountable with $\lambda < \kappa$. Then $SNR(\kappa, \lambda)$ *holds* if and only if there is $f : \kappa \longrightarrow \lambda$ such that for all $\alpha \in \kappa \cap \mathrm{cof}(\lambda)$ there is $C$ club in $\alpha$ such that $f \restriction C$ is strictly increasing.

This principle is true in $L$ for all $\kappa$ which are not weakly compact and all $\lambda < \kappa$. It implies that every stationary subset of $\kappa$ has a stationary subset which reflects at no point of cofinality $\lambda$.

Dzamonja and Shelah [12] have studied the cardinal $u(\lambda)$ which is defined to be the least $\kappa > \lambda$ such that $SNR(\kappa, \lambda)$ fails. In particular they have shown by an elaborate forcing argument that $u(\lambda)$ can be the successor of a singular cardinal. The following theorem, which is proved in the same way as Theorem 3.5, seems to have some bearing on problems of this type.

**Theorem 3.12** (Cummings [4]). *Suppose that $k < \omega$ and that $SNR(\aleph_n, \aleph_k)$ holds for all large $n < \omega$. Then there is $f : \aleph_{\omega+1} \longrightarrow \aleph_k$ such that for all good $\alpha \in \aleph_{\omega+1} \cap \mathrm{cof}(\aleph_k)$ there is $C$ club in $\alpha$ such that $f \restriction C \cap \{\gamma : \omega < \mathrm{cf}(\gamma) < \aleph_k\}$ is strictly increasing.*

One rather unsatisfying feature of some results in this section is the appearance of the Continuum Hypothesis among the hypotheses. CH is only being used to derive "all points of $\aleph_{\omega+1} \cap \mathrm{cof}(> \aleph_1)$ are good", and it is quite possible that this statement is actually true in ZFC. For some discussion of the difficulties associated with attempting to show that this statement is consistently false see [5].

## 4. Uniformity

In this section we develop the idea of *uniformity* for a structure. The main point is that sufficiently uniform structures can be reconstructed from a small amount of data.

We recall from Lemma 2.3 that if $M$ is an IA structure of size $\lambda$ with a continuous approaching sequence of length $\kappa$ then $\mathrm{cf}(M \cap \gamma) = \kappa$ for all ordinals $\gamma \in M$ such that $\mathrm{cf}(\gamma) > \lambda$. This is the prototype for the uniformity properties we will consider.

**Definition 4.1.** Let $M \prec (H_\theta, \in, <_\theta)$, let $K$ be a set of regular cardinals and let $\kappa$ be a cardinal. Then $M$ is *$\kappa$-uniform on $K$* if and only if $\mathrm{cf}(M \cap \eta) = \kappa$ for all $\eta \in K \cap M$, and is *weakly $\kappa$-uniform on $K$* if and only if $\mathrm{cf}(M \cap \eta) \geq \kappa$ for all $\eta \in K \cap M$.

This property arises naturally in the study of mutual stationarity. A structure which meets a sequence of stationary sets all consisting of ordinals of cofinality $\kappa$ will automatically be $\kappa$-uniform on the relevant set of regular cardinals.

A particularly interesting case for our purposes will occur when $K$ is an interval of cardinals and $M$ contains a large enough initial segment of the ordinals.

**Definition 4.2.** Let $M \prec (H_\theta, \in, <_\theta)$ and let $\kappa$, $\lambda$ and $\mu$ be cardinals with $\kappa \leq \lambda < \mu \leq \theta$. The structure $M$ is *$\kappa$-uniform (resp. weakly $\kappa$-uniform) between $\lambda$ and $\mu$* if and only if

1. $\lambda \subseteq M$.
2. The structure $M$ is $\kappa$-uniform (resp. weakly $\kappa$-uniform) on the interval of regular cardinals $\{\eta \in REG : \lambda < \eta < \mu\}$.

If $M$ is $\kappa$-uniform between $\lambda$ and $\theta$ we say that $M$ *is $\kappa$-uniform past $\lambda$,* and similarly for weak uniformity.

This kind of uniformity arises naturally in the study of IA structures. If $M$ is an IA structure with an approaching chain of length $\kappa$ and $\lambda = |M| \subseteq M$, then $M$ is $\kappa$-uniform past $\lambda$.

Let $M \prec (H_\theta, \in, <_\theta)$. If $\gamma$ is any limit ordinal in $M$, then $\mathrm{cf}(\gamma)$ is in $M$ and there exists in $M$ an increasing cofinal map $f$ from $\mathrm{cf}(\gamma)$ to $\gamma$. Restricting $f$ to $\mathrm{cf}(\gamma) \cap M$ we get a cofinal map from $\mathrm{cf}(\gamma) \cap M$ to $\gamma \cap M$, so that $\mathrm{cf}(\gamma) \cap M$ and $\gamma \cap M$ have the same cofinality. In particular if $M$ is $\kappa$-uniform on $K$ then $\mathrm{cf}(\gamma \cap M) = \kappa$ for all ordinals $\gamma \in M$ such that $\mathrm{cf}(\gamma) \in K$, and similarly for weak uniformity. If $M$ is $\kappa$-weakly uniform between $\lambda$ and $\mu$ and $\rho \in M$ is an ordinal of cofinality less than or equal to $\lambda$ it follows that $M \cap \rho$ is unbounded in $\rho$. If $cf(\rho)$ is strictly between $\lambda$ and $\mu$, then $cf(M \cap \rho) \geq \kappa$.

Before discussing the implications of weak uniformity, we recall a well-known variation on the closed unbounded filter. Given a regular cardinal $\kappa$ and a limit ordinal $\rho$ with $\kappa \leq \mathrm{cf}(\rho)$, we say that a set $A \subseteq \rho$ is a *$< \kappa$-club subset of $\rho$* if $A$ is unbounded in $\rho$, and $\sup(x) \in A$ for every $x \subseteq A$ with $|x| < \kappa$. If $\kappa$ is uncountable then the collection of $< \kappa$-club sets generates a $\mathrm{cf}(\rho)$-complete filter, which is in fact the restriction of the club filter to the set of points of cofinality less than $\kappa$.

**Lemma 4.3.** *Let $M$ be $\kappa$-weakly uniform between $\lambda$ and $\mu$. Then*

1. *For every bounded subset $x$ of $M \cap \mu$ with $|x| < \kappa$,* $\sup(x) \in M$.
2. *For every regular $\eta$ with $\lambda < \eta \leq \sup(M \cap \mu)$*
   (a) $\mathrm{cf}(M \cap \eta) \geq \kappa$.
   (b) *There exists $E \subseteq M \cap \eta$ which is $< \kappa$-club in* $\sup(M \cap \eta)$.

*(Note that in item 2, we do not require that $\eta \in M$.)*

*Proof.* For claim 1, we may as well assume that $x$ has limit order type. Let $\beta = \sup(x)$ and $\gamma = \min(M \setminus \beta)$, where clearly both $\beta$ and $\gamma$ are limit ordinals less than $\mu$ and $M \cap \gamma = M \cap \beta$. Suppose for a contradiction that $\gamma > \beta$. Then $\mathrm{cf}(\gamma) > \lambda$, because otherwise $M \cap \gamma$ would be unbounded in $\gamma$. Since $\gamma \in M$ and $\lambda < \mathrm{cf}(\gamma) < \mu$ we have $\mathrm{cf}(M \cap \gamma) \geq \kappa$, but this is impossible because $\mathrm{cf}(M \cap \beta) = \mathrm{cf}(\beta) \leq |x| < \kappa$ and $M \cap \beta = M \cap \gamma$.

For claim 2, we fix a regular $\eta$ with $\lambda < \eta \leq \sup(M \cap \mu)$. Suppose for a contradiction that $\mathrm{cf}(M \cap \eta) < \kappa$ and fix $x \subseteq M \cap \eta$ cofinal in $M \cap \eta$ with $|x| < \kappa$. Since $\kappa \leq \lambda < \eta$, $M \cap \eta$ is bounded in $\eta$. Since $\eta \leq \sup(M \cap \mu)$, $x$ is a bounded subset of $M \cap \mu$ and so by the first claim $\sup(x) = \sup(M \cap \eta) \in M$. This is a contradiction since $M \cap \eta$ is bounded in $\eta$. If we now fix any cofinal set $D \subseteq M \cap \eta$ and let $E$ be the closure of $D$ under suprema of size less than $\kappa$, then $E \subseteq M \cap \eta$ and $E$ is a $< \kappa$-club set as required. □

It is notable that claim 2) of Lemma 4.3 applies to regular cardinals which do not lie in $M$. Before we can exploit Lemma 4.3 we need some more or less standard facts about rebuilding structures.

The idea of Lemmas 4.4 and 4.5 first appears in the proof by Solovay that the Singular Cardinals Hypothesis holds above a strongly compact cardinal. Lemma 4.5 will find an immediate application in Theorem 4.6 where we show that sufficiently uniform structures are determined by their characteristic functions: the more general Lemma 4.4 will be useful later in the covering results of Section 7.

**Lemma 4.4.** *Let $\lambda$ be a cardinal, let $K$ be an interval of regular cardinals with* $\min(K) = \lambda^+$ *and let $\theta$ be a regular cardinal with* $\sup(K) < \theta$. *Let $M_0$ and $M_1$ be two elementary substructures of $(H_\theta, \in, <_\theta)$ such that*

- $\lambda \subseteq M_0 \cap M_1$.
- *For every* $\eta \in K$, $M_0 \cap M_1 \cap \eta$ *is cofinal in* $M_0 \cap \eta$.

*Then* $M_0 \cap \sup(K) \subseteq M_1 \cap \sup(K)$.

*Proof.* Clearly $M_0 \cap \lambda = M_1 \cap \lambda = \lambda$. We show by induction on $\eta \in K$ that $M_0 \cap \eta \subseteq M_1 \cap \eta$. There is nothing to do when $\eta$ is a limit cardinal, so let $\eta = \zeta^+$ where $M_0 \cap \zeta \subseteq M_1 \cap \zeta$ by induction. If $\alpha \in M_0 \cap \eta$ there is $\beta \in M_0 \cap M_1 \cap \eta$ with $\alpha < \beta$, and we fix $f$ the $<_\theta$-least bijection from $\beta$ to $|\beta|$. Since $\beta \in M_0 \cap M_1$ and $f$ is defined from $\beta$ we have $f \in M_0 \cap M_1$, and since $\beta < \zeta^+$ we have $|\beta| \leq \zeta$. So $f(\alpha) \in M_0 \cap \zeta$, and since $M_0 \cap \zeta \subseteq M_1 \cap \zeta$ we conclude that $f(\alpha) \in M_1 \cap \zeta$ and so $\alpha \in M_1$. Thus $M_0 \cap \eta \subseteq M_1 \cap \eta$ and the induction goes through. □

We note that we are not assuming that $K \subseteq M$ here.

**Lemma 4.5.** *If we strengthen the hypotheses of Lemma 4.4 by adding the demand that* $M_0 \cap M_1 \cap \eta$ *is also cofinal in* $M_1 \cap \eta$ *for all* $\eta \in K$, *then we may strengthen the conclusion to* $M_0 \cap \sup(K) = M_1 \cap \sup(K)$.

*Proof.* Immediate from Lemma 4.4. □

**Theorem 4.6.** *Let* $\kappa$ *be an uncountable cardinal. Let* $K$ *be an interval of regular cardinals with* $\min(K) = \lambda^+$ *and let* $\theta$ *be a regular cardinal with* $\sup(K) < \theta$. *Let* $M \prec (H_\theta, \in, <_\theta)$ *be* $\kappa$*-weakly uniform between* $\lambda$ *and* $\sup\{\eta^+ : \eta \in K\}$. *Then* $M \cap \sup(K)$ *is determined by* $\chi^K_M$ *and* $\sup(M \cap K)$.

*Proof.* Suppose that $M$ and $N$ are both substructures of $(H_\theta, \in, <_\theta)$ which are $\kappa$-weakly uniform between $\lambda$ and $\sup\{\eta^+ : \eta \in K\}$, with $\chi^K_M = \chi^K_N$ and $\sup(N \cap K) = \sup(M \cap K)$. Then $N \cap K = M \cap K$. By Lemma 4.3, for every $\eta \in K \cap M$ each of the sets $M \cap \eta$ and $N \cap \eta$ contains a set which is $< \kappa$-club in $\sup(M \cap \eta) = \sup(N \cap \eta)$. The intersection of two $< \kappa$-club sets is $< \kappa$-club, so $M \cap N \cap \eta$ is unbounded in $M \cap \eta$ and $N \cap \eta$. By Lemma 4.5 $M \cap \sup(K) = N \cap \sup(K)$. □

We will only be using Theorem 4.6 in the special case when $K \subseteq M$. It will be useful later to know that the process of reconstructing $M \cap \sup(K)$ from $\chi^K_M$ is simply definable.

**Lemma 4.7.** *Let* $K$, $\kappa$, $\lambda$, $\theta$ *be as in the last lemma and suppose that* $M, N \prec (H(\theta), \in, <_\theta)$ *with* $\chi^K_M \in N$. *Then* $M \cap \sup(K) \in N$.

*Proof.* Since $K = \mathrm{dom}(\chi^K_M)$ we see that $K \in N$, and thus $\sup(K) \in N$ and $\lambda \in N$. We may also find $\kappa^* \in N$ such that $M$ is $\kappa^*$-weakly uniform.

Let $\langle g_\beta : \beta < \sup(K) \rangle$ be such that $g_\beta$ is the $<_\theta$-least bijection from $\beta$ to $|\beta|$, and define partial functions $g$ and $h$ from $\sup(K) \times \sup(K)$ to $\sup(K)$ by

$$g(\beta, \alpha) = g_\beta(\alpha), \;\; h(\beta, \gamma) = g_\beta^{-1}(\gamma).$$

Since $g$ and $h$ are defined from parameters in $N$, they are members of $N$. The argument in the proof of Lemma 4.5 shows that $M \cap \sup(K)$ can be computed as follows: for each $\vec{C} = \langle C_\eta : \eta \in K \rangle$ such that $C_\eta$ is $< \kappa^*$-club in $\chi_M(\eta)$ for each $\eta$, compute the closure $X(\vec{C})$ of $\lambda \cup (\bigcup_\eta C_\eta)$ under $g$ and $h$, and then take the intersection of all the sets $X(\vec{C})$. It follows that $M \cap \sup(K) \in N$. □

## 5. Tight structures and PCF

Recall from definition 2.4 that if $K$ is a set of regular cardinals and $M$ is a structure with $K \in M$ and $K \subseteq M$, then $M$ is *tight for* $K$ if and only if $M \cap \prod K$ is cofinal in $\prod_{\kappa \in K} M \cap \kappa$. For the rest of this section we will make the following

**Blanket assumption:** We are given a set $K$ of regular cardinals and a structure $M \prec (H_\theta, \in, <_\theta)$ such that $K \in M$, $K \subseteq M$ and $|M| < \min(K)$.

We note that since $|M| < \min(K)$, we have $\chi^K_M \in \prod K$. Also since $K \subseteq M$ we have $|K| < \min(K)$ so that $K$ is a "progressive" set of regular cardinals, and the methods of PCF theory can be applied to $K$.

The main idea will be to analyse tightness for $M$ in terms of the PCF-theoretic properties of $K$. The key new points in the analysis (which are closely related) will be that under some reasonable circumstances we can discern whether a structure $M$ is tight by inspecting $M \cap ON$, and can reconstruct a tight structure $M$ from a finite set of ordinal parameters.

We summarize the main interest in tight structures by the following points:

(1) Tight structures are canonically determined by a finite number of canonical ordinal parameters, i.e. good ordinals. (Theorem 5.3) In particular, the stationary set of tight structures is canonically well-ordered. This is clearly not possible for arbitrary structures as they outnumber the possible collections of ordinal parameters.
(2) A tight elementary substructure of an $H(\theta)$ is determined by the its "cardinal structure". This is implicit in the statement and explicit in the proof of Theorem 7.3.
(3) Internally approachable structures share the two previous points (which is why they were used in the original PCF theory.) However, given a structure $N \prec (H(\theta), \in, <_\theta)$, to determine whether $N$ is tight (in an absolute way from knowledge of the regular cardinals) one must only know $\chi_N \restriction \aleph_{\omega+1}$ (i.e. $N$'s trace on the ordinals), but to determine whether $N \cap H(\aleph_{\omega+1})$ is internally approachable seems to require considerably more information and conceivably might not be an absolute question.

We summarize the last point by the slogan that being tight is an "exterior" question, but being internally approachable is an "internal" question.

The results of this section generalise a result by Foreman and Magidor [17, Section 7.3] which analyses uniform structures which are tight for $K = \{\aleph_n : n < \omega\}$ under the assumption that $\max \mathrm{pcf}(K) = \aleph_{\omega+1}$. They are also related to work by Shelah analysing the characteristic functions of certain IA structures in terms of PCF.

For our purposes the most important results are Theorems 5.2 and 5.3, which give a detailed analysis of tightness for $M$ under the assumptions that $\mathrm{pcf}(K) \subseteq M$ and $M$ is uniform on $K$. Theorem 5.5 shows that we can drop the assumption that $\mathrm{pcf}(K) \subseteq M$ and demand only weak uniformity for $M$ on $K$; the price we pay is that we need extra technical assumptions and we get a somewhat weaker conclusion. Theorem 5.6 shows that we can sometimes propagate tightness from $K$ to $\mathrm{pcf}(K)$.

As a warmup for the style of argument which we will be using in this section, we show that if $M$ is tight for $K$ then some uniformity properties can be propagated from $K$ to $\mathrm{pcf}(K)$.

**Theorem 5.1.** *Let $M$ be tight for $K$ and let $\eta$ be a cardinal with $\eta > |K|$. If $M$ is $\eta$-weakly uniform on $K$, then $M$ is $\eta$-weakly uniform on $\mathrm{pcf}(K)$. If $M$ is $\eta$-uniform on $K$, then $M$ is $\eta$-uniform on $\mathrm{pcf}(K)$.*

*Proof.* Suppose that $M$ is $\eta$-weakly uniform on $K$. Given $\lambda \in M \cap \mathrm{pcf}(K)$, let us fix $U \in M$ an ultrafilter on $K$ and $\langle f_\alpha : \alpha < \lambda \rangle \in M$ increasing and cofinal in $\prod K/U$. Suppose for a contradiction that $\mathrm{cf}(M \cap \lambda) < \eta$. We fix $A \subseteq M \cap \lambda$ cofinal with order type $\mathrm{cf}(M \cap \lambda)$, and note that if $\alpha \in A$ then $f_\alpha(\kappa) \in M \cap \kappa$ for all $\kappa \in K$. If we define $g(\kappa) = \sup_{\alpha \in A} f_\alpha(\kappa)$ for all $\kappa \in K$, then $g(\kappa) < \sup(M \cap \kappa)$ because $\mathrm{cf}(M \cap \kappa) \geq \eta > |A|$. Since $M$ is tight there is $h \in M \cap \prod K$ with $g < h$, and by elementarity there is $\alpha \in M \cap \lambda$ with $h <_U f_\alpha$. So $g <_U f_\alpha$, but this is impossible because by construction $f_\alpha \leq g$.

Now suppose that $M$ is $\eta$-uniform on $K$, and use this to find a pointwise increasing sequence $\langle g_i : i < \eta \rangle$ of functions whose ranges are contained in $M$ and whose pointwise supremum is $\chi_M$. As above let $\lambda \in M \cap \mathrm{pcf}(K)$ and fix $U \in M$ an ultrafilter on $K$ and $\langle f_\alpha : \alpha < \lambda \rangle \in M$ increasing and cofinal in $\prod K/U$. It is easy to see by tightness and elementarity that $\langle f_\alpha : \alpha \in M \cap \lambda \rangle$ is cofinally interleaved with $\langle g_i : i < \eta \rangle$, so $\mathrm{cf}(M \cap \lambda) = \eta$. □

We recall that we have a set $K$ of regular cardinals such that $|K| < \min(K)$. This is a situation in which Shelah's PCF theory gives an analysis of $\mathrm{pcf}(K)$ in terms of the *PCF generators,* and we give a brief review of this analysis.

As usual we let $J_{<\lambda}$ be the ideal of sets $A \subseteq K$ such that every ultrafilter $U$ containing $A$ gives an ultrapower $\prod K/U$ of cofinality less than $\lambda$. The sequence of ideals $J_{<\lambda}$ is increasing with $\lambda$ and is continuous at limit cardinals.

By standard results from PCF theory we fix a sequence of sets $\langle B_\lambda : \lambda \in \mathrm{pcf}(K) \rangle$ with $B_\lambda \subseteq K$ for each $\lambda$, such that $B_\lambda$ *generates* $J_{<\lambda^+}$ *over* $J_{<\lambda}$*:* that is to say that for $A \subseteq K$ we have $A \in J_{<\lambda^+} \iff A \setminus B_\lambda \in J_{<\lambda}$. An easy induction argument shows that $J_{<\lambda}$ is the ideal of sets which are covered by a finite union of sets in $\{B_\mu : \mu < \lambda\}$.

It is known that $\mathrm{pcf}(K)$ has a maximum element, and accordingly we will choose $B_{\max \mathrm{pcf}(K)} = K$. By standard facts from PCF theory we may fix a matrix of functions $\langle f^\lambda_\alpha : \alpha < \lambda,\, \lambda \in \mathrm{pcf}(K) \rangle$ such that $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ is a continuous scale in $\prod B_\lambda / J_{<\lambda}$ for every $\lambda \in \mathrm{pcf}(K)$.

We fix $\theta$ some sufficiently large regular cardinal and let $\mathcal{A}$ be the structure $(H_\theta, \in, <_\theta, \{K\}, \langle B_\lambda \rangle, \langle f^\lambda_\alpha \rangle)$. We will analyse tightness of $M$ for $K$ in terms of PCF theory, assuming that $M \prec \mathcal{A}$, $\mathrm{pcf}(K) \subseteq M$ and $M$ is uniform on $K$.

**Theorem 5.2.** *Let $\eta$ be regular and uncountable, and suppose that $|K| < \eta < \min(K)$. Suppose that $M \prec \mathcal{A}$ where $M$ is $\eta$-uniform on $K$ and $\mathrm{pcf}(K) \subseteq M$. Then the following are equivalent:*

1. *$M$ is tight for $K$.*
2. *For every $\lambda \in \mathrm{pcf}(K)$, $\sup(M \cap \lambda)$ is a good point of cofinality $\eta$ for $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ and $f^\lambda_{\sup(M \cap \lambda)} =_{J_{<\lambda}} \chi^{B_\lambda}_M$.*

*Proof.* For the forward direction, assume $M$ is tight. Since $\mathrm{cf}(M \cap \kappa) = \eta$ for all $\kappa \in K$, we may find a pointwise increasing sequence $\langle g_i : i < \eta \rangle$ in $\prod K$ whose pointwise supremum is $\chi^K_M$.

Fix $\lambda \in \mathrm{pcf}(K)$. We claim that $\langle g_i \restriction B_\lambda : i < \eta \rangle$ is cofinally interleaved with $\langle f^\lambda_\alpha : \alpha \in M \cap \lambda \rangle$ modulo $J_{<\lambda}$. For all $\alpha \in M \cap \lambda$, $f^\lambda_\alpha \in M$ and so $f^\lambda_\alpha$ is pointwise dominated by $\chi^{B_\lambda}_M$. Since $|K| < \eta$, there exists $i < \eta$ such that $f^\lambda_\alpha$ is dominated pointwise by $g_i \restriction B_\lambda$. Conversely if $i < \eta$ then it follows from the tightness of $M$ that $g_i$ is dominated pointwise by some function in $M$, and so by elementarity $g_i <_{J_{<\lambda}} f^\lambda_\alpha$ for some $\alpha \in M \cap \lambda$.

It follows that $\mathrm{cf}(M \cap \lambda) = \eta$ and $\sup(M \cap \lambda)$ is a good point of cofinality $\eta$. By Lemma 2.11 $f^\lambda_{\sup(M\cap\lambda)}$ agrees modulo $J_{<\lambda}$ with the pointwise supremum of the $g_i \restriction B_\lambda$, that is to say $f^\lambda_{\sup(M\cap\lambda)} =_{J_{<\lambda}} \chi^{B_\lambda}_M$.

For the converse direction, assume that for every $\lambda \in \mathrm{pcf}(K)$, $\sup(M \cap \lambda)$ is a good point of cofinality $\eta$ for $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ and $f^\lambda_{\sup(M\cap\lambda)} =_{J_{<\lambda}} \chi^{B_\lambda}_M$. Let $h$ be a function in $\prod K$ which is pointwise dominated by $\chi^K_M$. For every $\lambda$ in $\mathrm{pcf}(K)$ we have $h <_{J_{<\lambda}} f^\lambda_{\sup(M\cap\lambda)}$. Since $\sup(M \cap \lambda)$ is a good point it follows that $f^\lambda_{\sup(M\cap\lambda)}$ is an exact upper bound for $\langle f_\alpha : \alpha \in M \cap \lambda \rangle$. Hence for every $\lambda \in \mathrm{pcf}(K)$ there is $\alpha \in M \cap \lambda$ with $h <_{J_{<\lambda}} f^\lambda_\alpha$.

We will inductively build a decreasing sequence $\lambda_0 > \lambda_1 > \lambda_2 > \lambda_j$ of members of $\mathrm{pcf}(K)$ together with ordinals $\alpha_i \in M \cap \lambda_i$. We will also build a decreasing sequence of sets $D_0 \supseteq D_1 \supseteq D_2 \ldots$ such that $D_i \in J_{<\lambda_i}$, halting when we reach a stage with $D_i = \emptyset$.

We let $\lambda_0 = \max \mathrm{pcf}(K)$, and recall that $B_{\lambda_0} = K$. We choose $\alpha_0 \in M \cap \lambda_0$ such that $h <_{J_{<\lambda_0}} f^{\lambda_0}_{\alpha_0}$. Let $D_0 = \{\kappa : h(\kappa) \geq f^{\lambda_0}_{\alpha_0}(\kappa)\}$, so that $D_0 \in J_{<\lambda_0}$.

Suppose that we have defined $\lambda_i$, $\alpha_i$ and $D_i$ for $i \leq j$. We stop if $D_j = \emptyset$. Otherwise we choose $\lambda_{j+1}$ to be the unique member of $\mathrm{pcf}(K)$ with $D_j \in J_{<\lambda^+_{j+1}}$ and $D_j \notin J_{<\lambda_{j+1}}$, and choose $\alpha_{j+1} \in M \cap \lambda_{j+1}$ such that $h <_{J_{<\lambda_{j+1}}} f^{\lambda_{j+1}}_{\alpha_{j+1}}$. Now we let

$$D_{j+1} = \{\kappa \in D_j : \kappa \notin B_{\lambda_{j+1}} \text{ or } h(\kappa) \geq f^{\lambda_{j+1}}_{\alpha_{j+1}}(\kappa)\}.$$

Since $B_{\lambda_{j+1}}$ generates $J_{<\lambda^+_{j+1}}$ and $h <_{J_{<\lambda_{j+1}}} f^{\lambda_{j+1}}_{\alpha_{j+1}}$, we see that $D_{j+1} \in J_{<\lambda_{j+1}}$.

Since the descending sequence of $\lambda_i$ can only have finite length, the construction must terminate. Let $j$ be the last stage, so $D_j = \emptyset$. For each $\kappa \in K$, let $i$ be minimal with $\kappa \notin D_i$. By definition we must have $\kappa \in B_{\lambda_i}$ and $h(\kappa) < f^{\lambda_i}_{\alpha_i}(\kappa)$. We conclude that $h$ is pointwise dominated by the pointwise supremum of $\{f^{\lambda_i}_{\alpha_i} : i \leq j\}$, and since this function lies in $M$ we have proved that $M$ is tight. □

The following result will play an important role when we analyse the relationship between tightness and internal approachability in Section 6. It is a generalization of Shelah's analysis of internally approachable structures used to bound powers of singular cardinals. It is important for the purposes of that section to note that each of the sets $B_{\lambda_i} \setminus E_i$ is in $M$, and each of the indices $\sup(M \cap \lambda_i)$ is computed in a uniform way from $M$.

**Theorem 5.3.** *Let $\eta$ be regular and uncountable, and suppose that $|K| < \eta < \min(K)$. Suppose that $M$ is $\eta$-uniform on $K$ and $\mathrm{pcf}(K) \subseteq M$.*

*If $M$ is tight for $K$ then there exist $\lambda_0, \dots \lambda_j \in \mathrm{pcf}(K)$ and $E_0, \dots E_j$ with $E_i \in M \cap J_{<\lambda_i}$ such that $K = \bigcup_i (B_{\lambda_i} \setminus E_i)$ and $\chi_M$ is the pointwise supremum of the functions $f^{\lambda_i}_{\sup(N \cap \lambda_i)} \restriction (B_{\lambda_i} \setminus E_i)$ for $i \leq j$.*

*Proof.* By Theorem 5.2, $f^{\lambda}_{\sup(M \cap \lambda)} =_{J_{<\lambda}} \chi^{B_\lambda}_M$ for all $\lambda \in \mathrm{pcf}(K)$.

We argue in a way which parallels the last part of the proof of Theorem 5.2. As there we build a decreasing sequence $\lambda_0 > \lambda_1 > \dots$ of elements of $\mathrm{pcf}(K)$, and also sets $D_i$ with $D_i \in J_{<\lambda_i}$ which this time are not necessarily decreasing. We also build sets $E_i \in M \cap J_{<\lambda_i}$ such that $D_i \subseteq E_i$, and choose $D_{i+1}$ as a subset of $E_i$ rather than $D_i$.

Let $\lambda_0 = \max \mathrm{pcf}(K)$ and $D_0 = \{\kappa : \chi_M(\kappa) \neq f^{\lambda_0}_{\sup(M \cap \lambda_0)}(\kappa)\}$, so that $D_0 \in J_{<\lambda_0}$. Suppose now that we have constructed the ordinals $\lambda_0, \lambda_1 \dots \lambda_j$, together with the sets $D_0, \dots D_j$ and $E_0, \dots E_{j-1}$.

If $D_j = \emptyset$ we set $E_j = \emptyset$ and stop the construction. If $D_j \neq \emptyset$ we choose $\lambda_{j+1}$ as the unique member of $\mathrm{pcf}(K)$ such that $D_j \in J_{<\lambda^+_{j+1}}$ and $D_j \notin J_{<\lambda_{j+1}}$. We choose $E_j$ to be some finite union of sets in $\{B_\lambda : \lambda \leq \lambda_{j+1}\}$ such that $D_j \subseteq E_j$.

We note that $E_j \in J_{<\lambda^+_{j+1}} \cap M$, and that since $D_j \notin J_{<\lambda_{j+1}}$ we must have $B_{\lambda_{j+1}} \subseteq E_j$. Let

$$D_{j+1} = \{\kappa \in E_j : \kappa \notin B_{\lambda_{j+1}} \text{ or } \chi_M(\kappa) \neq f^{\lambda_{j+1}}_{\sup(M \cap \lambda_{j+1})}(\kappa)\}.$$

Since $E_j \setminus B_{\lambda_{j+1}}$ is covered by a finite union of sets from $\{B_\lambda : \lambda < \lambda_{j+1}\}$, and the functions $\chi_M$ and $f^{\lambda_{j+1}}_{\sup(M \cap \lambda_{j+1})}(\kappa)$ agree on a $J_{<\lambda_{j+1}}$-large subset of $B_{\lambda_{j+1}}$, we see that $D_{j+1} \in J_{<\lambda_{j+1}}$.

Since the sequence of $\lambda_i$ is decreasing we eventually reach a stage $j$ with $D_j = \emptyset$, and so we halt the construction after setting $E_j = \emptyset$. To finish we need to check that $\chi_M$ is the pointwise supremum of the functions $f^{\lambda_i}_{\sup(M \cap \lambda_i)} \restriction (B_{\lambda_i} \setminus E_i)$ for $i \leq j$.

Let $\kappa \in B_{\lambda_i} \setminus E_i$ for some $i \leq j$. Since $D_i \subseteq E_i$, $\kappa \in B_{\lambda_i} \setminus D_i$ . By the construction of $D_i$ we know that $\{\kappa \in B_{\lambda_i} : f^{\lambda_i}_{\sup(M \cap \lambda_i)}(\kappa) \neq \chi_M(\kappa)\}$ is a subset of $D_i$, so $f^{\lambda_i}_{\sup(M \cap \lambda_i)}(\kappa) = \chi_M(\kappa)$.

Given $\kappa \in K$, let $i$ be minimal such that $\kappa \notin E_i$. If $i = 0$ then $\kappa \in B_{\lambda_0}$ because we chose $B_{\lambda_0} = K$. If $i > 0$ then $\kappa \in E_{i-1}$, and since $\kappa \notin D_i$ we see that $\kappa \in B_{\lambda_i}$. It follows that every element of $K$ appears in $B_{\lambda_i} \setminus E_i$ for some $i$. □

**Remark 5.4.** For an application which we will make of Theorems 5.2 and 5.3 in Section 7, we note that we only needed the scales $\{f^\lambda_\alpha : \alpha < \lambda\}$ to be continuous at good points of cofinality $\eta$.

Now we give an alternative characterisation of tightness for $M$ in the style of Theorem 5.2, dropping the assumption that $\mathrm{pcf}(K) \subseteq M$ and weakening the uniformity assumption to weak uniformity. As we remarked earlier, this analysis owes a debt to Shelah's analysis of the characteristic functions of IA structures. For technical reasons we will not be using continuous scales, but rather scales with a technical property called $\omega$-club minimality.

Let us be given an index set $X$, an ideal $I$ on $X$ and a $<_I$-increasing sequence $\vec{f}$. If $\mathrm{cf}(\beta) > |X|$, we define a function $f^*_\beta$ by letting $f^*_\beta(\kappa)$ be the least ordinal of the form $\sup_{\alpha \in E} f_\alpha(\kappa)$ for $E$ an $\omega$-club subset of $\beta$. We note that if $E$ is some $\omega$-club subset of $\beta$ with $f^*_\beta(\kappa) = \sup_{\alpha \in E} f_\alpha(\kappa)$, then $f^*_\beta(\kappa) = \sup_{\alpha \in F} f_\alpha(\kappa)$ for any $F \subseteq E$ with $F$ an $\omega$-club subset of $\beta$.

Since the intersection of $|X|$-many $\omega$-club subsets of $\beta$ is $\omega$-club, we may fix a single $\omega$-club subset $F$ of $\beta$ such that $f^*_\beta(\kappa) = \sup_{\alpha \in F} f_\alpha(\kappa)$ for all $\kappa \in F$; it follows that $f^*_\beta$ is an upper bound modulo $I$ for $\langle f_\alpha : \alpha < \beta \rangle$. We say that the sequence $\vec{f}$ is *$\omega$-club minimal at $\beta$* if $f^*_\beta = f_\beta$. Returning to the context of this section, it is routine to check that if $I$ is an ideal on $K$ with $\mathrm{tcf}(\prod K/I) = \lambda$ then we may find a scale $\langle f_\alpha : \alpha < \lambda \rangle$ in $\prod K/I$ which is $\omega$-club minimal at every $\beta < \lambda$ such that $|K| < \mathrm{cf}(\beta) < \min(K)$; by a slight abuse of language we will say that such a scale is *$\omega$-club minimal in* $\prod K/I$.

We will fix a matrix of functions $\langle f^\lambda_\alpha : \alpha < \lambda,\ \lambda \in \mathrm{pcf}(K) \rangle$ such that $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ is a an $\omega$-club minimal scale in $\prod B_\lambda / J_{<\lambda}$ for every $\lambda \in \mathrm{pcf}(K)$. Let $\mathcal{A}$ be the structure $(H_\theta, \in, <_\theta, \{K\}, \langle B_\lambda \rangle, \langle f^\lambda_\alpha \rangle)$. We will assume that $M \prec \mathcal{A}$.

We are now ready to use our technical assumption of $\omega$-club minimality. Suppose that $M$ is $|K|^+$-weakly uniform between $\rho$ and $\sigma$, where all cardinals in $M \cap \mathrm{pcf}(K)$ are greater than $\rho$ and less than $\sigma$. Let $\lambda \in M \cap \mathrm{pcf}(K)$ and let $\gamma = \sup(M \cap \lambda)$. By Lemma 4.3 there is $E \subseteq M \cap \lambda$ which is $\omega$-club in $\gamma$. Since $|M| < \min(K)$ we see that $\mathrm{cf}(\gamma) < \min(K)$, and since $\lambda$ lies in an interval where $M$ is weakly uniform we also see that $\mathrm{cf}(\gamma) > |K|$. So by the assumption of $\omega$-club minimality, we may find $F \subseteq E$ such that $F$ is $\omega$-club in $\gamma$ and $f^\lambda_\gamma(\kappa) = \sup_{\alpha \in F} f^\lambda_\alpha(\kappa)$ for all $\kappa \in K$. Since $F \subseteq E \subseteq M$, $K \subseteq M$ and $\lambda \in M$ it follows easily that $f^\lambda_\gamma \leq \chi^{B_\lambda}_M$.

We can now give a characterisation of tightness.

**Theorem 5.5.** *Let $M$ be $|K|^+$-weakly uniform between $\rho$ and $\sigma$, where all cardinals in $M \cap \mathrm{pcf}(K)$ are greater than $\rho$ and less than $\sigma$. Then the following conditions on $M$ are equivalent.*

(1) *$M$ is tight for $K$.*
(2) *For every $\lambda \in M \cap \mathrm{pcf}(K)$, if $\gamma = \sup(M \cap \lambda)$ then*
   (a) $f^\lambda_\gamma \leq \chi^{B_\lambda}_M$.
   (b) *There is $A \subseteq B_\lambda$ such that $A \in M \cap J_{<\lambda}$ and $\{\kappa \in B_\lambda : f^\lambda_\gamma(\kappa) < \chi_M(\kappa)\} \subseteq A$.*
   (c) *$f^\lambda_\gamma$ is an exact upper bound for $\langle f^\lambda_\alpha : \alpha < \gamma \rangle$ modulo $J_{<\lambda}$ in the following strengthened sense: for all $f < f^\lambda_\gamma$ there exist $\beta \in M \cap \lambda$ and $B \in M \cap J_{<\lambda}$ such that $\{\kappa \in B_\lambda : f^\lambda_\beta(\kappa) \leq f(\kappa)\} \subseteq B$.*

*Proof.* 1) implies 2). We have already seen that the uniformity hypothesis on $M$ implies that $f^\lambda_\gamma \leq \chi^{B_\lambda}_M$. Since $M$ is tight we may find a function $g \in M \cap \prod K$ such that for all $\kappa \in K$, $f^\lambda_\gamma(\kappa) < \chi_M(\kappa)$ implies that $f^\lambda_\gamma(\kappa) < g(\kappa)$. By $\omega$-club minimality we know that $f^\lambda_\gamma$ is the pointwise supremum of $\{f^\lambda_\alpha : \alpha \in F\}$ for some $F \subseteq M$ which is $\omega$-club in $\gamma$, and we may find $\alpha \in F$ so large that $f^\lambda_\alpha$ dominates $g$ modulo $J_{<\lambda}$. Now since $f^\lambda_\alpha \leq f^\lambda_\gamma$ we see that for $\kappa \in K$

$$f^\lambda_\gamma(\kappa) < \chi_M(\kappa) \implies f^\lambda_\gamma(\kappa) < g(\kappa) \implies f^\lambda_\alpha(\kappa) < g(\kappa),$$

so that if we set $A = \{\kappa \in B_\lambda : f^\lambda_\alpha(\kappa) < g(\kappa)\}$ then $A$ is as required. Now let $f < f^\lambda_\gamma$. Since $f^\lambda_\gamma \leq \chi_M$ and $M$ is tight, we may find $h \in M$ such that $f < h$ and then find $\beta \in F$ such that $f^\lambda_\beta$ dominates $h$ modulo $J_{<\lambda}$. Now

$$f^\lambda_\beta(\kappa) \leq f(\kappa) \implies f^\lambda_\beta(\kappa) \leq h(\kappa),$$

and if we set $B = \{\kappa \in B_\lambda : f^\lambda_\beta(\kappa) \leq h(\kappa)\}$ then $B$ is as required.

2) implies 1). Let $f \in \prod K$ with $f < \chi^K_M$. The construction is very similar to that for Theorem 5.2. We construct a decreasing sequence of ordinals $\lambda_0 > \lambda_1 > \dots$ with $\lambda_i \in M \cap \mathrm{pcf}(K)$, together with ordinals $\alpha_i \in M \cap \lambda_i$ and sets $B_i \in M \cap J_{\lambda_i}$, in such a way that if $\kappa \in B_{i-1} \setminus B_i$ then $f(\kappa) < f^{\lambda_i}_{\alpha_i}(\kappa)$.

Let $\lambda_0 = \max \mathrm{pcf}(K)$ and $\gamma_0 = \sup(M \cap \lambda_0)$, and observe that by b) the functions $f^{\lambda_0}_{\gamma_0}$ and $\chi_M$ agree outside a set in $M \cap J_{<\lambda_0}$. By c) we can find a set $B_0 \in M \cap J_{<\lambda_0}$ such that $f(\kappa) < f^{\lambda_0}_{\gamma_0}(\kappa)$ for $\kappa \notin B_0$.

If at stage $i$ we have $B_i \neq \emptyset$, then choose $\lambda_{i+1}$ minimal with $B_i \notin J_{<\lambda_{i+1}}$, noting that $\lambda_{i+1} \in M$ because $B_i \in M$. Now we apply b) and c) to find $B_{i+1} \in J_{<\lambda_{i+1}}$ and $\alpha_{i+1} \in M \cap \lambda_{i+1}$ such that $f(\kappa) < f^{\lambda_{i+1}}_{\alpha_{i+1}}(\kappa)$ for $\kappa \in B_i \setminus B_{i+1}$.

As usual, the construction must terminate with $B_i = \emptyset$. Then $f$ is pointwise dominated by the supremum of the functions $f^{\lambda_j}_{\alpha_j}$ for $j < i$, and this function lies in $M$. □

Under certain circumstances we can do an analysis of tightness as in Theorem 5.5 under a weaker uniformity hypothesis. It is known that if $K$ is an interval of regular cardinals then $\mathrm{pcf}(K)$ is also an interval of cardinals. If $K$ is an interval, $M$ is tight for $K$ and $M$ is $|K|^+$-weakly uniform in some interval containing $K$ then it follows from Theorem 5.1 that $M$ is weakly uniform in an interval containing $\mathrm{pcf}(K)$, and so the analysis of Theorem 5.5 applies.

It follows from Theorem 5.1 that if $M$ is tight some information about cofinalities can be propagated from $K$ to $\mathrm{pcf}(K)$. The next result shows that under slightly stronger assumptions the same is true for the property of tightness itself. The idea of taking pointwise suprema of functions from many scales comes from Shelah's proof that $\mathrm{pcf}(\mathrm{pcf}(K)) = \mathrm{pcf}(K)$.

**Theorem 5.6.** *Let $\mathrm{pcf}(K) \subseteq M$ and let $M$ be $|\mathrm{pcf}(K)|^+$-weakly uniform on $K$. If $M$ is tight for $K$ then $M$ is tight for $\mathrm{pcf}(K)$.*

*Proof.* Let $L = \mathrm{pcf}(K)$, and let $F \in \prod_{\lambda \in L} M \cap \lambda$. We define $f \in \prod K$ by

$$f : \kappa \mapsto \sup_{\lambda \in L} f^\lambda_{F(\lambda)}(\kappa) + 1.$$

Since $L \subseteq M$, we see that $f^\lambda_{F(\lambda)} \in M$ and $f^\lambda_{F(\lambda)}(\kappa) \in M \cap \kappa$ for all $\lambda \in L$ and $\kappa \in K$. Since $\mathrm{cf}(M \cap \kappa) > |L|$ it follows that $f(\kappa) < \sup(M \cap \kappa)$ for all $\kappa \in K$.

Since $M$ is tight for $K$, there is $g \in M \cap \prod K$ which dominates $f$ pointwise. Since $\langle f^\lambda_\alpha : \alpha < \lambda, \lambda \in \mathrm{pcf}(K)\rangle \in M$ and the sequence $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$ is a scale for each $\lambda$, it follows by elementarity that there is $G \in M \cap \prod L$ such that $g <_{J_{<\lambda}} f^\lambda_{G(\lambda)}$ for all $\lambda \in L$. We now see that for each $\lambda \in L$ we have

$$f^\lambda_{F(\lambda)} < f < g <_{J_{<\lambda}} f^\lambda_{G(\lambda)},$$

so that $G$ dominates $F$ pointwise and we have shown that $M$ is tight for $L$. □

Theorem 5.6 allows us a practical alternative to Theorem 5.5:

**Corollary 5.7.** *Let $M$ be $|K|^+$-weakly uniform (resp. $\eta$-uniform with $|K| < \eta < \min(K)$) between $\rho$ and $\sigma$, where all cardinals in $M \cap \mathrm{pcf}(K)$ are greater than $\rho$ and less than $\sigma$. Let $M'$ be the Skolem hull of $M \cup \mathrm{pcf}(K)$ in $(H(\theta), \epsilon, <_\theta)$. Then the following conditions on $M$ are equivalent.*

(1) *$M$ is tight for $K$.*

(2) $M'$ *is tight for* $\mathrm{pcf}(K)$ *and* $|K|^+$*-weakly uniform (resp.* $\eta$*-uniform) on* $\mathrm{pcf}(K)$*.*
(3) *For every* $\lambda \in \mathrm{pcf}(K)$, $\sup(M' \cap \lambda)$ *is a good point of cofinality at least* $\kappa^+$ *(resp. of cofinality* $\eta$*) for* $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ *and* $f^\lambda_{\sup(M'\cap\lambda)} =_{J_{<\lambda}} \chi^{B_\lambda}_{M'}$*.*

*Proof.* By Fact 6.2 we know that $|\mathrm{pcf}(K)| < |K|^{+4}$. Thus $M' \subset sk^{H(\theta)}(M \cup |K|^{+3})$. Standard arguments show that for all $\kappa \in K \backslash |K|^{+3}$, $\sup(M \cap \kappa) = \sup(sk^{H(\theta)}(M \cup |K|^{+3}) \cap \kappa)$. Suppose now that $M$ is tight for $K$.

By the results in the last paragraph, $M'$ is tight for $K$ and by Theorem 5.6, $M'$ is tight for $\mathrm{pcf}(K)$. Moreover, by Theorem 5.1, $M'$ is $|K|^+$-weakly uniform (resp. $\eta$-uniform) on $\mathrm{pcf}(K)$. Hence 1 implies 2. Again by the previous paragraph it is clear that 2 implies 1

By Theorem 5.2, 2 and 3 are equivalent for $M'$. □

## 6. Tightness, approachability and reflection

In this section we prove a general covering theorem for tight structures, and use it to show that certain tight uniform structures are IA. We also discuss the connection between tightness and stationary reflection.

We start with some general discussion of covering properties of structures. Suppose that $\kappa$, $\lambda$ and $\theta$ are regular cardinals with $\kappa < \lambda < \theta$. We consider substructures $N \prec (H_\theta, \in, <_\theta, \{\kappa, \lambda\})$ such that $|N| = \kappa$ and $\kappa \subseteq N$.

Given a set $Z$ with $\kappa \subseteq Z$, we let $P_\kappa Z$ be the set of $x \subseteq Z$ such that $|x| < \kappa$ and $x \cap \kappa \in \kappa$. In increasing order of strength we may consider the following properties of $N$:

**Internally cofinal in** $\mathrm{P}_\kappa\lambda$**:** $N \cap P_\kappa(N \cap \lambda)$ is cofinal (in the inclusion ordering) in $P_\kappa(N \cap \lambda)$

**Internally stationary in** $\mathrm{P}_\kappa\lambda$**:** $N \cap P_\kappa(N \cap \lambda)$ is stationary in $P_\kappa(N \cap \lambda)$.

**Internally club in** $\mathrm{P}_\kappa\lambda$**:** $N \cap P_\kappa(N \cap \lambda)$ contains a club in $P_\kappa(N \cap \lambda)$.

**Internally approachable in** $\mathrm{P}_\kappa H_\lambda$**:** $N \cap H_\lambda$ is IA of length and cardinality $\kappa$.

The following easy lemma shows that some cardinal arithmetic assumptions simplify the picture.

**Lemma 6.1.** *Let* $\kappa$*,* $\lambda$ *and* $\theta$ *be regular cardinals with* $\kappa < \lambda < \theta$ *and let* $N \prec (H_\theta, \in, <_\theta, \{\kappa, \lambda\})$ *be such that* $|N| = \kappa$ *and* $\kappa \subseteq N$*.*

*If* $\kappa^{<\kappa} = \kappa$ *and* $N$ *is internally cofinal in* $\mathrm{P}_\kappa\lambda$*, then* $P_\kappa(N \cap \lambda) \subseteq N$*, from which it follows that* $N$ *is IA in* $\mathrm{P}_\kappa H_\lambda$*.*

For the rest of this section we will be studying structures $N$ which are tight for some *interval* $K$ of regular cardinals. One reason for this is that we wish to apply the results of Section 4 on recovering structures from their characteristic functions, and these results require an interval of regular cardinals. Another reason is that we can use the following result of Shelah.

**Fact 6.2** (Shelah)**.** *Let* $K$ *be an interval of regular cardinals with* $|K| < \min(K)$*. Then*

- $\mathrm{pcf}(K)$ *is an interval of regular cardinals, which has a largest element.*
- $|\mathrm{pcf}(K)| \leq |K|^{+3}$*.*
- $\mathrm{pcf}(\mathrm{pcf}(K)) = \mathrm{pcf}(K)$*.*

**Remark 6.3.** If $K$ is an infinite interval of regular cardinals with $|K| < \min(K)$, then it follows from Fact 6.2 that by deleting a suitable finite initial segment $k$ of $K$ we can obtain an interval $L$ with $|\operatorname{pcf}(L)| < \min(L)$.

**Remark 6.4.** It is a viable conjecture that $|\operatorname{pcf}(A)| = |A|$ for all $A$ with $|A| < \min(A)$.

Before stating and proving the very general Theorem 6.5, we digress briefly to consider what is perhaps the most interesting special case. Let $K = \{\aleph_n : 0 < n < \omega\}$, and suppose for simplicity that $\operatorname{pcf}(K)$ is countable, say $\operatorname{pcf}(K) = \{\aleph_{\alpha+1} : \alpha \leq \beta\}$ for some countable $\beta$.

Let $m > 0$ and let $\theta$ be a large regular cardinal. Let $N \prec H_\theta$ be such that $\aleph_m = |N| \subseteq N$, and let $N$ be $\aleph_m$-uniform and tight on $\{\aleph_n : m < n < \omega\}$. By Theorems 5.1 and 5.6 $N$ is $\aleph_m$-uniform and tight on $\{\aleph_{\alpha+1} : m \leq \alpha \leq \beta\}$.

Under these circumstances, it will follow from Theorem 6.5 that if $m > 1$ then $N$ is internally stationary in $P_{\aleph_m}(\aleph_{\beta+1})$. In conjunction with some extra cardinal arithmetic assumptions as in Lemma 6.1, it will follow that $N \cap H_{\aleph_{\beta+1}}$ is IA of length and cardinality $\aleph_m$. This is an instance of one of the motivating theses of this paper, that tightness plus uniformity is very close to internal approachability.

**Theorem 6.5.** *Let $\eta$ be an uncountable regular cardinal. Let $L$ be an interval of regular cardinals such that $\min(L) = \eta^{++}$, $|L| < \eta$, and $L = \operatorname{pcf}(L)$. Let $\theta$ be a large regular cardinal and let $N \prec (H_\theta, \in, <_\theta, \{L\})$ be such that $|N| = \eta^+$ and $N$ is $\eta^+$-uniform between $\eta^+$ and $\max(L)^+$ and $N$ is tight for $L$. Then $N$ is internally stationary in $P_{\eta^+}(\max(L))$.*

*Proof.* Fix some algebra $\mathcal{A}$ on $N \cap \max(L)$. We build an increasing sequence $\langle M_i : i < \eta \rangle$ such that for all $i$

1. $M_i \prec N$, with $\eta \subseteq M_i$ and $|M_i| = \eta$.
2. $M_i \cap \max(L) \prec \mathcal{A}$.
3. $M_i \cap \eta^+ < M_{i+1} \cap \eta^+$.
4. $\chi^L_{M_i}$ is pointwise dominated by some function in $M_{i+1} \cap \prod L$.

The last demand on $M_i$ is the crucial one, and it is possible to satisfy it because $N$ is tight and $\eta^+$-uniform. We let $M = \bigcup_i M_i$ and $L^* = \{\eta^+\} \cup L$, and note that $\operatorname{pcf}(L^*) = L^*$. As usual we will let $\langle B_\lambda : \lambda \in L^* \rangle$ be the $<_\theta$-minimal sequence of generators for $L^*$, $\langle f^\lambda_\alpha : \lambda \in L^*, \alpha < \lambda \rangle$ the $<_\theta$-least matrix of functions such that $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ is a continuous scale in $\prod B_\lambda / J_{<\lambda}$.

By construction $\chi^{L^*}_{M_i}$ is pointwise dominated by $\chi^{L^*}_{M_{i+1}}$, so that $M$ is $\eta$-uniform between $\eta^+$ and $\max(L)^+$. Since $|L^*| < \eta$ the construction also guarantees that $M$ is tight for $L^*$.

By Theorem 5.3 the function $\chi^{L^*}_M$ is the pointwise supremum of finitely many functions of the form $f^\lambda_{\sup(M \cap \lambda)} \restriction (B_\lambda \setminus E_\lambda)$, where $E_\lambda \in M$. By Lemma 4.3 $N \cap \max(L)$ is $\eta$-closed in $\sup(N \cap \max(L))$, so in particular $\sup(M \cap \lambda) \in N$ for all $\lambda$.

It follows that $\chi^{L^*}_M \in N$. By Lemma 4.7 we see that $M \cap \max(L) \in N$, and by construction $M \cap \max(L) \prec \mathcal{A}$ and $M \in P_{\eta^+}(N \cap \lambda)$. This shows that $N \cap P_{\eta^+}(N \cap \lambda)$ is stationary. $\square$

As we mentioned before proving Theorem 6.5, we can use the theorem to show that sometimes tight plus uniform equals IA. The following corollary gives the simplest interesting case.

**Corollary 6.6.** *Let $2^{\aleph_1} = \aleph_2$ and $2^{\aleph_\omega} = \aleph_{\omega+1}$. Let $\theta$ be a large regular cardinal and let $N \prec (H_\theta, \in, <_\theta)$ be $\aleph_2$-uniform between $\aleph_2$ and $\aleph_\omega$, and tight for $\{\aleph_n : 2 < n < \omega\}$. Then $N \cap H_{\aleph_{\omega+1}}$ is IA of length and cardinality $\aleph_2$.*

For structures of size $\aleph_1$ we have less than satisfactory results which we illustrate with the following example:

**Example 6.7.** *Suppose that $\aleph_\omega$ is a strong limit and $2^{\aleph_\omega} = \aleph_{\omega+1}$. Let $N \prec H(\aleph_{\omega+2})$ have cardinality $\aleph_1$ and uniform cofinality $\omega_1$. Suppose that $\sup(N \cap \aleph_{\omega+1})$ is an approachable ordinal $\gamma$ and $\chi_N =^* f_\gamma$. Then $N$ is internally approachable.*

*To see this, let $M$ be the internally approachable structure that has cardinality $\omega_1$ with $\chi_M =^* f_\gamma$. Then for some $\omega_l$, the Skolem hull of $N \cup \omega_l$ and the Skolem hull of $M \cup \omega_l$ have the same characteristic function and hence the two Skolem hulls have equal intersection in $\aleph_{\omega+1}$. In particular, the Skolem hull of $N \cup \aleph_l$ is closed under $\omega$-sequences below $\aleph_{\omega+1}$. One can sees inductively that if Skolem hull of $N \cup \aleph_{l-i}$ is closed under $\omega$-sequence then so is the Skolem hull of $N \cap \aleph_{l-(i+1)}$. In particular, $N$ is closed under $\omega$-sequences below $\aleph_{\omega+1}$ and is thus internally approachable below $\aleph_{\omega+1}$.*

Foreman and Magidor [15] showed that there is a close connection between internal approachability, uniformity and stationary reflection for sets of structures. In particular they showed that

- Let $\kappa$ be supercompact, let $\mu$ and $\nu$ be regular uncountable cardinals less than $\kappa$ with $\mu < \nu$. Let $G$ be generic for $Col(\nu, < \kappa)$, then in $V[G]$ the following stationary reflection principle holds:
  Every stationary set of IA substructures of $(H_\theta, \in, <_\theta)$ of length and cardinality $\mu$ reflects to some substructure of cardinality $\nu$.
- Let $\theta \geq \omega_3$ and let $S_{ij}$ be the set of $N \prec (H_\theta, \in, <_\theta)$ such that $|N| = \aleph_1$, $\aleph_1 \subseteq N$, $\mathrm{cf}(N \cap \aleph_2) = \aleph_i$ and $\mathrm{cf}(N \cap \aleph_3) = \aleph_j$.
  Then only $S_{11}$ can have the property that every stationary subset is reflecting.

The following result indicates that tightness is also relevant to problems about stationary reflection.

**Theorem 6.8.** *Let $K$ be a countable set of regular cardinals and let $S$ be a stationary set of elementary substructures of $(H_\theta, \in, <_\theta)$, such that every element of $S$ is tight for $K$. Let $M$ be such that $S$ reflects to $M$. Then $M \prec (H_\theta, \in, <_\theta)$ and $M$ is tight for $K$.*

*Proof.* Let $t$ be a Skolem term and let $\vec{a}$ be a finite sequence of parameters from $M$. There is $N \subseteq M$ such that $N \in S$ and all the members of $\vec{a}$ come from $N$. Since $N$ is closed under $t$ and $N \subseteq M$, $t(\vec{a}) \in M$ as required.

Note that since $K$ is countable, $K \subseteq N$ for all $N \in S$. Let $f \in \prod_{\kappa \in K} K \cap M$, and find $N \in S$ such that $N \subseteq M$ and $f \subseteq N$. Since $N$ is tight for $K$ there is $g \in N \cap \prod K$ which dominates $f$, and $g \in M$ since $N \subseteq M$. □

Foreman and Todorcevic [19] have defined a notion of tightness for countable structures, and have used this to investigate stationary reflection in $[H_\lambda]^{\aleph_0}$.

## 7. PCF absoluteness, covering and precipitous ideals

In this section we prove a version of the covering lemma and apply it to a problem about precipitous ideals. Our covering lemma states roughly that if $V$ and $W$ are inner models with $V \subseteq W$, and the PCF structures of $V$ and $W$ are similar enough then every set of ordinals in $W$ is covered by a set of the same size lying in $V$. The key idea of the proof is that the characteristic functions of certain IA structures in $W$ can (as in Section 5) be described in terms of the PCF structure of $W$, and so (by the PCF resemblance hypothesis) these characteristic functions will be elements of $V$.

We start with a result which says that sufficiently similar universes have similar PCF structures.

**Theorem 7.1.** *Let $V$ be an inner model of $W$, and in $V$ let $K$ be a set of regular cardinals with $|K| < \min(K)$. Assume that for all $\mu \in [\min(K), \max(\mathrm{pcf}(K))^V]$, if $\mu$ is regular in $V$ then $\mu$ is regular in $W$. Assume also that for every $f \in (\prod K)^W$ there is $g \in (\prod K)^V$ with $f < g$. Then*

1. *$J^V_{<\mu} = J^W_{<\mu} \cap V$ for all cardinals $\mu \geq \min(K)$.*
2. *Any sequence of PCF generators for $K$ in $V$ is still a sequence of PCF generators for $K$ in $W$.*
3. *$\mathrm{pcf}(K)^V = \mathrm{pcf}(K)^W$.*

*Proof.* We start by proving that $J^V_{<\mu} = J^W_{<\mu} \cap V$ for all $\mu \in [\min(K), \max(\mathrm{pcf}(K))^V]$, by induction on $\mu$. This is clear for $\mu = \min(K)$. For $\lambda$ singular we have $J_{<\lambda} = J_{<\lambda^+} = \bigcup_{\nu<\lambda} J_{<\nu}$, so it remains to show that if $\mu$ is regular and $J^V_{<\mu} = J^W_{<\mu} \cap V$ then $J^V_{<\mu^+} = J^W_{<\mu^+} \cap V$.

Suppose first that $A \in J^V_{<\mu^+}$. If $A \in J^V_{<\mu}$ then we are done, otherwise we fix a scale $\vec{f}$ of length $\mu$ in $(\prod A/J_{<\mu})^V$. By the induction hypothesis and our assumptions on $V$ and $W$, $\vec{f}$ is a scale in $(\prod A/J_{<\mu})^W$ and so $A \in J^W_{<\mu^+}$.

Now suppose that $A \in J^W_{<\mu^+} \cap V$. Let $D \in V$ be any $V$-ultrafilter on $K$ with $A \in D$, and in $W$ extend $D$ to $\bar{D}$ a $W$-ultrafilter. By our assumptions on $V$ and $W$ the map $\psi : [f]_D \mapsto [f]_{\bar{D}}$ is a cofinal order-preserving function from $(\prod A/D)^V$ to $(\prod A/\bar{D})^W$. Since $A \in J^W_{<\mu^+}$ the $W$-cofinality of $(\prod A/\bar{D})^W$ is at most $\mu$, and the existence of $\psi$ and our assumptions on $V$ and $W$ imply that the $V$-cofinality of $(\prod A/D)^V$ is at most $\mu$. Therefore $A \in J^V_{<\mu^+}$.

Next we show that PCF generators agree between $V$ and $W$. We note that we can identify the maximum of $\mathrm{pcf}(K)$ as the least $\lambda$ such that $K \in J_{<\lambda^+}$, so that $V$ and $W$ agree on the value of the maximum element in $\mathrm{pcf}(K)$. We call this common value $\lambda_{\max}$. In $V$ we fix a sequence $\langle B^V_\mu : \min(K) \leq \mu \leq \lambda_{\max} \rangle$ such that $B^V_\mu$ generates $J^V_{<\mu^+}$ over $J^V_{<\mu}$, and similarly we fix in $W$ a sequence $\langle B^W_\mu : \min(K) \leq \mu \leq \lambda_{\max} \rangle$ such that $B^W_\mu$ generates $J^W_{<\mu^+}$ over $J^W_{<\mu}$. We choose these generators so that $B^V_\mu = \emptyset$ for $\mu \notin \mathrm{pcf}(K)^V$, and similarly $B^W_\mu = \emptyset$ for $\mu \notin \mathrm{pcf}(K)^W$. We also choose $B^V_{\lambda_{\max}} = B^W_{\lambda_{\max}} = K$.

We will show by induction that $B^W_\mu$ and $B^V_\mu$ are equal modulo $J^W_{<\mu}$ for all $\mu$. Since $B^V_\mu$ is in $J^W_{<\mu^+}$ and $B^W_\mu$ is a generator, we see that $B^V_\mu$ is contained in $B^W_\mu$ modulo $J^W_{<\mu}$ for all $\mu$. Let $\lambda$ be least such that $B^W_\lambda$ and $B^V_\lambda$ are unequal modulo $J^W_{<\lambda}$. We now adjust the $B^W_\mu$, replacing $B^W_\mu$ by $B^V_\mu$ for $\mu < \lambda$; this is legitimate by the choice of $\lambda$. Let $C = B^W_\lambda \setminus B^V_\lambda$ so that $C \notin J^W_{<\lambda}$ but $C \in J^W_{<\lambda^+}$.

Now let $\mu$ be least such that $C$ is covered by some set $B^V_{\mu_0} \cup \ldots B^V_{\mu_n} \cup B^V_\mu$ with $\mu_0 < \ldots < \mu_n < \mu$; such a $\mu$ exists because $B^V_\lambda = K$. Suppose for a contradiction that $\mu \leq \lambda$; then $\mu < \lambda$ since $C$ is disjoint from $B^V_\lambda$. Now $B^V_\nu = B^W_\nu$ for $\nu < \lambda$, and it follows that $C \in J^W_{<\lambda}$, which is a contradiction: so $\mu > \lambda$. Note that $B^V_\mu \neq \emptyset$ so that $\mu \in \mathrm{pcf}(K)^V$.

Let $D = B^V_{\mu_0} \cup \ldots B^V_{\mu_n} \cup B^V_\mu$. Working in $V$ find $\langle g_\alpha : \alpha < \mu \rangle$ such that $g_\alpha \in (\prod D)^V$ and $\langle g_\alpha \restriction B^V_\mu : \alpha < \mu \rangle$ is increasing and cofinal in $(\prod B^V_\mu)^V$ modulo $J^V_{<\mu} \restriction B^V_\mu$. Working in $W$ find $\langle h_\beta : \beta < \lambda \rangle$ such that $h_\beta \in (\prod D)^V$ and $\langle h_\beta \restriction C : \beta < \lambda \rangle$ is increasing and cofinal in $(\prod C)^W$ modulo $J^W_{<\lambda} \restriction C$.

Since $\mu > \lambda$ we may find $h_\beta$ which dominates $g_\alpha$ on $C$ modulo $J^W_{<\lambda}$ for unboundedly many $\alpha < \mu$, and may then find $g_\alpha$ such that

(1) $h_\beta$ dominates $g_\alpha$ modulo $J^W_{<\lambda} \restriction C$.
(2) $g_\alpha$ dominates $h_\beta$ modulo $J^V_{<\mu} \restriction B^V_\mu$.

We find $\delta_1, \ldots \delta_n$ such that $\delta_1 < \ldots \delta_n < \lambda$ and $\{\kappa \in C : h_\beta(\kappa) \leq g_\alpha(\kappa)\} \subseteq B^W_{\delta_1} \cup \ldots \cup B^W_{\delta_n}$. Note that $B^W_{\delta_i} = B^V_{\delta_i}$. We also find $\epsilon_1, \ldots \epsilon_p$ such that $\epsilon_1 < \ldots \epsilon_p < \mu$ and $\{\kappa \in B^V_\mu : h_\beta(\kappa) \geq g_\alpha(\kappa)\} \subseteq B^V_{\epsilon_1} \cup \ldots \cup B^V_{\epsilon_p}$.

We claim that $C$ is covered by the union of the $B^V_{\mu_i}$, $B^V_{\delta_i}$ and $B^V_{\epsilon_i}$. To see this observe that if $\kappa \in C$ then at least one of the following must hold:

- $\kappa \in B^V_{\mu_i}$ for some $i$.
- $\kappa \in C \cap B^V_\mu$ and $h_\beta(\kappa) \leq g_\alpha(\kappa)$, in which case $\kappa \in B^V_{\delta_i}$ for some $i$.
- $\kappa \in C \cap B^V_\mu$ and $h_\beta(\kappa) \geq g_\alpha(\kappa)$, in which case $\kappa \in B^V_{\epsilon_i}$ for some $i$.

This contradicts the minimal choice of $\mu$. It follows that as we claimed the sets $B^V_\mu$ and $B^W_\mu$ agree modulo $J^W_\mu$ for all $\mu$. In particular the $B^V_\mu$ will serve as a sequence of generators in $W$.

It remains to be seen that $\mathrm{pcf}(K)^V = \mathrm{pcf}(K)^W$. This is immediate by the following computation: given any sequence $B_\nu$ such that $B_\nu$ generates $J_{<\nu^+}$ over $J_{<\nu}$, $\mathrm{pcf}(K)$ is the set of $\nu$ such that $B_\nu$ is not covered by a finite union of $B_\mu$ for $\mu < \nu$. □

A small technical difficulty is caused by the fact that the property of being the characteristic function of some structure is not obviously downwards absolute. We will resolve this difficulty using Lemma 4.4 and the following result.

**Lemma 7.2.** *Let $V$ and $W$ be inner models of set theory with $V \subseteq W$. In $V$ let $K$ be an interval of regular cardinals with $|K| < \mu$ and $\min(K) = \mu^+$ for some regular uncountable $\mu$. Assume that in $W$ the cardinal $\mu$ is still regular and $K$ is still an interval of regular cardinals. Let $f \in V$ be such that for all $\alpha$ with $\mu < \alpha < \sup(K)$, $f(\alpha, -)$ is a bijection between $\alpha$ and $|\alpha|$.*

*Let $\theta$ be some sufficiently large regular cardinal of $W$, and in $W$ let $N \prec (H_\theta, \in, <_\theta, \{K\}, f)$ be an IA structure of length and cardinality $\mu$. If $\chi^K_N \in V$ then there is $B \in V$ such that $N \cap \sup(K) \subseteq B \subseteq \sup(K)$ and $|B| = \mu$.*

*Proof.* Since $N$ is IA of length $\mu$, we may fix $\langle C_\kappa : \kappa \in K \rangle$ lying in $W$ with $C_\kappa \subseteq N \cap \kappa$ and $C_\kappa$ club in $\sup(N \cap \kappa)$ of order type $\mu$. Since $\chi^K_N \in V$ and cofinalities agree, we may find in $V$ a sequence $\langle D_\kappa : \kappa \in K \rangle$ with $D_\kappa$ club in $\sup(N \cap \kappa)$ of order type $\mu$.

Now let $M_0 = N \cap \sup(K)$ and note that $M_0 \prec (\sup(K), f)$. Let $M_1$ be the hull in $(\sup(K), f)$ of $\mu \cup \bigcup_\kappa D_\kappa$, and note that $M_1 \in V$ and $|M_1| = \mu$. The hypotheses of Lemma 4.4 are satisfied because for each $\kappa \in K$ the set $C_\kappa \cap D_\kappa$ is contained in $M_0 \cap M_1 \cap \kappa$ and is cofinal in $M_0 \cap \kappa$. It follows that $M_0 \subseteq M_1$ so we may set $B = M_1$. □

**Theorem 7.3.** *Let $V$ and $W$ be inner models of set theory with $V \subseteq W$. In $V$ let $\kappa$ be regular and let $\lambda > \kappa$ be a cardinal. Let $K = REG^V \cap [\kappa, \lambda]$ and suppose that $V$ and $W$ agree on regular cardinals in the interval $[\kappa, \max(\mathrm{pcf}(K)^V)]$, $K$ is progressive and $(\prod K)^V$ is cofinal in $(\prod K)^W$.*

*Let $A \subseteq \lambda$. Then there is $B \in V$ such that $A \subseteq B$ and $|B|^W \leq \max\{\kappa, |A|^W\}$.*

*Proof.* We will prove this by induction on $\sup(A)$. We way assume that $\sup(A) > \kappa$, since otherwise we may set $B = \kappa + 1$. For the rest of this proof cardinalities and cofinalities should be understood as computed in $W$, though we will tacitly use the agreement between cardinalities of $V$ and $W$ at several points.

**Case 0:** $\sup(A)$ is not a cardinal. Let $f \in V$ be a bijection between $\sup(A)$ and $|\sup(A)|$, and let $A^*$ be the image of $A$ under $f$. By induction we may find $B^* \subseteq |\sup(A)|$ such that $A^* \subseteq B^*$ and $|B^*| \leq \max\{\kappa, |A|\}$, and we may then let $B$ be the inverse image of $B^*$ under $f$.

**Case 1:** $\sup(A)$ is regular. In this case $|A| = \sup(A)$ and we let $B = \sup(A)$.

**Case 2:** $\sup(A)$ is singular. If $|A| = \sup(A)$ then we may set $B = \sup(A)$, so we now assume that $|A| < \sup(A)$. By our hypotheses on $K$ we may find $\mu$ regular with $\max\{\kappa, |A|, |K|^{+4}\} < \mu < \sup(A)$. We set $K_0 = (\mu^+, \sup(A)) \cap REG$.

Let $L = \mathrm{pcf}(K_0)$ where by Fact 6.2 $|L| < \mu$ and $L = \mathrm{pcf}(L)$. We fix in $V$ a sequence $\vec{B} = \langle B_\nu : \nu \in L \rangle$ of PCF generators for $K_0$, and a matrix of functions $\vec{f} = \langle f^\nu_i : \nu \in L, i < \nu \rangle$ such that $\langle f^\nu_i : i < \nu \rangle$ is a continuous scale in $\prod B_\nu / J_{<\nu}$.

By Theorem 7.1, $\vec{B}$ is still a sequence of PCF generators in $W$. It is clear that $\langle f^\nu_i : i < \nu \rangle$ is still a scale in $W$, and we claim that additionally this scale is still continuous at good points of cofinality $\mu$.

We use Fact 3.1, the Trichotomy Theorem of Shelah. Let $\nu \in L$, and in $W$ let $\rho < \nu$ be a point of cofinality $\mu$ which is good for the scale $\langle f^\nu_i : i < \nu \rangle$. We begin by arguing that in $V$ there must exist an eub for $\langle f^\nu_i : i < \rho \rangle$ whose values have cofinality at least $\kappa$.

If this is not the case then in $V$ we must be either in Case 2 or Case 3 from Fact 3.1. It is easy to see that the properties of being in Case 2 or Case 3 are upwards absolute from $V$ to $W$, in fact the witnesses from $V$ will work in $W$ as long as we extend the ultrafilter in $V$ for Case 2 to an ultrafilter in $W$. Since we have an eub for $\langle f^\nu_i : i < \rho \rangle$ of uniform cofinality $\mu$ in $W$, it follows from Fact 3.1 that in $W$ we are not in Case 2 or Case 3, so that in $V$ there is an eub for $\langle f^\nu_i : i < \rho \rangle$ whose values have cofinality at least $\kappa$.

By continuity in $V$, $f^\nu_\rho$ is an eub whose values have cofinality at least $\kappa$ almost everywhere in $V$. By the hypothesis on the resemblance between $V$ and $W$, $f^\nu_\rho$ retains these properties in $W$. It follows that the scale $\langle f^\nu_i : i < \nu \rangle$ is continuous at $\rho$.

We also fix in $V$ a function $f$ from $\sup(K_0)^2$ to $\sup(K_0)$ coding some information about cardinalities, as in Lemma 7.2.

We now build $N \prec (H_\theta, \in, <_\theta, \{K_0\}, f)$ where $A \subseteq N$ and $N$ is IA of length and cardinality $\mu$. By theorem 5.3 $\chi^{K_0}_N$ is the pointwise supremum of finitely many functions of the form $f^\nu_i \restriction (B_\nu \setminus E_\nu)$ where $E_\nu$ is the union of a finite subset of $\{B_\zeta : \zeta < \nu\}$.

It follows that $\chi^{K_0}_N \in V$. The hypotheses of Lemma 7.2 are satisfied so we may find $C$ in $V$ such that $|C| = \mu < \sup(A)$ and $A \subseteq C$. Now let $g \in V$ be a bijection between $C$ and $\mu$, and let $A^*$ be the image of $A$ under $g$. Now we may use the induction hypothesis to cover $A^*$ by a suitable $B^*$, and then pull back along $g$ to get a suitable $B$ covering $A$. $\square$

**Remark 7.4.** Theorem 7.3 is actually an equivalence. The covering statement in the conclusion implies that $V$ and $W$ agree on regular cardinals between $\kappa$ and $\lambda$, and also implies that $(\prod K)^V$ is cofinal in $(\prod K)^W$.

In order to be able to state the next result in a compact way, we make the following *ad hoc* definition.

**Definition 7.5.** Let $V$ and $W$ be inner models of set theory with $V \subseteq W$. In $V$, let $\kappa$ be regular and let $\lambda > \kappa$ be a cardinal. We say that $W$ *weakly resembles* $V$ *on* $[\kappa, \lambda)$ if and only if the hypotheses of Theorem 7.3 are satisfied.

For background on the theory of precipitous ideals we refer the reader to Foreman's survey paper [14]. The following result belongs to a genre of theorems in which we are given an ideal $I$ on a cardinal $\kappa$ and some information about preservation of cardinals when forcing with $P\kappa/I$, and we conclude that $I$ must be precipitous. This line of inquiry was begun by Baumgartner and Taylor[2], who proved for example that under GCH a countably closed ideal on $\aleph_1$ whose quotient algebra preserves $\aleph_2$ is necessarily precipitous.

**Theorem 7.6.** *Let $I$ be a countably complete ideal on $\aleph_1$. Let $\lambda = 2^{\aleph_1}$, and suppose it is forced by $P\aleph_1/I$ that $V[G]$ weakly resembles $V$ on $[\aleph_2, \lambda)$. Then $I$ is precipitous.*

*Proof.* Let $G$ be a generic ultrafilter on $(P\aleph_1)^V$ and let $M = Ult(V, G)$. We observe that by our hypotheses $\aleph_2^V = \aleph_1^{V[G]}$ and recall the standard fact that $\aleph_2^V$ is an initial segment of the well-founded part of $M$. It must be the case that $\aleph_2^V = \aleph_1^M$, for if not then in $M$ there is a surjection from $\omega$ onto some larger $M$-ordinal, contradicting the fact that $\aleph_2^V = \aleph_1^{V[G]}$.

Suppose for a contradiction that $M$ is not well-founded, and in $V[G]$ choose a sequence $\langle f_i : i \in \omega \rangle$ such that $f_i \in V$, $f_i : \aleph_1^V \longrightarrow ON$ and $f_{i+1} <_G f_i$ for all $i$. Since $P\aleph_1/I$ has the $\lambda^+$-c.c. we may find $C \in V$ such that $|C| \leq \lambda$ and $\text{range}(f_i) \subseteq C$ for all $i$.

By Theorem 7.3 we may find $B \subseteq C$ such that $B \in V$, $|B|^V = \aleph_2^V$ and $\text{range}(f_i) \subseteq B$ for all $i$. Working in $V$, we write $B = \bigcup_{j<\aleph_2} B_j$ where the $B_j$ are increasing and $|B_j| = \aleph_1$. In $V[G]$ this gives a representation of $B$ as an increasing union of countable sets. Since $\bigcup_i \text{range}(f_i)$ is countable in $V[G]$ we can fix a $j$ such that $\text{range}(f_i) \subseteq B_j$ for all $i$. Now let $\gamma$ be the order type of $B_j$, let $h : B_j \simeq \gamma$ be order preserving and let $f_i^* = h \circ f_i$. Then $\langle f_i^* : i \in \omega \rangle$ is a $G$-decreasing sequence of functions from $\aleph_1$ to $\gamma$, and so to get a contradiction we need only show that $j_G(\gamma)$ is well-founded for all $\gamma < \aleph_2^V$.

Suppose for a contradiction that $j_G(\gamma)$ is ill-founded, and choose a bijection $F \in M$ between $\aleph_1^M$ and $j_G(\gamma)$.

Since $\aleph_1^M = \aleph_1^{V[G]}$, we may find $\delta < \aleph_1^{V[G]}$ such that in $V[G]$ the set $F``\delta$ contains an infinite decreasing sequence of $M$-ordinals.

Working in$V[G]$, let $T$ be the tree of all finite sequences $(\alpha_0, \dots \alpha_i)$ from $\delta$ such that $F(\alpha_i) > F(\alpha_{i+1})$ for all $i$. The tree $T$ is in the well-founded part of $M$. Clearly $T$ is well-founded in $M$, and is not well-founded in $V[G]$.

The tree $T$ is countable in $M$, and so in $M$ there is a rank function $\rho : T \longrightarrow \aleph_1^M$. Since $\aleph_1^M$ is well-founded, this contradicts the existence of a branch of $T$ in $V[G]$. We conclude that $M$ is well-founded and so that $I$ is a precipitous ideal. □

## References


[1] U. Abraham and M. Magidor. Cardinal arithmetic. To appear in the Handbook of Set Theory.

[2] J. E. Baumgartner and A. D. Taylor. Saturation properties of ideals in generic extensions. II. *Transactions of the American Mathematical Society*, 271(2):587–609, 1982.

[3] M. Burke and M. Magidor. Shelah's pcf theory and its applications. *Annals of Pure and Applied Logic*, 50:207–254, 1990.

[4] J. Cummings. Compactness and incompactness in set theory. To appear in proceedings of Logic Colloquium 2001.

[5] J. Cummings. Collapsing successors of singulars. *Proceedings of the American Mathematical Society*, 125(9):2703–2709, 1997.

[6] J. Cummings, M. Džamonja, and S. Shelah. A consistency result on weak reflection. *Fundamenta Mathematicae*, 148(1):91–100, 1995.

[7] J. Cummings, M. Foreman, and M. Magidor. Canonical invariants for models of ZFC: Part Two. Submitted.

[8] J. Cummings, M. Foreman, and M. Magidor. Squares, scales and stationary reflection. *Journal of Mathematical Logic*, 1(1):35–99, 2001.

[9] J. Cummings, M. Foreman, and M. Magidor. The non-compactness of square. *Journal of Symbolic Logic*, 68:637–643, 2003.

[10] J. Cummings and S. Shelah. Some independence results on reflection. *Journal of the London Mathematical Soceity*, 59(1):37–49, 1999.

[11] M. Džamonja and S. Shelah. Saturated filters at successors of singular, weak reflection and yet another weak club principle. *Annals of Pure and Applied Logic*, 79(3):289–316, 1996.

[12] M. Dzamonja and S. Shelah. Weak reflection at the successor of a singular cardinal. *Journal of the London Mathematical Soceity*, 67:1–15, 2003.

[13] G. Fodor. Eine Bemerkung zur Theorie der regressiven Funktionen. *Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum.*, 17:139–142, 1956.

[14] M. Foreman. Ideals and generic embeddings. To appear in the Handbook of Set Theory.

[15] M. Foreman and M. Magidor. Large cardinals and definable counterexamples to the continuum hypothesis. *Annals of Pure and Applied Logic*, 76:47–97, 1995.

[16] M. Foreman and M. Magidor. A very weak square principle. *Journal of Symbolic Logic*, 62(1):175–196, 1997.

[17] M. Foreman and M. Magidor. Mutually stationary sequences and the non-saturation of the non-stationary ideal on $P_\kappa\lambda$. *Acta Mathematica*, 186(2):271–300, 2001.

[18] M. Foreman, M. Magidor, and S. Shelah. Martin's Maximum, saturated ideals and non-regular ultrafilters I. *Annals of Mathematics*, 127:1–47, 1988.

[19] M. Foreman and S. Todorcevic. A new Lowenheim-Skolem theorem. To appear.

[20] A. Hajnal, I. Juhász, and S. Shelah. Splitting strongly almost disjoint families. *Transactions of the American Mathematical Society*, 295(1):369–387, 1986.

[21] M. Magidor and S. Shelah. When does almost-free imply free? *Journal of the American Mathematical Society*, 7:769–830, 1994.

[22] W. J. Mitchell. $i[\omega_2]$ can be the non-stationary ideal on cofinality $\omega_1$. In preparation.

[23] W. J. Mitchell. Aronszajn trees and the independence of the transfer property. *Annals of Mathematical Logic*, 5:21–46, 1972.

[24] S. Shelah. On successors of singular cardinals. In M. Boffa, D. van Dalen, and K. McAloon, editors, *Logic Colloquium '78*, pages 357–380, Amsterdam, 1979. North-Holland.

[25] S. Shelah. Reflecting stationary sets and successors of singular cardinals. *Archive for Mathematical Logic*, 31(1):25–53, 1991.

[26] S. Shelah. *Cardinal Arithmetic*. Oxford University Press, Oxford, 1994.

[27] R. Solovay. Real-valued measurable cardinals. In *Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967)*, pages 397–428. Amer. Math. Soc., Providence, R.I., 1971.

CMU

U C Irvine

Hebrew University