Large cardinal properties of small cardinals

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1 Introduction

The fact that small cardinals (for example $\aleph_1$ and $\aleph_2$) can consistently have properties similar to those of large cardinals (for example measurable or supercompact cardinals) is a recurring theme in set theory. In these notes I discuss three examples of this phenomenon; stationary reflection, saturated ideals and the tree property.

These notes represent approximately the contents of a series of expository lectures given during the Set Theory meeting at CRM Barcelona in June 1996. None of the results discussed here is due to me unless I say so explicitly.

I would like to express my thanks to Joan Bagaria and Adrian Mathias for organising a very enjoyable meeting.

2 Large cardinals and elementary embeddings

We begin by reviewing the formulation of large cardinal properties in terms of elementary embeddings. See [40], [22] or [21] for more on this topic.

We will write "$j : V \longrightarrow M$" as a shorthand for the rather cumbrous assertion "M is transitive, j and M are classes of V and j is a non-trivial elementary embedding from V to M".

If $j : V \longrightarrow M$ then it is easy to see that j has a *critical point* κ. That is to say $j \restriction \kappa = \mathrm{id}_\kappa$ and $j(\kappa) > \kappa$. It turns out that many large cardinal properties can profitably be formulated in terms of elementary embeddings and their critical points.

The concept of a *measurable cardinal* was first considered by Ulam [42] in connection with problems in measure theory. Scott [35] initiated the study of elementary embedding formulations for large cardinals by proving

Theorem 2.1 (Scott [35]) *The following are equivalent.*

1. *κ is measurable (that is, there exists a normal measure on κ).*
2. *There exists $j : V \longrightarrow M$ such that* $\mathrm{crit}(j) = \kappa$.
3. *There exists $j : V \longrightarrow M$ such that* $\mathrm{crit}(j) = \kappa$ *and* ${}^\kappa M \subseteq M$.

Purists may worry about the quantification over proper classes in the statement "There exists $j : V \longrightarrow M \ldots$". These worries can be addressed either by regarding Theorem 2.1 as a theorem schema or by working in a theory which allows quantification over classes.

Other large cardinal properties can be defined by demanding that the "target model" M of the embedding should have some resemblance to V. Here are three popular large cardinal properties defined in terms of elementary embeddings:

Definition 2.2 *Let κ be a cardinal.*

1. *κ is* λ-strong *iff there exists $j : V \longrightarrow M$ such that* $\mathrm{crit}(j) = \kappa$, $\lambda < j(\kappa)$ *and $V_\lambda \subseteq M$.*

2. *κ is* λ-supercompact *iff there exists $j : V \longrightarrow M$ such that* $\mathrm{crit}(j) = \kappa$, $\lambda < j(\kappa)$ *and ${}^\lambda M \subseteq M$.*

3. *κ is* huge *iff there exists $j : V \longrightarrow M$ such that* $\mathrm{crit}(j) = \kappa$ *and ${}^{j(\kappa)}M \subseteq M$.*

It is worth noting that all of these large cardinal properties have equivalent definitions which do not involve elementary embeddings and just assert the existence of an appropriate set; see [30] for the case of "λ-strong cardinal" and [40] for the cases of "λ-supercompact cardinal" and "huge cardinal".

By Theorem 2.1 only cardinals at least as strong as a measurable cardinal can have this kind of definition as the critical point of $j : V \longrightarrow M$. However weakly compact cardinals can also be defined using a weaker form of embedding, and this will be useful later.

Fact 2.3 (Keisler) *κ is weakly compact iff κ is strongly inaccessible and for every transitive M such that $\kappa \in M$, ${}^{<\kappa}M \subseteq M$, $|M| = \kappa$ and M models* *enough set theory there exists $k : M \longrightarrow N$ an elementary embedding into some transitive set N with* $\mathrm{crit}(k) = \kappa$.

Hauser [19] has given similar formulations of many properties intermediate between weak compactness and measurability.

One advantage of formulating the large cardinal properties of a cardinal κ in terms of elementary embeddings with critical point κ is that it tends to make the "reflection properties" of κ very clear. We illustrate with an example which will be a paradigm for several later arguments.

Fact 2.4 *Let κ be measurable, let $S \subseteq \kappa$ be a stationary set. Then there exists a regular cardinal $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α (we say S* reflects at α*).*

Proof: Fix $j : V \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$. Now $j(S) \cap \kappa = S$, so $S \in M$. The statements "κ is regular" and "S is stationary" are downwards absolute (as they are expressed by Π_1 sentences). $\kappa < j(\kappa)$ because $\kappa = \mathrm{crit}(j)$. Hence

$$M \models \text{“}\kappa \text{ is regular and } j(S) \cap \kappa \text{ is stationary and } \kappa < j(\kappa)\text{”}.$$

By the elementarity of j,

$$V \models \text{“there is regular } \alpha < \kappa \text{ such that } S \cap \alpha \text{ is stationary”}.$$

♦

In fact the assumption that κ is weakly compact would suffice to prove Fact 2.4. The proof is very similar to the one we just gave, the key point being that for every S we may build an appropriate M with $S \in M$ and then apply Fact 2.3.

In what follows we will be concerned with a more general kind of elementary embedding. We'll write "$k : M \longrightarrow N$" to abbreviate "M, N are inner models of ZFC and k is a non-trivial elementary embedding from M to N". In this general setting we are not assuming that k and N are classes of M or even that $N \subseteq M$.

It's worthwhile to bear in mind the following differences between the special case $j : V \longrightarrow M$ and the general case $k : M \longrightarrow N$.

- If $j : V \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$ then
  - κ is measurable.
  - $V_{\kappa+1} \subseteq M$.
  - $V \neq M$.
  - $j \restriction V_\kappa = \mathrm{id}_{V_\kappa}$.
- In the general case there can be $k : M \longrightarrow N$ where (at one extreme) $M = N$, or (at the other extreme) where $\mathrm{crit}(k) = \aleph_1^M$ and $V_{\omega+1}^M \subsetneq V_{\omega+1}^N$.

We'll be particularly interested in the case of embeddings $j : V \longrightarrow M \subseteq V[G]$ where j, M are defined in $V[G]$, a generic extension of V. These are usually known as *generic elementary embeddings;* Foreman initiated the detailed study of generic embeddings and their applications in [9].

It will be convenient for us to assume that V-generic filters exist; it is possible to eliminate this assumption, using any of the standard methods. Our forcing terminology follows that of [25] for the most part. We write $Add(\kappa, \lambda)$ for the Cohen conditions to add λ subsets of κ, $Coll(\kappa, \lambda)$ for the Lévy conditions to collapse λ to have cardinality κ, and $Coll(\kappa, < \lambda)$ for the Lévy conditions to collapse each ordinal less than λ to have cardinality κ.

To build generic embeddings we will use the following basic result of Silver.

Fact 2.5 (Silver) *Let $k : M \longrightarrow N$, let $\mathbb{P} \in M$ be a forcing poset. Suppose that G is $\mathbb{P}$-generic over M, H is $k(\mathbb{P})$-generic over N and $k``G \subseteq H$. Then there exists a unique $k^* : M[G] \longrightarrow N[H]$ such that $k^* \restriction M = k$ and $k^*(G) = H$.*

Proof: If such a k^* exists then it must map τ^G to $k(\tau)^H$ for each $\mathbb{P}$-term $\tau \in M$. We need to check that this gives a well-defined elementary embedding.

Suppose that $\tau^G = \sigma^G$. Then there is $p \in G$ such that $p \Vdash^M_{\mathbb{P}} \sigma = \tau$, so by elementarity $k(p) \Vdash^N_{k(\mathbb{P})} k(\sigma) = k(\tau)$. Now $k(p) \in k``G \subseteq H$, so that $k(\sigma)^H = k(\tau)^H$ and we have proved that k^* is well-defined. The proof of elementarity is very similar.

♦

We list some ways to arrange that $k``G \subseteq H$ will hold. Fix $k : M \longrightarrow N$

and $\mathbb{P} \in M$.

1. If $\mathbb{P} \subseteq k(\mathbb{P})$, $k \restriction \mathbb{P} = \mathrm{id}_{\mathbb{P}}$, and $G = H \cap \mathbb{P}$ then clearly $k``G \subseteq H$.

2. Suppose $M \models$ "$\mathbb{P}$ is $< \lambda$-distributive" and $N = \{ k(F)(a_F) \mid F \in \mathcal{F} \}$ where $\mathcal{F} \subseteq M$ is a family of functions such that $\forall F \in \mathcal{F}$ $M \models |\mathrm{dom}(F)| < \lambda$.

   Then we claim that $k``G$ generates a filter H which is $k(\mathbb{P})$-generic over N. To see this let $D \in N$ be dense in $k(\mathbb{P})$, then $D = k(F)(a_F)$ where without loss of generality $F(x)$ is dense in $\mathbb{P}$ for each $x \in \mathrm{dom}(F)$. By distributivity $E = \bigcap_{x \in \mathrm{dom}(F)} F(x)$ is dense in $\mathbb{P}$, and of course $E \in M$, so let $p \in G \cap E$; then by elementarity $k(p) \in k(F)(a_F) = D$ and so $k``G \cap D \neq \emptyset$.

3. If $q \in k(\mathbb{P})$ and $\forall p \in G$ $q \leq k(p)$ then any $k(\mathbb{P})$-generic filter H such that $H \ni q$ will also have the property that $H \supseteq k``G$. Silver dubbed such a condition a *master condition.*

3 Stationary reflection

Recall that in the last section we proved that every stationary subset of a measurable (or even weakly compact) cardinal reflects. We now consider the possibilities for this kind of phenomenon in small cardinals like $\aleph_2$ and $\aleph_{\omega+1}$.

We introduce some convenient terminology for describing stationary sets.

Definition 3.1 $S^\lambda_\mu = \{ \alpha < \lambda \mid \mathrm{cf}(\alpha) = \mu \}$. $T^m_n = \{ \alpha < \aleph_m \mid \mathrm{cf}(\alpha) = \aleph_n \}$.

It is easy to see that full stationary reflection cannot hold at the successor of a regular cardinal λ. In fact $S^{\lambda^+}_\lambda$ is a stationary subset of λ^+, but if $\alpha < \lambda^+$ then $\mathrm{cf}(\alpha) \leq \lambda$ so we can choose C club in α such that $C \cap S^{\lambda^+}_\lambda = \emptyset$.

On the other hand it is consistent that stationary subsets of T^2_0 should all reflect. More precisely Baumgartner [2] proved the following

Theorem 3.2 (Baumgartner [2]) *If κ is weakly compact and G is generic over V for the Lévy collapse $Coll(\aleph_1, < \kappa)$ then*

$$V[G] \models \text{"If } S \subseteq T^2_0 \text{ is stationary, there is } \alpha \in T^2_1 \text{ with } S \cap \alpha \text{ stationary".}$$

Proof: For simplicity we assume that κ is measurable (and will indicate at the end of the proof how to weaken the assumption to weak compactness).

Fix $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$ and ${}^\kappa M \subseteq M$. Let $\mathbb{P} = Coll(\aleph_1, < \kappa)$. Then by the closure of M we have $j(\mathbb{P}) = Coll(\aleph_1, < j(\kappa))$, so in the natural way $j(\mathbb{P}) \simeq \mathbb{P} \times \mathbb{Q}$ where $\mathbb{Q} = Coll(\aleph_1, [\kappa, j(\kappa)))$. If G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over $V[G]$ then $G * H$ is $j(\mathbb{P})$-generic over V and so *a fortiori* is $j(\mathbb{P})$-generic over M.

What is more, for every $p \in G$ we have $j(p) = p \in G * H$, because $G \subseteq \mathbb{P} \subseteq V_\kappa$ and $j \restriction V_\kappa = \mathrm{id}$. It follows from Fact 2.5 that we may lift j to a new embedding

$$j : V[G] \longrightarrow M[G][H] \subseteq V[G][H].$$

Here we have denoted the new embedding by j also. There is no possibility of confusion because the new j extends the old one.

Notice that this embedding and its target are defined in $V[G][H]$, a generic extension of $V[G]$. This is our first example of the notion of *generic embedding* defined in the last section. Notice also that $\aleph_1 = \aleph_1^{V[G]}$ and $\kappa = \mathrm{crit}(j) = \aleph_2^{V[G]}$, while $j(\kappa) = \aleph_2^{M[G][H]}$.

Let $V[G] \models$ "S is a stationary subset of T^2_0". It is easy to see that the canonical name for S is a member of $V_{\kappa+1}$, and since $V_{\kappa+1} \subseteq M$ it follows that $S \in M[G]$. Since $M[G] \subseteq V[G]$ and stationarity is downwards absolute, $M[G] \models$ "S is stationary".

We also have that $j(S) \in M[G][H]$, and since $\mathrm{crit}(j) = \kappa$ it follows as in Fact 2.4 that $j(S) \cap \kappa = S$. What is more, it follows from the countable closure of $\mathbb{Q}$ in $M[G]$ that $M[G][H] \models \mathrm{cf}(\kappa) = \aleph_1$.

However we are still missing one thing; we need to know that S is stationary in $M[G][H]$ before we can complete the reflection argument using the elementarity of $j : V[G] \longrightarrow M[G][H]$. This problem is a very common one in arguments involving generic elementary embeddings, for example we will find ourselves in an exactly similar situation in the discussion of the tree property in section 5 of this survey.

To finish the argument we use the following fact (really a special case of the fact that countably closed forcings are proper).

Fact 3.3 *Let S be a stationary subset of S^κ_ω, where $\kappa = \mathrm{cf}(\kappa) > \omega$. Let $\mathbb{P}$ be countably closed. Then $\Vdash_{\mathbb{P}}$ "S is stationary".*

Proof: Let $p \in \mathbb{P}$ be any condition and suppose that $p \Vdash$ "$\dot{C}$ is club". Let θ

be some very large regular cardinal and let $<_\theta$ be a well ordering of H_θ. Find $N \prec (H_\theta, \in, <_\theta)$ such that $p, S, \kappa, \mathbb{P}, \dot{C} \in N$ and $\delta = N \cap \kappa \in S$. Now choose an increasing sequence $\langle \delta_i : i < \omega \rangle$ of elements of $N \cap \kappa$ which is cofinal in δ, and define $\langle p_i : i < \omega \rangle$ a decreasing sequence from $\mathbb{P} \cap N$ as follows; $p_0 = p$, and p_{i+1} is the $<_\theta$-least condition such that $p_{i+1} \leq p_i$ and p_{i+1} forces some ordinal larger than δ_i into $\dot{C}$.

Because p_{i+1} is defined from the parameters $\delta_i, p_i, \dot{C}, \mathbb{P}$, we can see (inductively) that each $p_i \in N$. If β_i is the least ordinal greater than δ_i which p_{i+1} forces into $\dot{C}$ then by a similar argument $\beta_i \in N$, so in fact $\beta_i \in N \cap \kappa = \delta$ and so $p_{i+1} \Vdash \dot{C} \cap (\delta_i, \delta) \neq \emptyset$.

Now use the countable closure of $\mathbb{P}$ to find p_ω such that $p_\omega \leq p_i$ for all $i < \omega$. Clearly $p_\omega \Vdash \delta \in \lim(\dot{C})$, so we have produced a refinement of p which forces a member of S into $\dot{C}$. It follows that $\Vdash_{\mathbb{P}}$ "S is stationary".

♦

Using this fact we can conclude that $M[G][H] \models$ "S is stationary", and then we can argue exactly as in Fact 2.4 that by elementarity

$$V[G] \models \text{"there is } \alpha \in T^2_1 \text{ such that } S \cap \alpha \text{ is stationary"}.$$

We promised at the start to show how the argument works from the weaker assumption that κ is weakly compact. To do this, suppose that $\Vdash_{\mathbb{P}}$ "$\dot{S}$ is a non-reflecting stationary subset of T^2_0" for some canonical name $\dot{S}$. Since $\dot{S} \in H_{\kappa^+}$ we may find M a model of enough set theory such that $|M| = \kappa$, ${}^{<\kappa}M \subseteq M$, $\dot{S} \in M$. We may also assume that for every $\alpha \in S^\kappa_{>\omega}$

the model M contains a $\mathbb{P}$-name for a club in α disjoint from $\dot{S}$.

Now by the weak compactness of κ there is $k : M \longrightarrow N$ with $\mathrm{crit}(k) = \kappa$. If G is $\mathbb{P}$-generic over V and H is $k(\mathbb{P})$-generic over V then in $V[G][H]$ we get an embedding $k : M[G] \longrightarrow N[G][H]$ by the same arguments as above. Let $S = \dot{S}^G$, then $V[G] \models$ "S is stationary". By Fact 3.3 we then get $V[G][H] \models$ "S is stationary" and so *a fortiori* $N[G][H] \models$ "S is stationary".

Now we can argue as before that $M[G] \models \exists \alpha \in T^2_1 \ S \cap \alpha$ is stationary. This is a contradiction, because we built names for clubs disjoint from $\dot{S} \cap \alpha$ into M for every $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \omega$.

♦

Notice that there is a problem with generalising the proof of Fact 3.3 to larger cofinalities, for example it is not obvious that $\aleph_2$-closed forcing will always preserve the stationarity of stationary subsets of $S^\kappa_{\aleph_1}$. The problem is that when we build the chain of conditions $\vec{p}$ we may wander out of the structure N at limit stages. A crude solution to this problem is to make a cardinal arithmetic assumption and then work with suitably closed substructures.

Fact 3.4 *If $\kappa = \lambda^+$ and $\lambda^{<\mu} = \lambda$ then every μ^+-closed forcing preserves the stationarity of stationary subsets of S^κ_μ.*

Proof:[Sketch] Build N containing everything relevant such that $|N| = \lambda$, ${}^{<\mu}\lambda = \lambda$ and $N \cap \kappa \in \kappa$. Then build a decreasing μ-sequence from $\mathbb{P} \cap N$ as in the proof of Fact 3.3; the closure of N makes the construction go through.

♦

It follows that Baumgartner's theorem generalises to any successor of a regular cardinal.

Theorem 3.5 (Baumgartner) *Let $\lambda = \mathrm{cf}(\lambda) < \kappa$ where κ is weakly compact. If G is $Coll(\lambda, < \kappa)$-generic then*

$V[G] \models$ *"If $S \subseteq S^{\lambda^+}_{<\lambda}$ is stationary, there is $\alpha \in S^{\lambda^+}_{\lambda}$ with $S \cap \alpha$ stationary".*

Proof:[Sketch] In $V[G]$ we have $\kappa = \lambda^+$ and $\lambda^{<\lambda} = \lambda$. Thus we can mimic the proof of Theorem 3.2, using Fact 3.4 in place of Fact 3.3.

♦

Notice that we are immediately in difficulties if we try to generalise these results to successors of singular cardinals; one problem is that there is no obvious analogue of the Lévy collapse to make a large cardinal become the successor of a singular cardinal, another is that since $\aleph_\omega^{\aleph_0} > \aleph_\omega$ the trick of working with closed substructures will no longer work. As we see shortly, the problem is not merely a technical one.

One subtle point is worth mentioning here. Inspection of the proof of Theorem 3.2 shows that actually the Lévy collapse of a weak compact to $\aleph_2$ gives a model in which any $\aleph_1$-sequence $\langle S_i : i < \aleph_1 \rangle$ of stationary subsets of T^2_0 reflect simultaneously (that is there is $\beta \in T^2_1$ such that $S_i \cap \beta$ is stationary for every i).

Jensen [20] proved that "every stationary subset of T^2_0 reflects to a point of T^2_1" requires a Mahlo cardinal, Magidor [28] showed that "every pair of stationary subsets of T^2_0 reflect simultaneously to a point of T^2_1" needs a weakly compact cardinal, and Harrington and Shelah [18] showed that consistency of "every stationary subset of T^2_0 reflects to a point of T^2_1" follows from that of a Mahlo cardinal.

It is also possible to show that instances of stationary reflection for different cofinalities are highly independent. In [5] models are constructed in which every stationary subset of T^3_0 reflects and every stationary subset of T^3_1 has a non-reflecting stationary subset, and *vice versa*.

The problem of stationary reflection has a much different flavour at successors of singular cardinals. Here it is possible for every stationary set to reflect, but this has a much higher consistency strength than that of a weak compact cardinal.

Fact 3.6 *If $\kappa < \mu = \mathrm{cf}(\mu)$ and κ is μ-supercompact then for every stationary $S \subseteq S^{\mu}_{<\kappa}$ there is $\alpha \in S^{\mu}_{<\kappa}$ such that $S \cap \alpha$ is stationary.*

Proof: Fix $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \mu$ and ${}^{\mu}M \subseteq M$. Let $\gamma = \sup(j``\mu)$, then $j``\mu \in M$ and thus $M \models \mathrm{cf}(\gamma) = \mu$. Since $\mu < j(\kappa) < j(\mu)$ and $j(\mu)$ is regular in M, $\gamma < j(\mu)$. We claim that $M \models$ "$j(S) \cap \gamma$ is stationary".

Let $C \subseteq \gamma$ be club in γ. Let $D = \{ \alpha < \mu \mid j(\alpha) \in C \}$, then it is routine to check that D is $< \kappa$-club in μ. Since $S \subseteq S^{\mu}_{<\kappa}$, this implies that $D \cap S \neq \emptyset$, and if $\alpha \in D \cap S$ then $j(\alpha) \in C \cap j``S \subseteq C \cap (j(S) \cap \gamma)$.

Since $M \models$ "$\mathrm{cf}(\gamma) < j(\kappa)$ and $j(S) \cap \gamma$ is stationary" it follows by elementarity that there is $\alpha \in S^{\mu}_{<\kappa}$ such that $S \cap \alpha$ is stationary.

♦

Actually, κ being μ-strongly compact would suffice here. As long as we are only concerned with stationary sets of cofinality ω ordinals the following result of Shelah [2] says that we can have reflection everywhere.

Fact 3.7 *Let κ be supercompact. Let $\mathbb{P} = Coll(\aleph_1, < \kappa)$. In $V^{\mathbb{P}}$, for every $\mu = \mathrm{cf}(\mu) \geq \kappa$ and every stationary subset of S^{μ}_{ω} there is $\alpha < \mu$ such that $\mathrm{cf}(\alpha) = \aleph_1$ and $S \cap \mu$ is stationary.*

The proof combines the ideas of Theorem 3.2 and Fact 3.6.

Fact 3.8 *Let λ be a singular limit of λ^+-supercompact cardinals, then every stationary subset of λ^+ reflects.*

Proof: λ is singular, so $\forall \alpha < \lambda^+ \ \mathrm{cf}(\alpha) < \lambda$. If $S \subseteq \lambda^+$ is stationary then there must be a λ^+-supercompact $\kappa < \lambda$ such that $S \cap S^{\lambda^+}_{<\kappa}$ is stationary, and then there is $\alpha \in S^{\lambda^+}_{<\kappa}$ such that $S \cap \alpha$ is stationary.

♦

It is natural to ask whether we need such strong hypotheses to get stationary reflection at the successor of a singular cardinal. The exact strength needed is still not known, but we will see that it must be considerable.

Jensen [20] introduced the combinatorial principle $\square_\lambda$: it states that there exists $\langle C_\alpha : \alpha < \lambda^+ \rangle$ such that C_α is club in λ, o.t.$(C_\alpha) \leq \lambda$ and $\beta \in \lim(C_\alpha) \implies C_\beta = C_\alpha \cap \beta$. The connection between $\square_\lambda$ and stationary reflection is the following useful fact.

Fact 3.9 *If $\square_\lambda$ holds and $S \subseteq \lambda^+$ is a stationary set, then there exists a stationary $T \subseteq \lambda^+$ such that $T \cap \alpha$ is nonstationary for every $\alpha < \lambda^+$.*

Proof: Find $T \subseteq S$ and β such that T is stationary and $\forall \alpha \in T$ o.t.$(C_\alpha) = \beta$. Now if $\alpha < \lambda^+$ and $\mathrm{cf}(\alpha) > \omega$, then $\lim(C_\alpha)$ is club in α and $\gamma \in \lim(C_\alpha) \implies C_\gamma = C_\alpha \cap \gamma$. It follows that $\gamma \longmapsto$ o.t.(C_γ) is 1-1 on $\lim(C_\alpha)$, hence $|\lim(C_\alpha) \cap T| \leq 1$ and so T is nonstationary in α.

♦

Jensen [6] proved that

- For all λ, $L \models \square_\lambda$.
- If $0^\sharp$ does not exist then for every singular λ we have $\lambda^+_V = \lambda^+_L$, from which it follows that $V \models \square_\lambda$ for every singular λ.

Combining these results, it follows that to have a singular cardinal such that every stationary subset of the successor reflects will require at least the strength of $0^\sharp$.

This argument has been greatly generalised by various workers in inner model theory. Combining fairly recent results of Mitchell, Schimmerling, Steel and Woodin we get

Fact 3.10 *If λ is singular and every stationary subset of λ^+ reflects then Projective Determinacy holds. In particular for every n there is an inner model of "ZFC + there exist n Woodin cardinals".*

Another natural question is whether small successors of singulars such as $\aleph_{\omega+1}$ can exhibit the phenomenon of stationary reflection. This question is answered by the following result from [28] Magidor originally had a more complex construction which involved more forcing, Shelah pointed out that the last step of Magidor's original construction was not necessary.

Theorem 3.11 (Magidor [28]) *Assume that $\langle \kappa_n : n < \omega \rangle$ be an increasing ω-sequence of supercompact cardinals. Define a forcing iteration: $\mathbb{P}_1 = Coll(\omega, < \kappa_0)$, $\mathbb{P}_{n+1} = \mathbb{P}_n * Coll(\kappa_{n-1}, < \kappa_n)_{V^{\mathbb{P}_n}}$ for $1 \leq n < \omega$, $\mathbb{P}_\omega$ is the inverse limit of the $\mathbb{P}_n$. Then*

$V^{\mathbb{P}_\omega} \models$ "If $S \subseteq \aleph_{\omega+1}$ is stationary, there is $\alpha < \aleph_{\omega+1}$ with $S \cap \alpha$ stationary".

We will sketch the proof and refer the reader who wants more details to [28].

Proof:[Sketch] Let $\lambda = \sup_n \kappa_n$; it is not hard to see that in $V[G_\omega]$ the cardinal κ_n becomes $\aleph_{n+1}$, λ becomes $\aleph_\omega$ and λ^+ becomes $\aleph_{\omega+1}$.

Let G_ω be $\mathbb{P}_\omega$-generic over V. For each n there is a generic extension $V[G_\omega][H_n]$ of $V[G_\omega]$ such that

- There is $k_n : V[G_\omega] \longrightarrow M_n \subseteq V[G_\omega][H_n]$ a generic embedding with critical point κ_n.

- H_n is generic for $\aleph_n$-closed forcing.

- $k_n \restriction \lambda^+ \in M$, $k_n(\kappa_n) > \lambda^+$.

In Foreman's terminology from [9], κ_n is *generically supercompact.* k_n is actually an extension of an embedding $j_n : V \longrightarrow M$ witnessing that κ_n is λ^+-supercompact.

To complete the argument in the style of Theorem 3.2 and Fact 3.8 we need to argue that in $V[G_\omega]$ the stationarity of a stationary subset of $T^{\omega+1}_{<k}$ is preserved by $\aleph_k$-closed forcing. This is false in general by results of Shelah [38], but fortunately it is true in $V[G_\omega]$. To see this we introduce Shelah's notion of an *approachable* set.

Definition 3.12 (Shelah [38]) *Let S be a subset of μ where $\mu = \mathrm{cf}(\mu) > \omega$. Then S is* approachable *iff there exists $\langle x_\alpha : \alpha < \mu \rangle$ and a closed unbounded set $C \subseteq \mu$ such that for every $\alpha \in S \cap C$ there is $c \subseteq \alpha$ club in α such that o.t.(c) $= \mathrm{cf}(\alpha)$ and $\forall \beta < \alpha\, \exists \gamma < \alpha\; c \cap \beta = x_\gamma$.*

Fact 3.13 (Shelah) *Let $\gamma = \mathrm{cf}(\gamma) < \mu = \mathrm{cf}(\mu)$. If $S \subseteq S^\mu_{<\gamma}$ is an approachable stationary set and $\mathbb{P}$ is γ-closed then S is still stationary in $V^{\mathbb{P}}$.*

Proof: Let $\vec{x}$ and C witness that S is approachable. Let p be any condition in $\mathbb{P}$, and suppose $p \Vdash$ "$\dot{D}$ is club in μ". Let $N \prec (H_\theta, \in, <_\theta)$ be a structure which contains everything relevant, with the property that $\alpha = N \cap \mu \in C \cap S$; fix $c \subseteq \alpha$ such that $\forall \beta < \alpha\, \exists \gamma < \alpha\; c \cap \beta = x_\gamma$.

The key point is that because $\vec{x} \in N$ and $\alpha \subseteq N$, we have $c \cap \beta \in N$ for all $\beta < \alpha$. Now we build a descending chain of conditions $\langle p_i : i < \mathrm{cf}(\alpha) \rangle$ such that $p_0 = p$ and p_j is the $<_\theta$-least condition such that

- $p_j \leq p_i$ for all $i < j$.
- p_j forces some ordinal greater than the j^{th} element of c into $\dot{D}$.

If $j < \mathrm{cf}(\alpha)$ then N can compute $\langle p_i : i < j \rangle$ from $c \cap \beta$ for β the j^{th} element of c, so $\langle p_i : i < j \rangle \in N$ and thus the sequence $\vec{p}$ never wanders out of N.

The proof now concludes exactly as the proof of Fact 3.3 does.

♦

Shelah observed that $\aleph_{\omega+1}$ is approachable in the model $V[G_\omega]$. Given this, we can finish the proof of the result as follows.

Let $V[G_\omega] \models$ "$S \subseteq \aleph_{\omega+1}$ is stationary". Then $S \cap T^{\omega+1}_{<n}$ is stationary for some $n < \omega$. Forcing with some $\aleph_n$-closed forcing we get a generic embedding $k_n : V[G_\omega] \longrightarrow M_n \subseteq V[G_\omega][H_n]$ such that $\mathrm{crit}(k_n) = \kappa_n$, $k_n \restriction \lambda^+ \in M_n$ and $k_n(\kappa_n) > \lambda^+$.

If we now let $\gamma = \sup k_n``\lambda^+$ then $\gamma < k_n(\lambda^+)$ and $M \models \mathrm{cf}(\gamma) = \aleph_n$. S is stationary in $V[G_\omega][H]$ because $\aleph_{\omega+1}$ is approachable in $V[G_\omega]$; it follows that $j(S) \cap \gamma$ is stationary in M_n and we can finish the argument exactly as in the proof of 3.3.

♦

It is worth noticing that $\square_\lambda$ implies that λ^+ is approachable. For more on the connections between squares and approachability see [13] and [4].

An important topic not touched on here is that of stationary reflection in $[X]^{\aleph_0}$, where X is an uncountable set and $[X]^{\aleph_0}$ is the set of its countable subsets. See [14] and [12] for more on this.

4 Saturated ideals

Suppose that $\kappa = \mathrm{cf}(\kappa) > \omega$. By an *ideal on* κ we always mean an ideal which is κ-complete, normal and uniform.

Definition 4.1 *Let I be an ideal on κ. Then I is* saturated *iff the Boolean algebra $\mathcal{P}\kappa/I$ has the κ^+-c.c.*

Saturated ideals are closely connected with generic elementary embeddings; the basic results are due to Solovay [39] and Kunen [23].

We start by outlining Solovay's analysis of a saturated ideal.

If I is any ideal then forcing with $\mathcal{P}\kappa/I$ adds U an ultrafilter on $\mathcal{P}\kappa \cap V$, with the property that $U \cap I = \emptyset$. The idea is to take an ultrapower of V by U, in essentially the same way that Scott [35] took an ultrapower of V by a measure on a measurable cardinal.

If f, g are two functions in V with domain κ, then we define $f \simeq_U g \iff \{ \alpha \mid f(\alpha) = g(\alpha) \} \in U$; this is an equivalence relation. Working in $V[U]$ we define V^κ/U to be the set of equivalence classes and also define $[f] E_U [g] \iff \{ \alpha \mid f(\alpha) \in g(\alpha) \} \in U$. The structure $(V^\kappa/U, E_U)$ is called the *generic ultrapower* of V by U, and the standard proof of Los' theorem shows that for any formula ϕ in the language of set theory

$$(V^\kappa/U, E_U) \models \phi([f_1]_U, \dots [f_n]_U) \iff \{ \alpha \mid \phi(f_1(\alpha), \dots f_n(\alpha)) \} \in U.$$

We get an elementary embedding $j : (V, \in) \longrightarrow (V^\kappa/U, E_U)$ by defining $j(x)$ to be the class of the constant function with value x.

For a general ideal I there is no guarantee that the structure $(V^\kappa/U, E_U)$ is well-founded. However if I is saturated then Solovay proved this will be the case, using the following key fact.

Fact 4.2 (Solovay) *If $\Vdash_{\mathcal{P}\kappa/I} \dot{f} \in V, \mathrm{dom}(\dot{f}) = \kappa$ then there is $g \in V$ such that $\Vdash \dot{f} \simeq_U g$.*

Using this it is possible to show

Fact 4.3 (Solovay) *Let I be a saturated ideal on κ. Let U be an ultrafilter added by forcing with $\mathcal{P}\kappa/I$. Then*

- *$(V^\kappa/U, E_U)$ is well-founded, so can be identified with its Mostowski collapse to give a generic embedding $j : V \longrightarrow M \subseteq V[U]$, where $M \simeq V^\kappa/U$.*
- $\mathrm{crit}(j) = \kappa$.
- *$V[U] \models {}^\kappa M \subseteq M$.*

In particular, if $\kappa = \aleph_1$ then we get an embedding such that $j(\aleph_1) = \aleph_2 = \aleph_1^M = \aleph_1^{V[U]}$, where $V[U] \models {}^\omega M \subseteq M$. Notice that here $V[U] \models {}^{<j(\aleph_1)} M \subseteq M$ (one might say that $\aleph_1$ is *generically almost huge*, see Definition 4.4).

Kunen showed [23] that it is possible to go in the other direction, and deduce the existence of a saturated ideal from that of an appropriate generic embedding. In particular he gave the first consistency proof for the existence of a saturated ideal on $\aleph_1$, starting from the consistency of a huge cardinal. Magidor [27] showed that Kunen's argument can be made to work from an "almost huge" cardinal, and we will outline this version.

Definition 4.4 *κ is* almost huge *iff there exists $j : V \longrightarrow M$ such that* $\mathrm{crit}(j) = \kappa$ *and* $V \models {}^{<j(\kappa)}M \subseteq M$.

Let κ be almost huge and fix $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$, ${}^{<\lambda}M \subseteq M$.

- We start by collapsing κ to $\aleph_1$ and λ to $\aleph_2$, to get a new model V_1 in which $2^{\aleph_0} = \aleph_1 = \kappa$ and $2^{\aleph_1} = \aleph_2 = \lambda$.
- In V_1 there is a 2-step forcing iteration $\mathbb{P} * \dot{\mathbb{Q}}$ such that
  - If $G * H$ is $\mathbb{P} * \mathbb{Q}$-generic then there is an extension of the original j,
  $$j : V_1 \longrightarrow M_1 \subseteq V_1[G][H].$$
  - $\mathbb{P}$ is $\aleph_2$-c.c. and $\lambda = \aleph_2^{V_1} = \aleph_1^{V_1^{\mathbb{P}}}$.
  - $\mathbb{Q}$ is $\aleph_1$-closed in $V_1^{\mathbb{P}}$.
- Using the closure of $\mathbb{Q}$ in $V_1^{\mathbb{P}}$, it is possible to show that in $V_1^{\mathbb{P}}$ we can define an ultrafilter on $\mathcal{P}\kappa \cap V_1$. The key points are that $V_1 \models 2^{\aleph_1} = \aleph_2$,

$V_1^{\mathbb{P}} \models |\mathcal{P}\kappa \cap V_1| = \aleph_1$, and $V^{\mathbb{P}} \models$ "$\mathbb{Q}$ is countably closed". Using this we can work in $V_1^{\mathbb{P}}$ and build a decreasing chain of conditions to decide "$\kappa \in j(X)$" for each $X \in \mathcal{P}\kappa \cap V_1$. Let $\dot{U}$ name this ultrafilter.

- Working in V_1, let $I = \{ X \subseteq \kappa \mid \Vdash_{\mathbb{P}} X \notin \dot{U} \}$. Using the $\aleph_2$-c.c. of $\mathbb{P}$ in V_1, it is possible to show that

$$V_1 \models \text{“}I \text{ is a saturated ideal on } \aleph_1\text{”}.$$

This style of argument will serve to get saturated ideals on many cardinals. The culmination of this line of development is Foreman's paper [8] in which it is proved to be consistent that every regular cardinal should carry a saturated ideal. Foreman and Laver [11] showed that is also possible to get stronger forms of chain condition for the quotient algebra.

However some questions were left open: for example

- How strong is the existence of a saturated ideal on $\aleph_1$?
- Can the non-stationary ideal on $\aleph_1$ be saturated?

For a time it was conjectured that an almost-huge cardinal was the right assumption to get a saturated ideal. Foreman, Magidor and Shelah's work [14] on the forcing axiom MM (Martin's Maximum) showed that this is not the case; in [14] it is shown (among other things) that

- Con(ZFC + there exists a supercompact cardinal) implies Con(ZFC + MM).

- MM implies that the nonstationary ideal on $\aleph_1$ is saturated.

The existence of an almost huge cardinal is known to be a much stronger assumption than the existence of a supercompact.

The question of the strength of a saturated ideal on $\aleph_1$ is now almost settled, in the light of the following results.

Fact 4.5 (Steel [41]) *If the nonstationary ideal on $\aleph_1$ is saturated and there exists a measurable cardinal then there is an inner model of "ZFC + there exists a Woodin cardinal".*

The assumption of the existence of a measurable cardinal is a technical device here. It is conjectured that the saturation of the nonstationary ideal should suffice.

Fact 4.6 (Shelah [37]) *If δ is Woodin then there is a forcing extension in which $\aleph_1$ is preserved, δ becomes $\aleph_1$, and the nonstationary ideal on $\aleph_1$ is saturated.*

We outline the proof of Shelah's result. We require Shelah's concepts of *semiproperness* and *revised countable support iteration,* for which we refer the reader to Goldstern's paper [17] in this volume.

Definition 4.7 *Let $\mathcal{A}$ be a maximal antichain in $\mathcal{P}\aleph_1/NS$, where NS is the nonstationary ideal on $\aleph_1$. Then we define a poset $\mathbb{S}(\mathcal{A})$ as follows: $(f, c) \in \mathbb{S}(\mathcal{A})$ iff*

- $\mathrm{dom}(f) = \max(c) < \aleph_1$.

- $\mathrm{rge}(f) \subseteq \mathcal{A}$.

- *c is closed.*

- $\forall \beta \in c\, \exists \alpha < \beta\ \beta \in f(\alpha)$.

The ordering is extension.

$\mathbb{S}(\mathcal{A})$ is defined in [14], and is used there to show that MM implies the saturation of NS. For any $\mathcal{A}$ it will be the case that $\mathbb{S}(\mathcal{A})$ is stationary preserving. $\mathbb{S}(\mathcal{A})$ makes $|\mathcal{A}| = \aleph_1$ and shoots a club through the diagonal union of $\mathcal{A}$, from which it follows that $\mathcal{A}$ will be a maximal antichain of size $\aleph_1$ in any extension of $V^{\mathbb{S}(\mathcal{A})}$ by stationary preserving forcing.

Definition 4.8 *$\mathcal{A}$ is a* semiproper antichain *iff $\mathbb{S}(\mathcal{A})$ is semiproper.*

We can now outline Shelah's argument; essentially the idea is to force exactly that fragment of MM which is needed to get the saturation of NS. We start by assuming that δ is a Woodin cardinal.

- The construction is a revised countable support iteration of length δ, where at stage α we force with $\mathbb{S}(\mathcal{A}_\alpha) * Coll(\aleph_1, 2^{2^\alpha})$ for some $\mathcal{A}_\alpha$ such that $V^{\mathbb{P}_\alpha} \models$ "$\mathcal{A}_\alpha$ is semiproper". The $\mathcal{A}_\alpha$ are chosen using some kind of diamond principle.

- At the end of the construction we have a semiproper forcing $\mathbb{P}_\delta$, which will preserve $\aleph_1$ and make δ into $\aleph_2$. $V^{\mathbb{P}_\delta} \models 2^{\aleph_1} = \aleph_2$. We need to check that the nonstationary ideal is saturated.

- Let $V^{\mathbb{P}_\delta} \models$ "$\langle A_\alpha : \alpha < \delta \rangle$ is an antichain in $\mathcal{P}\aleph_1/NS$". Applying the Woodin-ness of δ and the diamond principle used in defining the iteration we find $\kappa < \delta$ such that

$$V^{\mathbb{P}_\kappa} \models \text{“}\langle A_\alpha : \alpha < \kappa \rangle \text{ is a semiproper antichain”}$$

  and $\mathcal{A}_\kappa = \langle A_\alpha : \alpha < \kappa \rangle$. At stage κ the antichain $\langle A_\alpha : \alpha < \kappa \rangle$ is made maximal; it follows that every antichain in $\mathcal{P}\aleph_1/NS$ has size at worst $\aleph_1$.

At the heart of the argument lies the idea of a structure "catching an antichain" which comes from Foreman, Magidor and Shelah's work in [14]. Let $\mathcal{A} \in N \prec H_\theta$, where $\mathcal{A}$ is a maximal antichain in $\mathcal{P}\aleph_1/NS$ and N is countable. We say that $M \supseteq N$ "catches $\mathcal{A}$" iff

- $M \cap \aleph_1 = N \cap \aleph_1$ ($= \delta$ say).

- There is $A \in \mathcal{A} \cap M$ such that $\delta \in A$.

Assume that M catches $\mathcal{A}$, and that $A \in \mathcal{A}$ is such that $\delta \in A$. Suppose that we have some condition $(p, c) \in N \cap \mathbb{S}(\mathcal{A})$; then if $\epsilon = \mathrm{dom}(p)$ we have $\epsilon < \delta$. Working in the standard way we can build a decreasing ω-chain of conditions in $\mathbb{S}(\mathcal{A}) \cap M$ which meets every maximal antichain of $\mathbb{S}(\mathcal{A})$ lying in M; this sequence will have a lower bound because

- $\delta \in A$.
- $A \in M$ so that A gets enumerated before δ by the first entry of some condition in the chain.

The lower bound will be a weakly $(N, \mathbb{S}(\mathcal{A}))$-generic condition because $N \cap \omega_1 = M \cap \omega_1$.

It is worth remarking that the combinatorics of this argument resurfaces in Woodin's theory [43] of the stationary tower forcing.

We have only scratched the surface of the subject of saturated ideals here. We conclude by listing some of the other important results in the field.

Fact 4.9 (Woodin [44]) *If the non-stationary ideal on $\aleph_1$ is saturated and there is a measurable cardinal then $\delta^1_2 = \aleph_2$ (this is a strong form of the negation of the Continuum Hypothesis).*

Fact 4.10 (Shelah [36]) *If I is a saturated ideal on λ^+ then (as a corollary of a general result on changes of cofinality) $\{ \alpha < \lambda^+ \mid \mathrm{cf}(\alpha) \neq \mathrm{cf}(\lambda) \} \in I$.*

Fact 4.11 (Gitik and Shelah [16]) *If κ is weakly inaccessible then the non-stationary ideal on κ is not saturated. If κ is singular then the non-stationary ideal on κ^+ restricted to $\{ \alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \mathrm{cf}(\kappa) \}$ is not saturated.*

Fact 4.12 (Woodin [44]) *The following are equiconsistent*

1. *AD*

2. *There exists an* $\aleph_1$*-dense ideal.*

3. *The non-stationary ideal on* $\aleph_1$ *is* $\aleph_1$*-dense.*

Fact 4.13 (Foreman [10]) *From large enough cardinals it is consistent that there exist a* $\aleph_1$*-dense countably closed weakly normal ideal on* $\aleph_2$*.*

5 The tree property

We recall a few basic definitions about trees (see [26] for more details).

Let T be a tree, let κ be a regular cardinal.

1. T is a *κ-tree* iff $|T| = ht(T) = \kappa$ and $\forall \alpha < \kappa \; |T_\alpha| < \kappa$, where T_α is the α^{th} level of T.

2. T is a *κ-Aronszajn tree* iff T is a κ-tree with no cofinal branch.

3. T is a *special λ^+-Aronszajn tree* iff there exists $h : T \longrightarrow \lambda$ such that $x <_T y \Longrightarrow h(x) \neq h(y)$.

4. κ *has the tree property* iff there is no κ-Aronszajn tree.

The following easy argument gives a connection between elementary embeddings and the tree property. We write $T \restriction \beta$ for $\bigcup_{\alpha<\beta} T_\beta$.

Fact 5.1 *If κ is measurable then κ has the tree property.*

Proof: Let T be a κ-tree, and fix $j : V \longrightarrow M$ with $\text{crit}(j) = \kappa$. Then $j(T)$ is a $j(\kappa)$-tree in M, and what is more $j(T) \restriction \kappa$ is isomorphic to T (here we use the fact that each level T_α has size less than κ, so that $j(T_\alpha) = j``T_\alpha$).

$j(\kappa) > \kappa$ so $j(T)$ has at least one point on level κ. Looking at the points below this point we see that $j(T) \restriction \kappa$ has a cofinal branch in M; since $j(T) \restriction \kappa$ is isomorphic to κ and $M \subseteq V$, T has a cofinal branch.

♦

In fact κ being weakly compact would suffice here: just build T into an appropriate structure of size κ. It is known that the weakly compact cardinals are exactly those inaccessible cardinals which have the tree property. It is also known that if $\lambda^{<\lambda} = \lambda$ then there is a special λ^+-Aronszajn tree, in particular CH gives a special $\aleph_2$-Aronszajn tree.

It is natural to ask whether there can be a model with no $\aleph_2$-Aronszajn tree. A natural first try might be to take a measurable cardinal κ and Lévy collapse it to $\aleph_2$; this fails (because CH holds after doing the Lévy collapse) but it is instructive to see exactly what goes wrong.

Let κ be measurable, let $j : V \longrightarrow M$ be an elementary embedding with $\text{crit}(j) = \kappa$. Let G be $Coll(\omega_1, < \kappa)$-generic over V. Then as in Section 3 we get a generic embedding $j : V[G] \longrightarrow M[G][H] \subseteq V[G][H]$, where H is generic over $V[G]$ for the countably closed forcing $Coll(\aleph_1, j(\kappa) - \kappa)$. Now let us try and imitate the proof of Fact 5.1. If $V[G] \models$ "T is an $\aleph_2$-tree" we can argue as before that T has a cofinal branch in $M[G][H]$ and hence in $V[G][H]$. We would like to argue that T must have a cofinal branch in $V[G]$, but at this point the argument fails because it is quite possible in general for countably closed forcing to add a branch to an $\aleph_2$-tree. For example if $V = L$ there is a countably closed $\aleph_2$-Souslin tree S, and then $(S, \geq_S)$ is a countably closed poset which adds a branch through S.

Mitchell resolved the problem by proving

Fact 5.2 (Mitchell [31]) *The following are equiconsistent.*

1. *There exists a weakly compact cardinal.*

2. $\aleph_2$ *has the tree property.*

We will sketch Mitchell's argument, but we begin by stating a couple of useful facts about trees and forcing.

Fact 5.3 (Silver) *If* $2^{\aleph_0} > \aleph_1$*, countably closed forcing cannot add a new branch to an* $\aleph_2$*-tree.*

Fact 5.4 (Kunen and Tall [24]) *Let* $\mathbb{P}$ *have the property that for every* $\aleph_1$*-sequence of conditions from* $\mathbb{P}$ *there is a subsequence of length* $\aleph_1$ *of pairwise compatible conditions (*$\mathbb{P}$ *is* $\aleph_1$-Knaster*). let* T *be a tree of height* $\aleph_1$ *with no cofinal branch (not necessarily an* $\aleph_1$*-tree). Then forcing with* $\mathbb{P}$ *cannot add a cofinal branch through* T*.*

We now give an outline of Mitchell's argument (this way of presenting the argument appears in Abraham's paper [1]). Once again we will assume that κ is a measurable cardinal and indicate at the end how to weaken the assumption to weak compactness.

- Let $j : V \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$.

- Define $\mathbb{P}$ (we will not give the definition) a forcing with the following properties:

  1. $|\mathbb{P}| = \kappa$, $\mathbb{P}$ is κ-c.c. and $\mathbb{P} \subseteq V_\kappa$.
  2. If $\beta < \kappa$ is inaccessible then $\mathbb{P}_\beta =_{\text{def}} \mathbb{P} \cap V_\beta$ is a complete subforcing of $\mathbb{P}$, and in $V^{\mathbb{P}_\beta}$ the quotient $\mathbb{P}/\mathbb{P}_\beta$ is a projection of $Add(\omega, \kappa - \beta) \times \mathbb{Q}_\beta$ for some countably closed $\mathbb{Q}_\beta$. Moreover $V^{\mathbb{P}_\beta} \models \aleph_1 = \aleph_1^V, 2^{\aleph_0} = \aleph_2 = \beta$ and $V^{\mathbb{P}_\beta * \mathbb{Q}_\beta} \models \kappa = \aleph_2$.
  3. $V^{\mathbb{P}} \models \aleph_1 = \aleph_1^V, 2^{\aleph_0} = \aleph_2 = \kappa$.

- If G is $\mathbb{P}$-generic then it is possible to lift $j : V \longrightarrow M$ to get a generic embedding $j : V[G] \longrightarrow M[G][H]$. This is easy because $\mathbb{P} \subseteq V_\kappa$, $j \restriction \mathbb{P} = \text{id}_{\mathbb{P}}$, and $\mathbb{P}$ is κ-c.c.

- Suppose that $V[G] \models$ "T is an $\aleph_2$-Aronszajn tree". $T \in M[G]$ because T has a name in $V_{\kappa+1}$ and $V_{\kappa+1} \subseteq M$, and arguing as before T has a cofinal branch in $M[G][H]$.

- We work in $M[G]$ to do a factor analysis of $j(\mathbb{P})/G$. By elementarity we see that $M[G] \subseteq M[G][H] \subseteq M[G][h_1][h_2]$, where h_1 is $\mathbb{Q}$-generic over $M[G]$ for some $\mathbb{Q}$ which is countably closed in $M[G]$, and h_2 is $Add(\omega, j(\kappa))$-generic over $M[G][h_1]$.

- $\mathbb{Q}$ adds no branch to T because $M[G] \models 2^{\aleph_0} = \aleph_2$ and $\mathbb{Q}$ is countably closed. $\mathbb{Q}$ collapses κ to be an ordinal of cardinality and cofinality $\aleph_1$, so if we take an $\aleph_1$-sequence cofinal in κ and look at the corresponding levels of T we get a "squashed" tree T^* which has height $\aleph_1$ and no cofinal branch.

- Forcing with h_2 cannot add a cofinal branch to T^* because $Add(\omega, j(\kappa))$ has the $\aleph_1$-Knaster property. This is a contradiction because T has a cofinal branch in $M[G][H]$ and $M[G][H] \subseteq M[G][h_1][h_2]$.

- If κ is only weak compact the argument is similar. Suppose $\dot{T}$ is a canonical $\mathbb{P}$-name and $\Vdash$ "$\dot{T}$ is a κ-Aronszajn tree". Build $\dot{T}$ into an appropriate M, and let $k : M \longrightarrow N$ with critical point κ. As before we lift k to a new map $k : M[G] \longrightarrow N[G][H]$. $N[G] \subseteq V[G]$ so $N[G] \models$ "T has no cofinal branch". By the usual elementary embedding argument T has a branch in $N[G][H]$. This leads to a contradiction as before.

We conclude with a few remarks about other results on the tree property.

Fact 5.5 (Abraham [1]) *If κ is supercompact and $\lambda > \kappa$ is weakly compact, there is a forcing extension in which $2^{\aleph_0} = \kappa = \aleph_2$, $2^{\aleph_1} = \lambda = \aleph_3$, and both $\aleph_2$ and $\aleph_3$ have the tree property.*

Fact 5.6 (Foreman and Magidor) *If two successive cardinals have the tree property, there is an inner model with a strong cardinal.*

Fact 5.7 (Magidor and Shelah [29]) *From a very strong large cardinal hypothesis, it is consistent that $\aleph_{\omega+1}$ should have the tree property.*

Fact 5.8 (Cummings and Foreman [3]) *If it is consistent that there are ω supercompact cardinals, it is consistent that $\aleph_n$ has the tree property for every n with $2 \leq n < \omega$.*

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