ORGANIC AND TIGHT

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1. Introduction

This report is motivated by the combinatorics of $\aleph_\omega$ in L where there are canonical examples of scales, square sequences and other structures of PCF theory. There, too, concepts such as mutual stationarity and tightness have concrete reformulations in terms of fine structure. Several kinds of new questions come out of this analysis. First, surprisingly, there remain basic open problems regarding what is true in L. Second, it leads to reasonable conjectures about the combinatorics of $\aleph_\omega$ in ZFC alone without $V = L$. Third, it gives us expectations about arbitrary singular cardinals in extender models $L[E]$.

We use more or less standard terminology from PCF theory but only as it applies to $\aleph_\omega$. Thus, for any set X with $|X| < \aleph_\omega$, we have the characteristic function

$$\mathrm{char}_X : n \mapsto \sup(X \cap \aleph_n)$$

and we speak as if char_X is always a function in the product $\prod \aleph_n$ even though this is slightly incorrect. We say that X has eventual cofinality $\aleph_k$ iff

$$\mathrm{cf}(\mathrm{char}_X(n)) = \aleph_k$$

for all but finitely many $n < \omega$.

Recall that a sequence $\langle S_n \mid 1 < n < \omega \rangle$ is mutually stationary iff S_n is stationary in $\aleph_n$ for $1 < n < \omega$ and for every countable structure $\mathcal{A}$ whose universe includes $\aleph_\omega$, there exists $X \prec \mathcal{A}$ such that

$$\mathrm{char}_X(n) \in S_n$$

for all but finitely many $n < \omega$. The official definition of mutual stationarity says that there exists X whose characteristic function meets S_n for all $1 < n < \omega$ but this is equivalent. The following is one of several examples analyzed by Foreman and Magidor [FM2001]. Assume $V = L$. Define S^k_{n+1} to be the set of α between $\aleph_n$ and $\aleph_{n+1}$ such that

$$J_\alpha \models \text{every set has cardinality } \leq \aleph_n$$

and, if $\beta \geq \alpha$ is least such that

$$J_{\beta+1} \models |\alpha| = \aleph_n,$$

then

$$\aleph_n = \rho_{k+1}(J_\beta) < \alpha \leq \rho_k(J_\beta).$$

Then S^k_{n+1} is stationary for every pair $k, n < \omega$ and, for every function $e : \omega \to \omega$, the sequence $\langle S^{e(k)}_n \mid 1 < n < \omega \rangle$ is mutually stationary iff e is eventually constant.

A scale is a sequence $\langle f_\alpha \mid \alpha < \aleph_{\omega+1} \rangle$ that is increasing and unbounded in $\prod \aleph_n$ relative to the order $g <^* h$ iff $g(n) < h(n)$ for all but finitely many $n < \omega$. The scale is continuous iff for every limit $\beta < \aleph_{\omega+1}$ of uncountable cofinality, if there is an eub for $\langle f_\alpha \mid \alpha < \beta \rangle$, then f_β is such an eub. That is, for every $g <^* f_\beta$, there exists $\alpha < \beta$ such that $g <^* f_\alpha$.

Recall that a set X is tight iff $X \cap \prod \aleph_n$ is cofinal in $\prod (X \cap \aleph_n)$. Consider an arbitrary continuous scale $\langle f_\alpha \mid \alpha < \aleph_{\omega+1} \rangle$ in $\prod \aleph_n$. By [FM2001], if $X \prec H(\aleph_{\omega+1})$ and X has eventual cofinality $\aleph_m$ for some $m \geq 1$, then X is tight iff

$$\mathrm{char}_X =^* f_{\sup(X \cap \aleph_{\omega+1})}$$

and $\sup(X \cap \aleph_{\omega+1})$ is a good ordinal. This can be stated without reference to goodness if $\Box^*_{\aleph_\omega}$ holds since then the set of good ordinals contains a club in $\aleph_{\omega+1}$.

We introduce the following variant.

Definition 1. *Let $\langle f_\alpha \mid \alpha < \aleph_{\omega+1} \rangle$ be a continuous scale in $\prod \aleph_n$ and X be a set. Then X* is organic *to $\langle f_\alpha \mid \alpha < \aleph_{\omega+1} \rangle$ iff there exists $\delta < \aleph_{\omega+1}$ such that* $\mathrm{char}_X =^* f_\delta$.

In some contexts, the definition above is not sensitive to the choice of scale. (If $\langle f_\alpha \mid \alpha < \aleph_{\omega+1} \rangle$ and $\langle g_\alpha \mid \alpha < \aleph_{\omega+1} \rangle$ are continuous scales, then there is a club C in $\aleph_{\omega+1}$ such that $f_\alpha =^* g_\alpha$ for every $\alpha \in C$.) There are times, too, when the scale is clear from context. For example, when we work in L, we always mean the canonical scale that we will define. In such settings, we suppress mention of $\langle f_\alpha \mid \alpha < \aleph_{\omega+1} \rangle$.

The first two authors showed that the set of inorganic sets with eventual uncountable cofinality is stationary; see Section 2. One purpose of this report is to give practical characterizations of tightness and organicity in L in terms of fine structure. This is done in Section 3.

Recall that a mutually stationary sequence $\langle S_n \mid 1 < n < \omega \rangle$ is tightly stationary iff for every countable structure $\mathcal{A}$ whose universe includes $\aleph_\omega$, there exists a tight $X \prec \mathcal{A}$ such that

$$\mathrm{char}_X(n) \in S_n$$

for all but finitely many $n < \omega$. We introduce the following related concept.

Definition 2. *A mutually stationary sequence* $\langle S_n \mid 1 < n < \omega\rangle$ *is* organically stationary *iff for every countable structure* $\mathcal{A}$ *whose universe includes* $\aleph_\omega$*, there exists an organic* $X \prec \mathcal{A}$ *such that*

$$\mathrm{char}_X(n) \in S_n$$

for all but finitely many $n < \omega$*.*

Cummings, Foreman and Magidor [CFM2006] showed a certain forcing notion adds a mutually stationary sequence that is not tightly stationary. Since this is related to other purposes of this report, we give an overview. Donder, Jensen and Stanley [DJS] introduced a combinatorial principle called Coherent Squares and proved it holds in L. Coherent Squares involves $\square_n$-sequences $\langle C^n_\alpha \mid \alpha < \aleph_{n+1}\rangle$ for $n \leq \omega$ and a continuous scale in $\prod \aleph_n$ that are tied together in a particular way. Given Coherent Squares and a sequence of limit ordinals $\langle \gamma_n \mid 1 < n < \omega\rangle$ such that $\sup \gamma_n = \aleph_\omega$ and $\gamma_n < \aleph_n$, let

$$T_{n+1} = \{\alpha \in \aleph_{n+1} \cap \mathrm{Cof}(\aleph_1) \mid \mathrm{type}(C^n_\alpha) \geq \gamma_n\}.$$

In [CFM2006], it is shown that $\langle T_{n+1} \mid 1 < n < \omega\rangle$ is not tightly stationary. An easy modification of their proof shows that it is not organically stationary. They also showed that if the Coherent Squares was added by forcing, then $\langle T_{n+1} \mid 1 < n < \omega\rangle$ is mutually stationary. Therefore, it is consistent that is mutual and organic stationarity are different. In spite of what seems like progress, we have not answered the basic question of how mutual, organic and tight stationarity are related in L.

2. About V

A set X is said to be $\aleph_1$-uniform iff $\aleph_1 = |X| \subseteq X$ and

$$\mathrm{cf}(\sup(X \cap \aleph_n)) = \aleph_1$$

whenever $1 < n < \omega$. The following result is due to the first and second authors. The same technique was used for another purpose in [CFM2006].

Theorem 3. *Assume GCH. Then the set*

$$\{X \prec H(\aleph_{\omega+1}) \mid X \textit{ is } \aleph_1\textit{-uniform and inorganic}\}$$

is nonstationary.

Proof. The first claim is that if $X \prec H(\aleph_{\omega+1})$ is $\aleph_1$-uniform, then $X \cap \aleph_n$ is ω-closed. To see this, let $a \subseteq X \cap \aleph_n$ with $\mathrm{type}(a) = \omega$. Let $\alpha = \sup(a)$. Then $\alpha < \sup(X \cap \aleph_n)$. Let $\beta = \min(X - \alpha)$. Then

$\alpha \leq \beta$. For contradiction, suppose that $\alpha < \beta$. Then $\mathrm{cf}(\beta) > \aleph_1$ as $\aleph_1 \subset X$ and X is bounded in β. But now

$$\aleph_1 = \mathrm{cf}(X \cap \mathrm{cf}(\beta)) = \mathrm{cf}(X \cap \beta) = \mathrm{cf}(X \cap \alpha) = \aleph_0.$$

A corollary to the first claim is that, for every $\aleph_1$-uniform $X \prec H(\aleph_{\omega+1})$, there is a sequence $\langle C_n \mid 1 < n < \omega \rangle$ such that $\mathrm{type}(C_n) = \omega_1$ and C_n is a club subset of $X \cap \aleph_n$ whenever $1 < n < \omega$.

Let $\mathcal{A}$ be an expansion of $H(\aleph_{\omega+1})$. The second claim is that the set

$$\{\mathrm{char}_X \mid X \prec \mathcal{A} \text{ and } X \text{ is } \aleph_1\text{-uniform}\}$$

is closed in $\prod(\aleph_n \cap \mathrm{Cof}(\aleph_1))$ with the product of the discrete topologies on the coordinate spaces. Say

$$\lim_{i\to\infty} \mathrm{char}_{X_i} = \langle \alpha_n \mid 1 < n < \omega \rangle.$$

This means that

$$\lim_{i\to\infty} \sup(X_i \cap \aleph_n) = \alpha_n$$

whenever $1 < n < \omega$. In turn, this means that whenever $1 < n < \omega$, there exists $k(n) < \omega$ such that for all $i > k(n)$,

$$\sup(X_i \cap \aleph_n) = \alpha_n.$$

By the corollary to the first claim, we have C_n^i such that $\mathrm{type}(C_n^i) = \omega_1$ and C_n^i is a club subset of $X_i \cap \aleph_n$ for $i < \omega$ and $1 < n < \omega$. Let

$$D_n = \bigcap_{k(n)<i<\omega} C_n^i.$$

Then $\mathrm{type}(D_n) = \omega_1$ and D_n is a club subset of $X_i \cap \aleph_n$ whenever $1 < n < \omega$ and $k(n) < i < \omega$. In particular, $\sup(D_n) = \alpha_n$ for $1 < n < \omega$. Let

$$E = \bigcup_{1<n<\omega} D_n$$

and

$$X_\infty = \mathrm{Hull}^{\mathcal{A}}(E).$$

It is straightforward to verify that $\mathrm{char}_{X_\infty} = \langle \alpha_n \mid 1 < n < \omega \rangle$.

By the second claim, there is a tree T such that

$$[T] = \{\mathrm{char}_X \mid X \prec \mathcal{A} \text{ and } X \text{ is } \aleph_1\text{-uniform}\}.$$

The third claim is that T has a stationary branching subtree. To see this, play the following game.

I	$A_0 \subseteq \aleph_2 \cap \mathrm{Cof}(\aleph_1)$		$A_1 \subseteq \aleph_3 \cap \mathrm{Cof}(\aleph_1)$		$\cdots$
II		$\alpha_0 \notin A_0$		$\alpha_1 \notin A_1$	$\cdots$

where every A_n must be nonstationary and $\prod A_n \subseteq T$ or else player I loses. Also, $\langle \alpha_n \mid n < \omega \rangle \in [T]$ or else II loses. If both players survive, then II wins and I loses. If player II has a winning strategy, τ, then the set of sequences $\langle \alpha_i \mid i < n \rangle$ coming from plays according to τ is a stationary branching subtree of T. Since the game is determined, the other possibility is that player I has a winning strategy, σ. Let $X \prec \mathcal{A}$ be $\aleph_1$-uniform with $\sigma \in X$. Put

$$\alpha_n = \sup(X \cap \aleph_{n+2}).$$

Clearly,

$$\langle \alpha_n \mid n < \omega \rangle \in [T].$$

We will derive a contradiction by showing that player II can play these ordinals against σ and survive. The verification is by induction. Let $A_0 = \sigma(\langle\rangle)$. Then $A_0 \in X$. Since A_0 is nonstationary and $X \prec H(\theta)$, there exists $C \in X$ such that C is club in $\aleph_2$ and $A_0 \cap C = \emptyset$. Then $\alpha_0 \in C$, so $\alpha_0 \notin A_0$. Let $A_1 = \sigma(\langle \alpha_0 \rangle)$. Then

$$A_1 \subseteq \bigcup_{\beta \in \aleph_2 \cap \mathrm{Cof}(\aleph_1)} \sigma(\langle \beta \rangle).$$

The set on the right is an element of X and is a nonstationary subset of $\aleph_3$, so it does not have α_1 as an element. Hence $\alpha_1 \notin A_1$. Continuing in this way, we complete the proof of the third claim.

The final claim is that there is an $X \prec \mathcal{A}$ such that X is $\aleph_1$-uniform and organic. To see this, note that if X and Y are organic sets, then $\mathrm{char}_X \leq^* \mathrm{char}_Y$ or vice-versa. On the other hand, if S is a stationary branching subtree of T, then S has branches char_X and char_Y that are $\leq^*$-incomparable. □

Theorem 4. *Assume GCH. Then the set*

$$\{X \prec H(\aleph_{\omega+1}) \mid X \text{ is } \aleph_1\text{-uniform, organic but not tight}\}$$

is stationary.

Proof. We modify the proof of Zapletal's result (found in [FM2001]) that set of non-tight $X \prec H(\aleph_{\omega+1})$ with eventual uncountable cofinality is stationary. Let $\mathcal{A}$ be a structure whose underlying set is $H(\aleph_{\omega+1})$. Let $M \prec \mathcal{A}$ be transitive with $|M| = \aleph_\omega$. Let $\gamma = \sup(\mathrm{OR} \cap M)$. Let $\langle X_\alpha \mid \alpha < \omega_1 \rangle$ be a continuous chain of countable elementary submodels of M and $\langle \delta_\alpha \mid \alpha < \omega_1 \rangle$ be an increasing sequence of ordinals such that

$$\delta_0 \geq \gamma,$$

$$\mathrm{char}_{X_\alpha} <^* f_{\delta_\alpha}.$$

and

$$\mathrm{ran}(f_{\delta_\alpha}) \subseteq X_{\alpha+1}.$$

Let $\delta = \sup_{\alpha<\omega_1} \delta_\alpha$ and $X = \bigcup_{\alpha<\omega_1} X_\alpha$. Then char_X is an eub for $\langle f_\eta \mid \eta < \delta \rangle$ and hence so is f_δ. Therefore $\mathrm{char}_X =^* f_\delta$. $\square$

3. About L

Here is some of the fine structure notation that we will use in our discussion of L. If M is a structure and $n \leq \omega$, then $\rho_n(M)$ and $p_n(M)$ are the n-th projectum and the n-th standard parameter of M respectively. If κ is an ordinal but not a cardinal of L, then $\beta(\kappa)$ is the least $\beta \geq \kappa$ such that $\rho_\omega(J_\beta) < \kappa$, $n(\kappa)$ is the least $n < \omega$ such that $\rho_{n+1}(J_{\beta(\kappa)}) < \kappa$, and M_κ is the n-th mastercode structure for $J_{\beta(\kappa)}$. The following fact is useful.

Theorem 5. *Assume that $V = L$. Let $X \prec L_{\aleph_\omega}$ with $|X| < \aleph_\omega$. Then X has an eventual cofinality.*

Proof. $|X| = \aleph_\ell$. Let $\pi : L_{\kappa_\omega} \simeq X$. Say $\pi(\kappa_n) = \aleph_n$ for every $n < \omega$. Then

$$\mathrm{cf}(\kappa_n) = \mathrm{cf}(\sup(\pi''\kappa_n)) = \mathrm{cf}(\sup(X \cap \aleph_n)) = \mathrm{cf}(\mathrm{char}_X(n))$$

for every $n < \omega$. We mainly care about $n > \ell$ since $\mathrm{crit}(\pi) = \kappa_{\ell+1}$. Recall that $\mathrm{OR} \cap M_\kappa = \rho_n(J_{\beta(\kappa)})$ and $\rho_1(M_\kappa) = \rho_{n+1}(J_{\beta(\kappa)})$. In our case, we have $\ell \leq m < \omega$ such that $\rho_1(M_{\kappa_\omega}) = \kappa_m$. Then $M_{\kappa_\omega} = M_{\kappa_n}$ whenever $m < n < \omega$. By a standard fine structural calculation,

$$\mathrm{cf}(\kappa_n) = \mathrm{cf}(\mathrm{OR} \cap M_{\kappa_\omega})$$

whenever $m < n < \omega$, so we are done. $\square$

When working in L, it is often useful to consider the set Λ_{n+1} of local successors of $\aleph_n$. What we mean is that $\alpha \in \Lambda_{n+1}$ iff $\aleph_n < \alpha < \aleph_{n+1}$ and $\aleph_n$ is the largest cardinal of L_α. Since Λ_{n+1} is club in $\aleph_{n+1}$, we often slur over the difference between Λ_{n+1} and $\aleph_{n+1}$. Consider an arbitrary $\alpha \in \Lambda_{\omega+1}$. Then $\rho_1(M_\alpha) = \aleph_\omega$ and

$$M_\alpha = \mathrm{Hull}_1^{M_\alpha}(\aleph_\omega \cup p_1(M_\alpha)).$$

Define

$$f_\alpha(n+1) = \sup(\aleph_{n+1} \cap \mathrm{Hull}_1^{M_\alpha}(\aleph_n \cup p_1(M_\alpha))).$$

We will refer to $\langle f_\alpha \mid \alpha \in \Lambda_{\omega+1} \rangle$ as the canonical scale in L, which is justified by the following.

Theorem 6. *Assume $V = L$. Then $\langle f_\alpha \mid \alpha \in \Lambda_{\omega+1} \rangle$ is a continuous scale in $\prod \Lambda_{n+1}$.*

Proof. First we show that the sequence is $<^*$-increasing. Consider ordinals $\alpha < \beta$ in $\Lambda_{\omega+1}$. Pick $m < \omega$ so that

$$\alpha \in \mathrm{Hull}_1^{M_\beta}(\aleph_m \cup p_1(M_\beta)).$$

If $m \leq n < \omega$, then

$$f_\alpha(n+1) \in \mathrm{Hull}_1^{M_\beta}(\aleph_n \cup p_1(M_\beta))$$

so $f_\alpha(n+1) < f_\beta(n+1)$.

Next we show that the sequence is $<^*$-unbounded. Given $g \in \prod \Lambda_{n+1}$, pick $\alpha \in \Lambda_{\omega+1}$ with $g \in J_\alpha$. Let $m < \omega$ be large enough that

$$g \in \mathrm{Hull}_1^{M_\beta}(\aleph_m \cup p_1(M_\beta)).$$

Easily we see that $g(n+1) < f_\alpha(n+1)$ whenever $m \leq n < \omega$.

Finally, we show that the scale is continuous. Let β be a limit point of $\Lambda_{\omega+1}$ of uncountable cofinality. We must show that f_β is an eub for $\langle f_\alpha \mid \alpha \in \beta \cap \Lambda_{\omega+1} \rangle$. Assume that $g <^* f_\beta$. Let

$$\sigma = \sup(\mathrm{OR} \cap \mathrm{Hull}_1^{M_\beta}(\mathrm{ran}(g) \cup p_1(M_\beta)).$$

Then $\sigma < \mathrm{OR} \cap M_\beta$ because the ordinal height of M_β has the same uncountable cofinality as β. Let M be the Mostowski collapse of

$$\mathrm{Hull}_1^{M_\beta \restriction \sigma}(\aleph_\omega \cup p_1(M_\beta)).$$

Then $M = M_\alpha$ for some $\alpha \in \beta \cap \Lambda_{\omega+1}$ and $g <^* f_\alpha$. □

The following is our characterization of organic and tight.

Theorem 7. *Assume $V = L$. Let $X \prec L_{\aleph_{\omega+1}}$ such that $|X| < \aleph_\omega$ and X has eventual uncountable cofinality. Say*

$$\pi : L_{\kappa_{\omega+1}} \simeq X$$

and, for every $n \leq \omega$,

$$\pi(\kappa_n) = \aleph_n.$$

Let

$$Q = \mathrm{ult}(M_{\kappa_\omega}, \pi, \aleph_\omega).$$

Then

$$X \text{ is organic} \iff Q \text{ is wellfounded}$$

and

$$X \text{ is tight} \iff M_{\kappa_{\omega+1}} = M_{\kappa_\omega}.$$

Recall that if X has eventual uncountable cofinality and X is tight, then X is organic. Above, Q is the ultrapower of M_{κ_ω} by the extender of length $\aleph_\omega$ derived from π. It is the ultrapower formed by using coordinates

$$a \in [\aleph_\omega]^{<\omega}$$

and functions

$$f : [\kappa_n]^{|a|} \longrightarrow M_{\kappa_\omega}$$

with $n < \omega$ and $f \in M_{\kappa_\omega}$. This makes sense precisely because M_{κ_ω} has the same bounded subsets of κ_ω as $\mathrm{dom}(\pi) = L_{\kappa_{\omega+1}}$. Recall that there is a bounded subset of κ_ω that is not an element of M_{κ_ω} but is Σ_1 definable over M_{κ_ω}. Thus, in terms of the natural ordering on mastercode structures for levels of L, if $M_{\kappa_\omega} \triangleleft M$, then the extender of length $\aleph_\omega$ derived from π cannot be applied to M. In particular, note that $M_{\kappa_{\omega+1}} \trianglelefteq M_{\kappa_\omega}$.

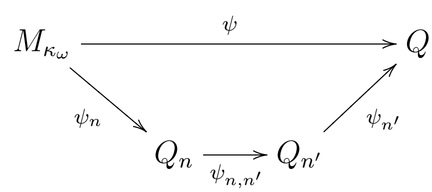
Proof. Let

$$\psi : M_{\kappa_\omega} \longrightarrow Q$$

be the ultrapower map. Also set

$$Q_n = \mathrm{ult}(M_{\kappa_\omega}, \pi, \aleph_n)$$

and define maps ψ_n and $\psi_{n,n'}$ according to the diagram



Take $m < \omega$ such that

$$\rho_1(M_{\kappa_\omega}) = \kappa_m.$$

Recall that M_{κ_ω} is sound, that is,

$$M_{\kappa_\omega} = \mathrm{Hull}_1^{M_{\kappa_\omega}}(\kappa_m \cup p_1(M_{\kappa_\omega})).$$

First we assume that Q is wellfounded and show that X is organic. This part of the proof does not use the hypothesis that X has eventual uncountable cofinality. Identify Q with its Mostowski collapse. Then ψ is a cofinal Σ_1-elementary embedding from M_{κ_ω} to Q and

$$\psi \upharpoonright \kappa_{\omega+1} = \pi \upharpoonright \kappa_{\omega+1}.$$

The soundness of M_{κ_ω} translates into that of Q, namely,

$$Q = \mathrm{Hull}_1^Q(\aleph_\omega \cup p_1(Q)).$$

Let δ be the cardinal successor of $\aleph_\omega$ in Q. Then $Q = M_\delta$. Observe that if λ is the cardinal successor of κ_ω in M_{κ_ω}, then $\delta = \sup(\psi''\lambda)$. Note too that $\kappa_\omega \leq \lambda$ and $\gamma \leq \delta$ where

$$\gamma = \sup(\pi''\kappa_{\omega+1}) = \sup(X \cap \aleph_{\omega+1}).$$

For $n < \omega$, we have that

$$\psi_n \restriction \kappa_{n+1} = \pi \restriction \kappa_{n+1}$$

and

$$\aleph_{n+1}^{Q_n} = \psi_n(\kappa_{n+1}) = \sup(\psi_n''\kappa_{n+1}) = \sup(\pi''\kappa_{n+1}) = \sup(X \cap \aleph_{n+1}) = \mathrm{char}_X(n+1).$$

If $m \leq n < \omega$, then Q_n is the Mostowski collapse of

$$\mathrm{Hull}_1^Q(\aleph_n \cup p_1(Q)).$$

Thus

$$f_\delta(n+1) = \sup(\aleph_{n+1} \cap \mathrm{Hull}_1^{M_\delta}(\aleph_n \cup p_1(M_\delta))) = \aleph_{n+1}^{Q_n}.$$

We have seen that for $m \leq n < \omega$,

$$\mathrm{char}_X(n+1) = \aleph_{n+1}^{Q_n} = f_\delta(n+1).$$

In particular, X is organic. Using the hypothesis that X has eventual uncountable cofinality, we have also seen that if Q is wellfounded, then

$$X \text{ is tight} \iff \delta = \gamma \iff \lambda = \kappa \iff M_{\kappa_\omega} = M_{\kappa_{\omega+1}}.$$

We remark that clause (2) of the theorem does not allow us to assume that Q is wellfounded for this characterization of tight; we will return to this point.

Now drop the assumption that Q is wellfounded. We claim that Q_n is wellfounded for every $n < \omega$. Suppose otherwise. Then we have a sequence of coordinates $\langle a_i \mid i < \omega \rangle$ from $[\aleph_n]^{<\omega}$ and a sequence of functions $\langle f_i \mid i < \omega \rangle$ from M_{κ_ω} such that

$$\mathrm{dom}(f_i) = [\kappa_n]^{|a_i|}$$

and

$$\psi(f_{i+1})(a_{i+1}) \in \psi(f_i)(a_i)$$

for $i < \omega$. By a standard calculation that we have used before,

$$\mathrm{cf}(X \cap \aleph_{n+1}) = \mathrm{cf}(\kappa_{n+1}) = \mathrm{cf}(\mathrm{OR} \cap M_{\kappa_\omega})$$

whenever $m \leq n < \omega$. Since X has eventual uncountable cofinality,

$$\mathrm{cf}(\mathrm{OR} \cap M_{\kappa_\omega}) > \omega.$$

Therefore, there exists $\sigma < \mathrm{OR} \cap M_{\kappa_\omega}$ large enough so that

$$\{f_i \mid i < \omega\} \subseteq \mathrm{Hull}_1^{M_{\kappa_\omega} \restriction \sigma}(\kappa_m \cup p_1(M_{\kappa_\omega})).$$

Let

$$G : \mathrm{Hull}_1^{M_{\kappa_\omega} \restriction \sigma}(\kappa_n \cup p_1(M_{\kappa_\omega})) \simeq M$$

be the Mostowski collapse. Then $M \in L_{\kappa_{n+1}} = \mathrm{dom}(\pi)$. We have a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & \pi(M) \\ & \overset{\bar{\psi}}{\searrow} & \uparrow \\ & & \mathrm{ult}(M, \pi, \aleph_n) \end{array}$$

that implies $\mathrm{ult}(M, \pi, \aleph_n)$ is wellfounded. On the other hand,

$$\mathrm{dom}(G(f_i)) = [\kappa_n]^{|a_i|}$$

and

$$\bar{\psi}(G(f_{i+1}))(a_{i+1}) \in \bar{\psi}(G(f_i))(a_i)$$

for $i < \omega$, which implies that $\mathrm{ult}(M, \pi, \aleph_n)$ is illfounded. This completes the proof of the claim.

If $M_{\kappa_\omega} = M_{\kappa_{\omega+1}}$, then

$$Q = \mathrm{ult}(M_{\kappa_{\omega+1}}, \pi, \aleph_\omega)$$

and the argument of the previous paragraph can be applied with $n = \omega$ to see that Q is wellfounded. Therefore, if $M_{\kappa_\omega} = M_{\kappa_{\omega+1}}$, then X is tight.

Finally, assume that X is organic. Say

$$\mathrm{char}_X =^* f_\delta.$$

We will conclude that Q is wellfounded by showing $Q = M_\delta$. Note that Q is the direct limit of the structures Q_n under the maps $\psi_{n,n'}$ for $n < n' < \omega$. Let

$$\widetilde{\psi} : \widetilde{Q}_n \to M_\delta$$

be the inverse of the Mostowski collapse of

$$\mathrm{Hull}_1^{M_\delta}(\aleph_n \cup p_1(M_\delta))$$

and

$$\widetilde{\psi}_{n,n'} : \widetilde{Q}_n \to \widetilde{Q}_{n'}$$

be the inverse of the Mostowski collapse of

$$\mathrm{Hull}_1^{\widetilde{Q}_{n'}}(\aleph_n \cup p_1(\widetilde{Q}_{n'})).$$

Then M_δ is the direct limit of the structures $\widetilde{Q}_n$ under the maps $\widetilde{\psi}_{n,n'}$ for $n < n' < \omega$. It is enough to show that

$$\widetilde{Q}_n = Q_n$$

and

$$\widetilde{\psi}_{n,n'} = \psi_{n,n'}$$

for all sufficiently large $n < n' < \omega$. Now Q_n is wellfounded,

$$\psi_n \restriction \kappa_{n+1} = \pi \restriction \kappa_{n+1}$$

and ψ_n is continuous at κ_{n+1}, so

$$\mathrm{char}_X(n{+}1) = \sup(X \cap \aleph_{n+1}) = \sup(\pi''\kappa_{n+1}) = \sup(\psi_n''\kappa_{n+1}) = \psi_n(\kappa_{n+1}) = \aleph_{n+1}^{Q_n}$$

for $n < \omega$. Because M_{κ_ω} is sound, Q_n is too whenever $m \le n < \omega$, i.e.,

$$Q_n = \mathrm{Hull}_1^{Q_n}(\aleph_n \cup p_1(Q_n)).$$

Thus,

$$Q_n = M_{\mathrm{char}_X(n+1)}$$

whenever $m \le n < \omega$. It also follows that $\psi_{n,n'}$ is the inverse of the Mostowski collapse of

$$\mathrm{Hull}_1^{Q_{n'}}(\aleph_n \cup p_1(Q_{n'}))$$

whenever $m \le n < n' < \omega$. In other words, $\psi_{n,n'}$ is determined by $Q_{n'}$ the same way that $\widetilde{\psi}_{n,n'}$ is determined by $\widetilde{Q}_{n'}$ for large enough $n < n' < \omega$. From the definition of f_δ it is clear that

$$f_\delta(n+1) = \aleph_{n+1}^{M_{f_\delta(n+1)}}$$

and

$$\widetilde{Q}_n = M_{f_\delta(n+1)}$$

for $n < \omega$. Since

$$\mathrm{char}_X(n+1) = f_\delta(n+1)$$

for all sufficiently large $n < \omega$, the result follows. □

There is are ZFC questions that comes out of the previous theorem. Given $X \prec H(\aleph_\omega)$, let

$$\pi : M \simeq X$$

be the Mostowski collapse and

$$\pi(\kappa_n) = \aleph_n$$

for $n < \omega$. Call X *firm* iff for every transitive model N of enough set theory, if

$$H(\kappa_\omega)^N = H(\kappa_\omega)^M,$$

then $\mathrm{ult}(N, \pi, \aleph_\omega)$ is wellfounded. In L, firm implies organic and with the right interpretation of "enough set theory" they are equivalent for X with eventual uncountable cofinality. What happens if $V \ne L$?

Theorem 8. *Assume $V = L$. Let $\langle C_\alpha \mid \alpha < \aleph_{n+1}\rangle$ be the canonical $\square_{\aleph_n}$-sequence for $n < \omega$. Suppose that $X \prec L_{\aleph_\omega}$ and $|X| < \aleph_\omega$. Assume that X has eventual uncountable cofinality. Then*

$$\mathrm{type}(C_{\mathrm{char}_X(n+1)})$$

is eventually constant for $n < \omega$.

By canonical we mean the one defined by Jensen. The following answers a question from [CFM2006].

Corollary 9. *Assume $V = L$. Let $\langle C_\alpha \mid \alpha < \aleph_{n+1}\rangle$ be the canonical $\square_{\aleph_n}$-sequence for $n < \omega$. Let $\langle \gamma_n \mid 1 < n < \omega\rangle$ be a sequence such that $\sup \gamma_n = \aleph_\omega$ and $\gamma_n \leq \aleph_n$. Put*

$$T_{n+1} = \{\alpha < \aleph_{n+1} \mid \mathrm{cf}(\alpha) > \omega \text{ and } \mathrm{type}(C_\alpha) \geq \gamma_n\}.$$

Then $\langle T_{n+1} \mid 1 < n < \omega\rangle$ is not mutually stationary.

For $\Gamma \subseteq \aleph_{n+1}$, we say that $\langle C_\alpha \mid \alpha \in \lim(\Gamma)\rangle$ is a $\square_{\aleph_n}(\Gamma)$-sequence iff every $\beta \in \lim(\Gamma)$,

(1) C_β is club in $\beta \cap \Gamma$,
(2) $\mathrm{type}(C_\beta) \leq \aleph_n$ and
(3) if $\alpha \in \lim(C_\beta)$, then $C_\alpha = C_\beta \cap \alpha$.

Then $\square_{\aleph_n}(\Gamma)$ is equivalent to $\square_{\aleph_n}$ for every club Γ in $\aleph_n$ via the Mostowski collapse $\Gamma \simeq \aleph_{n+1}$. In L, we usually work with the canonical $\square_{\aleph_n}(\Lambda_{n+1})$-sequence instead. In order to sketch the proof of the theorem, we must describe certain features of how this sequence is defined.

Fix $\alpha \in \lim(\Lambda_{n+1})$. The definition of C_α determined by M_α. Recall that α equals either $\aleph_{n+1}^{M_\alpha}$ or $\mathrm{OR} \cap M_\alpha$, and M_α is sound, so

$$M_\alpha = \mathrm{Hull}_1^{M_\alpha}(\aleph_n \cup p_1(M_\alpha)).$$

If α has uncountable cofinality, then there is a limit ordinal $\theta \leq \aleph_n$, a sequence of ordinals $d = \langle d_i \mid i < j\rangle$ with

$$\aleph_n > d_0 > \cdots > d_{j-1} \geq \theta$$

and a sequence of ordinals $s = \langle s_i \mid i < j\rangle$ with

$$s_0 < \cdots < s_{j-1} < \mathrm{OR} \cap M_\alpha$$

such that C_α is the image of θ under the non-decreasing function

$$\eta \mapsto \sup(\alpha \cap \mathrm{Hull}_1^{M_\alpha}(\eta \cup d \cup p_1(M_\alpha) \cup s)).$$

If α has countable cofinality, then there are two possibilities: either the definition of C_α has the same form as above or it is an arbitrary set of type ω unbounded in α. We will not explain the choice of d, s and θ and so we cannot explain why this works.

We are especially interested in $\alpha \in \Lambda_{n+1}$ whose associated M_α has a slightly rich cardinal structure. For $\ell \leq \omega$, define Λ^n_ℓ to be the set of α between $\aleph_n$ and $\aleph_{n+1}$ such that for every $k < \ell$,

$$M_\alpha \models \aleph_\ell \text{ exists.}$$

Then

$$\Lambda_{n+1} = \Lambda^n_{n+1} \supset \Lambda^n_{n+2} \supset \cdots \supset \Lambda^n_\omega.$$

Now we look again at the canonical $\Box_{\aleph_n}(\Lambda_{n+1})$-sequence for a fixed $\alpha \in \Lambda^n_\ell$ where $\ell \geq n+1$. For $k < \ell$, set

$$\kappa_k = \aleph_k^{M_\alpha}.$$

We also set

$$\kappa_\ell = \aleph_\ell^{M_\alpha}$$

or

$$\kappa_\ell = \mathrm{OR} \cap M_\alpha$$

depending on whether M_α has an ℓ-th infinite cardinal. Thus

$$\kappa_{n+1} = \alpha$$

and, for $n+1 \leq k \leq \ell$,

$$M_{\kappa_k} = M_\alpha.$$

Define functions F_{κ_k} with domain θ by

$$F_{\kappa_k} : \eta \mapsto \sup(\kappa_k \cap \mathrm{Hull}_1^{M_\alpha}(\eta \cup d \cup p_1(M_\alpha) \cup s))$$

and let

$$C_{\kappa_k} = \mathrm{ran}(F_{\kappa_k}).$$

Notice that the definition of C_{κ_k} is consistent with that of C_α described earlier in that

$$C_{\kappa_{n+1}} = C_\alpha$$

and $\kappa_k \notin \Lambda_{n+1}$ for every $k > n+1$. Fine structure calculations based on the definitions of s, d and θ we have omitted show that if $n+1 \leq k \leq \ell$, then C_{κ_k} is a club subset of $\kappa_k \cap \Lambda^n_\ell$. They also show that if $n+1 \leq k < k' \leq \ell$ and $\eta < \eta' < \theta$, then

$$F_{\kappa_k}(\eta) < F_{\kappa_k}(\eta') \iff F_{\kappa_{k'}}(\eta) < F_{\kappa_{k'}}(\eta'),$$

hence

$$\mathrm{type}(C_{\kappa_k}) = \mathrm{type}(C_{\kappa_{k'}}).$$

And they show coherence: if $\bar\kappa \in \lim(C_{\kappa_k})$, then

$$C_{\bar\kappa} = C_{\kappa_k} \cap \bar\kappa.$$

The case $k = n+1$ just repeats the coherence clause (3) for $\Box_{\aleph_{n+1}}(\Lambda_{n+1})$.

Now let $X \prec L_{\aleph_\omega}$ such that $|X| < \aleph_\omega$ and X has eventual uncountable cofinality. Let

$$\pi : L_{\kappa_\omega} \simeq X$$

and

$$\pi(\kappa_n) = \aleph_n$$

for $n < \omega$. Fix $m < \omega$ such that

$$\kappa_m = \rho_1(M_{\kappa_\omega}).$$

We will use the fact from the previous paragraph that for $m \leq n < \omega$,

$$\text{type}(C_{\kappa_{n+1}}) = \text{type}(C_{\kappa_{m+1}})$$

Let

$$Q_n = \text{ult}(M_{\kappa_\omega}, \pi, \aleph_n)$$

and

$$\psi_n : M_{\kappa_\omega} \to Q_n$$

be the ultrapower map. By ideas earlier in this section: the hypothesis of eventual uncountable cofinality implies that Q_n is wellfounded; since M_{κ_ω} is sound,

$$Q_n = \text{Hull}_1^{Q_n}(\aleph_n \cup p_1(Q_n))$$

and because

$$\psi_n \restriction \kappa_{n+1} = \pi \restriction \kappa_{n+1},$$

$$Q_n = M_{\text{char}_X(n+1)}$$

for $m \leq n < \omega$. Fine structure calculations using the definition of s, d and θ show that

$$F_{\text{char}_X(n+1)} = \psi_n \circ F_{\kappa_{n+1}},$$

hence

$$C_{\text{char}_X(n+1)} = \psi_n'' C_{\kappa_{n+1}}.$$

Therefore,

$$\text{type}(C_{\text{char}_X(n+1)}) = \text{type}(C_{\kappa_{n+1}}) = \text{type}(C_{\kappa_{m+1}})$$

for $m \leq n < \omega$ and the theorem follows.

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