A Descriptive View of Ergodic Theory

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Goal: Apply logical tools to some problems in Ergodic theory.

Approach initiated by Weiss, Kechris, Louveau et al:

> What kinds of results in ergodic theory can be carried through in a purely "Borel" context?

Led to a very fruitful study of such topics as Borel equivalence relations.

Discuss a different approach:

Work in usual context of ergodic theory and use logical tools to classify inherent complexity. The hope is that this can give some new invariants to distinguish between classes of measure preserving transformations and clarify some "classification" problems.

What do I mean by "Ergodic Theory"

Context:

$(X, \mathcal{B}, \mu)$ is a standard measure space.

$T : X \to X$ is an invertible measure preserving transformation.

Study T by studying its mixing and recurrence properties.

Example: Let $\{p_0, \ldots, p_{n-1}\}$ be a sequence of positive numbers summing to 1. This gives a natural finite measure space. (n-headed dice.)

We can study the long term properties of repeated trials in this measure space by considering:

$$n^{\mathbb{Z}} = \{\langle x_i \rangle : i \in \mathbb{Z} \text{ and } x_i \in \{0, \ldots, n-1\}\}$$

with the natural product measure.

This measure is invariant under the shift map $\langle x_n \rangle \to \langle x'_n \rangle$, where

$$x'_n = x_{n+1}$$

This example is called the **Bernoulli Shift** on the probabilities $\{p_0, \ldots, p_{n-1}\}$.

Example Let G be a compact abelian group, and $g \in G$. Let

$$T_g : G \to G$$

be defined by

$$T_g(h) = gh$$

Then T preserves the Haar measure on G.

Concretely: Let G be the unit circle $\mathbb{T}$ and α an irrational multiple of $2\pi i$. Then T_g is rotation by α radians.

Many “physical” systems have their statistical properties captured by measure preserving systems. For example many physical systems carry invariant measures such that their discrete time evolution is isomorphic to a Bernoulli shift. This is a rigorous way of discussing apparently random behavior in a deterministic system.

For this reason (among others) a lot of attention has gone into trying to classify, or at least understand the ergodic properties of measure preserving systems.

How can logic have anything to do with this?

Let X be a *Polish* space, i.e. a topological space with a compatible complete separable metric.

Then subsets of X can be distinguished by their logical complexity:

Recursive, Arithmetical, Borel, Analytical or Co-analytical, Projective etc...

(and "lightface" versions of these.)

Borel Sets

Let:

$$\underset{\sim}{\Sigma}^0_1 = \{\text{Open subsets of } X\}$$

and

$$\underset{\sim}{\Pi}^0_1 = \{X \backslash U : U \in \underset{\sim}{\Sigma}^0_1\} = \{\text{Closed subsets of } X\}.$$

More generally, for a countable ordinal $\alpha > 0$:

$$\underset{\sim}{\Sigma}^0_\alpha = \{\bigcup_n A_n : \text{for each } n \text{ there is a } \xi < \alpha, A_n \in \underset{\sim}{\Pi}^0_\xi\}$$

and

$$\underset{\sim}{\Pi}^0_\alpha = \{X \backslash A : A \in \underset{\sim}{\Sigma}^0_\alpha\},$$

$$\underset{\sim}{\Delta}^0_\alpha = \underset{\sim}{\Sigma}^0_\alpha \cap \underset{\sim}{\Pi}^0_\alpha.$$

Then the *Borel Sets* are defined to be:

$$\bigcup_{\alpha \in \omega_1} \underset{\sim}{\Sigma}^0_\alpha = \bigcup_{\alpha \in \omega_1} \underset{\sim}{\Pi}^0_\alpha$$

These collections form a proper hierarchy of sets:

Theorem In any uncountable Polish space all of the following inclusions are proper:

$$\begin{array}{ccccc} & & \underset{\sim}{\Sigma}^0_\alpha & & \\ & \subset & & \subset & \\ \underset{\sim}{\Delta}^0_\alpha & & & & \underset{\sim}{\Delta}^0_{\alpha+1} \\ & \subset & & \subset & \\ & & \underset{\sim}{\Pi}^0_\alpha & & \end{array}$$

Projective Sets

Let

$\underset{\sim}{\Sigma}{}^1_1 = \{A : A$ is a continuous image of a Borel set$\}$

and

$$\underset{\sim}{\Pi}{}^1_1 = \{X \backslash A : A \in \underset{\sim}{\Sigma}{}^1_1\}$$

In general for $n > 0$ let:

$\underset{\sim}{\Sigma}{}^1_{n+1} = \{A : A$ is a continuous image of a $\underset{\sim}{\Pi}{}^1_n$ set$\}$

and

$$\underset{\sim}{\Pi}{}^1_{n+1} = \{X \backslash A : A \in \underset{\sim}{\Sigma}{}^1_{n+1}\}$$

Let

$$\underset{\sim}{\Delta}{}^1_n = \underset{\sim}{\Sigma}{}^1_n \cap \underset{\sim}{\Pi}{}^1_n$$

Suslin (correcting a mistake of Lebesgue) showed:

Theorem(1917) Let X be an uncountable Polish Space, then:

- $\underset{\sim}{\Delta}^1_1 = \{\text{Borel Sets}\}$

- The following inclusions are all proper:

$$\begin{array}{ccccc} & & \underset{\sim}{\Sigma}^1_n & & \\ & \subset & & \subset & \\ \underset{\sim}{\Delta}^1_n & & & & \underset{\sim}{\Delta}^1_{n+1} \\ & \subset & & \subset & \\ & & \underset{\sim}{\Pi}^1_n & & \end{array}$$

In particular there are non-Borel $\underset{\sim}{\Sigma}^1_1$-sets.

Suslin also showed that the analytic sets have nice regularity properties, e.g. are universally measurable and have the property of Baire.

It was almost 70 years later (1983) that it was shown that projective sets were also regular (proof uses large cardinals).

The $\underset{\sim}{\Sigma}^1_1$-sets are called *analytic* sets and their complements the *co-analytic* sets.

As a heuristic:

Starting with closed sets

- Countable quantification yields Borel sets

- Quantification over Polish spaces gives projective sets

- Arbitrary use of uncountable transfinite induction or the Axiom of Choice, gives highly irregular sets.

Ergodic Theory in this context

Fix a measure space $X = [0, 1]$. Then a measure preserving transformation on X induces a unitary operator $U_T : L^2(X) \to L^2(X)$ by taking

$$U_T(f) = f \circ T.$$

The collection of unitary operators arising from measure preserving transformations forms a $\mathcal{G}_\delta$ subset of the Unitary operators with the WOT, and hence is a Polish group.

Explicitly, let I_n be an enumeration of the rational subintervals of $[0, 1]$ and define the following (semi)-metric:

$$d(S,T) = \sum_n [\mu(S(I_n)\Delta T(I_n))+$$

$$\mu(S^{-1}(I_n)\Delta T^{-1}(I_n))]$$

Then modulo sets of measure zero, d is a complete metric on the group of measure preserving transformations.

I will denote this group by $\mathcal{H}$.

An example: The entropy function is a Borel function on the space of measure preserving transformations.

To see this we need to code partitions as elements of a Polish space:

Note that a partition can be coded as a sequence of elements $\vec{\psi} \in L^1([0,1])^{\mathbb{N}}$ such that for all n:

- For all $n, \psi_n^2 = \psi_n$
- For all $n \neq m, \psi_n \perp \psi_m$
- $\sum \| \psi_n \|_1 = 1$

Note that this makes the collection of measurable partitions into a polish space $\mathcal{P}$. The partitions into finite sets are an $\mathcal{F}_\sigma$ subset of this space.

Entropy

Let $\qquad H : \mathcal{P} \to [0, \infty]$

be defined by $H(\vec{\psi}) = -\sum_n (\int \psi_n) log(\int \psi_n)$.

This formula gives the *entropy* or expected value of the "information" in a partition $\vec{\psi}$. H is clearly a Borel function on $\mathcal{P}$.

Let $T : [0, 1] \to [0, 1]$ be a measure preserving transformation, then T induces a homeomorphism of $\mathcal{P}$ by sending ψ_n to $\psi_n \circ T$. If $P \in \mathcal{P}$ is a finite partition, let $\bigvee_{i=k}^{l} T^i P$ be the coarsest partition refining each $T^i P$. This induces a Borel map from the finite partitions to finite partitions.

If P is a finite partition, then

$$h(T,P) =_{def} \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^i(P))$$

converges monotonically.

The *entropy* of the transformation T is defined by the formula:

$$h(T) = \sup_{Q \in \mathcal{Q}} h(T,Q)$$

where $\mathcal{Q}$ is the set of finite partitions into rational intervals.

Upshot: Entropy is a Borel function.

Corollary: $\{T : T \text{ has finite entropy}\}$ is a Borel subset of the measure preserving transformations.

Another example is the following theorem of Feldman.

Theorem:

$$\{\text{Bernoulli Shifts}\} \text{ is Borel.}$$

Feldman showed this by showing that the collection of Bernoulli shifts is both $\underset{\sim}{\Sigma}^1_1$ and $\underset{\sim}{\Pi}^1_1$ and hence Borel.

To see it is $\underset{\sim}{\Sigma}^1_1$ note that T is Bernoulli iff:

For some partition P:

- P is finite

- For all n, P is independent of $\bigvee_{i=-n}^{-1} T^i P$

- P "generates", i.e. the smallest σ-algebra containing all of the $\bigvee_{-n}^{n} T^i P$ contains all of the Borel sets (mod measure zero).

This latter statement is Borel since it is equivalent to the statement that all rational intervals can be approximated within ϵ by a union of elements of some $\bigvee_{-n}^{n} T^i P$.

How can “complexity” be used to separate classes of transformations?

A transformation T is:

- *ergodic* iff for all measurable sets A, B and all $\epsilon \in (0, 1)$ there is an n,

$$\mu(T^n(A) \cap B) > \epsilon\mu(A)\mu(B)$$

- *mixing* iff for all measurable sets A, B:

$$\lim_{n \to \infty} T^n(A) \cap B = \mu(A)\mu(B)$$

- *weakly mixing* iff for all measurable sets A, B:

$$\lim_{D} T^n(A) \cap B = \mu(A)\mu(B)$$

where D is the filter of subset of $\mathcal{N}$ which have density one.

Revisionist History

Theorem

- (Halmos) The weakly mixing transformations are a $\underset{\sim}{\Pi}^0_2$ (i.e. $\mathcal{G}_\delta$) subset of the measure preserving transformations.

- (Rohlin) The mixing transformations are a $\underset{\sim}{\Pi}^0_3$, non-$\underset{\sim}{\Pi}^0_2$ subset of the measure preserving transformations.

(This was the first proof that there was a difference between weakly mixing and mixing. It was proved this way, except that Halmos and Rohlin used "category" as the invariant.)

A Question

What is the complexity of:

$\{T : T$ is isomorphic to a smooth measure preserving transformation on a compact manifold$\}$

An easy calculation shows that it is $\underset{\sim}{\Sigma}^1_1$. If it is not Borel then there is a measure preserving transformation that has finite entropy that is not isomorphic to a smooth transformation on a compact manifold.

Is everything Borel?

Example 1 *G is a compact abelian group, and T is a translation on G by an element g.*

Example 2 *(* "Skew Product" *) Let G be an arbitrary compact group. Suppose $T : Y \to Y$ is a measure preserving transformation and $\phi : Y \to G$ be a measurable function. Define $T_\phi : Y \times G \to Y \times G$ by setting:*

$$T_\phi(y, g) = (Ty, \phi(y)g)$$

Concretely: Let $\mathbb{T}^2$ be the 2-torus and α be an irrational multiple of $2\pi i$. Define $T : \mathbb{T}^2 \to \mathbb{T}^2$ by:

$$T(\xi_0, \xi_1) = (\alpha\xi_0, \xi_0\xi_1)$$

Example 3 (Skew Products into homogeneous spaces)*Let G be a compact group. Let $H \subset G$ be a closed subset. Then G/H carries a Haar measure. Define $T'_\phi : Y \times G/H \to Y \times G/H$ by:*

$$T'_\phi(y, [g]_H) = (T_\phi y, [\phi(y)g]_H)$$

Example 4 *Iterations (possibly transfinite) of these operations. This gives the class of* measure distal *transformations.*

Example 5 *Let Z be a measure space, $T : Y \to Y$ be a measure preserving transformation and $\phi : Y \to \{$MPT's of $Z\}$ be Borel. Let $X = Y \times Z$ and define $T_\phi : X \to X$ by setting:*

$$T_\phi(y, z) = (Ty, \phi(y)(z)).$$

In this example, Y is called a *factor* of X and X an *extension* of Y.

Note that if Y is a factor of X, then

$$L^2(Y) \subset L^2(X)$$

in a natural way.

Example 6 *Suppose that* $X = Y \times Z$ *is an extension with transformation* T_ϕ*. Define the* relative product $T_\phi^2 : Y \times Z \times Z \to Y \times Z \times Z$ *by setting*

$$T_\phi^2(y, z_1, z_2) = (Ty, \phi(y)(z_1), \phi(y)(z_2))$$

Note that this defines an extension of Y.

Theorem $T : X \to X$ is weakly mixing iff $T^2 = T \times T : X \times X \to X \times X$ is ergodic.

Relativized Versions:

- X is a *relatively ergodic* extension of Y iff every invariant $f \in L^2(X)$ lies in $L^2(Y)$.

- X is a *relatively weakly mixing* extension of Y iff $T_\phi^2 : Y \times Z \times Z \to Y \times Z \times Z$ is a relatively ergodic extension of Y.

The Furstenberg Structure Theorem

Theorem Let $T : X \to X$ be an ergodic measure preserving transformation. Then X is a relatively weakly mixing extension of a measure distal transformation Y.

This (and its generalizations) are used to give ergodic theoretic proofs of combinatorial results such as Szemeredi's Theorem.

The combinatorial results are shown by establishing recurrence properties of measure preserving systems by induction on the ordinals required to build the measure distal transformation.

How do you connect sets of integers with ergodic theory?

Example 7

Let $A \subset \mathbb{Z}$ and $\chi_A : \mathbb{Z} \to \{0,1\}$ be its characteristic function. Let $T : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ be the shift map.

Let $X_A = cl\{T^n(\chi_A) : n \in \mathbb{Z}\}$. Then X_A is a shift invariant closed set.

If A has *positive upper Banach density* (i.e. for some $\epsilon > 0$ there are arbitrarily long intervals I such that $\frac{|I \cap A|}{|I|} > \epsilon$), then there is a shift invariant measure μ concentrating on $\{\langle x_n \rangle \in X_A : x_0 = 1\}$.

The recurrence properties of the resulting system give information about homogeneous sequences in A.

Theorem (Foreman-Beleznay) For any countable ordinal α there is an $A \subset \mathbb{Z}$ such that:

- X_A is measure distal
- X_A is not constructed with fewer than α iterated skew products.

Theorem (Foreman-Beleznay) The collection of measure distal transformations is not a Borel subset of the measure preserving transformations.