Ultrafilter Space Methods in Infinite Ramsey Theory

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Outline of Topics

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Λ-semigroups and colorings—a review
A **partial semigroup** is a set $S$ with a binary operation from a *subset* of $S \times S$ to $S$ such that, for $x, y, z \in S$, if one of the products $(xy)z, x(yz)$ is defined, then both are and are equal.

$\Lambda$ a set, $S$ a partial semigroup, and $X$ a set.

A **$\Lambda$-partial semigroup over $S$ based on $X$** is an assignment to each $\lambda \in \Lambda$ of a function from a subset of $X$ to $S$ such that for $s_0, \ldots, s_k \in S$ and $\lambda_0, \ldots, \lambda_k \in \Lambda$ there exists $x \in X$ with $s_0 \lambda_0(x), \ldots, s_k \lambda_k(x)$ defined.
Assume we have a Λ-partial semigroup over $S$ and based on $X$.

A sequence $(x_n)$ of elements of $X$ is **basic** if for all $n_0 < \cdots < n_l$ and $\lambda_0, \ldots, \lambda_l \in \Lambda$

$$\lambda_0(x_{n_0})\lambda_1(x_{n_1}) \cdots \lambda_l(x_{n_l})$$

is defined in $S$.

Assume we additionally have a point based Λ-semigroup $A$ over $(A, \land)$.

A coloring of $S$ is **$A$-tame on** $(x_n)$ if the color of elements of the form (1) with the additional condition $\lambda_k(\bullet) \land \cdots \land \lambda_l(\bullet) \in \Lambda(\bullet)$, for all $k \leq l$, depends only on

$$\lambda_0(\bullet) \land \lambda_1(\bullet) \land \cdots \land \lambda_l(\bullet) \in A.$$
\( \mathcal{A} \) and \( \mathcal{B} \) are \( \Lambda \)-semigroups with \( \mathcal{A} \) being over \( A \) and based on \( X \) and \( \mathcal{B} \) being over \( B \) and based on \( Y \).

A **homomorphism from** \( \mathcal{A} \) **to** \( \mathcal{B} \) **is** a pair of functions \( f, g \) such that \( f : X \to Y, \ g : A \to B \), \( g \) is a homomorphism of semigroups, and, for each \( x \in X \) and \( \lambda \in \Lambda \), we have

\[
\lambda(f(x)) = g(\lambda(x)).
\]
Theorem

Fix a finite set $\Lambda$. Let $S$ be a $\Lambda$-partial semigroup over $S$, and let $A$ be a point based $\Lambda$-semigroup. Let $(f, g): A \to \gamma S$ be a homomorphism. Then for each $D \in f(\bullet)$ and each finite coloring of $S$, there exists a basic sequence $(x_n)$ of elements of $D$ on which the coloring is $A$-tame.
The goal:
produce homomorphisms
from point based $\Lambda$-semigroups $\mathcal{A}$ to $\gamma S$ of interest.
New ones from old ones—tensor products
Fix a partial semigroup $S$.

$\Lambda_0$, $\Lambda_1$ finite sets

$S_i$, for $i = 0, 1$, $\Lambda_i$-partial semigroups over $S$ with $S_i$ is based on $X_i$
Put

\[ \Lambda_0 \star \Lambda_1 = \Lambda_0 \cup \Lambda_1 \cup (\Lambda_0 \times \Lambda_1). \]

Define

\[ S_0 \otimes S_1 \]

to be a \( \Lambda_0 \star \Lambda_1 \)-partial semigroup over \( S \) based on \( X_0 \times X_1 \) as follows: with

\[ \lambda_0, \lambda_1, (\lambda_0, \lambda_1) \in \Lambda_0 \star \Lambda_1 \]

associate partial functions \( X_0 \times X_1 \to S \) by letting

\[ \lambda_0(x_0, x_1) = \lambda_0(x_0), \]
\[ \lambda_1(x_0, x_1) = \lambda_1(x_1), \]
\[ (\lambda_0, \lambda_1)(x_0, x_1) = \lambda_0(x_0)\lambda_1(x_1). \]
$S_0 \otimes S_1$ is a $\Lambda_0 \ast \Lambda_1$-partial semigroup.
Proposition (S.)

Fix semigroups $A$ and $B$. For $i = 0, 1$, let $A_i$ and $B_i$ be $\land_i$-semigroups over $A$ and $B$, respectively. Let

$$(f_0, g): A_0 \to B_0 \quad \text{and} \quad (f_1, g): A_1 \to B_1$$

be homomorphisms. Then

$$(f_0 \times f_1, g): A_0 \otimes A_1 \to B_0 \otimes B_1$$

is a homomorphism.
Let $S_i$, $i = 0, 1$, be $\Lambda_i$-partial semigroups over $S$ based on $X_i$. Consider
\[
\gamma S_0 \otimes \gamma S_1 \text{ and } \gamma (S_0 \otimes S_1).
\]
Both are $\Lambda_0 \star \Lambda_1$-semigroups over $\gamma S$.
The first one is based on $\gamma X_0 \times \gamma X_1$, the second one on $\gamma (X_0 \times X_1)$.
There is a natural map $\gamma X_0 \times \gamma X_1 \to \gamma (X_0 \times X_1)$ given by
\[
(U, \mathcal{V}) \to U \times \mathcal{V},
\]
where, for $C \subseteq X_0 \times X_1$,
\[
C \in U \times \mathcal{V} \iff \{x_0 \in X_0 : \{x_1 \in X_1 : (x_0, x_1) \in C\} \in \mathcal{V}\} \in U.
\]
**Proposition (S.)**

Let $S$ be a partial semigroup. Let $S_i$, $i = 0, 1$, be $\Lambda_i$-partial semigroups over $S$. Then

$$(f, \text{id}_{\gamma S}): \gamma S_0 \otimes \gamma S_1 \to \gamma(S_0 \otimes S_1),$$

where $f(\mathcal{U}, \mathcal{V}) = \mathcal{U} \times \mathcal{V}$, is a homomorphism.
An application—Furstenberg–Katznelson Theorem for located words
A bit more of general theory
\( \mathcal{A} \) a point based \( \Lambda \)-semigroup over a semigroup \( A \)

Fix a natural number \( r \).

Associate with each \( \vec{\lambda} \in \Lambda_{<r} \) an element \( \vec{\lambda}(\bullet) \) of \( A \) by letting

\[
\vec{\lambda}(\bullet) = \lambda_0(\bullet) \land \cdots \land \lambda_m(\bullet),
\]

where \( m < r \) is the length of \( \vec{\lambda} \).

This way we get a finite set \( \Lambda_{<r}(\bullet) \subseteq A \).
$S$ a $\Lambda$-partial semigroup over a partial semigroup $S$

$(x_n)$ a basic sequence in $S$

A coloring of $S$ is $r$-$A$-tame on $(x_n)$ if the color of elements of the form

$$\lambda_0(x_{n_0}) \lambda_1(x_{n_1}) \cdots \lambda_l(x_{n_l})$$

for $n_0 < \cdots < n_l$ and $\lambda_0, \ldots, \lambda_l \in \Lambda$, with the additional condition

$$\lambda_k(\bullet) \wedge \cdots \wedge \lambda_l(\bullet) \in \Lambda_{<r}(\bullet) \text{ for all } k \leq l$$

depends only on

$$\lambda_0(\bullet) \wedge \lambda_1(\bullet) \wedge \cdots \wedge \lambda_l(\bullet) \in A.$$
The following corollary is an apparent generalization of the theorem.

**Corollary**

Fix a finite set $\Lambda$ and a natural number $r$. Let $S$ be a $\Lambda$-partial semigroup, $A$ a point based $\Lambda$-semigroup, and $(f, g): A \to \gamma S$ a homomorphism. Then for each $D \in f(\bullet)$ and each finite coloring of $S$, there exists a basic sequences $(x_n)$ of elements of $D$ on which the coloring is $r$-$A$-tame.
The corollary follows from the theorem and the two propositions.

**Proof.**

We have a homomorphism \((f, g)\) from \(A\) to \(S\).

There is a homomorphism \(A \otimes r \rightarrow (\gamma S) \otimes r\) equal to \((f^r, g)\) by the first proposition.

Note that \(D \times X^{r-1} \in f^r(\bullet)\).

Since, by the second proposition, there is a homomorphism \((\gamma S) \otimes r \rightarrow \gamma (S \otimes r)\), we have a homomorphism

\[ A \otimes r \rightarrow \gamma (S \otimes r), \]

and we are done by the theorem.
Katznelson–Furstenberg for located words
Recall the statement:

Fix a set $F$ of finitely many types. Color, with finitely many colors, all words from $\mathbb{N}$ to $M + N$. There exists a sequence of variable words $(x_n)$ from $\mathbb{N}$ to $M$ with $x_n < x_{n+1}$ and such that the color of words of the form

$$x_{n_0}[i_0] + x_{n_1}[i_1] + \cdots + x_{n_l}[i_l],$$

with $n_0 < n_1 < \cdots < n_l$, depends only on the type of the sequence obtained from $(i_0, \ldots, i_l)$ by deleting all entries less than $M$, provided this type belongs to $F$.

The type of $(j_0, \ldots, j_k)$ is the sequence obtained from $(j_0, \ldots, j_k)$ by shortening each run of identical numbers to a single number.
**Monoid** $\Lambda$:

$L, \Gamma$ finite disjoint sets, $e$ an element not in $L \cup \Gamma$.

$$\Lambda = L \cup \Gamma \cup \{e\},$$

with

$$\lambda_0 \cdot \lambda_1 = \begin{cases} 
\lambda_0, & \text{if } \lambda_1 = e; \\
\lambda_1, & \text{if } \lambda_1 \in L \cup \Gamma,
\end{cases}$$

is a monoid with the identity element $e$. 
**Semigroup** \( A \):
\[
\Gamma \text{ disjoint from } \{0, 1\}
\]
Let
\[
A
\]
be freely generated by \( \Gamma \cup \{0, 1\} \) subject to the relations
\[
a \land a = a \quad \text{and} \quad a \land 1 = 1 \land a = a.
\]
Point based $\Lambda$-semigroup $A$ over $A$:

Assignment to elements of $\Lambda$ of functions $\{ullet\} \rightarrow A$:
For $\lambda \in \Lambda$, let $\lambda(\bullet) \in A$ be

$$
\lambda(\bullet) = \begin{cases} 
0, & \text{if } \lambda = e; \\
1, & \text{if } \lambda \in L; \\
\lambda, & \text{if } \lambda \in \Gamma.
\end{cases}
$$

This defines a point based $\Lambda$-semigroup over $A$ called $A$. 

Proposition

\( U \) a compact semigroup, \( V \subseteq U \) a compact subsemigroup, \( H \subseteq V \) a compact two-sided ideal in \( V \). Assume \( \Lambda \) acts on \( U \) by continuous endomorphisms so that \( V \) is \( L \)-invariant.

Then there exists a homomorphism \( (f, g): \mathcal{A} \to U_\Lambda \) with \( f(\bullet) \in H \).
Partial semigroup:

\[ S = (L \cup \Gamma)\text{-words and variable } (L \cup \Gamma)\text{-words} \]
\[ T = L\text{-words and variable } L\text{-words} \]
\[ D = \text{variable } L\text{-words} \]

Note: \( D \subseteq T \subseteq S \), \( D \) a two-sided ideal in \( T \), \( T \) a subsemigroup of \( S \)

Action of \( \Lambda \) on \( S \):

\[
\lambda(x) = \begin{cases} 
  x, & \text{if } x \text{ is a } (L \cup \Gamma)\text{-word or } \lambda = e; \\
  x[\lambda], & \text{if } x \text{ is a variable } (L \cup \Gamma)\text{-word and } \lambda \in L \cup \Gamma.
\end{cases}
\]
Then $H = \gamma D$ is a compact two-sided ideal in $V = \gamma T$, which is a subsemigroup of $U = \gamma S$.

Note that $V$ is $L$-invariant.
So by the last theorem and the corollary:
given $r > 0$, there is a basic sequence $(x_n)$ in $D$ such that the color of
\[ \lambda_0(x_{n_0}) + \cdots + \lambda_l(x_{n_l}) = x_{n_0}[\lambda_0] + \cdots + x_{n_l}[\lambda_l] \]
depends only on
\[ \lambda_0(\bullet) \wedge \cdots \wedge \lambda_l(\bullet) \in A \]
as long as
\[ \lambda_0(\bullet) \wedge \cdots \wedge \lambda_l(\bullet) \in \Lambda_{<r}(\bullet). \]

Each finite set of types is included in $\Lambda_{<r}(\bullet)$ for some $r$. 
A sketch of an application—
the Hales–Jewett theorem for left-variable words
Monoid $\Lambda$:
\[ \Lambda = L \cup \{e\} \text{ with } e \notin L, \text{ with multiplication} \]
\[ \lambda_0 \cdot \lambda_1 = \begin{cases} 
\lambda_0, & \text{if } \lambda_1 = e; \\
\lambda_1, & \text{if } \lambda_1 \in L,
\end{cases} \]
is a monoid with the identity element $e$.

Semigroup $A$:
\[ A = \{0, 1\} \text{ with } i \wedge j = \min(i, j). \]

Assignment $\Lambda \to A$:
For $\lambda \in \Lambda$, let $\lambda(\bullet) \in A$ be
\[ \lambda(\bullet) = \begin{cases} 
0, & \text{if } \lambda = e; \\
1, & \text{if } \lambda \neq e.
\end{cases} \]
Proposition (S.)

$U$ a compact semigroup, $H$ a compact two-sided ideal in $U$, $G \subseteq H$ a right ideal. Assume $\Lambda$ acts on $U$ by continuous endomorphisms. Then there exist $u \in H$, a homomorphism $g : A \to U$, and $v \in G$ such that $\lambda(u) = g(\lambda(\bullet))$ and $uv = u$. 
Some questions
Λ a monoid
\( \hat{\Lambda} \) the semigroup generated freely by Λ subject to the relations
\[ e \leq \lambda = \lambda \leq e = e. \]

\( U \) a compact semigroup on which Λ acts by continuous endomorphisms
\( H \) a compact two sided ideal in \( U \)

**Question.** For what Λ, does there exist
\[ u \in H \text{ and a homomorphism } g: \hat{\Lambda} \to U \]
such that for each \( \lambda \in \Lambda \)
\[ \lambda(u) = g(\lambda)? \]
The question amounts to asking for what $\Lambda$ there exists a homomorphism

$$(f, g): \mathcal{A} \to U_\Lambda \text{ with } f(\bullet) \in H,$$

where $\mathcal{A}$ is the point based $\Lambda$-semigroup over $\hat{\Lambda}$ given by $\lambda(\bullet) = \lambda \in \hat{\Lambda}$. 
Fix $M > 0$. Let $E$ be the monoid with composition of all non-decreasing functions $s: M \to M$ such that

$$s(0) = 0 \text{ and } s(i + 1) \leq s(i) + 1, \text{ for all } i < M - 1.$$ 

**Question.** Does the question above have positive answer for $\Lambda = E$?
Some questions

$E$ as above acting on a compact semigroup $U$ with a compact two-sided ideal $H$

$A = M$ with $i \wedge j = \min(i, j)$

For $s \in E$, let

$$s(\bullet) = M - (1 + \max s) \in A.$$

**Question.** Do there exist $u \in H$ and a homomorphism $g: A \rightarrow U$

such that

$$s(u) = g(s(\bullet))?$$

A positive answer to this question implies the generalized Gowers’ theorem.