Ultrafilter Space Methods in Infinite Ramsey Theory

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Outline of Topics

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T. J. Carlson, *Some unifying principles in Ramsey theory*, 1988.

S. Todorcevic, *Introduction to Ramsey Spaces*, 2010.

S. Solecki, *Abstract approach to finite Ramsey theory and a self-dual Ramsey theorem*, 2013.

M. Gromov, *A number of questions*, 2014.

Examples

Words

Fix $M \in \mathbb{N}$, $M > 0$.

A **word** w (often called **located word**) is a function from $\mathbb{N}$ to M with finitely many non-zero values.

The **domain of** w is the finite set $\{n \in \mathbb{N} \colon w(n) > 0\}$.

For words, v, w, we write

$$v < w$$

if each element of the domain of v precedes each element of the domain of w.

By convention

$$v + w$$

is defined precisely when $v < w$ and is then equal to pointwise addition of v and w.

A **variable word** x (often called **located variable word**) is a finite non-empty set $d_x \subseteq \mathbb{N}$ and a function $f_x \colon \mathbb{N} \setminus d_x \to M$ with finitely many non-zero values

For $i \in \mathbb{N}$,

$$x[i]$$

is the word that is the union of f_x and the function constantly equal to i on d_x.

For variable words x, y, we write

$$x < y$$

if all elements of $\mathrm{domain}(f_x) \cup d_x$ precede all elements of $\mathrm{domain}(f_y) \cup d_y$.

Furstenberg–Katznelson theorem for located words

For a sequence of natural numbers $(i_0, \ldots, i_l)$, the **type of** $(i_0, \ldots, i_l)$ is the sequence obtained from $(i_0, \ldots, i_l)$ by shortening each run of identical numbers to a single number.

Katznelson–Furstenberg, S.(for located words)**:**

Fix a set F of finitely many types. Color, with finitely many colors, all words from $\mathbb{N}$ to $M + N$. There exists a sequence of variable words (x_n) from $\mathbb{N}$ to M with $x_n < x_{n+1}$ and such that the color of words of the form

$$x_{n_0}[i_0] + x_{n_1}[i_1] + \cdots + x_{n_l}[i_l],$$

with $n_0 < n_1 < \cdots < n_l$, depends only on the type of the sequence obtained from $(i_0, \ldots, i_l)$ by deleting all entries less than M, provided this type belongs to F.

For example:

one color if $i_p \leq i_q$, for $M \leq i_p, i_q$ and $p \leq q$,
another color if $i_p \geq i_q$, for $M \leq i_p, i_q$ and $p \leq q$.

Hales–Jewett Theorem for left variable words

A variable word x is **left-variable** if the minimal element of d_x is smaller than the minimal element of the domain of f_x.

Carlson–Simpson, Todorcevic:

Color, with finitely many colors, all words from $\mathbb{N}$ to M. There exist a word w and a sequence of left-variable words (x_n) from $\mathbb{N}$ to M with $w < x_n < x_{n+1}$ such that the color of words of the form

$$w + x_{n_0}[i_0] + x_{n_1}[i_1] + \cdots + x_{n_l}[i_l],$$

with

$$n_0 < n_1 < \cdots < n_l \;\; \text{and} \;\; i_0, \ldots, i_l < M,$$

is fixed.

Gowers' theorem

Let $T\colon \mathbb{N} \to \mathbb{N}$ be defined by

$$T(n) = \begin{cases} n-1, & \text{if } n > 0; \\ 0, & \text{if } n = 0. \end{cases}$$

Extend T to words by applying it pointwise.

For a sequence of natural numbers $(i_0, \ldots, i_l)$, the **type of** $(i_0, \ldots, i_l)$ is the number

$$\min(i_0, \ldots, i_l).$$

Gowers:

Color, with finitely many colors, all words from $\mathbb{N}$ to M. There exists a sequence of words (x_n) from $\mathbb{N}$ to M with

$$x_n < x_{n+1} \text{ and } \max x_n = M - 1$$

and such that the color of words of the form

$$T^{i_0}(x_{n_0}) + T^{i_1}(x_{n_1}) + \cdots + T^{i_l}(x_{n_l}),$$

with $n_0 < n_1 < \cdots < n_l$, depends only on the type of $(i_0, \ldots, i_l)$.

A strengthening of Gowers' theorem—not a theorem

Let E be the set of all non-decreasing functions $s\colon M \to M$ such that

$$s(0) = 0 \text{ and } s(i+1) \leq s(i) + 1, \text{ for all } i < M - 1.$$

Note that $T \in E$.

E acts on words pointwise.

For a sequence $(s_0, \ldots, s_l)$ of elements of E, define the **type of** $(s_0, \ldots, s_l)$ to be

$$\max(s_0[M], \ldots, s_l[M]).$$

Statement:

Color, with finitely many colors, all words from $\mathbb{N}$ to M. There exists a sequence of words (x_n) from $\mathbb{N}$ to M with

$$x_n < x_{n+1} \ \text{ and } \ \max x_n = M - 1$$

and such that the color of words of the form

$$s_0(x_{n_0}) + s_1(x_{n_1}) + \cdots + s_l(x_{n_l}),$$

with

$$n_0 < n_1 < \cdots < n_l \ \text{ and } \ s_0, \ldots, s_l \in E,$$

depends only on the type of $(s_0, \ldots, s_l)$.

The statement is **not** known to be true, but the finite version is true. This is a recent result of Bartošova and Kwiatkowska using some ideas of Tyros.

Structures

Λ-semigroups

A **partial semigroup** is a set S with a binary operation from a *subset* of $S \times S$ to S such that, for $x, y, z \in S$,
if one of the products $(xy)z$, $x(yz)$ is defined, then both are and are equal.

S, T partial semigroups
$h\colon S \to T$ is a **homomorphism** if, for $s_1, s_2 \in S$, whenever $s_1 s_2$ is defined, so is $h(s_1)h(s_2)$ and

$$h(s_1)h(s_2) = h(s_1 s_2).$$

A **semigroup** is a partial semigroup with total multiplication.

Λ a set, S a partial semigroup, and X a set

A **Λ-partial semigroup over S based on X** is an assignment to each $\lambda \in \Lambda$ of a function from a subset of X to S such that for $s_0, \ldots, s_k \in S$ and $\lambda_0, \ldots, \lambda_k \in \Lambda$ there exists $x \in X$ with $s_0\lambda_0(x), \ldots, s_k\lambda_k(x)$ defined.

A **Λ-semigroup over A based on X** is a Λ-partial semigroup over A based on X such that A a semigroup and the domain each $\lambda \in \Lambda$ is equal to X.

A Λ-semigroup is **point based** if X consist of one point, usually denoted by $\bullet$.

$\mathcal{A}$ and $\mathcal{B}$ are Λ-semigroups with $\mathcal{A}$ being over A and based on X and $\mathcal{B}$ being over B and based on Y.

A **homomorphism from $\mathcal{A}$ to $\mathcal{B}$** is a pair of functions f, g such that $f \colon X \to Y$, $g \colon A \to B$, g is a homomorphism of semigroups, and, for each $x \in X$ and $\lambda \in \Lambda$, we have

$$\lambda(f(x)) = g(\lambda(x)).$$

Colorings and Λ-semigroups

Assume we have a Λ-partial semigroup over S and based on X.

A sequence (x_n) of elements of X is **basic** if for all $n_0 < \cdots < n_l$ and $\lambda_0, \ldots, \lambda_l \in \Lambda$

$$\lambda_0(x_{n_0})\lambda_1(x_{n_1})\cdots\lambda_l(x_{n_l}) \tag{1}$$

is defined in S.

Assume we additionally have a point based Λ-semigroup $\mathcal{A}$ over $(A, \wedge)$.

A coloring of S is **$\mathcal{A}$-tame on (x_n)** if the color of elements of the form (1) with the additional condition $\lambda_k(\bullet) \wedge \cdots \wedge \lambda_l(\bullet) \in \Lambda(\bullet)$ for all $k \leq l$ depends only on

$$\lambda_0(\bullet) \wedge \lambda_1(\bullet) \wedge \cdots \wedge \lambda_l(\bullet) \in A.$$

A Λ-semigroup from a Λ-partial semigroup— following Bergelson, Blass, Hindman

$\mathcal{S}$ a Λ-partial semigroup over S based on X

γX is the set of all ultrafilters $\mathcal{V}$ on X such that for $s \in S$ and $\lambda \in \Lambda$

$$\{x \in X \colon s\lambda(x) \text{ is defined}\} \in \mathcal{V}.$$

γS is the set of all ultrafilters $\mathcal{U}$ on S such that for $s \in S$

$$\{t \in S \colon st \text{ is defined}\} \in \mathcal{U}.$$

γS is a semigroup with convolution: $(\mathcal{U}, \mathcal{V}) \to \mathcal{U} * \mathcal{V}$, where

$$C \in \mathcal{U} * \mathcal{V} \iff \{s \in S \colon \{t \in S \colon st \in C\} \in \mathcal{V}\} \in \mathcal{U}.$$

In other words,

$$C \in \mathcal{U} * \mathcal{V} \iff \forall^{\mathcal{U}} s \, \forall^{\mathcal{V}} t \, (st \in C).$$

Each λ induces a function from γX to γS by the formula

$$C \in \lambda(\mathcal{V}) \text{ iff } \lambda^{-1}(C) \in \mathcal{V}.$$

This procedure gives a Λ-**semigroup** $\gamma\mathcal{S}$ **over** γS **based on** γX.

Theorem

Theorem (S.)

Fix a finite set Λ. Let $\mathcal{S}$ be a Λ-partial semigroup over S, and let $\mathcal{A}$ be a point based Λ-semigroup. Let $(f, g)\colon \mathcal{A} \to \gamma\mathcal{S}$ be a homomorphism.

Then for each $D \in f(\bullet)$ and each finite coloring of S, there exists a basic sequence (x_n) of elements of D on which the coloring is $\mathcal{A}$-tame.

The goal:

produce homomorphisms
from point based Λ-semigroups $\mathcal{A}$ to $\gamma\mathcal{S}$ of interest.

A point based Λ-semigroup $\mathcal{A}$ over A is determined by an assignment

$$\Lambda \ni \lambda \to a_\lambda \in A.$$

If $\mathcal{S}$ is based on X and over S, a homomorphism from $\mathcal{A}$ to $\gamma\mathcal{S}$ is determined by an ultrafiler $\mathcal{V} \in \gamma X$ and a homomorphism $g \colon A \to \gamma S$ such that for $\lambda \in \Lambda$

$$\lambda(\mathcal{V}) = g(a_\lambda).$$

New ones from nothing—monoid actions and Gowers' theorem

Construction

A **monoid** is a semigroup with a distinguished element e, which is its identity.

e always acts as identity.

S a partial semigroup such that for all $s_1, \ldots, s_k$ there is $t \in S$ such that $s_1 t, \ldots, s_k t$ are defined.

Λ a monoid

Λ acts on S by endomorphisms so that, for $s, t \in S$, if st is defined, then so is $s\lambda(t)$ for each $\lambda \in \Lambda$.

Form a Λ-**partial semigroup** S_Λ **over** S**, based on** S, where each $\lambda \in \Lambda$ is interpreted as the function given by the action.

Example.

Partial semigroup S:

$S =$ the set of all words with $+$ as defined before

Monoid Λ:

$\Lambda = M$ with the following multiplication: for $i, j \in M$, let

$$i \cdot j = \min(i + j, M - 1)$$

Action of Λ on S:

Λ acts on S by

$$i(w) = T^i(w)$$

So Λ acts on γS by continuous endomorphisms.

Topology

A **compact semigroup** U is a semigroup whose underlying set is a compact space such that

$$U \ni u \to uv \in U$$

is continuous for each $v \in U$.

S a partial semigroup as above, γS is a semigroup

γS has a natural topology with basis consisting of sets of the form

$$\{\mathcal{U} \in \gamma S \colon C \in \mathcal{U}\},$$

where $C \subseteq S$.

γS is compact.

Multiplication on γS is continuous on the left.

So γS **is a compact semigroup**.

Each endomorphism

$$\lambda\colon S \to S$$

induces a **continuous endomorphism**

$$\lambda\colon \gamma S \to \gamma S$$

by the formula

$$C \in \lambda(\mathcal{U}) \iff \lambda^{-1}(C) \in \mathcal{U},$$

for $C \subseteq S$ and $\mathcal{U} \in \gamma S$.

Back to the construction

Given S and Λ as before, construct the Λ-partial semigroup S_Λ.

Then γS_Λ is a Λ-semigroup over γS based on γS, and each $\lambda \in \Lambda$ is a continuous endomorphism of γS.

Abstractly we have:

a compact semigroup U and a monoid Λ acting on U by continuous endomorphisms,

which we view as a Λ-semigroup U_Λ over U based on U, with each λ interpreted as the continuous endomorphism from the action.

A homomorphism from a point based Λ-semigroup to U_Λ:

If $\mathcal{A}$ is a pointed Λ-semigroup over A, then a homomorphism (f,g) from $\mathcal{A}$ to U_Λ is

$$f(\bullet) \in U \text{ and a homomorphism } g\colon A \to U$$

such that for $\lambda \in \Lambda$

$$\lambda(f(\bullet)) = g(\lambda(\bullet)).$$

So, after setting $u_\bullet = f(\bullet) \in U$ and $a_\lambda = \lambda(\bullet) \in A$,

$$\lambda(u_\bullet) = g(a_\lambda). \tag{2}$$

So a homomorphism from $\mathcal{A}$ to U_Λ is determined by

the assignment $\Lambda \ni \lambda \to a_\lambda \in A$,

an element $u_\bullet \in U$,

a homomorphism $g\colon A \to U$

such that (2) holds.

Example:

Semigroup A:

$A = M$ with the following multiplication: for $i, j \in M$, let

$$i \wedge j = \min(i, j)$$

Point based Λ-semigroup $\mathcal{A}$:

Assignment $\Lambda \to A$: the identity function.

This defines a point based Λ-semigroup $\mathcal{A}$: $i(\bullet) = i$.

Need: $u_\bullet \in \gamma S$ and a homomorphism $g\colon A \to \gamma S$ such that

$$i(u_\bullet) = g(i).$$

Set:

$D =$ the set of all $w \in S$ with $\max w = M - 1$

Note: D is a two-sided ideal in S

Need: $u_\bullet \in \gamma S$ and a homomorphism $g\colon A \to \gamma S$ such that

$$i(u_\bullet) = g(i) \text{ and } D \in u_\bullet.$$

Note that

$$D \in u_\bullet \iff u_\bullet \in H,$$

where

$$H = \{\mathcal{U} \in \gamma S\colon D \in \mathcal{U}\} = \gamma D.$$

H is a compact two-sided ideal in γS.

Proposition

U a compact semigroup, $\Lambda (= M)$ acts on U by continuous endomorphisms, $H \subseteq U$ be a compact two-sided ideal.
Then there exists $u_\bullet \in H$ and a homomorphism $g\colon A (= M) \to U$ such that for each $i \in M$

$$i(u_\bullet) = g(i).$$

Proposition

U a compact semigroup, Λ acts on U by continuous endomorphisms, $H \subseteq U$ a compact two-sided ideal. Then there exists a homomorphism $(f, g)\colon \mathcal{A} \to U_\Lambda$ with $f(\bullet) \in H$.

From the proposition and the theorem, we get a basic sequence (w_n) of elements of D such that the color of

$$T^{i_0}(w_{n_0}) + \cdots + T^{i_l}(w_{n_l}) = i_0(w_{n_0}) + \cdots + i_l(w_{n_l})$$

depends only on

$$i_0 \wedge \cdots \wedge i_l = \min(i_0, \ldots, i_l) = \text{type of } (i_0, \ldots, i_l).$$

This is Gowers' theorem.