A crash course in complex analysis

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0 Overview

The fundamental theorem of algebra reveals that complex polynomials enjoy certain advantages over real polynomials. It turns out that this is part of a more general phenomenon for differentiable maps defined on open sets of $\mathbb{C}$ and taking values in complex Banach spaces. The purpose of these notes is to give a brief introduction to the study of the special properties of these maps, which is known as complex analysis. The reader will need some knowledge of the differential calculus of maps between Banach spaces, the theory of power series, and the theory of the Cauchy integral of Banach-valued functions on a compact interval (the Bochner integral will also suffice, though that's a bit like saying it's okay to substitute a tank for a horse in battle). All of this can be found, for instance, in the fantastic book of Dieudonné [2]

As a warning, the title should be taken seriously: this is meant as a crash course and not a systematic study of complex analysis. Many (most, really) standard topics are completely ignored. The notes grew out of a smaller set of notes delivered during the last week of the honors course Mathematical Studies: Analysis II at Carnegie Mellon in the Spring of 2020. They are meant as an amuse bouche preceding a more serious course in complex analysis. For the latter the author recommends the books of Conway [1], Lang [3], and Needham [4] as well as the appropriate sections in Dieudonné's book [2].

In Section 1 we introduce holomorphic functions. We then define various types of special paths, including loops, roads, and circuits (it's a good thing this is all we need, as no one could keep straight a theory of avenues, boulevards, lanes, parkways, etc). We then develop some homotopy theory for these paths and record some approximation results that will be essential in our subsequent analysis. Most importantly, we introduce a version of a complex line integral along roads.

Section 2 is an ode to the Cauchy-Goursat theorem, which roughly speaking, shows that the integration theory of holomorphic functions along loops (paths that start and end at the same point) is horribly boring in the sense that such integrals always vanish. However, this vanishing has a number of truly remarkable consequences, some of which we then develop. These include the fact that holomorphic functions are analytic and the Cauchy integral formula, which essentially shows that a holomorphic function can be entirely reconstructed in an open set by its values on the boundary of the set. We also develop the acme of complex integration: integration on loop chains. Together with a brief study of loop chain homology, this then allows us to build the ultimate general form of Cauchy-Goursat for homologous loop chains, which is incredibly useful in practice.

In Section 3 we explore some more of the implications of the incredible rigidity of holomorphic functions. We study their zero sets and derive a number of powerful estimates. We prove the argument principle and Rouché's theorem and show how to use these to gain very useful information about polynomials, such as the fact that the roots of a polynomial depend continuously on its coefficients. We also enumerate some of the special properties of holomorphic maps from $\mathbb{C}$ to (shining) $\mathbb{C}$, including holomorphic versions of the inverse function and open mapping theorems.

Finally, in Section 4 we study Laurent series in annuli. This gives us a classification scheme for isolated singularities: removable, finite order poles, and essential. We then develop the basic theory of meromorphic functions, which are holomorphic functions away from a set of isolated poles. We complete the notes with the residue theorem and some applications in computing interesting integrals.

We conclude the overview with some remarks on notation used throughout the notes.

1. We write $B(z, r)$ for open balls and $B[z, r]$ for closed balls.

2. We write $\mathcal{L}(X, Y)$ for the set of bounded linear maps between two complex Banach spaces X

and Y. We write $\mathcal{L}^n(X;Y)$ for the set of bounded $n-$linear maps from X to Y.

3. Let X be a Banach space and $[a,b] \subset \mathbb{R}$ be a compact interval. The Cauchy integral is built out of the space of regulated maps,

$$\text{Reg}([a,b];X) = \{f : [a,b] \to X \mid f \text{ is regulated}\}, \tag{0.1}$$

where we say that $f : [a,b] \to X$ is regulated if it has left and right limits at each point in $[a,b]$. The key fact (which we won't prove here, but can be found for instance in [2]) is that $\text{Reg}([a,b];X)$ is the uniform closure of the space of step functions from $[a,b]$ to X. On the space of step functions, it's a trivial matter to define the integral (form Riemann sums!), and the resulting map is then easily shown to be Lipschitz. We can then extend the integral from the space of step functions to its uniform closure and arrive at the Cauchy integral

$$\int_a^b : \text{Reg}([a,b];X) \to X, \tag{0.2}$$

which is a bounded linear map satisfying all of the properties one would hope for, including versions of the fundamental theorems of calculus. Once again, we refer to [2] for a survey of this integral, which it seems was actually introduced by Bourbaki and just named after Cauchy.

1 Complex calculus

We begin by surveying some of the extra structures present in the calculus of functions defined on subsets of $\mathbb{C}$.

1.1 Holomorphic functions and the Cauchy-Riemann equations

The following definition gives a new name to the concept of differentiability. This new name is certainly not necessary, but is widely used in the literature, so we adopt the same convention here.

Definition 1.1. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open, X be a complex normed vector space, and $f : U \to X$. We say f is holomorphic at $z \in U$ if the limit*

$$f'(z) = \lim_{w \to z} \frac{f(z) - f(w)}{z - w} \in X \tag{1.1}$$

exists. We say f is holomorphic if it is holomorphic at each $z \in U$, in which case we define $f^{(1)} = f' : U \to X$. We inductively define $f^{(k+1)} = (f^{(k)})'$ as per usual, provided these exist. We say f is smooth if $f^{(k)} : U \to X$ exists for all $k \in \mathbb{N}$.

Remark 1.2. *Clearly, f is holomorphic at $z \in U$ if and only if f is differentiable at z, and in either case, $Df(z) \in \mathcal{L}(\mathbb{C};X)$ and $f'(z) \in X$ are related via*

$$Df(z)h = hf'(z) \text{ for } h \in \mathbb{C}. \tag{1.2}$$

Similarly, the above notion of smooth coincides with the usual one.

Let's consider some examples.

Example 1.3. Let X be a complex Banach space and let $x_0, \ldots, x_n \in X$. The complex polynomial $p : \mathbb{C} \to X$ given by $p(z) = \sum_{k=0}^{n} z^k x_k$ is holomorphic, and $p'(z) = \sum_{k=1}^{n} k z^{k-1} x_k$. Consequently, p' is holomorphic, and we can iterate to deduce that p is smooth. △

Example 1.4. Let X be a complex Banach space, $z_0 \in \mathbb{C}$, and $\{x_n\}_{n=0}^{\infty} \subseteq X$. Assume that

$$R = \left(\limsup_{n\to\infty} \|x_n\|_X^{1/n} \right)^{-1} \in (0, \infty]. \tag{1.3}$$

The maps $\mathbb{C} \ni z \mapsto z^n x_n \in X$ belong to $\mathcal{L}^n(\mathbb{C}; X)$ and have operator norms equal to $\|x_n\|_X$. Consequently, the theory of power series guarantees that the map $f : B(z_0, R) \to X$ given by

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n x_n \tag{1.4}$$

is smooth. In particular, for each $k \in \mathbb{N}$

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} (z - z_0)^{n-k} x_n, \tag{1.5}$$

where the series converges pointwise in $B(z_0, R)$ and uniformly in $B[z_0, S]$ for each $0 < S < R$. △

Example 1.5. The map $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}$ is not holomorphic. Indeed, for $h = re^{i\theta}$ we have

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{h}}{h} = e^{-2i\theta}, \tag{1.6}$$

from which we conclude that the difference quotient can have no limit. △

The next result builds an important bridge between holomorphic functions valued in $\mathbb{C}$ and differentiable vector fields on open subsets of $\mathbb{R}^2$.

Theorem 1.6 (Cauchy-Riemann)**.** *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$. Define the open set*

$$\tilde{U} = \{x \in \mathbb{R}^2 \mid x_1 + ix_2 \in U\} \tag{1.7}$$

and the vector field $F : \tilde{U} \to \mathbb{R}^2$ via $F(x) = (\operatorname{Re} f(x_1 + ix_2), \operatorname{Re} f(x_1 + ix_2))$. Let $z \in U$ be given by $z = x_1 + ix_2$ for $x \in \tilde{U}$. Then the following are equivalent.

1. *f is holomorphic at z.*

2. *F is differentiable at x and satisfies the Cauchy-Riemann equations:*

$$\partial_1 F_1(x) = \partial_2 F_2(x) \text{ and } \partial_2 F_1(x) = -\partial_1 F_2(x). \tag{1.8}$$

Proof. We begin with the proof that first item implies the second. Suppose that f is holomorphic at z. Note that $t \in (-\varepsilon, \varepsilon)\backslash\{0\}$ for $\varepsilon > 0$ sufficiently small, we have that

$$\lim_{t\to 0} \frac{f(z+t) - f(z)}{t} = f'(z) \text{ and } \lim_{t\to 0} \frac{f(z+it) - f(z)}{t} = i \lim_{t\to 0} \frac{f(z+it) - f(z)}{it} = if'(z). \tag{1.9}$$

Then we can compute

$$\partial_1 F(x) = \lim_{t\to 0} \frac{F(x + te_1) - F(x)}{t} = (\operatorname{Re} f'(x_1 + ix_2), \operatorname{Im} f'(x_1 + ix_2)) \tag{1.10}$$

and

$$\partial_2 F(x) = \lim_{t\to 0} \frac{F(x+te_2)-F(x)}{t} = (\operatorname{Re} if'(x_1+ix_2), \operatorname{Im} if'(x_1+ix_2))$$
$$= (-\operatorname{Im} f'(x_1+ix_2), \operatorname{Re} f'(x_1+ix_2)). \quad (1.11)$$

Hence

$$(\partial_1 F_1(x), \partial_1 F_2(x)) = (\partial_2 F_2(x), -\partial_2 F_1(x)), \quad (1.12)$$

which are the Cauchy-Riemann equations.

We now prove the converse. Let $r > 0$ be such that $B_{\mathbb{C}}(z,r) \subset U$. For $h \in \mathbb{C}$ with $0 < |h| < r$ write $h = \eta_1 + i\eta_2 \in \mathbb{C}$ for $\eta \in \mathbb{R}^2$ with $0 < |\eta| < r$. Set $w = \partial_1 F_1(x) + i\partial_1 F_2(x) \in \mathbb{C}$, and note that the Cauchy-Riemann condition requires that

$$\begin{aligned} hw &= (\eta_1\partial_1 F_1(x) - \eta_2\partial_1 F_2(x)) + i(\eta_1\partial_1 F_2(x) + \eta_2\partial_1 F_1(x)) \\ &= (\eta_1\partial_1 F_1(x) + \eta_2\partial_2 F_1(x)) + i(\eta_1\partial_2 F_2(x) + \eta_2\partial_2 F_2(x)) = \nabla F_1(x)\cdot\eta + i\nabla F_2(x)\cdot\eta. \end{aligned} \quad (1.13)$$

Then

$$f(z+h) - f(z) - hw = (F_1(x+\eta) - F_1(x) - \nabla F_1(x)\cdot\eta) + i(F_2(x+\eta) - F_2(x) - \nabla F_2(x)\cdot\eta) \quad (1.14)$$

and hence

$$\begin{aligned} \left|\frac{f(z+h)-f(z)}{h} - w\right|^2 &= \left|\frac{F_1(x+\eta) - F_1(x) - \nabla F_1(x)\cdot\eta}{|\eta|}\right|^2 \\ &\quad + \left|\frac{F_2(x+\eta) - F_2(x) - \nabla F_2(x)\cdot\eta}{|\eta|}\right|^2 \to 0 \text{ as } h \to 0. \end{aligned} \quad (1.15)$$

Thus f is holomorphic at z and $f'(z) = w$. □

Let's consider an important example of how the Cauchy-Riemann equations are used to find holomorphic maps.

Example 1.7. Consider the negative real axis $N = \{z \in \mathbb{C} \mid \operatorname{Re} z \le 0 \text{ and } \operatorname{Im} z = 0\}$, and set $U = \mathbb{C}\backslash N$. For each $z \in U$ there exists a unique $0 < r < \infty$ and $-\pi < \theta < \pi$ such that

$$z = re^{i\theta}. \quad (1.16)$$

Clearly $r = |z|$. To compute θ conveniently we use the tangent half-angle formula

$$\tan(\theta/2) = \frac{\sin(\theta)}{1+\cos(\theta)} = \frac{r\sin(\theta)}{r + r\cos(\theta)} = \frac{\operatorname{Im} z}{|z| + \operatorname{Re} z} \quad (1.17)$$

to arrive at the expression

$$\theta = 2\arctan\left(\frac{\operatorname{Im} z}{|z| + \operatorname{Re} z}\right) \text{ for } z \in U. \quad (1.18)$$

Now define the map $L : U \to \mathbb{C}$ via

$$L(z) = \log|z| + 2i\arctan\left(\frac{\operatorname{Im} z}{|z| + \operatorname{Re} z}\right). \quad (1.19)$$

If we define $\theta(z)$ and $r(z)$ as above, then L satisfies

$$e^{L(z)} = z \text{ for } z \in U. \tag{1.20}$$

We claim that L is holomorphic in U. To see this let $\tilde{U} = \{x \in \mathbb{R}^2 \mid x_1 + ix_2 \in U\}$ and $F : \tilde{U} \to \mathbb{R}^2$ via $F(x) = (\operatorname{Re} L(x_1 + ix_2), \operatorname{Im} L(x_1 + ix_2))$. Then

$$F_1(x) = \log|x| \text{ and } F_2(x) = 2\arctan\left(\frac{x_2}{|x| + x_1}\right), \tag{1.21}$$

which is smooth in $\tilde{U}$ and satisfies

$$\partial_1 F_1(x) = \frac{x_1}{|x|^2} \text{ and } \partial_2 F_1(x) = \frac{x_2}{|x|^2} \tag{1.22}$$

as well as

$$\partial_1 F_2(x) = \frac{2}{1 + \left(\frac{x_2}{|x|+x_1}\right)^2}\left(\frac{-x_2(1 + \frac{x_1}{|x|})}{(x_1 + |x|)^2}\right) = -\frac{2(x_1 + |x|)^2}{2|x|(x_1 + |x|)} \cdot \frac{x_2(x_1 + |x|)}{|x|(x_1 + |x|)^2} = -\frac{x_2}{|x|^2} \tag{1.23}$$

and

$$\partial_2 F_2(x) = \frac{2}{1 + \left(\frac{x_2}{|x|+x_1}\right)^2}\left(\frac{(x_1 + |x|) - x_2^2/|x|}{(x_1 + |x|)^2}\right) = \frac{2(x_1 + |x|)^2}{2|x|(x_1 + |x|)} \cdot \frac{x_1(x_1 + |x|)}{|x|(x_1 + |x|)^2} = \frac{x_1}{|x|^2}. \tag{1.24}$$

Then F satisfies the Cauchy-Riemann equations in $\tilde{U}$, and so L is holomorphic. △

This suggests some notation.

Definition 1.8. *Let* $N = \{z \in \mathbb{C} \mid \operatorname{Re} z \le 0$ *and* $\operatorname{Im} z = 0\}$. *The holomorphic function* $\log : \mathbb{C}\backslash N \to \mathbb{C}$ *is defined via*

$$\log(z) = \log|z| + 2i\arctan\left(\frac{\operatorname{Im} z}{|z| + \operatorname{Re} z}\right). \tag{1.25}$$

1.2 Paths, loops, roads, and circuits

We now seek to define a version of complex line integrals. We do so in a manner that allows for functions taking values in complex Banach spaces, and for this we employ the Cauchy integral. We begin by introducing some notation related to this integral.

Definition 1.9. *Suppose* $a, b \in \mathbb{R}$ *with* $a < b$ *and let* X *be a Banach space. We say that* $F \in C^0([a,b];X)$ *is a primitive if there exists* $f \in \operatorname{Reg}([a,b];X)$ *such that*

$$F(x) = F(a) + \int_a^x f \text{ for all } x \in [a,b]. \tag{1.26}$$

From the fundamental theorems of calculus we have the following result.

Theorem 1.10. *Suppose* $a, b \in \mathbb{R}$ *with* $a < b$ *and let* X *be a Banach space. The following are equivalent for* $F \in C^0([a,b];X)$.

1. *F is a primitive.*

2. *F is differentiable on $[a,b]\backslash E$, where $E \subset [a,b]$ is countable, and $F' = f$ on $[a,b]\backslash E$ for $f \in \operatorname{Reg}([a,b];X)$.*

In either case, F is Lipschitz.

Proof. These follow immediately from the first and second fundamental theorems of calculus. □

We now introduce some refinements of the idea of paths and path connectedness.

Definition 1.11. *Let $\varnothing \neq U \subseteq \mathbb{C}$.*

1. *A path in U is a continuous map $\gamma : [a,b] \to U$, where $a, b \in \mathbb{R}$ and $a < b$. We define its range to be the compact set $\operatorname{ran}(\gamma) = \gamma([a,b]) \subset U$. We call $\gamma(a)$ the start of γ and $\gamma(b)$ the end of γ. We say γ is a loop if $\gamma(a) = \gamma(b)$, i.e. the start and end of γ agree.*

2. *If $\gamma : [a,b] \to U$ is a path in U, its reversal is the path $\check{\gamma} : [a,b] \to U$ given by $\check{\gamma}(t) = \gamma(a+b-t)$.*

3. *Let $\gamma_i : [a_i, b_i] \to \mathbb{C}$, $i = 1, 2$, be two paths in U. We say γ_1 meets γ_2 if $\gamma_1(b_2) = \gamma_2(a_2)$, in which case we define their concatenation to be the path $\gamma_1 \vee \gamma_2 : [a_1, b_1 + b_2 - a_2] \to U$ defined by*
$$\gamma_1 \vee \gamma_2(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a_1, b_1] \\ \gamma_2(t - b_1 + a_2) & \text{if } t \in [b_1, b_1 + b_2 - a_2]. \end{cases} \tag{1.27}$$

4. *Two paths $\gamma_i : [a_i, b_i] \to \mathbb{C}$, $i = 1, 2$, are equivalent, written $\gamma_1 \sim \gamma_2$, if there exists an increasing bijection $\varphi : [a_1, b_1] \to [a_2, b_2]$ such that φ and φ^{-1} have primitives and $\gamma_1 = \gamma_2 \circ \varphi$. This is easily seen to be an equivalence relation.*

5. *A road in U is a path in U that is a primitive. A circuit in U is a loop that is a road.*

Remark 1.12. *Suppose $\gamma : [a,b] \to U \subseteq \mathbb{C}$ is a path. Then we can define the path $\beta : [0,1] \to U$ via*
$$\beta(t) = \gamma(a + t(b-a)). \tag{1.28}$$
Clearly $\beta \sim \gamma$. As such, up to equivalence, it's not a loss of generality to restrict our attention to paths that are defined on the compact domain $[0,1]$.

The following two propositions encode some basic properties of these definitions. The first deals with path equivalence and its relation to other ideas.

Proposition 1.13. *Let $\varnothing \neq U \subseteq \mathbb{C}$. Let γ_1 and γ_2 be two paths in U such that $\gamma_1 \sim \gamma_2$. Then the following hold.*

1. $\operatorname{ran}(\gamma_1) = \operatorname{ran}(\gamma_2)$.

2. *γ_1 is a loop if and only if γ_2 is a loop.*

3. *γ_1 is a road if and only if γ_2 is a road.*

Proof. Exercise. □

The second result deals with concatenation.

Proposition 1.14. *Let $\varnothing \neq U \subseteq \mathbb{C}$. Let γ_1 and γ_2 be two paths in U such that γ_1 meets γ_2. Then following hold.*

1. *If $\gamma_i \sim \beta_i$ for $i = 1, 2$, then β_1 meets β_2 and $\gamma_1 \vee \gamma_2 \sim \beta_1 \vee \beta_2$.*

2. *If γ_1 and γ_2 are roads, then the path $\gamma_1 \vee \gamma_2$ is a road.*

3. *If γ_1 and γ_2 are loops, then $\gamma_1 \vee \gamma_2$ is a loop.*

Proof. Exercise. □

We will use counterclockwise circuits around circles so often that they merit some special notation.

Definition 1.15. *Given $z \in \mathbb{C}$ and $r > 0$ we define the circuit $\partial B(z, r) : [0, 1] \to \mathbb{C}$ via*

$$(\partial B(z, r))(t) = z + re^{2\pi it}. \tag{1.29}$$

Clearly, $\operatorname{ran}(\partial B(z, r)) = \partial B(z, r)$.

It will also be useful to introduce rectangular circuits.

Definition 1.16. *Given a closed rectangle*

$$R = \{z \in \mathbb{C} \mid a \leq \operatorname{Re} z \leq a + w \text{ and } b \leq \operatorname{Im} z \leq b + h\} \subset \mathbb{C} \tag{1.30}$$

for $a, b, w, h \in \mathbb{R}$ with $w, h > 0$, we define the circuit $\partial R : [0, 2w + 2h] \to \mathbb{C}$ via

$$\partial R(t) = \begin{cases} a + t + ib & \text{if } 0 \leq t < w \\ a + w + i(b + t - w) & \text{if } w \leq t < w + h \\ a + w - (t - w - h) + i(b + h) & \text{if } w + h \leq t < 2w + h \\ a + i(b + l - (t - 2w - h)) & \text{if } 2w + h \leq t \leq 2w + 2h. \end{cases} \tag{1.31}$$

Clearly, $\operatorname{ran}(\partial R) = \partial R$.

1.3 Homotopy

We now have the tools needed to introduce the idea of loop homotopy, which is a way of continuously deforming one loop into another.

Definition 1.17. *Let $\varnothing \neq U \subseteq \mathbb{C}$.*

1. *Consider γ_1 and γ_2 be two loops in U. We say they are (loop) homotopic if there exists a continuous map $H : [0, 1] \times [0, 1] \to U$ such that*

   (a) $H(\cdot, 0) \sim \gamma_1$ *and* $H(\cdot, 1) \sim \gamma_2$,

   (b) $H(0, s) = H(1, s)$ *for all* $s \in [0, 1]$.

2. *We say that a loop γ in U is homotopic to a point $z \in U$ if γ is homotopic to the trivial loop $[0, 1] \ni t \mapsto z \in U$.*

3. *We say that U is simply connected if it is path connected and every loop γ in U is homotopic to some point in U.*

Some remarks are in order.

Remark 1.18. *The condition $H(0,s) = H(1,s)$ for $s \in [0,1]$ is equivalent to requiring that the paths $H(\cdot, s) : [0,1] \to U$ are loops for each $s \in [0,1]$.*

Remark 1.19. *In order for a path connected set $U \subseteq \mathbb{C}$ to be simply connected, every loop in U has to be homotopic to a point $z \in U$. However, since U is path connected, this is equivalent to being homotopic to any other point $w \in U$. Thus the specific choice of the point is irrelevant in definition.*

Homotopy defines an equivalence relation on the set of loops in U.

Proposition 1.20. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open. Then homotopy of loops is an equivalence relation, i.e. if γ_1, γ_2, and γ_3 are loops in U, then the following hold.*

1. *γ_1 is homotopic to γ_1.*
2. *If γ_1 is homotopic to γ_2, then γ_2 is homotopic to γ_1.*
3. *If γ_1 is homotopic to γ_2, and γ_2 is homotopic to γ_3, then γ_1 is homotopic to γ_3.*

Proof. Exercise. □

Let's consider some examples.

Example 1.21. Suppose that $U \subseteq \mathbb{C}$ is star-shaped with respect to $z \in U$. Then U simply connected. Indeed, let γ be a loop in U with $[0,1]$ as its parameterization domain. Fix $z \in U$ and define the continuous map $H : [0,1] \times [0,1] \to U$ via

$$H(t,s) = sz + (1-s)\gamma(t), \tag{1.32}$$

which takes values in U since U is star-shaped. This is readily verified to be a homotopy, and so γ is homotopic to the point z. The star-shaped condition also shows that U is path connected, so U is simply connected. △

We can push this a bit further.

Example 1.22. Suppose that $U \subseteq \mathbb{C}$ is star-shaped with respect to $z \in U$. Let β and γ be loops in U with $[0,1]$ as their parameterization domains. Define the continuous map $H : [0,1] \times [0,1] \to U$ via

$$H(t,s) = \begin{cases} 2sz + (1-2s)\gamma(t) & \text{if } 0 \le s \le 1/2 \\ 2(1-s)z + 2(s-1/2)\beta(t) & \text{if } 1/2 \le s \le 1 \end{cases} \tag{1.33}$$

which takes values in U since U is star-shaped. This is readily verified to be a homotopy between γ and β. △

Example 1.23. Every convex set $C \subseteq \mathbb{C}$ is star-shaped and hence simply connected. Moreover, any pair of loops in C are homotopic. △

Example 1.24. Let $\varnothing \neq U \subseteq \mathbb{C}$ and $\gamma : [a,b] \to U$ be a path in U. Consider the loop $\gamma \vee \check{\gamma}$. Then β is homotopic to the point $\gamma(a)$ in U. Indeed, the map $H : [0,1] \times [0,1] \to U$ given by

$$H(t,s) = \gamma(2\,|1/2 - t|\, a + (1 - 2\,|1/2 - t|)(sa + (1-s)b)) \tag{1.34}$$

is a homotopy from $\gamma \vee \check{\gamma}$ to $\gamma(a)$ in U. △

It will be convenient to introduce a notion of homotopy for certain types of path as well.

Definition 1.25. *Let $\varnothing \neq U \subseteq \mathbb{C}$ and let γ_1 and γ_2 be two paths in U that start and end at the same points. We say they are (path) homotopic if there exists a continuous map $H : [0,1] \times [0,1] \to U$ such that*

1. *$H(\cdot, 0) \sim \gamma_1$ and $H(\cdot, 1) \sim \gamma_2$,*
2. *$H(0,s) = H(0,0)$ and $H(1,s) = H(1,0)$ for all $s \in [0,1]$.*

Remark 1.26. *If γ_1 and γ_2 are loops that are path homotopic, then they are loop homotopic, so there is no conflict in our definitions. However, the requirements for the path homotopy are slightly more rigid in this case. In referring to homotopy, if we state that the two paths are loops, then we always mean loop homotopy, and if we only state that they are paths then we mean path homotopy.*

Let's consider an example.

Example 1.27. If $\varnothing \neq U \subseteq \mathbb{C}$ is convex, then any two paths with common start and end points are homotopic. △

We have the same notion of equivalence as with loop homotopy.

Proposition 1.28. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open. Then homotopy of paths with common start and end points is an equivalence relation, i.e. if γ_1, γ_2, and γ_3 are paths in U with the same start and end points, then the following hold.*

1. *γ_1 is homotopic to γ_1.*
2. *If γ_1 is homotopic to γ_2, then γ_2 is homotopic to γ_1.*
3. *If γ_1 is homotopic to γ_2, and γ_2 is homotopic to γ_3, then γ_1 is homotopic to γ_3.*

Proof. Exercise. □

Our next result is an essential technical lemma that allows us to approximate general homotopies with nicer maps.

Lemma 1.29. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open. Suppose that $H : [0,1] \times [0,1] \to U$, and that either*

(a) γ_0 and γ_1 are two loops and H is a loop homotopy of γ_0 and γ_1, or

(b) γ_0 and γ_1 are two paths with the same start and end points, and H is a path homotopy of γ_0 and γ_1.

Then for every $\varepsilon > 0$ there exists a Lipschitz map $L : [0,1] \times [0,1] \to \mathbb{C}$ such that the following hold.

1. *$L([0,1]^2) \subset U$, and $\|H - L\|_{C_b^0} < \varepsilon$.*

2. *$L(\cdot, s)$ is a road for every $s \in [0,1]$, and $L(t, \cdot)$ is a road for every $t \in [0,1]$. Moreover, in case (a) we have that $L(\cdot, s)$ is a circuit for every $s \in [0,1]$, and in case (b) we have that the paths $L(\cdot, s)$ have the same start and end points for every $s \in [0,1]$.*

3. *L is a homotopy of $\beta_0 = L(\cdot, 0)$ and $\beta_1 = L(\cdot, 1)$ in U. Moreover, β_0 is homotopic to γ_0 in U and β_1 is homotopic to γ_1 in U.*

4. *In case (a), if γ_1 and γ_2 are circuits, then $L(\cdot, 0) = H(\cdot, 0)$ and $L(\cdot, 1) = H(\cdot, 1)$. In case (b), if γ_1 and γ_2 are roads, then $L(\cdot, 0) = H(\cdot, 0)$ and $L(\cdot, 1) = H(\cdot, 1)$.*

Proof. The set $H([0,1]^2) \subset U$ is compact and U^c is closed, so we can choose $\varepsilon_0 > 0$ such that

$$K = \{z \in \mathbb{C} \mid \operatorname{dist}(z, H([0,1]^2)) \leq \varepsilon_0\} \subset U. \tag{1.35}$$

Let $0 < \varepsilon < \varepsilon_0$. Since $[0,1]^2$ is compact, the Stone-Weierstrass theorem provides a polynomial $P_0 : [0,1]^2 \to \mathbb{C}$ such that

$$\|H - P_0\|_{C_b^0} < \frac{\varepsilon}{6}. \tag{1.36}$$

Consider now case (a), i.e. γ_0 and γ_1 are loops, in which case H is a loop homotopy and $H(0, s) = H(1, s)$ for $s \in [0,1]$. We then have that

$$\sup_{s\in[0,1]} |P_0(0,s) - P_0(1,s)| \leq \sup_{s\in[0,1]} |P_0(0,s) - H(0,s)| + \sup_{s\in[0,1]} |H(1,s) - P_0(1,s)| < \frac{2\varepsilon}{6}. \tag{1.37}$$

Define the polynomial $P : [0,1]^2 \to \mathbb{C}$ via

$$P(t,s) = P_0(t,s) - t(P_0(1,s) - P_0(0,s)) \tag{1.38}$$

and note that $P(0,s) = P_0(0,s) = P(1,s)$ for all $s \in [0,1]$. Also,

$$|H(t,s) - P(t,s)| \leq |H(t,s) - P_0(t,s)| + |P_0(1,s) - P_0(0,s)|, \tag{1.39}$$

so

$$\|H - P\|_{C_b^0} < \frac{\varepsilon}{6} + \frac{2\varepsilon}{6} = \frac{\varepsilon}{2}. \tag{1.40}$$

Then $P([0,1]^2) \subseteq K \subset U$.

When one of the γ_j loops fails to be a circuit we set $L = P$. In this case it remains only to prove that β_j is homotopic to γ_j for $j = 0, 1$. Since γ_0 is homotopic to γ_1 and β_0 is homotopic to β_1, it suffices to prove only that β_0 is homotopic to γ_0. Define the continuous map $\eta : [0,1] \times [0,1] \to \mathbb{C}$ via

$$\eta(t,s) = s\beta_0(t) + (1-s)H(t,0). \tag{1.41}$$

For $t, s \in [0,1]$ we have that

$$|\eta(t,s) - H(t,0)| = s\,|H(t,0) - P(t,0)| < \varepsilon, \tag{1.42}$$

and so $\eta([0,1]^2) \subseteq K \subset U$. Since $\eta(\cdot, 0) = H(\cdot, 0) \simeq \gamma_0$ and $\eta(\cdot, 1) = \beta_0$, we then conclude that η is the desired homotopy.

Now suppose that γ_0 and γ_1 are circuits, in which case $H(\cdot, 0)$ and $H(\cdot, 1)$ are circuits as well. Since H is uniformly continuous we can pick $\delta > 0$ such that if $t_j, s_j \in [0,1]$ for $j = 0, 1$ and $|t_0 - t_1| + |s_0 - s_1| < \delta$, then $|H(t_0, s_0) - H(t_1, s_1)| < \varepsilon/4$. Define $\chi, \chi_0, \chi_1 \in \operatorname{Reg}([0,1]; \mathbb{R})$ via

$$\chi_0(s) = \begin{cases} 1 - 2s/\delta & \text{if } 0 \leq s \leq \delta/2 \\ 0 & \text{if } \delta/2 < s \leq 1, \end{cases} \tag{1.43}$$

$\chi_1(s) = \chi_0(1-s)$, and $\chi = 1 - \chi_0 - \chi_1$. Note that each of these is valued in $[0,1]$. We then define the Lipschitz map $L : [0,1] \times [0,1] \to \mathbb{C}$ via

$$L(t,s) = \chi(s)P(t,s) + \chi_0(s)H(t,0) + \chi_1(s)H(t,1). \tag{1.44}$$

The construction of χ, χ_0, and χ_1 and the choice of δ allow us to estimate

$$\|H - L\|_{C_b^0} \leq \sup_{s,t\in[0,1]} \chi(s)\,|P(t,s) - H(s,t)| + \sup_{0\leq s\leq \delta/2}\sup_{t\in[0,1]} \chi_0(s)\,|H(t,0) - H(t,s)| \\ + \sup_{1-\delta/2<s\leq 1}\sup_{t\in[0,1]} \chi_1(s)\,|H(t,1) - H(t,s)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \tag{1.45}$$

In particular, this means that $L([0,1]^2) \subseteq K \subset U$. Moreover, $L(0,s) = L(1,s)$ and $L(\cdot,s)$ is a circuit for $s \in [0,1]$, and $L(t,\cdot)$ is a road for $t \in [0,1]$. Finally, $L(\cdot,0) = H(\cdot,0) \sim \gamma_0$ and $L(\cdot,1) = H(\cdot,1) \sim \gamma_1$, so L is a homotopy from γ_0 to γ_1. This completes the construction of L in case (a).

Now consider case (b). Define $P : [0,1] \times [0,1] \to \mathbb{C}$ via

$$L(t,s) = P_0(t,s) + t(H(1,s) - P_0(1,s)) + (1-t)H(0,s) - P_0(0,s)). \tag{1.46}$$

We may then argue as above to show that $L = P$ satisfies all of the stated properties when γ_1 or γ_2 is not a road. If both are roads, then we modify P to construct L as above. We leave it as an exercise to check the details. □

Next we consider another approximation result by rectangular roads, which we now define.

Definition 1.30. *We say a road $\gamma : [a,b] \to \mathbb{C}$ is rectangular if there exist finite sets $E \subset [a,b]$ and*

$$D \subset \{z \in \mathbb{C} \mid \operatorname{Re} z = 0 \text{ or } \operatorname{Im} z = 0\} \tag{1.47}$$

such that γ is differentiable on $[a,b]\backslash E$ and $\gamma'(t) \in D$ for $t \in [a,b]\backslash E$. In other words, rectangular roads are piecewise-differentiable and have derivatives parallel to the real and imaginary axes.

The next technical lemma approximates general paths by rectangular roads.

Lemma 1.31. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and γ be a path in U. Then there exists a rectangular path β in U that is homotopic to γ in U. Moreover, if γ is loop chain, then so is β.*

Proof. Since $\operatorname{ran}(\gamma)$ is compact, we can choose $\varepsilon > 0$ such that

$$K = \{z \in \mathbb{C} \mid \operatorname{dist}(z, \operatorname{ran}(\gamma)) \leq 2\varepsilon\} \subset U. \tag{1.48}$$

Let $[a,b]$ be the domain of γ. Since γ is uniformly continuous we can pick $\delta > 0$ such that $t,s \in [a,b]$ and $|s-t| < \delta$ implies $|\gamma(s) - \gamma(t)| < \varepsilon$. Let $a = t_0 < t_1 < \cdots < t_n = b$ be such that $0 < t_{j+1} - t_j < \delta$ for $0 \leq j \leq n-1$. Define $\beta : [a,b] \to \mathbb{C}$ via

$$\beta(t) = \gamma(t_j) + 2\frac{t - t_j}{t_{j+1} - t_j} \operatorname{Re}(\gamma(t_{j+1}) - \gamma(t_j)) \text{ if } t_j \leq t \leq \frac{t_j + t_{j+1}}{2} \tag{1.49}$$

and

$$\beta(t) = \operatorname{Re}(\gamma(t_{j+1})) + i\operatorname{Im}(\gamma(t_j)) + \left(\frac{2t - t_{j+1} - t_j}{t_{j+1} - t_j}\right) i\operatorname{Im}(\gamma(t_{j+1}) - \gamma(t_j)) \text{ if } \frac{t_j + t_{j+1}}{2} \leq t \leq t_{j+1} \tag{1.50}$$

for $0 \leq j \leq n-1$. Clearly, β is a rectangular road with the same start and end points as γ. Moreover, by construction, for $t_j \leq t \leq t_{j+1}$ we have that

$$|\beta(t) - \gamma(t)| \leq |\gamma(t_j) - \gamma(t)| + |\gamma(t_{j+1}) - \gamma(t_j)| < 2\varepsilon, \tag{1.51}$$

which in particular means that $\beta([a,b]) \subseteq K \subset U$.

Now define $H : [0,1] \times [0,1] \to \mathbb{C}$ via

$$H(s,t) = s\beta(t) + (1-s)\gamma(t). \tag{1.52}$$

By the above estimate, we have that

$$|H(s,t) - \gamma(t)| \leq s\,|\beta(t) - \gamma(t)| < 2\varepsilon \text{ for } s,t \in [0,1], \tag{1.53}$$

so H is a path homotopy of γ and β in U.

□

1.4 Road length and road integrals

We now define the length of a road.

Definition 1.32. *Let $\gamma : [a,b] \to \mathbb{C}$ be a road. We define the length of γ to be*

$$\operatorname{len}(\gamma) = \int_a^b |\gamma'| \in [0,\infty) \tag{1.54}$$

which is well-defined because $|\gamma'| \in \operatorname{Reg}([a,b];\mathbb{R})$.

The basic properties of length are recorded in the following result.

Proposition 1.33. *Let $\varnothing \neq U \subseteq \mathbb{R}^n$. The following hold.*

1. *If γ_1 and γ_2 are two roads in U such that $\gamma_1 \sim \gamma_2$, then $\operatorname{len}(\gamma_1) = \operatorname{len}(\gamma_2)$.*
2. *If γ_1 and γ_2 are two roads in U such that γ_1 meets γ_2, then*

$$\operatorname{len}(\gamma_1 \vee \gamma_2) = \operatorname{len}(\gamma_1) + \operatorname{len}(\gamma_2). \tag{1.55}$$

3. *$\operatorname{len}(\gamma) = \operatorname{len}(\check{\gamma})$.*

Proof. Exercise. □

We now have all the tools needed to define complex road integrals.

Definition 1.34. *Let $\varnothing \neq U \subseteq \mathbb{C}$ and X be a complex Banach space. Let $\gamma : [a,b] \to U$ be a road in U and $f \in C^0(\operatorname{ran}(\gamma); X)$. We define*

$$\int_\gamma f = \int_a^b \gamma' f \circ \gamma \in X, \tag{1.56}$$

which is well-defined since $\gamma' f \circ \gamma \in \operatorname{Reg}([a,b];X)$. This induces a linear map $\int_\gamma : C^0(\operatorname{ran}(\gamma);X) \to X$. We sometimes write

$$\int_\gamma f(z)dz = \int_\gamma f \tag{1.57}$$

to emphasize the variable of integration.

The basic properties of the integral are recorded in the following result.

Theorem 1.35. *Let $\varnothing \neq U \subseteq \mathbb{C}$ and X be a complex Banach space. Let γ be a road in U and $f \in C^0(\mathrm{ran}(\gamma); X)$. The following hold.*

1. *If β is road in U such that $\gamma \sim \beta$, then*
$$\int_\gamma f = \int_\beta f. \tag{1.58}$$

2. *We have that*
$$\int_\gamma f = -\int_{\check{\gamma}} f. \tag{1.59}$$

3. *If β is a road in U that meets γ and f extends to a function $f \in C^0(\mathrm{ran}(\beta) \cup \mathrm{ran}(\gamma); X)$, then*
$$\int_{\beta \vee \gamma} f = \int_\beta f + \int_\gamma f. \tag{1.60}$$

4. *If $f = gx$ for $g \in C^0(\mathrm{ran}(\gamma); \mathbb{C})$ and $x \in X$, then*
$$\int_\gamma f = \left(\int_\gamma g \right) x. \tag{1.61}$$

5. *We have the bound*
$$\left\| \int_\gamma f \right\|_X \leq \mathrm{len}(\gamma) \max_{z \in \mathrm{ran}(\gamma)} \|f(z)\|_X \,. \tag{1.62}$$
In particular, the map $\int_\gamma : C^0_b(U; X) \to X$ is bounded and linear.

6. *If Y is a complex Banach space and $T \in \mathcal{L}(X, Y)$, then*
$$T \int_\gamma f = \int_\gamma Tf. \tag{1.63}$$

7. *If $\{f_n\}_{n=\ell}^\infty \subseteq C^0(\mathrm{ran}(\gamma), X)$ is such that $f_n \to f$ uniformly as $n \to \infty$, then*
$$\int_\gamma f_n \to \int_\gamma f \text{ as } n \to \infty. \tag{1.64}$$

Proof. These follow immediately from the properties of the Cauchy integral. □

We also have a version of the fundamental theorem of calculus for complex road integrals.

Theorem 1.36 (FTC for complex road integrals)**.** *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open, X be a complex Banach space, and $f \in C^1(U; X)$. If $\gamma : [a, b] \to U$ is a road in U, then*
$$\int_\gamma f' = f(\gamma(b)) - f(\gamma(a)). \tag{1.65}$$

Proof. By the chain rule and the second fundamental theorem of calculus we have that

$$\int_\gamma f' = \int_a^b (f \circ \gamma)' = f(\gamma(b)) - f(\gamma(a)). \tag{1.66}$$

□

Let's consider some examples.

Example 1.37. For $k \in \mathbb{N}$ define $f_k : U \to \mathbb{C}$ via $f(z) = z^k$. Let $\gamma : [a,b] \to U$ be a road in U. Then $f'_{k+1} = (k+1)f_k$, and so the fundamental theorem of calculus implies that

$$\int_\gamma z^k dz = \int_\gamma f_k = \int_\gamma \frac{f'_{k+1}}{k+1} = \frac{f_{k+1}(\gamma(b)) - f_{k+1}(\gamma(a))}{k+1} = \frac{(\gamma(b))^{k+1} - (\gamma(a))^{k+1}}{k+1}. \tag{1.67}$$

In particular, if γ is a circuit, then

$$\int_\gamma z^k dz = 0. \tag{1.68}$$

△

Example 1.38. Let $\varnothing \neq U \subseteq \mathbb{C}$ and let γ be a circuit in U. Let X be a complex Banach space and $p : U \to X$ be the polynomial $p(z) = \sum_{k=0}^n z^k x_k$ for $x_0, \ldots, x_n \in X$. Then

$$\int_\gamma p = \sum_{k=0}^n \int_\gamma z^k x_k dz = \sum_{k=0}^n \left(\int_\gamma z^k dz \right) x_k = 0 \tag{1.69}$$

since γ is a circuit. △

Example 1.39. Consider the setting of Example 1.4 with $f : B(z_0, R) \to X$ defined by

$$f(z) = \sum_{n=0}^\infty (z - z_0)^n x_n. \tag{1.70}$$

Define $F : B(z_0, R) \to X$ via

$$F(z) = \sum_{n=0}^\infty \frac{(z - z_0)^{n+1}}{n+1} x_n. \tag{1.71}$$

This is well-defined since

$$\limsup_{n\to\infty} \left(\frac{\|x_n\|_X}{n+1} \right)^{1/n} = \limsup_{n\to\infty} \|x_n\|_X^{1/n}, \tag{1.72}$$

and in fact F is smooth and $F' = f$.

If $\gamma : [a,b] \to \mathbb{C}$ is any road in $B(z_0, R)$, then

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a)). \tag{1.73}$$

In particular, if γ is a circuit, then

$$\int_\gamma f = 0. \tag{1.74}$$

△

Example 1.40. Let $\varnothing \neq U \subseteq \mathbb{C}$ and let $\gamma : [a,b] \to \mathbb{C}$ be a circuit in U. Fix $k \in \mathbb{N}\backslash\{1\}$ and $z_0 \in U\backslash \operatorname{ran}(\gamma)$, and define $f_k : U\backslash\{z\} \to \mathbb{C}$ via $f_k(z) = (z - z_0)^{-k}$. Then $f_k' = -kf_{k+1}$, so the fundamental theorem of calculus implies that if $k \geq 2$, then

$$\int_\gamma \frac{1}{(z-z_0)^k} dz = 0. \tag{1.75}$$

△

The above examples have only produced trivial circuit integrals. We now show that it's possible to get something other than 0.

Example 1.41. Let $z_0 \in \mathbb{C}$ and define $f : \mathbb{C}\backslash\{z_0\} \to \mathbb{C}$ via $f(z) = (z - z_0)^{-1}$. It is a simple matter to check that f is holomorphic in $\mathbb{C}\backslash\{z_0\}$. Let $r > 0$ and define $\gamma : [0, 2\pi] \to \mathbb{C}\backslash\{z_0\}$ via $\gamma(t) = z_0 + re^{it}$. Then

$$\int_\gamma f = \int_0^{2\pi} \frac{ire^{it}}{\gamma(t) - z_0} dt = i \int_0^{2\pi} \frac{re^{it}}{re^{it}} dt = 2\pi i. \tag{1.76}$$

This shows that there is something special about the functions $\mathbb{C}\backslash\{z_0\} \ni z \mapsto 1/(z - z_0)$. We will see this again. △

2 The Cauchy-Goursat theorems and their implications

We saw in Example 1.39 that $\int_\gamma f = 0$ when f is a holomorphic function given by a power series in some ball and γ is a circuit in the ball. We now aim to show that this vanishing is a much more general phenomenon. In turn, this vanishing has some truly remarkable consequences for holomorphic maps.

2.1 Cauchy-Goursat for circuits and roads

We begin with a key technical lemma that shows how the maps from Lemma 1.29 interact with holomorphic functions. First we introduce some notation.

Definition 2.1. *Given a cube $Q = [a, a + l] \times [b, b + l]$ for $a, b, l \in \mathbb{R}$ with $l > 0$, we define the map $\omega_Q \in \operatorname{Reg}([0, 4l]; \partial Q)$ via*

$$\omega_Q(t) = \begin{cases} (a+t, b) & \text{if } 0 \leq t < l \\ (a+l, b+t) & \text{if } l \leq t < 2l \\ (a+l-t, b+l) & \text{if } 2l \leq t < 3l \\ (a, b+l-t) & \text{if } 3l \leq t \leq 4l. \end{cases} \tag{2.1}$$

We can now present the lemma.

Lemma 2.2. *Let X be a complex Banach space, $\varnothing \neq U \subseteq \mathbb{C}$ be open, and $f : U \to X$ be holomorphic. Suppose that $L : [0,1]^2 \to U$ is Lipschitz and is such that $L(\cdot, s)$ is a circuit for all $s \in [0,1]$ and $L(t, \cdot)$ is a road for all $t \in [0,1]$. Let $\alpha : [0,4] \to U$ be the circuit given by $\alpha = L \circ \omega_{[0,1]^2}$, where $\omega_{[0,1]^2}$ is as in Definition* (2.1). *Then*

$$\int_\alpha f = 0. \tag{2.2}$$

Proof. Define

$$\int_\alpha f = x \in X. \tag{2.3}$$

We claim that there exists a sequence of cubes $\{Q_n\}_{n=0}^\infty$ such that Q_n has side length 2^{-n}, $Q_{n+1} \subseteq Q_n \subseteq [0,1]^2$, and if we write $\omega_n = \omega_{Q_n}$ as in Definition 2.1, then

$$\frac{\|x\|_X}{4^n} \le \left\| \int_{L\circ\omega_n} f \right\|_X. \tag{2.4}$$

To prove the claim we proceed inductively. For the base case we set $Q_0 = [0,1]^2$ and note that $\alpha = \omega_0$, which means that (2.4) with $n = 0$ follows from (2.3). Now suppose that we have cubes $Q_n \subseteq Q_{n-1} \subseteq Q_0$ such that Q_m has side length 2^{-m} and

$$\frac{\|x\|_X}{4^m} \le \left\| \int_{L\circ\omega_m} f \right\|_X \text{ for } 0 \le m \le n. \tag{2.5}$$

Write $Q_{n,j}$ for $j \in \{0,1,2,3\}$ for four cubes of side length 2^{-n-1} such that $Q_n = \bigcup_{j=0}^3 Q_{n,j}$. By Theorem 1.35 we have that

$$\int_{L\circ\omega_n} f = \sum_{j=0}^3 \int_{L\circ\omega_{Q_{n,j}}} f, \tag{2.6}$$

and so there must exist $j \in \{0,1,2,3\}$ such that

$$\frac{1}{4}\left\| \int_{L\circ\omega_n} f \right\|_X \le \left\| \int_{L\circ\omega_{Q_{n,j}}} f \right\|_X. \tag{2.7}$$

Setting $Q_{n+1} = Q_{n,j}$ and employing (2.5), we find that

$$\frac{\|x\|_X}{4^{n+1}} = \frac{1}{4}\frac{\|x\|_X}{4^n} \le \left\| \int_{L\circ\omega_{Q_{n+1}}} f \right\|_X, \tag{2.8}$$

which is (2.5) with $m = n+1$. The claim then follows by strong induction.

Now let $\lambda \in [0,\infty)$ be the Lipschitz constant for the map L. For $t, s \in [0, 4\cdot 2^{-n}]$ with $t \ne s$ we can estimate

$$\left| \frac{L(\omega_n(t)) - L(\omega_n(s))}{t-s} \right| \le \lambda \left| \frac{\omega_n(t) - \omega_n(s)}{t-s} \right| \tag{2.9}$$

to see that

$$|(L\circ\omega_n)'(t)| \le \lambda\, |\omega_n'(t)| \tag{2.10}$$

for the all but countably many $t \in [0, 4\cdot 2^{-n}]$ where $L\circ\omega_n$ is differentiable. From this estimate we then deduce that

$$\operatorname{len}(L\circ\omega_n) \le \lambda \operatorname{len}(\omega_n) = 4\lambda 2^{-n}. \tag{2.11}$$

Since each Q_n is compact with side length 2^{-n} and $Q_{n+1} \subseteq Q_n$, we may use the shrinking closed set property to find $z \in U$ with $\{z\} = \bigcap_{n=0}^\infty Q_n$. Define $R : U \to X$ via $R(z) = 0$ and

$$R(w) = \frac{f(w) - f(z)}{w - z} - f'(z). \tag{2.12}$$

Since f is holomorphic at z we have that R is continuous, and

$$\varepsilon_n = \sup_{w \in Q_n} \|R(w)\|_X \to 0 \text{ as } n \to \infty. \tag{2.13}$$

Since $L \circ \omega_n$ is a circuit, Example 1.39 shows that

$$\int_{L\circ\omega_n} (f(z) + (w-z)f'(z))dw = 0, \tag{2.14}$$

so

$$\int_{L\circ\omega_n} f = \int_{L\circ\omega_n} (w-z)R(w)dw. \tag{2.15}$$

Using this, Theorem 1.35, and (2.11), we arrive at the bound

$$\left\| \int_{L\circ\omega_n} f \right\|_X \leq \text{len}(L\circ\omega_n)\varepsilon_n \operatorname{diam}(Q_n) \leq 4\lambda 2^{-n}\varepsilon_n\sqrt{2}2^{-n} = \frac{2^{5/2}\lambda\varepsilon_n}{4^n}. \tag{2.16}$$

We now combine (2.4) and (2.16) to see that

$$\frac{\|x\|_X}{4^n} \leq \left\| \int_{L\circ\omega_n} f \right\|_X \leq \frac{2^{5/2}\lambda\varepsilon_n}{4^n}. \tag{2.17}$$

Thus

$$\|x\|_X \leq 2^{5/2}\lambda\varepsilon_n \to 0 \text{ as } n \to \infty, \tag{2.18}$$

and so $x = 0$. □

We now have all of the tools needed to show that the previously mentioned vanishing phenomenon is completely general. The importance of the following theorem in complex analysis cannot be understated: it is the essential ingredient in nearly every result to come. It would not be unfair to call it the fundamental theorem of holomorphic functions.

Theorem 2.3 (Cauchy-Goursat, circuit version)**.** *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open, X be a complex Banach space, and $f : U \to X$ be holomorphic. The following hold*

1. *If γ_0 and γ_1 are circuits in U that are homotopic, then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f. \tag{2.19}$$

2. *If γ is a circuit in U that is homotopic to a point in U, then*

$$\int_\gamma f = 0. \tag{2.20}$$

3. *If U is simply connected and γ is a circuit in U, then*

$$\int_\gamma f = 0. \tag{2.21}$$

Proof. We begin with the proof of the first item. Let $H : [0,1]^2 \to U$ be a homotopy from γ_0 to γ_1. Let $\varepsilon = 1$ and pick $L : [0,1]^2 \to U$ as in Lemma 1.29. Let α be the circuit associated to L as in Lemma 2.2 and note that α is the concatenation of the circuits β, ζ, $\check{\tau}$, and $\check{\zeta}$, where

$$\beta = H(\cdot, 0) \sim \gamma_0, \zeta = L(1, \cdot), \text{ and } \tau = H(\cdot, 1) \sim \gamma_1 \tag{2.22}$$

Then from Lemma 2.2 and Theorem 1.35 we have that

$$0 = \int_\alpha f = \int_\beta f + \int_\zeta f + \int_{\check{\tau}} f + \int_{\check{\zeta}} f = \int_{\gamma_0} f + \int_\zeta f - \int_{\gamma_1} f - \int_\zeta f = \int_{\gamma_0} f - \int_{\gamma_1} f, \tag{2.23}$$

and so

$$\int_{\gamma_0} f = \int_{\gamma_1} f. \tag{2.24}$$

This proves the first item.

We now prove the second item. Pick $z \in U$ such that γ is homotopic to the constant map $\beta : [0,1] \to U$ with $\beta(t) = z$ for all $t \in [0,1]$. Then the first item implies that

$$\int_\gamma f = \int_\beta f = \int_0^1 \beta' f \circ \beta = 0. \tag{2.25}$$

This proves the second item, and the third follows immediately from the second. □

We also have a version of Cauchy-Goursat for roads.

Theorem 2.4 (Cauchy-Goursat, road version)**.** *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open, X be a complex Banach space, and $f : U \to X$ be holomorphic. If γ_0 and γ_1 are roads in U with the same start and end points, and γ_0 and γ_1 are homotopic in U, then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f. \tag{2.26}$$

Proof. The proof is similar to that of the circuit version except that we use case (b) from Lemma 1.29. We leave the details as an exercise. □

As a first glimpse of the power of the Cauchy-Goursat theorem, we prove a remarkable formula known as Cauchy's integral formula.

Theorem 2.5 (Cauchy's integral formula, ball version)**.** *Let X be a complex Banach space, $\varnothing \neq U \subseteq \mathbb{C}$ be open, and $B[z_0, r] \subset U$. Let $\partial B(z_0, r)$ be the circuit from Definition 1.15. If $f : U \to X$ is holomorphic and $z \in B(z_0, r)$, then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(w)}{w - z} dw. \tag{2.27}$$

Proof. Fix $z \in B(z_0, r)$. Write $\gamma = \partial B(z_0, r)$. For $0 < \varepsilon \leq \frac{1}{2}(r - |z - z_0|)$ let $\gamma_\varepsilon : [0,1] \to \partial B(z, \varepsilon) \subset B(z_0, r)$ be the circuit given by $\gamma_\varepsilon(t) = z + \varepsilon e^{2\pi i t}$. The map $H_\varepsilon : [0,1] \times [0,1] \to U \backslash \{z\}$ given by

$$H(t, s) = (1 - s)\gamma(t) + s\gamma_\varepsilon(t) \tag{2.28}$$

is a homotopy of γ and γ_ε in $U\backslash\{z\}$. The function $U\backslash\{z\} \ni w \mapsto f(w)/(w-z) \in X$ is holomorphic, so the Cauchy-Goursat theorem then implies that

$$\int_\gamma \frac{f(w)}{w-z}dw = \int_{\gamma_\varepsilon} \frac{f(w)}{w-z}dw. \tag{2.29}$$

We then compute

$$\begin{aligned}\int_{\gamma_\varepsilon} \frac{f(w)}{w-z}dw &= \left(\int_{\gamma_\varepsilon} \frac{1}{w-z}dw\right) f(z) + \int_{\gamma_\varepsilon} \frac{f(w)-f(z)}{w-z}dw \\ &= 2\pi i f(z) + \int_0^1 2\pi i\varepsilon e^{2\pi it} \frac{f(z+\varepsilon e^{2\pi it}) - f(z)}{\varepsilon e^{2\pi t}}dw \\ &= 2\pi i f(z) + 2\pi i \int_0^1 [f(z+\varepsilon e^{2\pi it}) - f(z)]dw. \end{aligned} \tag{2.30}$$

Hence, by the continuity of f at z,

$$\int_\gamma \frac{f(w)}{w-z}dw = \lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} \frac{f(w)}{w-z}dw = 2\pi i f(z). \tag{2.31}$$

□

Cauchy's integral formula shows just how special holomorphic functions are. Indeed, the formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(w)}{w-z}dw \text{ for } z \in B(z_0,r) \tag{2.32}$$

shows that the values of f in the entirety of the ball $B[z_0,r]$ are encoded in the values on the circle $\partial B(z_0,r)$. This shows a first glimpse of the rigidity of holomorphic functions: it is not possible to modify f in the interior of the ball without simultaneously changing the values on the boundary, and vice-versa.

2.2 Analyticity

We now recall the definition of analytic functions from open sets of $\mathbb{C}$ to complex Banach spaces.

Definition 2.6. *Let X be a complex Banach space, $\varnothing \neq U \subseteq \mathbb{C}$ be open, and $f : U \to X$ be smooth. We say f is analytic if for each $z_0 \in U$ there exists $R > 0$ with $B(z_0,R) \subseteq U$ such that f can be written as a power series with radius of convergence $r \geq R$.*

Obviously, analytic functions are holomorphic. Remarkably, holomorphic functions are also analytic.

Theorem 2.7. *Let X be a complex Banach space, $\varnothing \neq U \subseteq \mathbb{C}$ be open, and $f : U \to X$. Then f is holomorphic if and only if f is analytic. In either case, if $B[z_0,r] \subset \Omega$, then*

$$\limsup_{n\to\infty} \left(\frac{\left\|f^{(n)}(z_0)\right\|_X}{n!}\right)^{1/n} \leq \frac{1}{r}. \tag{2.33}$$

Proof. If f is analytic, then it is trivially holomorphic. Suppose, then, that f is holomorphic. Fix $z_0 \in U$ and $r > 0$ such that $B[z_0, r] \subset U$. Define the circuit $\partial B(z_0, r)$ from Definition 1.15. For $z \in B(z_0, r)$ we set $0 < \delta(z) = |z - z_0| / r < 1$ and note that for $w \in \partial B(z_0, r)$

$$\left| \frac{z - z_0}{w - z_0} \right| = \frac{|z - z_0|}{r} = \delta(z) < 1, \tag{2.34}$$

which allows us to write

$$\frac{1}{w - z} = \frac{1}{w - z_0 - (z - z_0)} = \frac{1}{w - z_0} \frac{1}{1 - (z - z_0)/(w - z_0)} \\ = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^n} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{1+n}}. \tag{2.35}$$

Here the series converges uniformly on $\partial B(z_0, r)$ since $\delta(z) < 1$. Using this and Theorem 1.35 we can then compute

$$2\pi i f(z) = \int_{\partial B(z_0,r)} \frac{f(w)}{w - z} dw = \sum_{n=0}^{\infty} \int_{\partial B(z_0,r)} \frac{(z - z_0)^n}{(w - z_0)^{1+n}} f(w) dw \\ = \sum_{n=0}^{\infty} (z - z_0)^n \int_{\partial B(z_0,r)} \frac{f(w)}{(w - z_0)^{1+n}} f(w) dw. \tag{2.36}$$

For $n \in \mathbb{N}$ set

$$x_n = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(w)}{(w - z_0)^{1+n}} f(w) dw \in X \tag{2.37}$$

and note that Theorem 1.35 provides the estimate

$$\|x_n\|_X \leq \frac{2\pi r}{2\pi r^{1+n}} \max_{w \in \partial B(z_0,r)} \|f(w)\|_X = r^{-n} \max_{w \in \partial B(z_0,r)} \|f(w)\|_X . \tag{2.38}$$

Hence

$$\limsup_{n \to \infty} \|x_n\|_X^{1/n} \leq 1/r \text{ and } r \leq \left(\limsup_{n \to \infty} \|x_n\|_X^{1/n} \right)^{-1}. \tag{2.39}$$

We deduce from these that

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n x_n \tag{2.40}$$

and that this power series converges pointwise in $B(z_0, r)$ and uniformly absolutely in $B[z_0, s]$ for $s < r$. Since $z_0 \in U$ was arbitrary we deduce that f is analytic. Finally, we have that

$$\frac{1}{n!} f^{(n)}(z_0) = x_n, \tag{2.41}$$

so (2.33) follows from (2.39).

□

Remark 2.8. *The estimate* (2.33) *shows that if a ball $B[z_0, r]$ is contained in U, then we have the power series expansion*

$$f(z) = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) \text{ for } z \in B(z_0, r) \tag{2.42}$$

with absolute converge in $B(z_0, s)$ for every $0 < s < r$. This estimate for the radius of convergence is often useful.

This is another massive difference between complex differentiability and real differentiability. Indeed, the existence of one complex derivative is enough to guarantee that all derivatives exist and that power series expansions converge. In particular, this means that there is no reason to introduce the $C^k(U;X)$ spaces when $\varnothing \neq U \subseteq \mathbb{C}$ is open, X is a complex Banach space, and $1 \leq k < \infty$ since they all coincide with $D^1(U;X)$, the space of differentiable (holomorphic) maps.

2.3 Holomorphic path integrals

Our construction of the road integral requires that we integrate on roads. Using Cauchy-Goursat, we now aim to extend the definition of the integral to all paths, provided the integrand is holomorphic in an open set containing the path. This might seem a somewhat strange goal, as it's not clear how to define such an integral without being able to differentiate the path. We will accomplish this with the use of homotopy. We begin with a technical lemma.

Lemma 2.9. *Let γ be a path in $\mathbb{C}$ and $U \subseteq \mathbb{C}$ be an open set such that* $\operatorname{ran}(\gamma) \subseteq U$. *Then the following hold.*

1. *There exists a road β in U with the same start and end points as γ such that β and γ are homotopic in U.*
2. *Suppose that X is a complex Banach space and $f : U \to X$ is holomorphic. If β_1 and β_2 are two roads in U with the same start and end points as γ, both homotopic to γ in U, then*

$$\int_{\beta_1} f = \int_{\beta_2} f. \tag{2.43}$$

Proof. The first item follows from Lemma 1.29. To prove the second we note that by the transitivity of homotopy, β_1 and β_2 are homotopic. The stated identity then follows directly from Cauchy-Goursat.

□

The lemma allows us to define path integrals of holomorphic functions.

Definition 2.10. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open, X be a complex Banach space, and $f : U \to X$ be holomorphic. If γ is a path in U, then we define*

$$\int_\gamma f = \int_\beta f \in X, \tag{2.44}$$

where β is any road in U, with the same start and end points as γ, that is homotopic to γ. This is well-defined by Lemma 2.9. We call this the integral of f on the path γ.

It may appear at first glance that the integral defined above depends on the choice of the open set U. Our next result shows that this is not the case

Lemma 2.11. *Let X be a complex Banach space and γ be a path in $\mathbb{C}$. Suppose that $U, V \subseteq \mathbb{C}$ are open sets such that* $\operatorname{ran}(\gamma) \subset U \cap V$. *Write*

$$\int_\gamma^{U\cap V}, \int_\gamma^{U}, \text{ and } \int_\gamma^{V} \tag{2.45}$$

for the integrals defined above, relative to the open sets $U \cap V$, U, and V, respectively. Suppose that $f : U \cap V \to X$, $g : U \to X$, and $h : V \to X$ are holomorphic and $f = g = h$ on $U \cap V$. Then

$$\int_\gamma^{U\cap V} f = \int_\gamma^{U} g = \int_\gamma^{V} h. \tag{2.46}$$

Proof. Since γ is a path in $U \cap V$ we can pick a road β in $U \cap V$ with the same start and end points as γ that is homotopic to γ in $U \cap V$, in which case

$$\int_\gamma^{U\cap V} f = \int_\beta f. \tag{2.47}$$

On the other hand, β is homotopic to γ in both U and V, so

$$\int_\gamma^{U} g = \int_\beta g = \int_\beta f \text{ and } \int_\gamma^{V} h = \int_\beta h = \int_\beta f. \tag{2.48}$$

□

The next result shows that the properties of the road integral carry over to the holomorphic path integral.

Theorem 2.12. *Let $\varnothing \neq U \subseteq \mathbb{C}$ and X be a complex Banach space. Let γ be a path in U and $f : U \to X$ be holomorphic. The following hold.*

1. *If β is path in U such that $\gamma \sim \beta$, then*

$$\int_\gamma f = \int_\beta f. \tag{2.49}$$

2. *We have that*

$$\int_\gamma f = -\int_{\check{\gamma}} f. \tag{2.50}$$

3. *If $g : U \to X$ is holomorphic and $a, b \in \mathbb{C}$, then*

$$\int_\gamma (af + bg) = a\int_\gamma f + b\int_\gamma g. \tag{2.51}$$

4. *(Cauchy-Goursat for loops) Suppose γ is a loop. If β is a loop in U that is homotopic to γ, then*

$$\int_\gamma f = \int_\beta f. \tag{2.52}$$

If γ is homotopic to a point in U, then

$$\int_\gamma f = 0. \tag{2.53}$$

Moreover, if U is simply connected, then

$$\int_\gamma f = 0. \tag{2.54}$$

5. *(Cauchy-Goursat for paths) If β is a path in U that has the same start and end points as γ and is homotopic to γ in U, then*
$$\int_\gamma f = \int_\beta f. \tag{2.55}$$

6. *If $f = gx$ for $g : U \to \mathbb{C}$ holomorphic and $x \in X$, then*
$$\int_\gamma f = \left(\int_\gamma g\right) x. \tag{2.56}$$

7. *If Y is a complex Banach space and $T \in \mathcal{L}(X;Y)$, then*
$$T\int_\gamma f = \int_\gamma Tf. \tag{2.57}$$

Proof. Exercise. □

2.4 Loop indices

Next we need the idea of the index of a loop.

Lemma 2.13. *Let γ be a loop in $\mathbb{C}$. Define the map $\operatorname{ind}(\gamma,\cdot) : \mathbb{C}\backslash \operatorname{ran}(\gamma) \to \mathbb{C}$ via*
$$\operatorname{ind}(\gamma,z) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z}, \tag{2.58}$$
which is well-defined since γ is loop in $\mathbb{C}\backslash\{z\}$, where the integrand is holomorphic. Then the following hold.

1. $\operatorname{ind}(\gamma,\cdot)$ *is continuous.*
2. $\operatorname{ind}(\gamma,z) \in \mathbb{Z}$ *for each* $z \in \mathbb{C}\backslash \operatorname{ran}(\gamma)$.
3. $\operatorname{ind}(\gamma,\cdot)$ *is constant on each connected component of* $\mathbb{C}\backslash \operatorname{ran}(\gamma)$.

Proof. Fix $z \in \mathbb{C}\backslash \operatorname{ran}(\gamma)$. It suffices to prove the first two items under the assumption that γ is a circuit in $\mathbb{C}\backslash\{z\}$. Assume this.

Set $\delta = \operatorname{dist}(z, \operatorname{ran}(\gamma)) > 0$. For $h \in \mathbb{C}$ and $|h| < \delta/2$, then $\operatorname{dist}(z+h, \operatorname{ran}(\gamma)) > 0$, so $z + h \in \mathbb{C}\backslash \operatorname{ran}(\gamma)$. In turn this allows us to estimate
$$|w-z-h| \geq |w-z| - |h| \geq \delta - |h| \geq \frac{\delta}{2} \tag{2.59}$$
for all $w \in \operatorname{ran}(\gamma)$, and hence
$$|\operatorname{ind}(\gamma,z+h) - \operatorname{ind}(\gamma,z)| = \frac{1}{2\pi}\left|\int_\gamma \left(\frac{1}{w-z-h} - \frac{1}{w-z}\right) dw\right|$$
$$= \frac{1}{2\pi}\left|\int_\gamma \left(\frac{h}{(w-z-h)(w-z)}\right) dw\right| \leq |h| \frac{\operatorname{len}(\gamma)}{2\pi} \sup_{w\in\operatorname{ran}(\gamma)} \frac{1}{|w-z-h|\,|w-z|} \leq |h| \frac{\operatorname{len}(\gamma)}{\pi\delta^2} \tag{2.60}$$
from which we deduce that $\operatorname{ind}(\gamma,\cdot)$ is continuous at z. The first item is proved.

We now prove the second item. Let $[a, b]$ be the domain of γ and define $g : [a, b] \to \mathbb{C}$ via

$$g(t) = \int_a^t \frac{\gamma'}{\gamma - z_0}. \tag{2.61}$$

This is well-defined because $z_0 \notin \operatorname{ran}(\gamma)$, so the integrand is regulated. The first fundamental theorem of calculus shows that g is Lipschitz on $[a, b]$ and differentiable on $(a, b)\backslash E$ for some countable set E with $g'(t) = \gamma'(t)/(\gamma(t) - z_0)$ for these t. Then the function $h : [a, b] \to \mathbb{C}$ defined by

$$h(t) = e^{-g(t)}(\gamma(t) - z_0) \tag{2.62}$$

is continuous and differentiable outside a countable set with

$$h'(t) = e^{-g(t)} \left(-g'(t)(\gamma(t) - z_0) + \gamma'(t)\right) = 0 \tag{2.63}$$

for points of differentiability t. Then $h(t) = h(a) = \gamma(a) - z_0$ for all $t \in [a, b]$, which means that

$$e^{g(t)} = \frac{\gamma(t) - z_0}{\gamma(a) - z_0} \text{ for all } t \in [a, b]. \tag{2.64}$$

In particular,

$$\exp\left(\int_\gamma \frac{dz}{z - z_0}\right) = \exp\left(\int_a^b \frac{\gamma'}{\gamma - z_0}\right) = e^{g(b)} = \frac{\gamma(b) - z_0}{\gamma(a) - z_0} = 1 \tag{2.65}$$

since γ is a loop. In turn this implies that

$$\int_\gamma \frac{dz}{z - z_0} = 2\pi i n \text{ for some } n \in \mathbb{Z}, \tag{2.66}$$

and the second item is proved.

The third item follows from the first two since the set $U \backslash \operatorname{ran}(\gamma)$ is open and so its connected components are the same as the path connected components.

□

This suggests some notation.

Definition 2.14. *For any loop γ in $\mathbb{C}$ and $z \in \mathbb{C}\backslash \operatorname{ran}(\gamma)$ we define the index of γ relative to z to be*

$$\operatorname{ind}(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z} \in \mathbb{Z}. \tag{2.67}$$

If γ is a loop, we say a point $z \in \mathbb{C}$ is enclosed by γ if $\operatorname{ind}(\gamma, z) \neq 0$. We say a loop γ is counterclockwise simple if $\operatorname{ind}(\gamma, z) \in \{0, 1\}$ for all $z \in \mathbb{C}\backslash \operatorname{ran}(\gamma)$.

The index is sometimes called the winding number. This terminology is justified by the following example.

Example 2.15. For $m \in \mathbb{Z}\backslash\{0\}$ set $\gamma_m : [0, 1] \to \mathbb{C}$ via $\gamma_m(t) = z_0 + re^{2\pi imt}$ for some fixed $z_0 \in \mathbb{C}$ and $r > 0$. Then for $z \in B(z_0, r)$ we have

$$\operatorname{ind}(\gamma, z) = \operatorname{ind}(\gamma, z_0) = \frac{1}{2\pi i} \int_0^1 \frac{2\pi imre^{2\pi imt}}{z_0 + re^{2\pi imt} - z_0} dt = m. \tag{2.68}$$

On the other hand, for $z \in B[z_0, r]^c$ we have

$$\operatorname{ind}(\gamma, z) = \operatorname{ind}(\gamma, z_0 + 2r) = m \int_0^1 \frac{e^{2\pi imt}}{e^{2\pi imt} - 2} dt = -m \int_0^1 \frac{e^{2\pi imt}}{2} \frac{1}{1 - e^{2\pi imt}/2} dt$$
$$= -m \int_0^1 \sum_{n=0}^{\infty} \left(\frac{e^{2\pi imt}}{2} \right)^{n+1} dt = -m \sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^1 e^{2\pi imnt} dt = 0, \quad (2.69)$$

where here uniform convergence justifies pulling the sum out of the integral. Thus, the index is counting the number of times γ winds around the point z.

△

Our next result records some basic properties of the index.

Proposition 2.16. *Let $z \in \mathbb{C}$ and γ_1 and γ_2 be loops in $\mathbb{C} \backslash \{z\}$. Then the following hold.*

1. $\operatorname{ind}(\check{\gamma}_1, z) = -\operatorname{ind}(\gamma_1, z)$.
2. *If γ_1 meets γ_2, then* $\operatorname{ind}(\gamma_1 \vee \gamma_2, z) = \operatorname{ind}(\gamma_1, z) + \operatorname{ind}(\gamma_2, z)$.
3. *If γ_1 and γ_2 are homotopic in $\mathbb{C} \backslash \{z\}$, then* $\operatorname{ind}(\gamma_1, z) = \operatorname{ind}(\gamma_2, z)$.
4. *If* $\operatorname{ran}(\gamma) \subset B[z_0, R]$ *and* $|z - z_0| > R$, *then* $\operatorname{ind}(\gamma, z) = 0$.

Proof. The first two items follow from Theorem 2.12. For the third item we note that since $\mathbb{C} \backslash \{z\} \ni w \mapsto (w - z)^{-1} \in \mathbb{C}$ is holomorphic, the result follows immediately from the Cauchy-Goursat theorem applied in $\mathbb{C} \backslash \{z\}$.

Now suppose that $\operatorname{ran}(\gamma) \subset B[z_0, R]$ and $|z - z_0| > R$. Write $[a, b]$ for the domain of γ. Then the continuous map $H : [0,1] \times [0,1] \to \mathbb{C}$ given by

$$H(t, s) = s\gamma(a + t(b - a)) + (1 - s)(z_0 + Re^{2\pi it}) \quad (2.70)$$

is continuous and satisfies

$$|H(t, s) - z_0| \leq s |\gamma(a + t(b - a)) - z_0| + (1 - s) \left| Re^{2\pi it} \right| \leq sR + (1 - s)R = R, \quad (2.71)$$

so H is a homotopy in $\mathbb{C} \backslash \{z\}$ between γ and the circuit $\partial B(z_0, r)$ given in Definition 1.15. The third item and Example 2.15 then imply that

$$\operatorname{ind}(\gamma, z) = \operatorname{ind}(\partial B(z_0, r), z) = 0. \quad (2.72)$$

This proves the fourth item. □

2.5 Chains, homology, and the general form of Cauchy-Goursat

With the notion of loop indices in hand, we now aim to prove an even stronger version of Cauchy-Goursat. We begin by introducing the concept of path and loop chains. In the following definition, given a set $X \neq \varnothing$, we write

$$\mathcal{F}_{fin}(X; \mathbb{Z}) = \{ f : X \to \mathbb{Z} \mid f^{-1}(\mathbb{Z} \backslash \{0\}) \text{ is finite} \} \quad (2.73)$$

for the set of functions from X to $\mathbb{Z}$ with finite support. For $f \in \mathcal{F}_{fin}(X; \mathbb{Z})$ we write

$$\operatorname{spt}(f) = \{ x \in X \mid f(x) \neq 0 \}. \quad (2.74)$$

We endow this set with the algebraic structure known as a $\mathbb{Z}-$module: if $f, g \in \mathcal{F}_{fin}(X; \mathbb{Z})$ and $a, b \in \mathbb{Z}$, then $af + bg \in \mathcal{F}_{fin}(X; \mathbb{Z})$ is defined by

$$(af + bg)(x) = af(x) + bg(x) \in \mathbb{Z} \text{ for } x \in X. \tag{2.75}$$

We are now ready to make the definition.

Definition 2.17. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open.*

1. *Let $P(U)$ denote the set of paths in U and $L(U) \subset P(U)$ denote the set of loops in U.*
2. *A path chain is an element of $\mathcal{F}_{fin}(P(U); \mathbb{Z})$. A loop chain is an element of $\mathcal{F}_{fin}(L(U); \mathbb{Z}) \subset \mathcal{F}_{fin}(P(U); \mathbb{Z})$.*
3. *We define the range of a path chain γ to be the set*

$$\operatorname{ran}(\gamma) = \bigcup_{\beta \in \operatorname{spt}(\gamma)} \operatorname{ran}(\beta) \subset U. \tag{2.76}$$

4. *We write path chains γ as*

$$\gamma = m_1\beta_1 + \cdots + m_n\beta_n, \tag{2.77}$$

for $\operatorname{spt}(\gamma) \subseteq \{\beta_1, \ldots, \beta_n\}$ and $m_j = \gamma(\beta_j)$ for $1 \leq j \leq n$. In other words, rather than write path chains as functions, we write them as formal linear combinations of paths with integer coefficients corresponding to the value of the path chain on that path. The path chain 0 is the unique path chain taking only the value $0 \in \mathbb{Z}$, which we will also write as

$$0 = 0\beta_1 + \cdots 0\beta_n \tag{2.78}$$

for any finite number of paths $\beta_1, \ldots, \beta_n \in P(U)$.

5. *Addition and multiplication by $\mathbb{Z}$ for path chains is written similarly: if $\gamma = \sum_{j=1}^J m_j\gamma_j$ and $\beta = \sum_{j=1}^J n_j\gamma_j$, then for $b, c \in \mathbb{Z}$,*

$$b\beta + c\gamma = \sum_{j=1}^J (bn_j + cm_j)\gamma_j. \tag{2.79}$$

6. *If X is a complex Banach space, $f : U \to X$ is holomorphic, and $\gamma = \sum_{j=1}^J n_j\gamma_j$ is a path chain, we define*

$$\int_\gamma f = \sum_{j=1}^J n_j \int_{\gamma_j} f \in X. \tag{2.80}$$

Our next result records the basic properties of the chain integral.

Theorem 2.18. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open, X be a complex Banach space, and $f : U \to X$ be holomorphic. Let γ be a path chain in U.*

1. *If β is a path chain in U, then*

$$\int_{\gamma+\beta} f = \int_\gamma f + \int_\beta f. \tag{2.81}$$

2. *We have that*

$$\int_0 f = 0 \tag{2.82}$$

and

$$\int_{-\gamma} f = -\int_\gamma f. \tag{2.83}$$

3. *If $g : U \to X$ is holomorphic and $a, b \in \mathbb{C}$, then*

$$\int_\gamma (af + bg) = a\int_\gamma f + b\int_\gamma g. \tag{2.84}$$

4. *If $f = gx$ for $g : U \to \mathbb{C}$ holomorphic and $x \in X$, then*

$$\int_\gamma f = \left(\int_\gamma g\right) x. \tag{2.85}$$

5. *If Y is a complex Banach space and $T \in \mathcal{L}(X;Y)$, then*

$$T\int_\gamma f = \int_\gamma Tf. \tag{2.86}$$

Proof. Exercise. □

We now define loop chain indices in the obvious way. Note that the following definition is the principal reason we use coefficients $\mathbb{Z}$ in the definition of path chains rather than the more natural choice of $\mathbb{C}$.

Definition 2.19. *Let γ be a loop chain in $\mathbb{C}$. We define $\operatorname{ind}(\gamma,\cdot) : \mathbb{C}\backslash \operatorname{ran}(\gamma) \to \mathbb{Z}$ via*

$$\operatorname{ind}(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w - z}. \tag{2.87}$$

This general version of the index inherits the same basic properties of the loop version.

Proposition 2.20. *Let γ be a loop chain in $\mathbb{C}$. Then the following hold.*

1. *If $\gamma = \sum_{j=1}^J n_j\beta_j$, where $n_j \in \mathbb{Z}$ and β_j is a loop chain for $1 \le j \le J$, then*

$$\operatorname{ind}(\gamma, z) = \sum_{j=1}^J n_j \operatorname{ind}(\beta_j, z) \text{ for } z \in \mathbb{C}\backslash \bigcup_{j=1}^J \operatorname{ran}(\beta_j). \tag{2.88}$$

2. *$\operatorname{ind}(\gamma,\cdot)$ is constant on each connected component of $\mathbb{C}\backslash \operatorname{ran}(\gamma)$.*

3. *If $\operatorname{ran}(\gamma) \subset B[z_0, R]$ and $|z - z_0| > r$, then $\operatorname{ind}(\gamma, z) = 0$.*

Proof. These follow immediately from Proposition 2.16 and Theorem 2.18. □

Our aim in introducing the path chain integral is to prove a version of Cauchy-Goursat for this new general integral. Along the way it is convenient to give a name to the property we want to investigate.

Definition 2.21. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open. We say that two path chains β and γ in U are Cauchy-Goursat equivalent if for every complex Banach space X and every holomorphic map $f : U \to X$ we have that*

$$\int_\beta f = \int_\gamma f. \tag{2.89}$$

Employing our previous work, we have two very simple sufficient conditions for Cauchy-Goursat equivalence.

Proposition 2.22. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and suppose that $\beta = \sum_{j=1}^J m_j \beta_j$ and $\gamma = \sum_{j=1}^J n_j \gamma_j$ are path chains in U.*

1. *If for each $1 \leq j \leq J$ we have that either $n_j = m_j$ and $\gamma_j \sim \beta_j$ or else $n_j = -m_j$ and $\gamma_j \sim \check{\beta}_j$, then then γ and β are Cauchy-Goursat equivalent.*

2. *If for each $1 \leq j \leq J$ we have that either $n_j = m_j$ and γ_j is homotopic to β_j in U or else $n_j = -m_j$ and γ_j is homotopic to $\check{\beta}_j$ in U, then γ and β are Cauchy-Goursat equivalent.*

Proof. These follow from immediately from Theorem 2.12. □

Next we seek to find a necessary condition for the Cauchy-Goursat equivalence in the case of loop chains.

Proposition 2.23. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and suppose that β and γ are Cauchy-Goursat equivalent loop chains in U. Then*

$$\operatorname{ind}(\beta, z) = \operatorname{ind}(\gamma, z) \text{ for each } z \in U^c. \tag{2.90}$$

Proof. For each $z \in U^c$ the function $f_z : U \to \mathbb{C}$ given by $f_z(w) = (2\pi i(w - z))^{-1}$ is holomorphic. Thus

$$\operatorname{ind}(\beta, z) = \int_\beta f_z = \int_\gamma f_z = \operatorname{ind}(\gamma, z). \tag{2.91}$$

□

We now give this necessary condition a name.

Definition 2.24. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open.*

1. *We say two loop chains γ and β in U are homologous in U if*

$$\operatorname{ind}(\gamma, z) = \operatorname{ind}(\beta, z) \text{ for all } z \in U^c. \tag{2.92}$$

2. *We say a loop chain γ in U is homologous to zero in U if γ is homologous to the chain loop 0, i.e.*

$$\operatorname{ind}(\gamma, z) = 0 \text{ for all } z \in U^c. \tag{2.93}$$

Remark 2.25. *Homology induces an equivalence relation on the set of loop chains in U.*

If two loops are homotopic, then they are homologous.

Proposition 2.26. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open. If γ and β are homotopic loops in U, then γ and β are homologous.*

Proof. If $z \in U^c$, then γ and β are homotopic in $\mathbb{C}\backslash\{z\}$. Proposition 2.16 then shows that $\operatorname{ind}(\gamma, z) = \operatorname{ind}(\beta, z)$. □

However, the converse fails, so homology is weaker than homotopy.

Example 2.27. Let $U = \mathbb{C}\backslash\{-i, i\}$. Define $\alpha, \beta : [0,1] \to U$ via $\alpha(t) = i - ie^{2\pi it}$ and $\beta(t) = -i + ie^{2\pi it}$. Then the circuit $\gamma = \alpha \vee \beta \vee \breve{\alpha} \vee \breve{\beta}$ is homologous to zero in U. However, it can be shown, using the tools of algebraic topology, that the circuit is not homotopic to a point in U. △

Example 2.28. The circuit $3\partial B(0,10)$ is homologous to the loop chain $-\partial B(1,2) + 5\partial B(i,2) - \partial B(-1,2)$ in $B(0,20)\backslash\{0\}$. △

Example 2.29. The circuit $-2000\partial B(0,10)$ is homologous to the loop chain

$$100\partial B(1,2) - 5\partial B(i,2) + 10\partial B(-1,2) \tag{2.94}$$

in $B(0,20)$ since $B(0,20)$ is convex. △

Next we establish two technical lemmas. The first concerns approximation by rectangular path chains, which we now define.

Definition 2.30. *We say a path chain γ is rectangular if $\gamma = \sum_{j=1}^{J} n_j\gamma_j$, where γ_j is a rectangular road for $1 \leq j \leq J$, as defined in Definition 1.30.*

We now state our first technical lemma.

Lemma 2.31. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and γ be a path chain in U. Then there exists a rectangular path chain β in U that is Cauchy-Goursat equivalent to γ. Moreover, if γ is loop chain, then so is β.*

Proof. Lemma 1.31 shows that if γ is a path in U, then there exists a rectangular road β with the same start and end points, such that γ and β are homotopic in U. Thus Cauchy-Goursat for paths implies that β and γ are Cauchy-Goursat equivalent. This proves the result when γ is a path.

Now suppose that $\gamma = \sum_{j=1}^{J} n_j\gamma_j$, where γ_j is a path in U. Applying the above, we produce a rectangular road β_j in U that is Cauchy-Goursat equivalent to γ_j. Then $\beta = \sum_{j=1}^{J} n_j\beta_j$ is a rectangular path chain that is Cauchy-Goursat equivalent to γ. Moreover, the construction shows that if γ is a loop chain, then so is β.

□

The second technical lemma establishes a connection between rectangular loop chains and linear combinations of rectangular boundary circuits, as defined in Definition 1.16.

Lemma 2.32. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and γ be a rectangular loop chain in U that is homologous to zero in U. Then there exist nondegenerate rectangles $R_1, \ldots, R_n \subset \mathbb{C}$ and $m_1, \ldots, m_n \in \mathbb{Z}$ such that γ is Cauchy-Goursat equivalent to the loop chain $\sum_{j=1}^{n} m_j\partial R_j$, where ∂R_j is the circuit defined in Definition 1.16 and $R_j \subset U$ if $m_j \neq 0$.*

Proof. Since γ is a rectangular loop chain, Proposition 2.22 clearly shows that it is Cauchy-Goursat equivalent to a rectangular path chain of the form

$$\gamma' = \sum_{j=1}^{j} m_j h_j + \sum_{k=1}^{K} n_k v_k, \tag{2.95}$$

where $h_j, v_k : [0,1] \to \mathbb{C}$ via $h_j(t) = a_j + t\lambda_j$ and $v_k(t) = b_k + it\mu_k$ for $\{a_1, \ldots, a_J, b_1, \ldots, b_K\} \subset \mathbb{C}$ and $\{\lambda_1, \ldots, \lambda_J, \mu_1, \ldots, \mu_K\} \subset (0, \infty)$. Proposition 2.23 shows that γ' is also homologous to zero.

Let $r > 0$ be such that $\operatorname{ran}(\gamma) \subset B[0, r/2]$ and define the cube

$$Q = \{z \in \mathbb{C} \mid -r \leq \operatorname{Re}(z) \leq r \text{ and } -r \leq \operatorname{Im}(z) \leq r\}. \tag{2.96}$$

The horizontal lines $a_j + \lambda_j\mathbb{R}$ and vertical lines $b_k + \mu_k\mathbb{R}$ form a grid partition on Q. Call the resulting grid of nondegenerate rectangles $\{R_1, \ldots, R_M\}$. Increasing to $J' \geq J$ and $K' \geq K$ if necessary, we may assume that all of the horizontal and vertical line segments forming this grid are given by h_j for $1 \leq j \leq J'$ and v_k for $1 \leq k \leq K'$ of the same form as above. Consequently, we may again use Proposition 2.22 to see that each circuit ∂R_m from Definition 1.16 is Cauchy-Goursat equivalent to a sum

$$\rho_m = h_j + v_k - h_{j'} - v_{k'} \text{ for } j, j' \in \{1, \ldots, J'\} \text{ and } k, k' \in \{1, \ldots, K'\}. \tag{2.97}$$

Write $\Sigma = \{h_j\}_{j=1}^{J'} \cup \{v_k\}_{k=1}^{K'}$, let $z_m \in R_m^\circ$ denote the center of the rectangle R_m, and set

$$W = \{1 \leq m \leq M \mid \operatorname{ind}(\gamma, z_m) \neq 0\}. \tag{2.98}$$

Consider $m \in W$. Suppose, by way of contradiction, that there exists $z \in R_m \cap U^c$. Since R_k is convex, z and z_m lie in the same connected component of $\mathbb{C}\backslash \operatorname{ran}(\gamma)$, so $\operatorname{ind}(\gamma, z) = \operatorname{ind}(\gamma, z_m)$. Since γ is homologous to zero, $\operatorname{ind}(\gamma, z) = 0$, contradicting the inclusion $m \in W$. Thus $R_m \subset U$ for each $m \in W$.

Now define the loop chain

$$\beta = \sum_{m=1}^{M} \operatorname{ind}(\gamma, z_m)\partial R_m \tag{2.99}$$

and the Cauchy-Goursat equivalent path chain

$$\beta' = \sum_{m=1}^{M} \operatorname{ind}(\gamma, z_m)\rho_m. \tag{2.100}$$

Suppose, by way of contradiction, that $\gamma' \neq \beta'$. Note that $\operatorname{spt}(\beta'), \operatorname{spt}(\gamma') \subseteq \Sigma$, and so $\operatorname{spt}(\gamma' - \beta') \subseteq \Sigma$. We may then select $\sigma \in \Sigma$ and $m \in \mathbb{Z}\backslash\{0\}$ such that $(\gamma' - \beta')(\sigma) = m$ (where here we view $\gamma' - \beta'$ as a function from Σ to $\mathbb{Z}$). Pick $1 \leq \mu \leq M$ such that σ is one of the terms appearing in ρ_μ and consider the loop chain

$$\delta = \gamma - \beta - m\partial R_\mu. \tag{2.101}$$

If σ belongs to both R_μ and R_λ for $\lambda \neq \mu$, let $z_* = z_\lambda$. Otherwise select $z_* \in \mathbb{C}$ to be any point on the opposite side of σ, lying outside the cube Q, on a common line with z_μ.

For $z \notin \operatorname{ran}(\delta)$ we can compute

$$2\pi i \operatorname{ind}(\delta, z) = \int_{\gamma-\beta} \frac{dw}{w-z} - m\int_{\partial R_\mu} \frac{dw}{w-z} = \int_{\gamma'-\beta'-m\sigma} \frac{dw}{w-z} - m\int_{\rho_\mu - \sigma} \frac{dw}{w-z}. \tag{2.102}$$

For $z \in \sigma((0,1)) \subset \operatorname{ran}(\sigma)$ we have that $z \notin \operatorname{ran}(\gamma' - \beta' - m\sigma) \cup \operatorname{ran}(\rho_\mu - \sigma)$, so we can use the latter expression and continuity to see that $\operatorname{ind}(\delta, \cdot)$ may be extended continuously to $\operatorname{ran}(\delta) \cup \sigma((0,1))$. Consequently,

$$\operatorname{ind}(\delta, z_\mu) = \operatorname{ind}(\delta, z_*). \tag{2.103}$$

If $z_* = z_\lambda$, then

$$-m = \operatorname{ind}(\gamma, z_\mu) - \sum_{m=1}^{m} \operatorname{ind}(\gamma, z_m) \operatorname{ind}(\partial R_m, z_\mu) - m \operatorname{ind}(\partial R_\mu, z_\mu) = \operatorname{ind}(\delta, z_\mu) = \operatorname{ind}(\delta, z_\lambda)$$
$$= \operatorname{ind}(\gamma, z_\lambda) - \sum_{m=1}^{m} \operatorname{ind}(\gamma, z_m) \operatorname{ind}(\partial R_m, z_\lambda) - m \operatorname{ind}(\partial R_\mu, z_\lambda) = 0, \quad (2.104)$$

while if z_* lies outside Q, then

$$-m = \operatorname{ind}(\delta, z_\mu) = \operatorname{ind}(\delta, z_*) = \operatorname{ind}(\gamma, z_*) = 0 \quad (2.105)$$

due to Proposition 2.20 and the fact that $|z_*| > r/2$. This contradicts the fact that $m \neq 0$, and so we deduce that $\gamma' = \beta'$, which in turn means that γ is Cauchy-Goursat equivalent to β. To conclude, we write

$$\beta = \sum_{m=1}^{M} \operatorname{ind}(\gamma, z_m) \partial R_m = \sum_{m \in W} \operatorname{ind}(\gamma_m, z_m) \partial R_m, \quad (2.106)$$

and note that $R_m \subset U$ if $m \in W$. □

With the previous two technical lemmas in hand, we are now ready to prove the most general version of the Cauchy-Goursat theorem, which shows that the Cauchy-Goursat equivalence of two loop chains is the same as the chains being homologous. In other words, the necessary condition identified in Proposition 2.23 is also sufficient.

Theorem 2.33 (Cauchy-Goursat for loop chains). *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and suppose that β and γ are loop chains in U. Then β and γ are homologous if and only if they are Cauchy-Goursat equivalent.*

Proof. Suppose that β and γ are homologous. According to Lemma 2.31, we can choose rectangular loop chains β' and γ' such that β is Cauchy-Goursat equivalent to β' and γ is Cauchy-Goursat equivalent to γ'. Consider the rectangular loop chain $\alpha = \gamma' - \beta'$. Proposition 2.23 shows that $\operatorname{ind}(\alpha, z) = 0$ for every $z \in U^c$, i.e. α is homologous to zero in U. Using Lemma 2.32, we then find nondegenerate rectangles $R_1, \ldots, R_n \subset U$ and $m_1, \ldots, m_n \in \mathbb{Z}$ such that α is Cauchy-Goursat equivalent to the loop chain $\sum_{j=1}^n m_j \partial R_j$ (if the lemma produces the 0 chain then we pick the rectangles arbitrarily and set each $m_j = 0$). If X is a complex Banach space and $f : U \to X$ is holomorphic, then

$$m_j \int_{\partial R_j} f = 0 \text{ for each } 1 \leq j \leq J \quad (2.107)$$

by Cauchy-Goursat for circuits. Hence,

$$\int_\gamma f - \int_\beta f = \int_{\gamma'} f - \int_{\beta'} f = \int_\alpha f = \sum_{j=1}^{J} m_j \int_{\partial R_j} f = 0, \quad (2.108)$$

and we deduce that β and γ are Cauchy-Goursat equivalent. Proposition 2.23 shows that the converse also holds. □

Remark 2.34. *In particular, this general form of Cauchy-Goursat shows that if γ is homologous to zero in U, then*

$$\int_\gamma f = 0 \quad (2.109)$$

for every holomorphic $f : U \to X$, where X is a complex Banach space.

2.6 The general Cauchy integral formula

We are now in a position to prove the general form of Cauchy's integral formula.

Theorem 2.35 (Cauchy's integral formula, general version)**.** *Let X be a complex Banach space and $\varnothing \neq U \subseteq \mathbb{C}$ be open. Let γ be a loop chain in U that is homologous to zero in U. If $f : U \to X$ is holomorphic, $z \in U \backslash \operatorname{ran}(\gamma)$, and $n \in \mathbb{N}$, then*

$$\operatorname{ind}(\gamma, z) f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} dw. \tag{2.110}$$

Proof. Fix $z \in U \backslash \operatorname{ran}(\gamma)$ and define $F : U \to X$ via

$$F(w) = \frac{1}{(w-z)^{n+1}} \left(f(w) - \sum_{m=0}^{n} \frac{(w-z)^m}{m!} f^{(m)}(z) \right) \tag{2.111}$$

for $w \neq z$ and

$$F(z) = \frac{f^{(n+1)}(z)}{(n+1)!}. \tag{2.112}$$

Clearly, F is holomorphic in $U \backslash \{z\}$. We claim that F is also holomorphic at z.

To prove the claim we use the fact that f is analytic to write

$$f(w) = \sum_{m=0}^{\infty} \frac{(w-z)^m}{m!} f^{(m)}(z) \tag{2.113}$$

for $w \in B(z,r) \subset U$. Then

$$F(w) - F(z) = \frac{1}{(w-z)^{n+1}} \sum_{m=n+1}^{\infty} \frac{(w-z)^m}{m!} f^{(m)}(z) - \frac{f^{(n+1)}(z)}{(n+1)!} = \sum_{m=n+2}^{\infty} \frac{(w-z)^{m-n-1}}{m!} f^{(m)}(z), \tag{2.114}$$

where the latter series continues to converge in $B(z,r)$. Thus

$$\lim_{w \to z} \frac{F(w) - F(z)}{w - z} = \frac{f^{(n+2)}(z)}{(n+2)!}, \tag{2.115}$$

which proves the claim.

Since γ is homologous to zero, we may apply Cauchy-Goursat to F to see that

$$0 = \int_\gamma F = \int_\gamma \frac{f(w)}{(w-z)^{1+n}} dw - \sum_{m=0}^{n} \frac{1}{m!} \left(\int_\gamma (w-z)^{m-n-1} dw \right) f^{(m)}(z). \tag{2.116}$$

Rearranging and employing the calculations from Example 1.40, we then find that

$$\begin{aligned} \int_\gamma \frac{f(w)}{(w-z)^{1+n}} dw = \int_\gamma \frac{f(w)}{(w-z)^{1+n}} dw &= \sum_{m=0}^{n} \frac{1}{m!} \left(\int_\beta (w-z)^{m-n-1} dw \right) f^{(m)}(z) \\ &= \frac{1}{n!} \left(\int_\gamma \frac{dw}{w-z} \right) f^{(n)}(z) = \frac{2\pi i \operatorname{ind}(\gamma, z)}{n!} f^{(n)}(z). \end{aligned} \tag{2.117}$$

This is the desired formula. □

This form of Cauchy's integral formula shows that not only can f be recovered from its values on a loop chain homologous to zero, but the values of its derivatives can be recovered as well. Moreover, if we formally differentiate the formula

$$\operatorname{ind}(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw. \tag{2.118}$$

n times in z by differentiating under the integral, we arrive at the stated form of the integral formula.

3 Rigidity of holomorphic functions

We have seen above some of the first glimpses of the extreme rigidity of holomorphic maps. We now turn our attention to a deeper study of this rigidity.

3.1 Zeros

We begin by studying the zeros of holomorphic functions. Our first result is a restatement of a general fact about analytic functions that we saw before.

Lemma 3.1. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and connected, X be a complex Banach space, and $f : U \to X$ be holomorphic. If $z \in U$ is such that $f^{(n)}(z) = 0$ for all $n \in \mathbb{N}$, then $f = 0$ in U.*

Proof. Define the set

$$E = \{w \in U \mid f^{(n)}(w) = 0 \text{ for all } n \in \mathbb{N}\}. \tag{3.1}$$

We will prove that E is relatively open and relatively closed in U. Then since U is connected and $z \in E$ we conclude that $E = U$, so $f = 0$ in U.

Suppose $\{w_m\}_{m=\ell}^\infty \subseteq E$ and $w_m \to w \in U$ as $m \to \infty$. For $n \in \mathbb{N}$ we have that $f^{(n)}(w_m) = 0$, and upon sending $m \to \infty$ and using the continuity of $f^{(n)}$ we deduce that $f^{(n)}(w) = 0$. Hence $w \in E$, and so E is relatively closed in U.

On the other hand, suppose $w_0 \in E$. Since f is analytic we can pick $r > 0$ such that $B(w_0, r) \subseteq U$ and the power series

$$f(w) = \sum_{n=0}^\infty \frac{(w-w_0)^n}{n!} f^{(n)}(w_0) \tag{3.2}$$

converges in $B(w_0, r)$. Since $w_0 \in E$ the series sums to 0, and we deduce that $f = 0$ in $B(w_0, r)$. Hence $B(w_0, r) \subseteq E$, and we deduce that E is relatively open in U. □

Using this lemma we can prove more remarkable facts about holomorphic functions. The first shows that the zeros of holomorphic maps are isolated and have a well-defined order.

Theorem 3.2. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and connected, X be a complex Banach space, and $f : U \to X$ be holomorphic and nontrivial. Define the zero set $Z = \{z \in U \mid f(z) = 0\} \subset U$. Then the following hold.*

1. *For each $z \in Z$ there exist $1 \leq n \in \mathbb{N}$, $r > 0$ such that $B(z, r) \subseteq U$, and a holomorphic function $g : B(z, r) \to X$ such that $g(w) \neq 0$ and $f(w) = (w - z)^n g(w)$ for $w \in B(z, r)$.*
2. *$Z' \cap U = \varnothing$, i.e. Z has no limit points in U, or equivalently, all of the zeros of f are isolated.*
3. *If $K \subset U$ is compact, then $Z \cap K$ is finite.*

Proof. We begin with the proof of the first item. Consider $z \in Z$. If $f^{(m)}(z) = 0$ for all $m \in \mathbb{N}$, then Lemma 3.1 implies that $f = 0$, a contradiction. Thus we can select a minimal $n \in \mathbb{N}$ such that $f^{(n)}(z) \neq 0$. Then using analyticity, we can write

$$f(w) = \sum_{m=n}^{\infty} \frac{(w-z)^m}{m!} f^{(m)}(z) = (w-z)^m \sum_{m=n}^{\infty} \frac{(w-z)^{m-n}}{m!} f^{(m)}(z), \tag{3.3}$$

with the series converging in $B(z, R) \subseteq U$ for some $R > 0$. Then the function $g : B(z, R) \to X$ defined by the power series

$$g(w) = \sum_{m=n}^{\infty} \frac{(w-z)^{m-n}}{m!} f^{(m)}(z) \tag{3.4}$$

converges to a holomorphic function such that $g(z) = f^{(m)}(z)/m! \neq 0$. By continuity, there exists $0 < r < R$ such that $g \neq 0$ in $B(z, r)$. This prove the first item.

To prove the second item we suppose, by way of contradiction, that $z \in Z' \cap U$. Letting n, g, and r be as in the first item, we pick a point $w \in B(z, r) \cap Z \backslash \{z\}$, which means that

$$0 = f(w) = (w-z)^n g(w) \neq 0, \tag{3.5}$$

a contradiction. Hence $Z' \cap U = \varnothing$, and the second item is proved.

We now turn to the proof of the third item. If $K \subset U$ is compact then $Z \cap K$ is as well. Consequently, if $Z \cap K$ is infinite, then by the Bolzano-Weierstrass totally bounded limit point property (every infinite subset of a totally bounded metric space has a limit point) it has a limit point, which means $Z' \cap U \neq \varnothing$, a contradiction. Thus $Z \cap K$ is finite.

□

This suggests some notation.

Definition 3.3. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open, X be a complex Banach space, and $f : U \to X$ be nontrivial and holomorphic. For each $z \in U$ such that $f(z) = 0$ the order of the zero is $1 \leq n \in \mathbb{N}$ from the first item of Theorem 3.2, applied to the restriction of f to the connected set $B(z, r) \subset U$. We write* $\operatorname{ord}(f, z) = n$. *Note that by construction,*

$$n = 1 + \min\{m \in \mathbb{N} \mid f^{(m)}(z) = 0\}. \tag{3.6}$$

Remark 3.4. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and suppose that $f(w) = 0$ and $n \geq 1$ is the order of w, i.e. $f(z) = (z-w)^n g(z)$ for $z \in B(w, r)$. Then by construction, the map*

$$U \ni z \mapsto \begin{cases} f(z)/(z-w)^n & \text{if } z \neq w \\ g(w) & \text{if } z = w \end{cases} \tag{3.7}$$

is holomorphic. We often slightly abuse notation by saying that $z \mapsto f(z)/(z-w)^n$ is holomorphic, with the understanding that the value at w has to be recovered from g or by taking the limit as $z \to w$.

The next remarkable result shows that two holomorphic functions cannot agree on a set without agreeing on a larger set.

Corollary 3.5. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and connected, X be a complex Banach space, and $f, g : U \to X$ be holomorphic. If $f = g$ on a set with a limit point in U, then $f = g$ in all of U. In particular, if $\varnothing \neq V \subseteq U$ is open and $f = g$ in V, then $f = g$ in U.*

Proof. The first assertion follows from Theorem 3.2 applied to $f - g$, and the second assertion follows since nonempty open sets have limit points. $\square$

Next we show a special property of non-vanishing holomorphic maps with values in $\mathbb{C}$.

Theorem 3.6. *Suppose that $\varnothing \neq U \subseteq \mathbb{C}$ is open and simply connected and $g : U \to \mathbb{C}$ is holomorphic and non-vanishing, i.e. $g(z) \neq 0$ for $z \in U$. There there exists a holomorphic $h : U \to \mathbb{C}$ such that $g = e^h$.*

Proof. Note that g'/g is holomorphic in U since g never vanishes. Fix $w \in U$ and consider a point $z \in U$. Since U is open and connected, it is polygonally connected, and so there exists a road in U that starts at w and ends at z. If γ_1 and γ_2 are any roads in U such that γ_1 and γ_2 both start at w and end at z, then $\gamma_1 \vee \check{\gamma}_2$ is a circuit and so Cauchy-Goursat implies

$$0 = \int_{\gamma_1 \vee \check{\gamma}_2} \frac{g'}{g} = \int_{\gamma_1} \frac{g'}{g} - \int_{\gamma_2} \frac{g'}{g}, \tag{3.8}$$

which means that

$$\int_{\gamma_1} \frac{g'}{g} = \int_{\gamma_2} \frac{g'}{g}. \tag{3.9}$$

We may thus define $H : U \to \mathbb{C}$ via

$$H(z) = \int_{\gamma_z} \frac{g'}{g}, \tag{3.10}$$

where γ_z is any road in U that starts at w and ends at z. This is well-defined by the above.

Let $z \in U$ and $r > 0$ be such that $B[z, r] \subset U$. Let $\eta \in \mathbb{C}$ with $0 < |\eta| < r$. Define the road $\beta : [0, 1] \to U$ via $\beta(t) = z + t\eta$. Fix a road γ_z from w to z and note that $\gamma_z \vee \beta$ is a road from w to $z + \eta$. Then

$$H(z + \eta) - H(z) = \int_{\beta} \frac{g'}{g} = \int_0^1 \eta \frac{g'(z + t\eta)}{g(z + t\eta)} dt \tag{3.11}$$

and so

$$\frac{H(z + \eta) - H(z)}{\eta} - \frac{g'(z)}{g(z)} = \int_0^1 \left(\frac{g'(z + t\eta)}{g(z + t\eta)} - \frac{g'(z)}{g(z)} \right) dt \to 0 \tag{3.12}$$

as $\eta \to 0$ by continuity. Thus H is holomorphic and $H' = g'/g$. Then

$$(e^{-H} g)' = e^{-H}(g' - gH') = 0 \text{ in } U, \tag{3.13}$$

but U is connected, so $e^{-H} g$ is constant. We have $H(w) = 0$ by construction, so

$$e^{-H} g = e^{-H(w)} g(w) = g(w) \neq 0 \tag{3.14}$$

since g doesn't vanish. Writing $g(w) = e^{R+it}$ for $R = \log |g(w)|$ and $t \in [0, 2\pi)$, and defining the holomorphic function $h = H + R + it$, we deduce that

$$g = e^{R+it+H} = e^h, \tag{3.15}$$

the desired equality. $\square$

Remark 3.7. *The formula $g = e^h$ suggests that we should simply set $h = \log g$ for* log *the complex logarithm from Definition 1.8. However, we do not know that $g(U) \subseteq \mathbb{C} \backslash N$ (where N is as in the definition), so this strategy cannot work in general.*

3.2 Estimates of holomorphic functions

Next we explore a number of remarkable estimates associated with the rigidity of holomorphic functions. The first shows that if a holomorphic map $f : \mathbb{C} \to X$ has a bounded derivative of some order or grows no faster than a polynomial, then it must actually be a polynomial.

Theorem 3.8 (Liouville). *Let X be a complex Banach space, $f : \mathbb{C} \to X$ be holomorphic, and $n \in \mathbb{N}$. Then the following are equivalent.*

1. *$f^{(n)} : \mathbb{C} \to X$ is bounded.*
2. *f is a polynomial of degree at most n.*
3. *There exists a constant $C \in (0, \infty)$ such that $\|f(z)\|_X \leq C(1 + |z|^n)$ for $z \in \mathbb{C}$.*

In any case,

$$f(z) = \sum_{m=0}^{n} \frac{z^m}{m!} f^{(m)}(0). \tag{3.16}$$

In particular, the bounded holomorphic functions from $\mathbb{C}$ to X are precisely the constant functions.

Proof. Assume $f^{(n)}$ is bounded. Consider the circuit $\partial B(z, r)$ from Definition 1.15. According to Theorem 2.7, we can write

$$f^{(n+1)}(z) = \frac{1}{2\pi i} \int_{\partial B(z,r)} \frac{f^{(n)}(w)}{(w - z)^2} dw. \tag{3.17}$$

Then

$$\left\|f^{(n+1)}(z)\right\|_X \leq \frac{\operatorname{len}(\partial B(z, r))}{2\pi r^2} \sup_{w \in \mathbb{C}} \left\|f^{(n)}(w)\right\|_X = \frac{1}{r} \sup_{w \in \mathbb{C}} \left\|f^{(n)}(w)\right\|_X \to 0 \text{ as } r \to \infty. \tag{3.18}$$

Since z was arbitrary, we deduce that $f^{(n+1)} = 0$ on $\mathbb{C}$. The second item then follows by applying Lemma 3.1 to the holomorphic function $F : \mathbb{C} \to X$ defined by

$$F(z) = f(z) - \sum_{m=0}^{n} \frac{z^m}{m!} f^{(m)}(0), \tag{3.19}$$

which satisfies $F^{(m)}(0) = 0$ for all $m \in \mathbb{N}$. This proves the first item implies the second.

The second item trivially implies the third. Suppose, then that the third holds. Then Cauchy's integral formula implies that

$$\begin{aligned} \left\|f^{(n)}(z)\right\|_X &= \left\|\frac{n!}{2\pi i} \int_{\partial B(z,R)} \frac{f(w)}{(w - z)^{n+1}} dw\right\|_X \leq \frac{n! 2\pi R}{2\pi R^{n+1}} \max_{|z-w|=R} C(1 + |w|^n) \\ &\leq \frac{Cn!}{R^n} \max_{|z-w|=R} (1 + (|w - z| + |z|)^n) \leq \frac{Cn!}{R^n} (1 + (R + |z|)^n) \to Cn! \text{ as } R \to \infty. \end{aligned} \tag{3.20}$$

Hence, $\left\|f^{(n)}(z)\right\|_X \leq Cn!$ for all $z \in \mathbb{C}$, and the first item is proved. □

Liouville's theorem is often used in conjunction with contradiction arguments to show the existence of points of interest in $\mathbb{C}$. We demonstrate this now with the following generalization of the fundamental theorem of algebra.

Theorem 3.9. *Let X be a unital Banach algebra over $\mathbb{C}$ and suppose that $x_0, \ldots, x_n \in X$ for some $1 \le n \in \mathbb{N}$, where x_n is invertible. Consider the polynomial $p : \mathbb{C} \to X$ given by $p(z) = \sum_{k=0}^{n} z^k x_k$. Set $R = 2\left(1 + \|x_n^{-1}\|_X \sum_{k=0}^{n-1} \|x_k\|_X\right)$. Then the following hold.*

1. *If $|z| > R$, then $p(z)$ is invertible.*

2. *There exists $z \in B[0, R]$ such that $p(z)$ is not invertible.*

Proof. First note that for $z \neq 0$ we can compute

$$\frac{1}{z^n} p(z) - x_n = \frac{1}{z^n} \sum_{k=0}^{n-1} z^k x_k \tag{3.21}$$

in order to estimate, for $|z| \ge 1$,

$$\left\| \frac{1}{z^n} p(z) - x_n \right\|_X \le \frac{|z|^{n-1}}{|z|^n} \sum_{k=0}^{n-1} \|x_k\|_X = \frac{1}{|z|} \sum_{k=0}^{n-1} \|x_k\|_X . \tag{3.22}$$

The above then shows that for $|z| > R$ we have that

$$\left\| \left(\frac{1}{z^n} p(z) - x_n \right) x_n^{-1} \right\|_X \le \frac{1}{|z|} \left\| x_n^{-1} \right\|_X \sum_{k=0}^{n-1} \|x_k\|_X < \frac{1}{2}, \tag{3.23}$$

and hence the identity

$$p(z) = z^n x_n + (p(z) - z^n x_n) = z^n \left(I - \left(\frac{1}{z^n} p(z) - x_n \right) x_n^{-1} \right) x_n \tag{3.24}$$

implies that $p(z)$ is invertible with

$$p(z)^{-1} = z^{-n} x_n^{-1} \sum_{k=0}^{\infty} \left[\left(\frac{1}{z^n} p(z) - x_n \right) x_n^{-1} \right]^k \tag{3.25}$$

and

$$\left\| p(z)^{-1} \right\|_X \le \frac{\|x_n^{-1}\|_X}{|z|^n} \frac{1}{1 - 1/2} = \frac{2 \, \|x_n^{-1}\|_X}{|z|^n}. \tag{3.26}$$

Thus, $p(z)$ is always invertible for $|z| > R$, and the inverse $p(z)^{-1}$ is bounded there.

Suppose then, by way of contradiction, that $p(z)$ is invertible for all $z \in \mathbb{C}$. Since inversion is continuous, we deduce that

$$\max_{|z| \le R} \left\| p(z)^{-1} \right\|_X < \infty. \tag{3.27}$$

Hence the map $\mathbb{C} \ni z \mapsto p(z)^{-1} \in X$ is holomorphic (because inversion is differentiable) and bounded. Liouville's theorem then implies that it's constant, which is readily shown to be a contradiction. We deduce, then, that there exists at least one $z \in \mathbb{C}$ such that $p(z)$ fails to be invertible. □

Remark 3.10. *Taking $X = \mathbb{C}$ in the previous theorem provides a proof of the fundamental theorem of algebra since the only non-invertible element of $\mathbb{C}$ is 0.*

Next we show that holomorphic functions cannot achieve their maximal norm without having constant norm.

Theorem 3.11 (Strong maximum principle)**.** *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and connected, X be a complex Banach space, and $f : U \to X$ be holomorphic. Suppose that there exists $z \in U$ such that*

$$\|f(z)\|_X = \max_{w\in U} \|f(w)\|_X . \tag{3.28}$$

Then $\|f\|_X$ is constant in U. Moreover, if $X = \mathbb{C}$, then f is constant in U.

Proof. Write $M = \|f(z)\|_X$ and $E = \{w \in U \mid \|f(w)\|_X = M\}$. The set E is relatively closed since $\|f(\cdot)\|_X$ is continuous. We claim that E is relatively open. Once this is established, we have that $E = U$ since U is connected and $E \neq \varnothing$ by assumption.

We now prove the claim. Let $w \in M$ and pick $R > 0$ such that $B[w, R] \subset U$. For $0 < r < R$ consider the circuit $\gamma_r : [0, 2\pi] \to U$ given by $\gamma_r(t) = w + re^{it}$. Then by Cauchy's integral formula we have

$$\|f(w)\|_X = \frac{1}{2\pi}\left\|\int_{\gamma_r} \frac{f(\zeta)}{\zeta - w} d\zeta\right\|_X = \frac{1}{2\pi}\left\|\int_0^{2\pi} \frac{ire^{it}}{w + re^{it} - w} f(w + re^{it})dt\right\|_X$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} \left\|f(w + re^{it})\right\|_X dt, \tag{3.29}$$

and hence (3.28) implies that

$$0 = \|f(w)\|_X - M \leq \frac{1}{2\pi}\int_0^{2\pi} \left(\left\|f(w + re^{it})\right\|_X - M\right) dt \leq 0. \tag{3.30}$$

We deduce from this that the continuous map $\varphi : [0, 2\pi] \to (-\infty, 0]$ given by

$$\varphi(t) = \left\|f(w + re^{it})\right\|_X - M \tag{3.31}$$

satisfies

$$\int_0^{2\pi} \varphi = 0, \tag{3.32}$$

which can only happen if $\varphi = 0$ on $[0, 2\pi]$. Hence $\|f(w + re^{it})\|_X = M$ for all $t \in [0, 2\pi]$ and $0 \leq r < R$, and we deduce that $B(z, R) \subset E$. Thus E is relatively open, and the claim is proved.

It remains to show that f is constant when $X = \mathbb{C}$. In this case we return to the context of the Cauchy-Riemann theorem, Theorem 1.6, and identify the holomorphic map $f : U \to \mathbb{C}$ with the vector field $F : \tilde{U} \to \mathbb{R}^2$. The theorem shows that F obeys the Cauchy-Riemann equations

$$\partial_1 F_1(x_1, x_2) = \partial_2 F_2(x_1, x_2) \text{ and } \partial_2 F_1(x_1, x_2) = -\partial_1 F_2(x_1, x_2). \tag{3.33}$$

Since f is smooth, F is as well, and we know from the above that $|F(x_1, x_2)| = M$ for all $(x_1, x_2) \in \tilde{U}$. If $M = 0$, then $F = 0$ identically, and so $f = 0$ identically as well, so we may reduce to the case $M > 0$. In this case we then compute

$$0 = \partial_1(F_1^2 + F_2^2) = 2F_1\partial_1 F_1 + 2F_2\partial_1 F_2 \text{ and } 0 = \partial_2(F_1^2 + F_2^2) = 2F_1\partial_2 F_1 + 2F_2\partial_2 F_2, \tag{3.34}$$

which combine with the Cauchy-Riemann equations to show that

$$0 = F_1\partial_2 F_2 + F_2\partial_1 F_2 \text{ and } 0 = -F_1\partial_1 F_2 + F_2\partial_2 F_2. \tag{3.35}$$

We then multiply the the first equation by F_1 and the second by F_2 and sum to see that

$$0 = (F_1^2 + F_2^2)\partial_2 F_2 = M^2 \partial_2 F_2, \tag{3.36}$$

which implies that $\partial_2 F_2 = \partial_1 F_1 = 0$. Similarly multiplying the first equation by F_2 and the second by F_1 and subtracting shows that

$$0 = (F_1^2 + F_2^2)\partial_1 F_2 = M^2 \partial_1 F_2, \tag{3.37}$$

which implies that $\partial_1 F_2 = \partial_2 F_1 = 0$. Thus, $\nabla F_1 = \nabla F_2 = 0$, and since U is connected we conclude that F is constant, which then implies that f is as well. □

In the strong maximum principle the assertion that f itself is constant can fail in the case $X \neq \mathbb{C}$, as we show in the next example.

Example 3.12. Consider the Banach space $\mathbb{C}^2$ with the norm $\|(z_1, z_2)\|_{\mathbb{C}^2} = \max\{|z_1|, |z_2|\}$. Consider the holomorphic map $f : U \to \mathbb{C}^2$ given by $f(z) = (1, z)$. Then $\|f(z)\|_{\mathbb{C}^2} = 1$ for all $z \in B(0,1)$ but f is not constant. △

The strong maximum principle has a somewhat weaker variant that applies to holomorphic functions that extend continuously to the boundary of bounded open sets. This weak maximum principle is extremely useful for deriving further estimates.

Theorem 3.13 (Weak maximum principle)**.** *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and bounded and X be a complex Banach space. Suppose that $f \in C^0(\bar{U}; X)$ and that f is holomorphic in U. Then*

$$\max_{z \in \bar{U}} \|f(z)\|_X = \max_{z \in \partial U} \|f(z)\|_X . \tag{3.38}$$

Proof. Suppose initially that U is connected. Both $\bar{U}$ and ∂U are compact, and so

$$\max_{\bar{U}} \|f\|_X = \max\{\sup_U \|f\|_X , \max_{\partial U} \|f\|_X\}. \tag{3.39}$$

If $\sup_U \|f\|_X > \max_{\partial U} \|f\|_X$, then $\|f\|_X$ achieves its maximum in U, and so the strong maximum principle implies that $\|f\|_X$ is a constant in U, and hence in $\bar{U}$ by the continuity of f, which contradicts the fact that $\sup_U \|f\|_X > \max_{\partial U} \|f\|_X$. Thus $\sup_U \|f\|_X \leq \max_{\partial U} \|f\|_X$, and we deduce that

$$\max_{\bar{U}} \|f\|_X = \max_{\partial U} \|f\|_X . \tag{3.40}$$

Now consider the general case in which U is only assumed to be bounded and open. Decomposing U into its connected components, we may then write $U = \bigcup_{k \in K} U_k$, where U_k is a nonempty open connected component of U and K is countable and nonempty. Applying the above analysis to each nonempty U_k, we see that

$$\max_{\bar{U}_k} \|f\|_X = \max_{\partial U_k} \|f\|_X , \tag{3.41}$$

which then implies that

$$\max_{\bar{U}} \|f\|_X = \max_{k \in K} \max_{\bar{U}_k} \|f\|_X = \max_{k \in K} \max_{\partial U_k} \|f\|_X = \max_{\partial U} \|f\|_X . \tag{3.42}$$

□

The maximum principles can be parlayed into striking quantitative estimates. Our first of these shows that it is possible to interpolate estimates for a holomorphic function in a strip-like domain.

Theorem 3.14 (Hadamard's three lines estimate)**.** *Let* $R = \{z \in \mathbb{C} \mid 0 \le \mathrm{Re}(z) \le 1\}$*,* X *be a complex Banach space, and suppose that* $f \in C^0_b(R;X)$ *is holomorphic in* R°*. Then for every* $x \in [0,1]$ *we have that*

$$\sup_{y\in\mathbb{R}} \|f(x+iy)\|_X \le \left(\sup_{y\in\mathbb{R}} \|f(0+iy)\|_X\right)^{1-x} \left(\sup_{y\in\mathbb{R}} \|f(1+iy)\|_X\right)^{x}. \tag{3.43}$$

Proof. Let $0 < M_0, M_1 < \infty$ be such that

$$\sup_{y\in\mathbb{R}} \|f(0+iy)\|_X \le M_0 \text{ and } \sup_{y\in\mathbb{R}} \|f(1+iy)\|_X \le M_1. \tag{3.44}$$

For $z = x+iy \in R$ we have that

$$\left|M_0^{1-z} M_1^z\right| = M_0^{1-x} M_1^x \ge \min\{M_0, M_1\} > 0. \tag{3.45}$$

This allows us to define the functions $g, g_n \in C^0_b(R;X)$ (here $1 \le n \in \mathbb{N}$) via

$$g(z) = \frac{f(z)}{M_0^{1-z} M_1^z} \text{ and } g_n(z) = g(z) e^{(z^2-1)/n}. \tag{3.46}$$

The boundedness of g follows from the boundedness of f and (3.45), while the boundedness of g_n follows since for $z = x+iy$ we have that

$$\|g_n(z)\|_X = \frac{\|f(z)\|_X}{M_0^{1-x} M_1^x} e^{(x^2-y^2-1)/n} \le \frac{\|f\|_{C^0_b(R;X)}}{\min\{M_0, M_1\}} e^{-y^2/n}. \tag{3.47}$$

This estimate also tells us that

$$\|g_n(z)\|_X \le 1 \text{ for } z \in \partial R \tag{3.48}$$

and that for each $n \ge 1$ there exists $r_n > 0$ such that

$$|\mathrm{Im}(z)| \ge r_n \Rightarrow \|g_n(z)\|_X \le 1. \tag{3.49}$$

Clearly g and g_n for $n \ge 1$ are holomorphic in R°. Fix $n \ge 1$. For each $m \ge 1$ define the rectangle $R_m = \{z \in R^\circ \mid |\mathrm{Im}(z)| < r_n + m\}$. According to the estimates (3.48) and (3.49) we have that for $n \in \mathbb{N}$

$$\max_{\partial R_m} \|g_n\|_X \le 1, \tag{3.50}$$

and so the weak maximum principle guarantees that

$$\max_{R_m} \|g_n\|_X \le 1. \tag{3.51}$$

Sending $m \to \infty$ then shows that

$$\sup_{R} \|g_n\|_X \le 1. \tag{3.52}$$

Finally, since $g_n(z) \to g(z)$ as $n \to \infty$ for each $z \in R$, we deduce that $\|g(z)\|_X \le 1$ on R, and hence

$$\|f(x+iy)\|_X \le M_0^{1-x} M_1^x. \tag{3.53}$$

Since this holds for all such M_0, M_1, we conclude that (3.43) holds.

□

Hadamard's three lines estimate gives rise to a corresponding three circle estimate for holomorphic functions defined on annuli.

Theorem 3.15 (Hadamard's three circles estimate)**.** *Let $0 < r_0 < r_1 < \infty$ and consider the annulus $A = \{z \in \mathbb{C} \mid r_0 \le |z| \le r_1\}$. Let X be a complex Banach space, and suppose that $f \in C_b^0(A;X)$ is holomorphic in A°. Then for every $s \in [0,1]$ we have that*

$$\sup_{z \in \partial B(0, r_0^{1-s} r_1^s)} \|f(z)\|_X \le \left(\sup_{z \in \partial B(0,r_0)} \|f(z)\|_X \right)^{1-s} \left(\sup_{z \in \partial B(0,r_1)} \|f(z)\|_X \right)^{s}. \tag{3.54}$$

Proof. Let $R = \{z \in \mathbb{C} \mid 0 \le \operatorname{Re}(z) \le 1\}$ and define the holomorphic map $\Phi : R \to \mathbb{C}$ via

$$\Phi(z) = r_0 \exp(z \log(r_1/r_0)). \tag{3.55}$$

Note that

$$\Phi(s+it) = r_0^{1-s} r_1^s e^{is\log(r_1/r_0)}, \tag{3.56}$$

so $\Phi(R) = A$, $\Phi(R^\circ) = A^\circ$, and $\Phi(\{z \in R \mid \operatorname{Re} z = s\}) = \partial B(0, r_0^{1-s} r_1^s)$. Define the function $F \in C_b^0(R;X)$ via $F = f \circ \Phi$, which is clearly holomorphic in R°. Hadamard's three lines estimate shows that

$$\begin{aligned} \sup_{z \in \partial B(0, r_0^{1-s} r_1^s)} \|f(z)\|_X = \sup_{y \in \mathbb{R}} \|F(s+iy)\|_X &\le \left(\sup_{y\in\mathbb{R}} \|F(0+iy)\|_X \right)^{1-s} \left(\sup_{y\in\mathbb{R}} \|F(1+iy)\|_X \right)^{s} \\ &= \left(\sup_{z \in \partial B(0,r_0)} \|f(z)\|_X \right)^{1-s} \left(\sup_{z \in \partial B(0,r_1)} \|f(z)\|_X \right)^{s}, \end{aligned} \tag{3.57}$$

which is the desired bound. □

Hadamard's three circles estimate provides rigid estimates for holomorphic maps defined in annuli. We now derive estimates for holomorphic maps defined in balls.

Theorem 3.16 (Schwarz estimate)**.** *Suppose that $R, S > 0$, X is a complex Banach space, $x_0 \in X$, and $z_0 \in \mathbb{C}$. Suppose that $f : B(z_0, R) \to B_X(x_0, S)$ is holomorphic and $f(z_0) = x_0$. Then*

$$\|f(z) - x_0\|_X \le \frac{S}{R} |z - x_0| \text{ for all } z \in B(z_0, R). \tag{3.58}$$

Moreover, if either $\|f'(z_0)\|_X = S/R$ or there exists $z \in B(z_0,R)\backslash\{z_0\}$ such that $\|f(z) - x_0\|_X = \frac{S}{R}|z - x_0|$, then

$$\|f(z) - x_0\|_X = \frac{S}{R} |z - x_0| \text{ for all } z \in B(z_0, R), \tag{3.59}$$

and if $X = \mathbb{C}$ then there exists $x \in \mathbb{C}$ with $|x| = 1$ such that

$$f(z) = x_0 + \frac{S}{R}(z - z_0)x \text{ for all } z \in B(z_0, R). \tag{3.60}$$

Proof. We first prove the result under the extra assumption that $z_0 = 0$, $x_0 = 0$, and $R = S = 1$. Then $f : B(0,1) \to B_X(0,1)$ is holomorphic and $f(0) = 0$. Since f has a zero at 0, we may consider the holomorphic function $F : B(0,1) \to X$ defined by $F(z) = f(z)/z$ for $z \neq 0$ and $F(0) = f'(0)$. Applying the weak maximum principle to the restriction of F to $B[0,r]$, for $0 < r < 1$, we find that

$$\max_{z\in B[0,r]} \|F(z)\|_X = \max_{z\in\partial B(0,r)} \frac{\|f(z)\|_X}{|z|} \leq \max_{z\in\partial B(0,r)} \frac{1}{|z|} = \frac{1}{r}. \tag{3.61}$$

Sending $r \to 1$, we deduce that

$$\sup_{z\in B(0,1)} \|F(z)\|_X \leq 1, \tag{3.62}$$

and hence that $\|f(z)\|_X \leq |z|$ for all $x \in B(0,1)$. Moreover, if $\|f'(0)\|_X = 1$ or there exists $z \in B(0,1)\backslash\{0\}$ such that $\|f(z)\|_X = |z|$, then $\|F(w)\|_X = 1$ for some $w \in B(0,1)$, and so the strong maximum principle implies that $\|F\|_X$ is constant and that if $X = \mathbb{C}$ then there exists $x \in \mathbb{C}$ with $|x| = 1$ such that $F(z) = x$ for all $z \in B(0,1)$, which means $f(z) = zx$. This proves the result in the special case.

In the general case of $f : B(z_0, R) \to B_X(x_0, S)$ we define the holomorphic function $g : B(0,1) \to B_X(0,1)$ via

$$g(z) = \frac{f(z_0 + Rz) - x_0}{S} \tag{3.63}$$

and note that $g(0) = 0$. Applying the specialized result to g then yields the general result. □

3.3 The argument principle and Rouché's theorem

The following result is a further generalization of Cauchy's integral formula that is often called the argument principle.

Theorem 3.17 (Argument principle). *Let X be a complex Banach space, $\varnothing \neq U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ be nontrivial and holomorphic, and $g : U \to X$ be holomorphic. Write $Z(f) = \{z \in U \mid f(z) = 0\} \subset U$ and recall that* $\operatorname{ord}(f,z)$ *denotes the order of the zero $z \in Z(f)$. If γ is a loop chain in U that is homologous to zero in U and* $\operatorname{ran}(\gamma) \cap Z(f) = \varnothing$, *then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} g = \sum_{z\in Z(f)} \operatorname{ord}(f,z) \operatorname{ind}(\gamma,z) g(z), \tag{3.64}$$

where the sum is finite due to the compactness of $\operatorname{ran}(\gamma)$*, Proposition 2.20, and Theorem 3.2, and the integral is well-defined because gf'/f is holomorphic in $U\backslash Z(f)$.*

Proof. Pick $R > 0$ such that $\operatorname{ran}(\gamma) \subseteq B[0,R]$. Since $\operatorname{ind}(\gamma,z) = 0$ for $|z| > R$, we deduce that γ is homologous to zero in $U \cap B(0,2R)$. Theorem 3.2 implies that the compact set $K = Z(f) \cap B[0,3R]$ is finite. Again appealing to Theorem 3.2, we find that the function $h : U \to \mathbb{C}$ given by

$$h(z) = f(z) \prod_{w\in K} (z-w)^{-\operatorname{ord}(f,w)} \tag{3.65}$$

is holomorphic and does not vanish in $U \cap B(0,2R)$. Then

$$f(z) = h(z) \prod_{w\in K} (z-w)^{\operatorname{ord}(f,w)} \text{ for } z \in U \cap B(0,2R), \tag{3.66}$$

and hence the product rule implies that

$$\frac{f'(z)}{f(z)} = \frac{h'(z)}{h(z)} + \sum_{w\in K} \frac{\operatorname{ord}(f,w)}{z-w} \text{ for } z \in U \cap B(0,2R)\backslash K. \tag{3.67}$$

Since gh'/h is holomorphic in $U \cap B(0,2R)$ and γ is homologous to zero in $U \cap B(0,2R)$, we then deduce from Cauchy-Goursat and Cauchy's integral formula that

$$\int_\gamma \frac{f'}{f} g = \int_\gamma \frac{h'}{h} g + \sum_{w \in K} \operatorname{ord}(f,w) \int_\gamma \frac{g(z)}{z-w} dz = \sum_{w \in K} \operatorname{ord}(f,w) 2\pi i \operatorname{ind}(\gamma, w) g(w)$$
$$= 2\pi i \sum_{w \in Z(f)} \operatorname{ord}(f,w) \operatorname{ind}(f,w) g(w). \quad (3.68)$$

This yields the stated identity. □

Remark 3.18. *By taking $f(w) = w - z$ for some $z \in U$, we recover Cauchy's integral formula from the argument principle.*

The argument principle has a particularly nice corollary (which is actually the origin of its name) when $X = \mathbb{C}$ and $g = 1$. First we need a lemma.

Lemma 3.19. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic. Let $\gamma = \sum_{j=1}^J m_j \gamma_j$ be a loop chain in U and define the loop chain $f \circ \gamma = \sum_{j=1}^J m_j f \circ \gamma_j$. Then for $w_0 \in \mathbb{C} \backslash f(\operatorname{ran}(\gamma))$,*

$$\operatorname{ind}(f \circ \gamma, w_0) = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{dw}{w - w_0} = \frac{1}{2\pi i} \int_\gamma \frac{f'}{f - w_0}. \quad (3.69)$$

Proof. Suppose first that γ is a lop in U. Since $\operatorname{ran}(\gamma) \subseteq U \backslash f^{-1}(\{w_0\})$ we can pick $\beta : [a,b] \to U \backslash \{z \in U \mid f(z) = w_0\}$ to be a circuit homotopic to γ. If $H : [0,1]^2 \to U \backslash f^{-1}(\{w_0\})$ is a homotopy of β and γ, then $f \circ H : [0,1]^2 \to \mathbb{C} \backslash \{w_0\}$ is a homotopy of $f \circ \beta$ and $f \circ \gamma$. Since $f \circ \beta$ is a circuit, we may then compute

$$2\pi i \operatorname{ind}(f \circ \gamma, w_0) = \int_{f \circ \gamma} \frac{dw}{w - w_0} = \int_{f \circ \beta} \frac{dw}{w - w_0} = \int_a^b \frac{(f \circ \beta)'}{f \circ \beta - w_0}$$
$$= \int_a^b \frac{f' \circ \beta}{f \circ \beta - w_0} \beta' = \int_\gamma \frac{f'}{f - w_0}, \quad (3.70)$$

which is the stated identity when γ is a loop.

Now let $\gamma = \sum_{j=1}^J m_j \gamma_j$ for $m_j \in \mathbb{Z}$ and γ_j a loop in U. Then

$$\operatorname{ind}(f \circ \gamma, w_0) = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{dw}{w - w_0} = \sum_{j=1}^J \frac{m_j}{2\pi i} \int_{f \circ \gamma_j} \frac{dw}{w - w_0} = \sum_{j=1}^J \frac{m_j}{2\pi i} \int_{\gamma_j} \frac{f'}{f - w_0} = \int_\gamma \frac{f'}{f - w_0}, \quad (3.71)$$

which is the stated identity in the general case.

□

With the lemma in hand, we get a nice identity related to the index of $f \circ \gamma$ for γ a loop and f holomorphic.

Corollary 3.20. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic. Let γ be a chain loop in U that is homologous to zero in U, and let $w_0 \in \mathbb{C} \backslash f(\operatorname{ran}(\gamma))$. Write $Z(f - w_0) = \{z \in U \mid f(z) = w_0\} \subset U$ and define the chain loop $f \circ \gamma$ in $\mathbb{C}$ as in Lemma 3.19. Then $w_0 \notin \operatorname{ran}(f \circ \gamma)$, and*

$$\operatorname{ind}(f \circ \gamma, w_0) = \sum_{z \in Z(f - w_0)} \operatorname{ord}(f - w_0, z) \operatorname{ind}(\gamma, z). \quad (3.72)$$

Proof. The identity follows from Lemma 3.19 and the argument principle applied to $g = 1$ and $f - w_0$, which is nontrivial since $w_0 \notin f(\mathrm{ran}(\gamma))$. □

Another nice result based on the argument principle is Rouché's theorem, which gives a tool for counting the zeros of holomorphic functions.

Theorem 3.21 (Rouché's theorem)**.** *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and let $f, g : U \to \mathbb{C}$ be holomorphic. Suppose γ is a loop chain in U that is homologous to zero in U and that*

$$|f(z) - g(z)| < |f(z)| + |g(z)| \text{ for all } z \in \mathrm{ran}(\gamma). \tag{3.73}$$

Then for $Z(f) = f^{-1}(\{0\})$ and $Z(g) = g^{-1}(\{0\})$ we have $(Z(f) \cup Z(g)) \cap \mathrm{ran}(\gamma) = \varnothing$, and

$$\sum_{z \in Z(f)} \mathrm{ord}(f, z) \,\mathrm{ind}(\gamma, z) = \sum_{z \in Z(g)} \mathrm{ord}(g, z) \,\mathrm{ind}(\gamma, z). \tag{3.74}$$

Proof. The condition (3.73) implies that $g(z) \neq 0$ and $f(z) \neq 0$ for $z \in \mathrm{ran}(\gamma)$. Consequently, $0 \notin \mathrm{ran}(f \circ \gamma)$ and $0 \notin \mathrm{ran}(g \circ \gamma)$. We may then define $h : U \backslash (Z(f) \cup Z(g)) \to \mathbb{C}$ via $h = f/g$, which is holomorphic and does not vanish since $Z(h) = Z(f)$. Then γ is a loop chain in $U \backslash (Z(h) \cup Z(g))$ such that $0 \notin \mathrm{ran}(h \circ \gamma)$, and so Lemma 3.19 implies that

$$\mathrm{ind}(h \circ \gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{h'}{h}. \tag{3.75}$$

However,

$$\frac{h'}{h} = \frac{g}{f} \frac{(f'g - fg')}{g^2} = \frac{f'}{f} - \frac{g'}{g}, \tag{3.76}$$

so the argument principle implies that

$$\mathrm{ind}(h \circ \gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'}{f} - \frac{1}{2\pi i} \int_\gamma \frac{g'}{g} = \sum_{z \in Z(f)} \mathrm{ord}(f, z) \,\mathrm{ind}(\gamma, z) - \sum_{z \in Z(g)} \mathrm{ord}(g, z) \,\mathrm{ind}(\gamma, z). \tag{3.77}$$

Returning to (3.73), we find that

$$|h(z) - 1| < |h(z)| + 1 \text{ for all } z \in \mathrm{ran}(\gamma). \tag{3.78}$$

A bit of algebra shows that this is equivalent to

$$-\,\mathrm{Re}\, h(z) < |h(z)| \text{ for all } z \in \mathrm{ran}(\gamma), \tag{3.79}$$

and hence

$$N = \{w \in \mathbb{C} \mid \mathrm{Re}\, w \leq 0 \text{ and } \mathrm{Im}\, w = 0\} \subseteq \mathrm{ran}(h \circ \gamma)^c. \tag{3.80}$$

Let $r > 0$ be such that $\mathrm{ran}(h \circ \gamma) \subset B[0, r]$. Since N is connected, 0 and $-2r \in N$ belong to the same connected component of $\mathbb{C} \backslash \mathrm{ran}(h \circ \gamma)$, so Proposition 2.20 then implies that

$$\mathrm{ind}(h \circ \gamma, 0) = \mathrm{ind}(h \circ \gamma, -2r) = 0. \tag{3.81}$$

The stated equality then follows by plugging this into (3.77) and rearranging.

□

Let's consider some examples.

Example 3.22. Rouché's theorem is particularly appealing when, in addition to the stated hypotheses, γ is assumed to be a counter-clockwise simple loop. Then the theorem says that if we define $E(\gamma) = \{z \in U \mid \gamma \text{ encloses } z\} = (\operatorname{ind}(\gamma, \cdot))^{-1}(\{1\})$, then

$$\sum_{z \in Z(f) \cap E(\gamma)} \operatorname{ord}(f, z) = \sum_{z \in Z(g) \cap E(\gamma)} \operatorname{ord}(g, z). \tag{3.82}$$

In other words, counting with multiplicity, the number of zeros of f enclosed by γ equals the number of zeros of g enclosed by γ. △

Example 3.23. Let $n \geq 3$, $a, b \in \mathbb{C}$ with $|a|, |b| \leq 1$, and consider the polynomial $f(z) = z^n + az + b$. We claim that all of the roots of f lie within $B(0, 3/2)$. Indeed, $20 < 27$, so $5/2 < 27/8$ and hence for $|z| = 3/2$,

$$|f(z) - z^n| \leq |z| + 1 = \frac{5}{2} < \frac{27}{8} = |z|^3 \leq |z|^n. \tag{3.83}$$

Rouché's theorem with $\gamma(t) = (3/2)e^{2\pi i t}$ for $t \in [0, 1]$ then shows that f has the same number or roots as z^n in $B(0, 3/2)$, but this is n. Hence all roots of f lie in $B(0, 3/2)$.

△

The requirement that the codomain be $\mathbb{C}$ cannot be relaxed in Rouché's theorem.

Example 3.24. Rouché's theorem fails in general for $X \neq \mathbb{C}$. Let $f, g : \mathbb{C} \to \mathbb{C}^2$ via $f(z) = (z, 1)$ and $g(z) = (2z, 0)$. Let $\gamma : [0, 1] \to \partial B(0, 1)$ via $\gamma(t) = e^{2\pi i t}$. On $\operatorname{ran}(\gamma)$ we have that $|z| = 1$, so

$$|f(z) - g(z)| = \sqrt{1 + |z|^2} = \sqrt{2} < 2 = |g(z)|. \tag{3.84}$$

However, $Z(f) = \varnothing$, $Z(g) = \{0\}$, $\operatorname{ord}(g, 0) = 1$, and $\operatorname{ind}(\gamma, 0) = 1$, so

$$\sum_{z \in Z(f)} \operatorname{ord}(f, z) \operatorname{ind}(\gamma, z) = 0 \neq 1 = \sum_{z \in Z(g)} \operatorname{ord}(g, z) \operatorname{ind}(\gamma, z). \tag{3.85}$$

△

Rouché's theorem easily yields a stronger form of the fundamental theorem of algebra that comes with an estimate for the locations of all of the roots.

Theorem 3.25 (Fundamental theorem of algebra, quantitative version)**.** *Let $1 \leq n \in \mathbb{N}$ and $p : \mathbb{C} \to \mathbb{C}$ be the polynomial $p(z) = \sum_{m=0}^{n} a_m z^m$ with $a_n \neq 0$. Then p has n roots in $B[0, r]$, where*

$$r = \max\{1, \frac{1}{|a_n|} \sum_{m=0}^{n-1} |a_m|\}. \tag{3.86}$$

Proof. Let $q : \mathbb{C} \to \mathbb{C}$ via $q(z) = a_n z^n$. Then for $|z| = R > r \geq 1$ we have

$$|p(z) - q(z)| \leq \sum_{m=0}^{n-1} |a_m| R^m \leq R^{n-1} \sum_{m=0}^{n-1} |a_m| \leq |a_n| r R^{n-1} < |a_n| R^n = |q(z)| \leq |q(z)| + |p(z)| \tag{3.87}$$

by the choice of R. Let $\gamma : [0, 1] \to \partial B(0, R)$ via $\gamma(t) = Re^{2\pi i t}$. Rouché's theorem then implies that no roots of p lie on $\partial B(0, R)$ and that p has n roots in $B(0, R)$ since q does. This holds for every $R > r$, so we conclude that all roots of p lie inside $B[0, r]$. □

The implicit function theorem can be used to show that the roots of a complex polynomial depend smoothly on the coefficients of the polynomial, provided all of the roots are distinct. If there are repeated roots, then smoothness breaks down, but we can still hope for continuity. We now prove this with the help of Rouché's theorem. To formulate continuity of the set of roots we will use the Hausdorff metric h on the set of nonempty compact subsets of $\mathbb{C}$, $\mathfrak{K}(\mathbb{C})$. Recall that, given a metric space X, the Hausdorff metric space $\mathfrak{H}(X)$ consists of all nonempty closed subsets of X and that

$$\mathfrak{K}(X) = \{\varnothing \neq K \subseteq X \mid K \text{ is compact}\} \subseteq \mathfrak{H}(X) \tag{3.88}$$

is the subspace of nonempty compact sets in X.

Theorem 3.26 (Continuous dependence of polynomial roots on coefficients). *Let $1 \leq n \in \mathbb{N}$ and define the open set $U = \{a = (a_0, \ldots, a_n) \in \mathbb{C}^{n+1} \mid a_n \neq 0\}$. For $a \in U$ let $p_a : \mathbb{C} \to \mathbb{C}$ be the polynomial $p_a(z) = \sum_{k=0}^n a_k z^k$. Define the map $\Phi : U \to \mathfrak{K}(\mathbb{C})$ via $\Phi(a) = Z(p_a) = p_a^{-1}(\{0\})$. Then Φ is continuous.*

Proof. Let $a \in U$ and pick $\delta_0 > 0$ such that $B(a, \delta_0) \subset U$. Write $Z(p_a) = \{z_1, \ldots, z_m\}$ with $z_j \neq z_k$ for $j \neq k$, and let $\sigma_j = \text{ord}(p_a, z_j)$ for $1 \leq j \leq m$. Then we can write

$$p_a(z) = a_n \prod_{j=1}^m (z - z_j)^{\sigma_j} \tag{3.89}$$

with $n = \sum_{j=1}^m \sigma_j$. If $m = 1$ set $r = 1$; otherwise set

$$0 < r = \min\{|z_j - z_k| \mid 1 \leq j, k \leq m \text{ and } j \neq k\}. \tag{3.90}$$

Let $0 < \varepsilon < r/2$ and set

$$\delta = \min\{\delta_0, |a_n| \, \varepsilon^n \left(\sum_{k=0}^n (|z_j| + \varepsilon)^k \right)^{-1} \}. \tag{3.91}$$

Assume $b \in B(a, \delta) \subset U$. Let $1 \leq j \leq m$ and $z \in \partial B(z_j, \varepsilon)$. Then

$$|p_a(z) - p_b(z)| \leq \sum_{k=0}^n |a_k - b_k| \, |z|^k < \delta \sum_{k=0}^n (|z_j| + \varepsilon)^k \leq |a_n| \, \varepsilon^n. \tag{3.92}$$

Also, if $m > 1$ we can estimate, for each $k \neq j$,

$$|z - z_k| \geq |z_j - z_k| - |z - z_j| \geq r - \varepsilon \geq \frac{r}{2} > \varepsilon. \tag{3.93}$$

Thus,

$$|p_a(z)| = |a_n| \, |z - z_j|^{\sigma_j} \prod_{j \in \{1,\ldots,m\} \setminus \{k\}} |z - z_k|^{\sigma_k} \geq |a_n| \, \varepsilon^{\sigma_j} \varepsilon^{n - \sigma_j} = |a_n| \, \varepsilon^n. \tag{3.94}$$

Combining these estimates, we find that

$$|p_a(z) - p_b(z)| < |a_n| \, \varepsilon^n \leq |p_a(z)| \leq |p_a(z)| + |p_b(z)| \tag{3.95}$$

for all $z \in \partial B(z_j, \varepsilon)$. Rouché's theorem then implies that no roots of p_b lie on $\partial B(z_j, \varepsilon)$ and that p_b has the same number of roots as p_a in $B(z_j, \varepsilon)$, which is σ_j. Since $n = \sum_{j=1}^m \sigma_j$ and p_b has n roots, we deduce from this that

$$Z(p_b) \subseteq \bigcup_{j=1}^m B(z_j, \varepsilon), \tag{3.96}$$

which in turn implies that $h(Z(p_a), Z(p_b)) < \varepsilon$, where h is the Hausdorff metric. Thus, Φ is continuous.

□

It's essential that we restrict to polynomials of the same degree in this result, as we now show.

Example 3.27. The polynomials $p, q : \mathbb{C} \to \mathbb{C}$ given by $p(z) = z$ and $q(z) = z - \varepsilon z^2$ for $\varepsilon > 0$ are such that $Z(p) = \{0\}$ and $Z(q) = \{0, 1/\varepsilon\}$ but $|(0,1,0) - (0,1,-\varepsilon)| = \varepsilon$. Thus the coefficients are arbitrarily close, but the roots of q are not confined to nearby the roots of p. △

3.4 Special properties of complex-valued holomorphic functions

We now turn our attention to some properties of complex-valued holomorphic functions. Our first result is a holomorphic version of the inverse function theorem.

Theorem 3.28 (Holomorphic inverse function theorem)**.** *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic. Suppose that $z_0 \in U$ and $f'(z_0) \neq 0$. Then there exists an open set $V \subseteq U$ with $z_0 \in V$ such that $f : V \to f(V)$ is a homeomorphism with $f^{-1} : f(V) \to V$ holomorphic.*

Proof. Let $\tilde{U} = \{x \in \mathbb{R}^2 \mid x_1 + ix_2 \in U\}$ and $F : \tilde{U} \to \mathbb{R}^2$ be the smooth function given by $F(x) = (\operatorname{Re} f(x_1 + ix_2), \operatorname{Im} f(x_1 + ix_2))$. Let $y \in U$ be such that $z = y_1 + iy_2$. Note that since $f'(z) \neq 0$ we have that

$$|\partial_1 F_1(y)|^2 + |\partial_2 F_2(y)|^2 = |f'(z)|^2 \neq 0. \tag{3.97}$$

Since F satisfies the Cauchy-Riemann system, we may compute

$$DF(y) = \begin{pmatrix} \partial_1 F_1(y) & \partial_2 F_1(y) \\ -\partial_2 F_1(y) & \partial_1 F_1(y) \end{pmatrix}, \tag{3.98}$$

and so $DF(y)$ is invertible with

$$(DF(y))^{-1} = \frac{1}{|\partial_1 F_1(y)|^2 + |\partial_2 F_2(y)|^2} \begin{pmatrix} \partial_1 F_1(y) & -\partial_2 F_1(y) \\ \partial_2 F_1(y) & \partial_1 F_1(y) \end{pmatrix}. \tag{3.99}$$

The inverse function theorem provides a set $\tilde{V} \subseteq \tilde{U}$ with $y \in \tilde{U}$ such that $F : \tilde{V} \to F(\tilde{V})$ is a smooth diffeomorphism. Set $V = \{x_1 + ix_2 \in U \mid x \in V\}$. Then $f : V \to f(V)$ is a homeomorphism and $f^{-1} : f(V) \to V$ is related to F^{-1} via $F^{-1}(x) = (\operatorname{Re} f^{-1}(x_1 + ix_2), \operatorname{Im} f^{-1}(x_1 + ix_2))$. From (3.99) and the identity $DF^{-1}(F(y)) = (DF(y))^{-1}$ we then deduce that F^{-1} satisfies the Cauchy-Riemann equations, and hence f^{-1} is holomorphic on $f(V)$. □

Next we prove that non-constant complex-valued holomorphic functions are open maps.

Theorem 3.29 (Open mapping)**.** *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic and not constant. Then f is an open map, i.e. if $V \subseteq U$ is open, then $f(V)$ is open.*

Proof. Let $V \subseteq U$ be open. If $V = \varnothing$, there's nothing to prove, so we may assume that $V \neq \varnothing$. Let $z_0 \in V$ and $w_0 = f(z_0)$. The holomorphic function $F : V \to \mathbb{C}$ defined by $F(z) = f(z) - w_0$ has a zero at z_0, and its zeros must be isolated, so we can pick $r > 0$ such that $B[z_0, r] \subset V$ and if $z \in B[z_0, r]$, then $f(z) \neq w_0$. Define

$$\delta = \min\{|f(z) - w_0| \mid z \in \partial B(z_0, r)\} > 0. \tag{3.100}$$

Now let $w \in B(w_0, \delta/2)$, and suppose, by way of contradiction, that $f(z) \neq w$ for any $z \in V$. We may then define the holomorphic function $g : U \to \mathbb{C}$ via

$$g(z) = \frac{1}{f(z) - w}. \tag{3.101}$$

By the weak maximum principle,

$$\max_{z \in B[z_0, r]} |g(z)| = \max_{z \in \partial B(z_0, r)} \frac{1}{|f(z) - w_0|}. \tag{3.102}$$

For $z \in \partial B(z_0, r)$ we may compute

$$|f(z) - w| \geq |f(z) - w_0| - |w_0 - w| \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}, \tag{3.103}$$

so

$$\max_{z \in \partial B(z_0, r)} \frac{1}{|f(z) - w_0|} \leq \frac{2}{\delta}. \tag{3.104}$$

On the other hand,

$$|g(z_0)| = \frac{1}{|f(z_0) - w|} = \frac{1}{|w_0 - w|} > \frac{2}{\delta} \tag{3.105}$$

and so $2/\delta < 2/\delta$, a contradiction. Thus, $B(w_0, \delta/2) \subseteq f(V)$, and we conclude that $f(V)$ is open for every open $V \subseteq U$.

□

Our next result shows that injective holomorphic functions with values in $\mathbb{C}$ have holomorphic inverses.

Theorem 3.30. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic and injective. Then $\varnothing \neq f(U) \subseteq \mathbb{C}$ is open and the inverse map $f^{-1} : f(U) \to \mathbb{C}$ is holomorphic.*

Proof. First note that $f(U)$ is open due to the open mapping theorem, Theorem 3.29. This also shows that f^{-1} is continuous since $f(V) = (f^{-1})^{-1}(V)$ is open for every open $V \subseteq U$, again by the open mapping theorem. It remains to prove that f^{-1} is holomorphic. In light of Theorem 3.29, it suffices to prove that $f' : U \to \mathbb{C}$ never vanishes.

Suppose, by way of contradiction, that $f'(z_0) = 0$ for some $z_0 \in U$. Then $f - f(z_0)$ has a zero of order at least two at z_0, so in a ball $B(z_0, r) \subset U$ we can write $f(z) = f(z_0) + (z - z_0)^n g(z)$ for $n \geq 2$ and $g : B(z_0, r) \to \mathbb{C}$ a holomorphic function that does not vanish. By Theorem 3.6 we can write $g = e^h$ for $h : B(z_0, r) \to \mathbb{C}$ holomorphic. Define the holomorphic function $H : B(z_0, r) \to \mathbb{C}$ via $H(z) = (z - z_0)e^{h(z)/n}$. We can then write

$$f(z) = f(z_0) + (H(z))^n \tag{3.106}$$

for all $z \in B(z_0, r)$. We have that $H(z_0) = 0$, and since H is holomorphic the set $H(B(z_0, r))$ is open, so we can pick $\delta > 0$ such that $B(0, 2\delta) \subseteq H(B(z_0, r))$. In particular, $w_1 = \delta$ and $w_2 = \delta e^{2\pi i/n} \neq w_1$ are such that $w_1, w_2 \in H(B(z_0, r))$. Then there exist $z_1, z_2 \in B(z_0, r)$ such that $H(z_j) = w_j$ for $j = 1, 2$, and so

$$f(z_j) = f(z_0) + (H(z_j))^n = f(z_0) + \delta^n \text{ for } j = 1, 2, \tag{3.107}$$

contradicting the injectivity of f. Hence f' does not vanish in U. □

We next construct a special holomorphic homeomorphism from the unit ball to itself.

Example 3.31. Consider $a \in B(0,1)$. Then for $z \in \mathbb{C}$ we have that

$$|z| \le 1 \Leftrightarrow |z|^2 \le \frac{1-|a|^2}{1-|a|^2} \Leftrightarrow |z|^2 + |a|^2 \le 1 + |a|^2 |z|^2 \Leftrightarrow |z|^2 - \bar{a}z - a\bar{z} + |a|^2 \le 1 - \bar{a}z - a\bar{z} + |a|^2 |z|^2$$
$$\Leftrightarrow |z-a|^2 = (z-a)(\bar{z}-\bar{a}) \le (1-\bar{a}z)(1-a\bar{z}) = |1-\bar{a}z|^2 . \quad (3.108)$$

Moreover, if $|z| \le 1$, then $|\bar{a}z| = |a|\,|z| < 1$, so $1 - \bar{a}z \neq 0$ We may thus define $f_a : B[0,1] \to B[0,1]$ via

$$f_a(z) = \frac{a-z}{1-\bar{a}z}. \quad (3.109)$$

The above shows that this is well-defined and yields a holomorphic function in $B(0,1)$ such that $f_a(0) = a$ and $f_a(a) = 0$. For $w \in B[0,1]$ we compute

$$w = f_a(z) \Leftrightarrow w - \bar{a}wz = a - z \Leftrightarrow z(1-\bar{a}w) = a - w \Leftrightarrow z = f_a(w), \quad (3.110)$$

which reveals that f_a is a homeomorphism with $f_a^{-1} = f_a$. In particular, f_a^{-1} is also holomorphic in $B(0,1)$. Moreover, since $|z| = 1$ if and only if $|f_a(z)| = 1$, the restriction of f_a to $B(0,1)$ is also a homeomorphism. △

Remarkably, this is essentially the only example of a holomorphic homeomorphism between balls.

Theorem 3.32. *Suppose that $R, S > 0$, $z_0, w_0 \in \mathbb{C}$, and $f : B(z_0,R) \to B(w_0,S)$ is a homeomorphism. Then the following are equivalent.*

1. *f is holomorphic.*
2. *There exist $a, u \in \mathbb{C}$ with $|u| = 1$ and $|a| < 1$ such that*

$$f(z) = w_0 + Su\left(\frac{Ra - (z-z_0)}{R - \bar{a}(z-z_0)}\right). \quad (3.111)$$

Proof. We first prove the result under the extra assumptions that $z_0 = w_0 = 0$ and $R = S = 1$, in which case $f : B(0,1) \to B(0,1)$ is a homeomorphism.

Suppose that f is holomorphic, in which case Theorem 3.30 shows that f^{-1} is also holomorphic. Let $a = f^{-1}(0)$. Define the holomorphic bijection $f_a : B(0,1) \to B(0,1)$ as in Example 3.31. Then $F = f \circ f_a : B(0,1) \to B(0,1)$ and $F^{-1} = f_a \circ f^{-1}$ are holomorphic and $F(0) = 0 = F^{-1}(0)$. Consequently, the Schwarz estimate, Theorem 3.16, shows that

$$|z| = \left|F^{-1}(F(z))\right| \le |F(z)| \le |z| \text{ for } z \in B(0,1), \quad (3.112)$$

and so $|F(z)| = |z|$ for $z \in B(0,1)$. Again appealing to the Schwarz estimate, we conclude that $F(z) = uz$ for some $u \in \mathbb{C}$ with $|u| = 1$. Then $f \circ f_a(z) = uz$, and hence

$$f(z) = uf_a^{-1}(z) = uf_a(z) = u\frac{a-z}{1-\bar{a}z}. \quad (3.113)$$

Conversely, if $f = uf_a$, then f is holomorphic.

This proves the theorem in the special case. In the general case we set $g : B(0,1) \to B(0,1)$ via

$$g(z) = \frac{f(z_0 + Rz) - w_0}{S}, \quad (3.114)$$

which is a homeomorphism. If f is holomorphic, then g is as well, and the special result then proves that f has the stated form. On the other hand, if f has the stated special form, then $g = uf_a$, and so g is holomorphic, which shows that f is as well. □

4 Laurent series, singularities, and meromorphic functions

Given an open set $\varnothing \neq U \subseteq \mathbb{C}$ and two holomorphic functions $f, g : U \to \mathbb{C}$, the quotient f/g is not defined on $Z(g)$ but is perfectly holomorphic on $U \backslash Z(g)$. Theorem 3.2 shows that the points in $Z(g)$ are isolated, so the singularities of f/g are isolated as well. We now turn our attention to the study of mappings of this type, i.e. maps that are holomorphic away from a set of isolated singularities. Along the way we develop a classification for such singularities and extend many of the above results to a special subset of these maps, which are called meromorphic.

4.1 Laurent series and classification of isolated singularities

We have seen that holomorphic functions are analytic, and can therefore be expanded in power series in balls. It turns out that there is a more general version of this expansion that is valid for maps that are holomorphic in a ball with its center deleted. We now prove this in the context of holomorphic functions defined on an annulus.

Theorem 4.1. *Let X be a complex Banach space. Let $z_0 \in \mathbb{C}$, $0 \leq r_0 < r_1 \leq \infty$, and define the open annulus*

$$A(z_0; r_0, r_1) = \{z \in \mathbb{C} \mid r_0 < |z - z_0| < r_1\}. \tag{4.1}$$

Let $r_0 < r < r_1$ and $f : A(z_0; r_0, r_1) \to X$. Then the following are equivalent.

1. *f is holomorphic in $A(z_0; r_0, r_1)$.*
2. *There exist $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=0}^{\infty} \subseteq X$ such that*

$$\limsup_{n\to\infty} \|x_n\|_X^{1/n} \leq r_0 \text{ and } \limsup_{n\to\infty} \|y_n\|_X^{1/n} \leq \frac{1}{r_1}, \tag{4.2}$$

and

$$f(z) = \sum_{n=1}^{\infty} (z - z_0)^{-n} x_n + \sum_{n=0}^{\infty} (z - z_0)^n y_n \text{ for } z \in A, \tag{4.3}$$

where the series converge in $A(z_0; r_0, r_1)$ and uniformly absolutely in $A(z_0; s_0, s_1)$ for $r_0 < s_0 < s_1 < r_1$.

In either case, for any $r_0 < r < r_1$ we have that

$$x_n = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} (w - z_0)^{n-1} f(w) dw \text{ for } 1 \leq n \in \mathbb{N} \tag{4.4}$$

and

$$y_n = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(w)}{(w - z_0)^{n+1}} dw \text{ for } n \in \mathbb{N}. \tag{4.5}$$

Proof. Suppose initially that the second item holds. Then

$$\limsup_{n\to\infty} \left(\frac{\|x_n\|_X}{|z - z_0|^n} \right)^{1/n} \leq \frac{r_0}{|z - z_0|} \text{ and } \limsup_{n\to\infty} \left(|z - z_0|^n \|y_n\|_X\right)^{1/n} \leq \frac{|z - z_0|}{r_1}, \tag{4.6}$$

which shows that the series in (4.3) converge and defines a holomorphic function f in $A(z_0; r_0, r_1)$. Moreover, for $r_0 < r < r_1$ and $n \in \mathbb{Z}$, the we can use the uniform convergence of the series to compute

$$\frac{1}{2\pi i}\int_{\partial B(z_0,r)} (w-z_0)^{n-1} f(w)dw = \sum_{m=1}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial B(z_0,r)} (w-z_0)^{n-m-1}dw\right) x_m \\ + \sum_{m=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial B(z_0,r)} (w-z_0)^{n+m-1}dw\right) y_m = \begin{cases} x_n & \text{if } n \geq 1 \\ y_n & \text{if } n \leq 0. \end{cases} \quad (4.7)$$

This proves that the second item implies the first.

We now prove the converse. It suffices to prove that the first item implies the second under the extra assumption that $z_0 = 0$. Indeed, in the general case we define the holomorphic map $g : B(0, r_1)\backslash B[0, r_0] \to X$ via $g(z) = f(z_0 + z)$ and apply the special case to arrive at the result. Assume, then, that $z_0 = 0$.

Let $z \in A = B(0, r_1)\backslash B[0, r_0]$ and pick $\varepsilon > 0$ such that $r_0 < |z| - \varepsilon < |z| + \varepsilon < r_1$. Also let $\theta \in [0, 2\pi)$ be such that $z = |z| e^{i\theta}$. We then define the map $H : [0, 1] \times [0, 1] \to \mathbb{C}$ via

$$H(t,s) = \begin{cases} [|z| + (1-s)\varepsilon] \exp(i\theta + i\pi(1-s)(8t-1)) & \text{if } 0 \leq t < \frac{1}{4} \\ [(2-4t)(|z| + (1-s)\varepsilon) + (4t-1)(|z| - (1-s)\varepsilon)] \exp(i\theta + i\pi(1-s)) & \text{if } \frac{1}{4} \leq t < \frac{1}{2} \\ [|z| - (1-s)\varepsilon] \exp(i\theta - i\pi(1-s)(8t-5)) & \text{if } \frac{1}{2} \leq t < \frac{3}{4} \\ [(4-4t)(|z| - (1-s)\varepsilon) + (4t-3)(|z| + (1-s)\varepsilon)] \exp(i\theta - i\pi(1-s)) & \text{if } \frac{3}{4} \leq t \leq 1. \end{cases} \quad (4.8)$$

We leave it as an exercise to check that H is continuous, $H([0, 1]^2) \subseteq B[0, |z|+\varepsilon]\backslash B(0, |z|-\varepsilon) \subset A$, $\beta = H(\cdot, 0)$ is a circuit, $H(t, 1) = z$ for all $t \in [0, 1]$, and $H(0, s) = H(1, s)$ for all $s \in [0, 1]$. Thus, H defines a homotopy from β to the point $z \in A$.

By construction, we have that $\beta = H(\cdot, 0)$ is the concatenation of β_0, β_1, β_2, and β_3, where $\beta_j : [0, 1/4] \to A$ for $j \in \{0, 1, 2, 3\}$ are given by

$$\begin{aligned} \beta_0(t) &= [|z| + \varepsilon] \exp(i\theta + i\pi(8t-1)) \\ \beta_1(t) &= -[(1-4t)(|z| + \varepsilon) + 4t(|z| - \varepsilon)]e^{i\theta} \\ \beta_2(t) &= [|z| - \varepsilon] \exp(i\theta - i\pi(8t-1)) \\ \beta_3(t) &= -[4t(|z| + \varepsilon) + (1-4t)(|z| - \varepsilon)]e^{i\theta}. \end{aligned} \quad (4.9)$$

From this we see that $z \notin \operatorname{ran}(\beta)$ and that $\beta_3 = \check{\beta}_1$. We may then apply the Cauchy integral formula and Theorem 1.35 to see that

$$2\pi i \operatorname{ind}(\beta, z) f(z) = \int_\beta \frac{f(w)}{w-z}dw = \int_{\beta_0} \frac{f(w)}{w-z}dw + \int_{\beta_2} \frac{f(w)}{w-z}dw. \quad (4.10)$$

We will now compute the terms on the left and right.

First we compute, using Proposition 1.35,

$$\operatorname{ind}(\beta, z) = \operatorname{ind}(\beta_0, z) + \operatorname{ind}(\beta_1, z) + \operatorname{ind}(\beta_2, z) - \operatorname{ind}(\beta_1, z) = \operatorname{ind}(\beta_0, z) + \operatorname{ind}(\beta_2, z). \quad (4.11)$$

Note that $\operatorname{ran}(\beta_0) = \partial B(0, |z| + \varepsilon)$ and $\operatorname{ran}(\beta_2) = \partial B(0, |z| - \varepsilon)$. Using this, and arguing as in Example 2.15, we find that $\operatorname{ind}(\beta_0, z) = 1$ and $\operatorname{ind}(\beta_2, z) = 0$. Hence,

$$\operatorname{ind}(\beta, z) = 1. \quad (4.12)$$

For $w \in \mathrm{ran}(\beta_0)$ we have that $|z| < |z| + \varepsilon = |w|$, so we can write

$$\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-z/w} = \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}, \tag{4.13}$$

where the series converges uniformly absolutely on $\mathrm{ran}(\beta_0)$. Similarly, $w \in \mathrm{ran}(\beta_2)$ we have that $|w| = |z| - \varepsilon < |z|$, so we can write

$$\frac{1}{w-z} = -\frac{1}{z} \cdot \frac{1}{1-w/z} = -\sum_{n=0}^{\infty} \frac{w^n}{z^{n+1}}, \tag{4.14}$$

where the series converges uniformly absolutely on $\mathrm{ran}(\beta_2)$. On the other hand, β_0 and $\check{\beta}_2$ are both homotopic in A to $\partial B(0,r)$. Thus, another application of Cauchy's integral formula and Theorem 1.35 shows that

$$\int_{\beta_0} \frac{f(w)}{w-z} dw = \sum_{n=0}^{\infty} z^n \int_{\beta_0} \frac{f(w)}{w^{n+1}} dw = \sum_{n=0}^{\infty} z^n \int_{\partial B(0,r)} \frac{f(w)}{w^{n+1}} dw \tag{4.15}$$

and

$$\int_{\beta_2} \frac{f(w)}{w-z} dw = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_{\beta_2} w^n f(w) dw = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_{\partial B(0,r)} w^n f(w) dw. \tag{4.16}$$

Combining (4.10), (4.12), (4.15), and (4.16) then shows that

$$f(z) = \sum_{n=0}^{\infty} z^n \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{f(w)}{w^{n+1}} dw + \sum_{n=1}^{\infty} z^{-n} \frac{1}{2\pi i} \int_{\partial B(0,r)} w^{n-1} f(w) dw. \tag{4.17}$$

This proves everything except for the estimates, but these follow easily from the above expressions for x_n and y_n. This proves the second item holds in the case $z_0 = 0$. □

This suggests some notation.

Definition 4.2. *The series expansion* (4.3) *is called the Laurent series expansion of f in A. We often write*

$$f(z) = \sum_{n \in \mathbb{Z}} (z - z_0)^n \hat{f}(n) \tag{4.18}$$

as shorthand for the two series in (4.3)*, where*

$$\hat{f}(n) = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} (w - z_0)^{-n-1} f(w) dw \in X \tag{4.19}$$

for $n \in \mathbb{Z}$ and any $r_0 < r < r_1$.

The coefficient map $\hat{f} : \mathbb{Z} \to X$ carries a lot of interesting information about the size of f, at least when X is a Hilbert space. We now investigate this.

Theorem 4.3. *Let X be a complex Hilbert space with inner-product $\langle \cdot, \cdot \rangle_X$. Let $z_0 \in \mathbb{C}$, $0 \le r_0 < r_1 \le \infty$, and $A(z_0; r_0, r_1) \subseteq \mathbb{C}$ be the annulus as in Theorem 4.1. Suppose that $f, g : A(z_0; r_0, r_1) \to X$ are holomorphic and have Laurent series*

$$f(z) = \sum_{n \in \mathbb{Z}} (z - z_0)^n \hat{f}(n) \text{ and } g(z) = \sum_{n \in \mathbb{Z}} (z - z_0)^n \hat{g}(n) \text{ for } z \in A(z_0; r_0, r_1). \tag{4.20}$$

Then for each $r_0 < r < r_1$ the following hold.

1. The maps $\hat{f}, \hat{g} : \mathbb{Z} \to X$ satisfy

$$\sum_{n\in\mathbb{Z}} r^{2n} \left\| \hat{f}(n) \right\|_X^2 + \sum_{n\in\mathbb{Z}} r^{2n} \left\| \hat{g}(n) \right\|_X^2 < \infty. \tag{4.21}$$

2. We have the Plancherel identity:

$$\int_0^1 \left\| f(z_0 + re^{2\pi it}) \right\|_X^2 dt = \sum_{n\in\mathbb{Z}} r^{2n} \left\| \hat{f}(n) \right\|_X^2. \tag{4.22}$$

3. We have the Parseval identity:

$$\int_0^1 \langle f(z_0 + re^{2\pi it}), g(z_0 + re^{2\pi it}) \rangle_X dt = \sum_{n\in\mathbb{Z}} r^{2n} \langle \hat{f}(n), \hat{g}(n) \rangle_X. \tag{4.23}$$

Proof. For $N \in \mathbb{N}$ define the holomorphic functions $f_N, g_N : A(z_0; r_0, r_1) \to X$ via

$$f_N(z) = \sum_{n=-N}^{N} (z - z_0)^n \hat{f}(n) \text{ and } g_N(z) = \sum_{n=-N}^{N} (z - z_0)^n \hat{g}(n). \tag{4.24}$$

Fix $r \in (r_0, r_1)$. For any z such that $|z - z_0| = r$ we have that

$$\overline{z - z_0} = \frac{r^2}{z - z_0}, \tag{4.25}$$

and hence

$$\langle f_N(z), g_N(z) \rangle_X = \sum_{m,n=-N}^{N} (z - z_0)^n (\overline{z - z_0})^m \langle \hat{f}(n), \hat{g}(m) \rangle_X = \sum_{m,n=-N}^{N} r^{2m} (z - z_0)^{n-m} \langle \hat{f}(n), \hat{g}(m) \rangle_X. \tag{4.26}$$

Using this, we compute

$$\int_0^1 \langle f_N(z_0 + re^{2\pi it}), g_N(z_0 + re^{2\pi it}) \rangle_X dt = \int_{\partial B(z_0,r)} \langle f_N(z), g_N(z) \rangle_X \frac{dz}{2\pi i (z - z_0)}$$
$$= \sum_{m,n=-N}^{N} \left(\frac{1}{2\pi i} \int_{\partial B(z_0,r)} (z - z_0)^{n-m-1} dz \right) r^{2m} \langle \hat{f}(n), \hat{g}(m) \rangle_X = \sum_{n=-N}^{N} r^{2n} \langle \hat{f}(n), \hat{g}(n) \rangle_X. \tag{4.27}$$

Using (4.27) with g_N replaced by f_N, we see that

$$\int_0^1 \left\| f_N(z_0 + re^{2\pi it}) \right\|_X^2 dt = \sum_{n=-N}^{N} r^{2n} \left\| \hat{f}(n) \right\|_X^2, \tag{4.28}$$

and so upon sending $N \to \infty$ and using the uniform convergence of the Laurent series, we find that

$$\int_0^1 \left\| f(z_0 + re^{2\pi it}) \right\|_X^2 dt = \sum_{n\in\mathbb{Z}} r^{2n} \left\| \hat{f}(n) \right\|_X^2. \tag{4.29}$$

A similar argument shows that

$$\int_0^1 \left\|g(z_0 + re^{2\pi it})\right\|_X^2 dt = \sum_{n\in\mathbb{Z}} r^{2n} \left\|\hat{g}(n)\right\|_X^2 . \tag{4.30}$$

This proves the first item and the Plancherel identity.

Returning now to (4.27) and sending $N \to \infty$, we deduce that

$$\int_0^1 \langle f(z_0 + re^{2\pi it}), f(z_0 + re^{2\pi it})\rangle_X dt = \sum_{n\in\mathbb{Z}} r^{2n} \langle \hat{f}(n), \hat{g}(n)\rangle_X. \tag{4.31}$$

This proves Parseval's identity.

□

The results of Theorem 4.3 are particularly striking when $z_0 = 0$ and $r = 1$.

Example 4.4. Suppose that X is a complex Hilbert space and $0 < r_0 < 1 < r_1$. Let $F, G : A(0; r_0, r_1) \to X$ be holomorphic and define $f, g : [0,1] \to X$ via $f(t) = F(e^{2\pi it})$ and $g(t) = G(e^{2\pi it})$. Then we can take $r = 1$ to compute

$$\hat{F}(n) = \frac{1}{2\pi i} \int_0^1 (e^{2\pi it})^{-n-1} 2\pi i e^{2\pi it} F(e^{2\pi it}) dt = \int_0^1 e^{-2\pi int} F(e^{2\pi it}) dt = \int_0^1 e^{-2\pi int} f(t) dt \tag{4.32}$$

and, similarly,

$$\hat{G}(n) = \int_0^1 e^{-2\pi int} g(t) dt. \tag{4.33}$$

Moreover,

$$\int_0^1 \left\|F(e^{2\pi it})\right\|_X^2 dt = \int_0^1 \|f(t)\|_X^2 dt, \text{ and } \int_0^1 \langle F(e^{2\pi it}), G(e^{2\pi it})\rangle_X dt = \int_0^1 \langle f(t), g(t)\rangle_X dt. \tag{4.34}$$

Theorem 4.3 then says that

$$\int_0^1 \|f(t)\|_X^2 dt = \sum_{n\in\mathbb{Z}} \left\| \int_0^1 e^{-2\pi int} f(t) dt \right\|_X^2 \tag{4.35}$$

and

$$\int_0^1 \langle f(t), g(t)\rangle_X dt = \sum_{n\in\mathbb{Z}} \langle \int_0^1 e^{-2\pi int} f(t) dt, \int_0^1 e^{-2\pi int} g(t) dt\rangle_X. \tag{4.36}$$

To see this from a higher-level perspective, define

$$H = \{f : [0,1] \to X \mid f(t) = F(e^{2\pi it}) \text{ for } t \in [0,1], \\ \text{where } F : A(0; r_0, r_1) \to X \text{ is holomorphic for some } 0 < r_0 < 1 < r_1\}. \tag{4.37}$$

It is a simple matter to verify that H is a vector subspace of the smooth functions from $[0,1]$ to X and that if $f \in H$ then $f^{(n)}(0) = f^{(n)}(1)$ for all $n \in \mathbb{N}$, i.e. the functions in H are $1-$periodic. We endow H with the inner product defined by

$$\langle f, g\rangle_H = \int_0^1 \langle f(t), g(t)\rangle_X dt. \tag{4.38}$$

For $f \in H$ define $\hat{f} : \mathbb{Z} \to X$ via

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n t} f(t) dt. \tag{4.39}$$

According to the above, this induces a linear map $\hat{\cdot} : H \to \ell^2(\mathbb{Z}; X)$ satisfying

$$\langle f, g \rangle_{H(X)} = \langle \hat{f}, \hat{g} \rangle_{\ell^2} \text{ and } \|f\|_H = \left\| \hat{f} \right\|_{\ell^2}. \tag{4.40}$$

In particular $\hat{\cdot}$ is an isometric embedding, which we call the Fourier transform. Additionally, we have the identity

$$f(t) = F(e^{2\pi i t}) = \sum_{n \in \mathbb{Z}} e^{2\pi i n t} \hat{f}(n) \text{ for } t \in [0,1], \tag{4.41}$$

where the series converges uniformly absolutely. This shows that the functions in H are actually linear combinations of the smooth and $1-$periodic functions $[0,1] \ni t \mapsto e^{2\pi i n t} x \in X$ for $x \in X$ a constant.

△

The Laurent series provide an extremely useful tool for classifying the isolated singularities of an otherwise holomorphic map. In the following definition we will use the simple fact that if X is a metric space and $E \subseteq X$ is such that $E' = \varnothing$, then E is closed.

Definition 4.5. *Let X be a complex Banach space, $\varnothing \neq U \subseteq \mathbb{C}$ be open, and let $\varnothing \neq E \subset U$ be isolated, i.e. $E' \cap U = \varnothing$. Let $f : U \backslash E \to X$ be holomorphic. For any $0 < R \leq \infty$ such that $B(z_0, R) \subseteq U$ and $B(z_0, R) \cap E = \{z_0\}$, Theorem 4.1 allows us to write*

$$f(z) = \sum_{n=1}^{\infty} (z - z_0)^{-n} x_n + \sum_{n=0}^{\infty} (z - z_0)^n y_n \text{ for } z \in B(z_0, R) \backslash \{z_0\}, \tag{4.42}$$

where $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=0}^{\infty} \subseteq X$, and the series converge uniformly absolutely in $B(z_0, r_1) \backslash B[z_0, r_0]$ for every $0 < r_0 < r_1 < R$.

1. *If $x_n = 0$ for $1 \leq n \in \mathbb{N}$, then we say f has a removable singularity at z_0.*
2. *If there exists $1 \leq N \in \mathbb{N}$ such that $x_N \neq 0$ and $x_n = 0$ for $n \geq N+1$, then we say f has a pole of order N at z_0, and we write* $\operatorname{ord}(f, z_0) = N$. *We say the pole is simple if* $\operatorname{ord}(f, z_0) = 1$.
3. *If $x_n \neq 0$ for infinitely many $1 \leq n \in \mathbb{N}$, then we say f has an essential singularity at z_0.*

Remark 4.6. *It's clear from the definition that since E isolated, the classification of the singularities is local, i.e. the classification is the same if we view $f : U \backslash E \to X$ or if we consider the restriction $f : B(z_0, R) \backslash \{z_0\} \to X$.*

Let's consider some examples.

Example 4.7. Let $f : \mathbb{C} \backslash \{0, i, -i\} \to \mathbb{C}$ via

$$f(z) = \frac{z+2}{z^5(z^2+1)}. \tag{4.43}$$

Then f has simple poles at $\pm i$, and $\operatorname{ord}(f, 0) = 5$. △

Example 4.8. Define $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ via $f(z) = e^{1/z}$. For $n \in \mathbb{N}$ we compute

$$\int_{\partial B(0,r)} z^n f(z)dz = \int_0^1 2\pi i r^{n+1} e^{2\pi i(n+1)t} \exp(r^{-1}e^{-2\pi it})dt$$
$$= 2\pi i \sum_{k=0}^{\infty} \frac{1}{k!} r^{n+1-k} \int_0^1 e^{2\pi i(n+1-k)t}dt = \frac{2\pi i}{(n+1)!}, \quad (4.44)$$

which means, by way of Theorem 4.1, that f has an essential singularity at 0. △

The name removable singularity is justified by the following result, which shows that removable singularities are not singularities at all and may simply be removed to form a holomorphic function.

Theorem 4.9. *Let X be a complex Banach space, $\varnothing \neq U \subseteq \mathbb{C}$ be open, and let $\varnothing \neq E \subset U$ be such that $E' \cap U = \varnothing$. Let $f : U\backslash E \to X$ be holomorphic. If f has a removable singularity at $z_0 \in E$, then the limit $y_0 = \lim_{z\to z_0} f(z) \in X$ exists, and the map $F : (U\backslash E) \cup \{z_0\} \to X$ defined by*

$$F(z) = \begin{cases} f(z) & \text{if } z \neq z_0 \\ y_0 & \text{if } z = z_0 \end{cases} \quad (4.45)$$

is holomorphic.

Proof. Since z_0 is a removable singularity, Theorem 4.1 allows us to write

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n y_n \text{ for } z \in B(z_0, R)\backslash\{z_0\}, \quad (4.46)$$

where $\{y_n\}_{n=0}^{\infty} \subseteq X$ and $B(z_0, R) \subseteq U$ with $B(z_0, R) \cap E = \{z_0\}$. The result follows immediately. □

4.2 Poles and meromorphic functions

We now turn our attention to a more thorough characterization of poles.

Theorem 4.10. *Let X be a complex Banach space, $z_0 \in \mathbb{C}$, $0 < R \leq \infty$, and $f : B(z_0, R)\backslash\{z_0\} \to X$ be holomorphic. Let $1 \leq N \in \mathbb{N}$. Then the following are equivalent.*

1. *f has a pole of order N at z_0.*
2. *There exist $x_1, \dots, x_N \in X$ with $x_N \neq 0$, and a holomorphic function $g : B(z_0, R) \to X$ such that*
$$f(z) = \sum_{n=1}^{N} \frac{1}{(z - z_0)^n} x_n + g(z) \text{ for } z \in B(z_0, R)\backslash\{z_0\}. \quad (4.47)$$
3. *For $0 < r < R$ we have that*
$$\int_{\partial B(z_0,r)} (z - z_0)^n f(z)dz = 0 \text{ for } n \geq N, \quad (4.48)$$
and
$$\int_{\partial B(z_0,r)} (z - z_0)^{N-1} f(z)dz \neq 0. \quad (4.49)$$

4. *There exists a holomorphic map $h : B(z_0, R) \to X$ such that $h(z_0) \neq 0$ and*

$$f(z) = \frac{h(z)}{(z - z_0)^N} \text{ for all } z \in B(z_0, R)\backslash\{z_0\}. \tag{4.50}$$

5. *There exists holomorphic maps $H : B(z_0, R) \to X$, $d : B(z_0, R) \to \mathbb{C}$ such that $H(z_0) \neq 0$, $Z(d) = \{z_0\}$, $\operatorname{ord}(d, z_0) = N$, and*

$$f(z) = \frac{H(z)}{d(z)} \text{ for all } z \in B(z_0, R)\backslash\{z_0\}. \tag{4.51}$$

In any case, we have that

$$x_N = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} (z - z_0)^{N-1} f(z)dz = h(z_0) = \lim_{z \to z_0} \frac{(z - z_0)^N H(z)}{d(z)} \tag{4.52}$$

Proof. We will prove that (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (1).

Proof of (1) ⇒ (2)**:** This follows immediately from the definition and the fact that analytic functions are holomorphic.

Proof of (2) ⇒ (3)**:** Let $0 < r < R$, so that $B[z_0, r] \subset U$. By Cauchy-Goursat, we then have that if $n \geq N$ then

$$\int_{\partial B(z_0,r)} (z - z_0)^n f(z)dz = \sum_{k=1}^{N} \left(\int_{\partial B(z_0,r)} (z - z_0)^{n-k} dz \right) x_k + \int_{\partial B(z_0,r)} (z - z_0)^n g(z)dz = 0. \tag{4.53}$$

Similarly,

$$\begin{aligned} \int_{\partial B(z_0,r)} (z - z_0)^{N-1} f(z)dz &= \sum_{k=1}^{N} \left(\int_{\partial B(z_0,r)} (z - z_0)^{N-1-k} dz \right) x_k + \int_{\partial B(z_0,r)} (z - z_0)^{N-1} g(z)dz \\ &= \left(\int_{\partial B(z_0,r)} (z - z_0)^{-1} dz \right) x_N = 2\pi i x_N \neq 0. \end{aligned} \tag{4.54}$$

Proof of (3) ⇒ (4)**:** We have that f is holomorphic in the open annulus $A = B(z_0, R)\backslash\{z_0\}$, so for $0 < r < R$ Theorem 4.1 implies that

$$\begin{aligned} f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \left(\frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(w)}{(w - z_0)^{n+1}} dw \right) \\ + \sum_{n=1}^{\infty} (z - z_0)^{-n} \left(\frac{1}{2\pi i} \int_{\partial B(z_0,r)} (w - z_0)^{n-1} f(w)dw \right), \end{aligned} \tag{4.55}$$

where the series converge in A. Define the holomorphic map $H : A \to X$ via

$$\sum_{n=0}^{\infty} (z - z_0)^n \left(\frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(w)}{(w - z_0)^{n+1}} dw \right). \tag{4.56}$$

If $N \leq 0$, then $f = H$ and so z_0 is not a pole, a contradiction. Thus $N \geq 1$ and

$$f(z) = H(z) + \sum_{n=1}^{N} (z - z_0)^{-n} \left(\frac{1}{2\pi i} \int_{\partial B(z_0,r)} (w - z_0)^{n-1} f(w) dw \right)$$
$$= \frac{1}{(z - z_0)^N} \left((z - z_0)^N H(z) + \sum_{n=1}^{N} (z - z_0)^{N-n} \left(\frac{1}{2\pi i} \int_{\partial B(z_0,r)} (w - z_0)^{n-1} f(w) dw \right) \right), \quad (4.57)$$

and since the term in parentheses is analytic and doesn't vanish at z_0, the fourth item is proved.

Proof of (4) $\Rightarrow$ (5)**:** Trivial.

Proof of (5) $\Rightarrow$ (1)**:** In light of Theorem 3.2, we can write

$$d(z) = (z - z_0)^N D(z) \quad (4.58)$$

for $D : B(z_0, R) \to \mathbb{C}\backslash\{0\}$ holomorphic. Define the holomorphic map $h : B(z_0, R) \to X$ via $h(z) = H(z)/D(z)$, and note that $h(z_0) \neq 0$. Then $f(z) = h(z)/(z - z_0)^n$ for $z \in B(z_0, R)\backslash\{z_0\}$.

Since h is holomorphic in $B(z_0, R)$, Theorem 2.7 shows that we can write

$$h(z) = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!} h^{(n)}(z_0), \quad (4.59)$$

with the series converging pointwise in $B(z_0, R)$ and uniformly absolutely in $B(z_0, r)$ for every $0 < r < R$. Then

$$f(z) = \frac{h(z)}{(z - z_0)^N} = \sum_{n=0}^{N-1} \frac{(z - z_0)^{n-N}}{n!} h^{(n)}(z_0) + \sum_{n=N}^{\infty} \frac{(z - z_0)^{n-N}}{n!} h^{(n)}(z_0)$$
$$= \sum_{k=1}^{N} \frac{(z - z_0)^{-k}}{(N - k)!} h^{(N-k)}(z_0) + \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(n + N)!} h^{(n+N)}(z_0) \quad (4.60)$$

for $z \in B(z_0, R)\backslash\{z_0\}$, and so f has a pole of order N at z_0 since $h^{(0)}(z_0) = h(z_0) \neq 0$. $\square$

We now define a special class of nearly holomorphic maps that are allowed to have isolated singularities, provided they are only poles. We will call such functions meromorphic.

Definition 4.11. *Let X be a complex Banach space and $\varnothing \neq U \subseteq \mathbb{C}$ be open.*

1. *We write $X_\infty = X \cup \{\infty\}$.*
2. *For $f : U \to X_\infty$ we write $P(f) = f^{-1}(\{\infty\})$ for the polar set of f and $Z(f) = f^{-1}(\{0\})$ for the zero set.*
3. *Let $f : U \to X_\infty$. We say that f is meromorphic if $P(f)' \cap U = \varnothing$ (which means $P(f)$ is relatively closed in U), f is holomorphic in $U\backslash P(f)$, and if $z_0 \in P(f)$ then f has a pole at z_0.*

Remark 4.12. *If f is meromorphic, then by definition all points in $P(f)$ are isolated, so $P(f) \cap K$ is finite for every compact set $K \subset \mathbb{C}$. This is analogous to the behavior of $Z(f)$ when f is holomorphic, but here it is built into the definition.*

Let's consider some examples

Example 4.13. The function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^{1/z}$ for $z \neq 0$ and $f(z) = \infty$ for $z = 0$ is not meromorphic since it has an essential singularity at 0 $\triangle$

Example 4.14. Let $f : \mathbb{C} \to \mathbb{C}$ via

$$f(z) = \frac{z+2}{z^5(z^2+1)} \tag{4.61}$$

for $z \notin \{0, i, -i\}$ and $f(z) = \infty$ otherwise. Then f is meromorphic since the singularities are poles. $\triangle$

Example 4.15. Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and X be a complex Banach space. Let $h : U \to X$ and $g : U \to \mathbb{C}$ be holomorphic. Then $f : U \to X_\infty$ defined by

$$f(z) = \begin{cases} h(z)/g(z) & \text{if } z \notin Z(g) \\ \infty & \text{if } z \in Z(g) \end{cases} \tag{4.62}$$

is meromorphic due to Theorems 3.2 and 4.10. $\triangle$

The previous two examples show a nice analogy. Holomorphic functions behave locally like polynomials, and meromorphic functions behave locally like rational functions (ratios of polynomials).

Example 4.16. Let $P(f) = \{n\pi \mid n \in \mathbb{Z}\}$ and define $f : \mathbb{C} \to \mathbb{C}^{2\times 2}$ via

$$f(z) = \begin{pmatrix} 1/z & z^2 - 2 \\ z^4 e^z & 1/\sin(z) \end{pmatrix} \tag{4.63}$$

for $z \notin P(f)$ and $f(z) = \infty$ otherwise. Then f is meromorphic and each pole in $P(f)$ is simple. $\triangle$

There is a generalization of the argument principle for meromorphic functions. We present this now.

Theorem 4.17 (Meromorphic argument principle)**.** *Let X be a complex Banach space, $\varnothing \neq U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}_\infty$ be nontrivial and meromorphic, and $g : U \to X$ be holomorphic. If γ is a loop chain in U that is homologous to zero in U and* $\operatorname{ran}(\gamma) \cap Z(f) \cap P(f) = \varnothing$, *then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} g = \sum_{z \in Z(f)} \operatorname{ord}(f,z)\operatorname{ind}(\gamma,z)g(z) - \sum_{z\in P(f)} \operatorname{ord}(f,z)\operatorname{ind}(\gamma,z)g(z), \tag{4.64}$$

where the sums are finite due to the compactness of $\operatorname{ran}(\gamma)$, *Proposition 2.20, and the fact that all points is $Z(f)$ and $P(f)$ are isolated, and the integral is well-defined because gf'/f is holomorphic in $U\backslash(Z(f) \cup P(f))$.*

Proof. Pick $R > 0$ such that $\operatorname{ran}(\gamma) \subseteq B[0,R]$. Since $\operatorname{ind}(\gamma, z) = 0$ for $|z| > R$, we deduce that γ is homologous to zero in $U \cap B(0,2R)$. The compact sets $K_Z = Z(f) \cap B[0,3R]$ and $K_P = P(f) \cap B[0,3R]$ are finite, so we can define the function $h : U \cap B(0,2R) \to \mathbb{C}$ via

$$h(z) = f(z) \prod_{w\in K_P} (z-w)^{\operatorname{ord}(f,w)} \prod_{w \in K_Z} (z-w)^{-\operatorname{ord}(f,w)} \tag{4.65}$$

and

$$h(w) = \lim_{z\to w} h(z) \text{ if } w \in K_P \cup K_Z, \tag{4.66}$$

which exists due to Theorems 3.2 and 4.10. These results also show that h is holomorphic in $U \cap B(0, 2R)$ and does not vanish there. Then

$$f(z) = h(z) \prod_{w \in K_P} (z-w)^{-\operatorname{ord}(f,w)} \prod_{w \in K_Z} (z-w)^{\operatorname{ord}(f,w)} \text{ for } z \in U \cap B(0,2R) \backslash K_P, \tag{4.67}$$

and hence the product rule implies that

$$\frac{f'(z)}{f(z)} = \frac{h'(z)}{h(z)} + \sum_{w \in K_Z} \frac{\operatorname{ord}(f,w)}{z-w} - \sum_{w \in K_P} \frac{\operatorname{ord}(f,w)}{z-w} \text{ for } z \in U \cap B(0,2R) \backslash K_P. \tag{4.68}$$

Since gh'/h is holomorphic in $U \cap B(0, 2R)$ and γ is homologous to zero in $U \cap B(0, 2R)$, we then deduce from Cauchy-Goursat and Cauchy's integral formula that

$$\begin{aligned} \int_\gamma \frac{f'}{f} g &= \int_\gamma \frac{h'}{h} g + \sum_{w \in K_Z} \operatorname{ord}(f,w) \int_\gamma \frac{g(z)}{z-w} dz - \sum_{w \in K_P} \operatorname{ord}(f,w) \int_\gamma \frac{g(z)}{z-w} dz \\ &= \sum_{w \in K_Z} \operatorname{ord}(f,w) 2\pi i \operatorname{ind}(\gamma,w) g(w) - \sum_{w \in K_P} \operatorname{ord}(f,w) 2\pi i \operatorname{ind}(\gamma,w) g(w) \\ &= 2\pi i \sum_{w \in Z(f)} \operatorname{ord}(f,w) \operatorname{ind}(f,w) g(w) - 2\pi i \sum_{w \in P(f)} \operatorname{ord}(f,w) \operatorname{ind}(f,w) g(w). \end{aligned} \tag{4.69}$$

This yields the stated identity. □

We then have the following variant of Corollary 3.20.

Corollary 4.18. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}_\infty$ be meromorphic. Let γ be a chain loop in U that is homologous to zero in U and satisfies* $\operatorname{ran}(\gamma) \cap P(f) \cap Z(f) = \varnothing$. *Define the chain loop $f \circ \gamma$ in $\mathbb{C}$ as in Lemma 3.19. Then*

$$\operatorname{ind}(f \circ \gamma, 0) = \sum_{z \in Z(f)} \operatorname{ord}(f,z) \operatorname{ind}(\gamma,z) - \sum_{z \in P(f)} \operatorname{ord}(f,z) \operatorname{ind}(\gamma,z). \tag{4.70}$$

Proof. The identity follows from Lemma 3.19 and the meromorphic argument principle applied with $g = 1$ and f, which is nontrivial since $0 \notin f(\operatorname{ran}(\gamma))$. □

4.3 The residue theorem

Now that we have the concept of a meromorphic function, it is natural to investigate whether a variant of Cauchy-Goursat holds. In answering this question we will need the concept of a residue, which we now define.

Definition 4.19. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open, X be a complex Banach space, and $f : U \to X_\infty$ be meromorphic. For $z_0 \in P(f)$ we define the residue of f at z_0 via*

$$\operatorname{Res}(f, z_0) = x_1 \in X, \tag{4.71}$$

where x_1 is as in the second item of Theorem 4.10.

The residue may be computed in different ways using Theorem 4.10.

Proposition 4.20. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open, X be a complex Banach space, and $f : U \to X_\infty$ be meromorphic. For $z_0 \in P(f)$ we have that*

$$\begin{aligned} \operatorname{Res}(f, z_0) &= \frac{1}{(\operatorname{ord}(f, z_0) - 1)!} \lim_{z \to z_0} \left(\frac{d}{dz} \right)^{\operatorname{ord}(f,z_0)-1} \left((z - z_0)^{\operatorname{ord}(f,z_0)} f(z) \right) \\ &= \frac{1}{(\operatorname{ord}(f, z_0) - 1)!} h^{(\operatorname{ord}(f,z_0)-1)}(z_0), \end{aligned} \tag{4.72}$$

where in the latter equality h is as in the fourth item of Theorem 4.10. In particular, if z_0 is a simple pole, then

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z) = h(z_0) = \frac{H(z_0)}{d'(z_0)}, \tag{4.73}$$

where in the latter equality $f = H/d$ for H and d as in the fifth item of Theorem 4.10.

Proof. These are immediate from the second, fourth, and fifth items of Theorem 4.10. □

The utility of the residue concept is evident in the following simple lemma.

Lemma 4.21. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open, X be a complex Banach space, and $f : U \to X_\infty$ be meromorphic. Suppose that $z_0 \in P(f)$ and $r > 0$ is such that $B[z_0, r] \subseteq U$ and $B(z_0, r) \cap P(f) = \{z_0\}$. Then*

$$\frac{1}{2\pi i} \int_{\partial B(z_0,r)} f = \operatorname{Res}(f, z_0). \tag{4.74}$$

Proof. Using Theorem 4.10, we can pick $0 < R < r$ so that we can write

$$f(z) = \sum_{k=1}^{\operatorname{ord}(f,z_0)} \frac{1}{(z - z_0)^k} x_k + g(z) \text{ for all } z \in B(z_0, R) \backslash \{z_0\} \tag{4.75}$$

for $g : B(z_0, R) \to X$ holomorphic. From Cauchy-Goursat and Examples 1.40 and 1.41 we then have that

$$\int_{\partial B(z_0,r)} f = \int_{\partial B(z_0,R/2)} f = 2\pi i x_1. \tag{4.76}$$

□

This lemma shows that Cauchy-Goursat does not hold for meromorphic functions with poles. However, it suggests that there may be a simple replacement utilizing residues. Remarkably, this holds. We now prove one of the most useful theorems in all of complex analysis, the residue theorem. This is the natural generalization of Cauchy-Goursat to meromorphic functions, and in fact reduces to Cauchy-Goursat for functions without poles.

Theorem 4.22 (Residue theorem)**.** *Let X be a complex Banach space, $\varnothing \neq U \subseteq \mathbb{C}$ be open, and $f : U \to X_\infty$ be meromorphic. Suppose that γ is a loop chain in U that is homologous to zero in U. If* $\operatorname{ran}(\gamma) \cap P(f) = \varnothing$*, then*

$$\frac{1}{2\pi i} \int_\gamma f = \sum_{z \in P(f)} \operatorname{ind}(\gamma, z) \operatorname{Res}(f, z), \tag{4.77}$$

where the sum is finite since only finitely many $z \in P(f)$ satisfy $\operatorname{ind}(\gamma, z) \neq 0$*, and the integral is well-defined since f is holomorphic in $U \backslash P(f)$.*

Proof. Pick $R > 0$ such that $\operatorname{ran}(\gamma) \subseteq B[0, R]$. Since $\operatorname{ind}(\gamma, z) = 0$ for $|z| > R$, we deduce that γ is homologous to zero in $U \cap B(0, 2R)$. Note that the set $K = P(f) \cap B[0, 2R]$ is finite. If $K = \varnothing$, then f is holomorphic in $U \cap B(0, 2R)$, and so

$$\frac{1}{2\pi i} \int_\gamma f = 0 = \sum_{z \in P(f)} \operatorname{ind}(\gamma, z) \operatorname{Res}(f, z) \tag{4.78}$$

since $\operatorname{ind}(f, z) = 0$ for any $z \in P(f) \backslash K$. This proves the result in the case $K = \varnothing$. Assume then that $K \neq \varnothing$ and write $K = \{z_1, \dots, z_n\}$. For $1 \le j \le n$ pick $0 < r_j$ such that $B(z_j, r_j) \subset U \cap B(0, 2R)$, $B(z_j, r_j) \cap P(f) = \{z_j\}$.

Consider the loop chain $\delta = \gamma - \sum_{j=1}^n \operatorname{ind}(\gamma, z_j) \partial B(z_j, r_j)$. If $z \in (U \cap B(0, 2R))^c$, then Example 2.15 shows that

$$\operatorname{ind}(\delta, z) = \operatorname{ind}(\gamma, z) = 0 \tag{4.79}$$

since γ is homologous to zero in $U \cap B(0, 2R)$. On the other hand, Example 2.15 also shows that

$$\operatorname{ind}(\delta, z_k) = \operatorname{ind}(\gamma, z_k) - \sum_{j=1}^n \operatorname{ind}(\gamma, z_j) \operatorname{ind}(\partial B(z_j, r_j), z_k) = \operatorname{ind}(\gamma, z_k) - \operatorname{ind}(\gamma, z_k) = 0. \tag{4.80}$$

Thus, δ is homologous to 0 in $U \cap B(0, 2R) \backslash K$, so Cauchy-Goursat implies that

$$0 = \int_\delta f = \int_\gamma f - \sum_{j=1}^n \operatorname{ind}(f, z_j) \int_{\partial B(z_j, r_j)} f. \tag{4.81}$$

From this and Lemma 4.21 we then compute

$$\frac{1}{2\pi i} \int_\gamma f = \sum_{j=1}^n \operatorname{ind}(f, z_j) \frac{1}{2\pi i} \int_{\partial B(z_j, r_j)} f = \sum_{j=1}^n \operatorname{ind}(\gamma, z_j) \operatorname{Res}(f, z_j) = \sum_{z \in P(f)} \operatorname{ind}(\gamma, z) \operatorname{Res}(f, z). \tag{4.82}$$

□

The residue theorem gives an incredible toolbox for computing integrals. We briefly demonstrate this with the following examples.

Example 4.23. From the monotone convergence theorem we know that

$$\int_{\mathbb{R}} \frac{dx}{1 + x^4} = \lim_{R \to \infty} \int_{-R}^R \frac{dx}{1 + x^4}. \tag{4.83}$$

For $R > 1$ set

$$I_R = \int_{-R}^R \frac{dx}{1 + x^4}. \tag{4.84}$$

The function $f : \mathbb{C} \to \mathbb{C}_\infty$ defined by $f(z) = 1/(1 + z^4)$ where the denominator doesn't vanish and ∞ otherwise has simple poles at $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$ and $e^{i7\pi/4}$. Consider the loop $\Gamma_R = \gamma_1 \vee \gamma_2$, where $\gamma_j : [0, 1] \to \mathbb{C}$ via

$$\gamma_1(t) = -R + 2Rt \text{ and } \gamma_2(t) = Re^{i\pi t}. \tag{4.85}$$

Then

$$\int_{\gamma_1} \frac{dz}{1 + z^4} = \int_0^1 \frac{2R}{1 + (-R + 2Rt)^4} dt = \int_{-R}^R \frac{dx}{1 + x^4} = I_R. \tag{4.86}$$

On the other hand, if $R > 1$, then

$$\left| \int_{\gamma_2} \frac{dz}{1+z^4} \right| \leq \frac{\pi R}{R^4 - 1}. \tag{4.87}$$

Thus, by the residue theorem, if we set $z_0 = e^{i\pi/4}$ and $z_1 = e^{i3\pi/4}$, then

$$\int_{\Gamma_R} \frac{dz}{1+z^4} = 2\pi i \operatorname{Res}(f, z_0) + 2\pi i \operatorname{Res}(f, z_1). \tag{4.88}$$

However,

$$\left| \int_{\Gamma_R} \frac{dz}{1+z^4} - I_R \right| = \left| \int_{\gamma_2} \frac{dz}{1+z^4} \right| \to 0 \text{ as } R \to \infty, \tag{4.89}$$

so

$$\lim_{R\to\infty} I_R = 2\pi i \operatorname{Res}(f, z_0) + 2\pi i \operatorname{Res}(f, z_1). \tag{4.90}$$

Write $g(z) = 1 + z^4$. Then by Proposition 4.20 we have that

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} (z - z_0) f(z) = \lim_{z\to z_0} \frac{z - z_0}{g(z) - g(z_0)} = \frac{1}{g'(z_0)} = \frac{1}{4z_0^3} = \frac{e^{-i3\pi/4}}{4} \tag{4.91}$$

and

$$\operatorname{Res}(f, z_1) = \lim_{z\to z_1} (z - z_1) f(z) = \frac{1}{g'(z_1)} = \frac{e^{-i9\pi/4}}{4} = \frac{e^{-i\pi/4}}{4}. \tag{4.92}$$

Hence

$$\int_{\mathbb{R}} \frac{dx}{1+x^4} = \lim_{R\to\infty} I_R = \frac{2\pi}{4} \left(e^{-i\pi/4} + e^{i\pi/4} \right) = \pi \cos(\pi/4) = \frac{\pi}{\sqrt{2}}. \tag{4.93}$$

△

Example 4.24. Define $I : (-1, 1) \to \mathbb{R}$ via

$$I(a) = \int_0^{2\pi} \frac{\cos(\theta)}{1 + a\cos(\theta)} d\theta. \tag{4.94}$$

Note that the change of variables $\theta = \varphi + \pi$ shows that

$$I(-a) = \int_0^{2\pi} \frac{\cos(\theta)}{1 - a\cos(\theta)} d\theta = \int_0^{2\pi} \frac{\cos(\varphi)}{1 + a\cos(\varphi)} d\varphi = I(a), \tag{4.95}$$

so I is odd, and we can restrict our attention to $a \in (0, 1)$.

Define $\gamma : [0, 2\pi] \to \mathbb{C}$ via $\gamma(\theta) = e^{i\theta}$ and note that

$$\gamma'(\theta) = ie^{i\theta} \Rightarrow 1 = \frac{\gamma'(\theta)}{i\gamma(\theta)}. \tag{4.96}$$

Define the meromorphic function $f : \mathbb{C} \to \mathbb{C}_\infty$ via

$$f(z) = \frac{(z + z^{-1})}{2} \left(1 + a\frac{z + z^{-1}}{2} \right)^{-1} = \frac{z^2 + 1}{az^2 + 2z + a} = \frac{z^2 + 1}{a(z - r_+)(z - r_-)}, \tag{4.97}$$

where

$$r_\pm = \frac{-1 \pm \sqrt{1 - a^2}}{a}. \tag{4.98}$$

Note that

$$\sqrt{1-a} < \sqrt{1+a} \Rightarrow \sqrt{1-a^2} < 1+a \Rightarrow 0 < r_+ < 1, \tag{4.99}$$

and $r_- < -1$. Write $g : \mathbb{C} \to \mathbb{C}_\infty$ for the meromorphic function $g(z) = f(z)/z$. Then g has simple poles at 0, r_- and r_+. Then the residue theorem implies that

$$I(a) = \int_\gamma f(z)\frac{dz}{iz} = \frac{1}{ia}\int_\gamma \frac{z^2+1}{z(z-r_+)(z-r_-)}dz = \frac{2\pi}{a}\left(\operatorname{Res}(g,0) + \operatorname{Res}(g,r_+)\right). \tag{4.100}$$

We then compute

$$\operatorname{Res}(g,0) = \lim_{z\to 0} zg(z) = \frac{1}{a} \tag{4.101}$$

and

$$\operatorname{Res}(g,r_+) = \lim_{z\to 0}(z-r_+)g(z) = \frac{r_+^2+1}{ar_+(r_+-r_-)} = -\frac{1}{a\sqrt{1-a^2}}. \tag{4.102}$$

Hence for $0 < a < 1$ we have that

$$I(a) = \frac{2\pi}{a}\left(1 - \frac{1}{\sqrt{1-a^2}}\right), \tag{4.103}$$

since this expression is also odd, we deduce that

$$\int_0^{2\pi} \frac{\cos(\theta)}{1+a\cos(\theta)}d\theta = \frac{2\pi}{a}\left(1 - \frac{1}{\sqrt{1-a^2}}\right) \text{ for all } a \in (-1,1). \tag{4.104}$$

△

Example 4.25. Recall that hyperbolic cosine is given by $\cosh(x) = \frac{e^x+e^{-x}}{2}$. It's easy to verify that for any $\xi \in \mathbb{R}$, the function $\mathbb{R} \ni x \mapsto \cos(\xi x)/\cosh(x) \in \mathbb{R}$ is integrable, and the dominated convergence theorem shows that

$$\int_{\mathbb{R}} \frac{\cos(\xi x)}{\cosh(x)}dx = \lim_{R\to\infty} I_R \tag{4.105}$$

for

$$I_R = \int_{-R}^{R} \frac{\cos(\xi x)}{\cosh(x)}dx = \int_{-R}^{R} \frac{2\cos(\xi x)e^x}{e^{2x}+1}dx. \tag{4.106}$$

Define the meromorphic function $f : \mathbb{C} \to \mathbb{C}$ via

$$f(z) = \frac{2e^{i\xi z}e^z}{e^{2z}+1} \tag{4.107}$$

and note that f has simple poles at $z = \pi(n+1/2)$ for $n \in \mathbb{Z}$. Further note that

$$f(z+i\pi) = -e^{-\pi\xi}f(z) \text{ for all } z \in \mathbb{C}\backslash P(f). \tag{4.108}$$

Define $\gamma_0, \gamma_2 : [-R,R] \to \mathbb{C}$ and $\gamma_1, \gamma_3 : [0,\pi] \to \mathbb{C}$ via

$$\gamma_0(t) = t, \gamma_2(t) = -t + i\pi \tag{4.109}$$

and

$$\gamma_1(t) = R + it, \gamma_3(t) = -R + (\pi - t)i. \tag{4.110}$$

Consider the counterclockwise simple circuit $\gamma = \gamma_0 \vee \gamma_1 \vee \gamma_2 \vee \gamma_3$, which avoids $P(f)$ and encloses exactly one pole, namely $i\pi/2$.

We now compute, using the fact that sin is odd and cosh is even,

$$\int_{\gamma_0} f(z)dz = \int_{-R}^{R} \frac{2e^{it\xi}e^t}{e^{2t}+1}dt = \int_{-R}^{R} \frac{2\cos(\xi t)e^t}{e^{2t}+1}dt = I_R \tag{4.111}$$

and

$$\int_{\gamma_2} f(z)dz = \int_{-R}^{R} -f(-t+i\pi)dt = e^{-\pi\xi}\int_{-R}^{R} f(-t)dt = e^{-\pi\xi}I_R. \tag{4.112}$$

On the other hand, we may readily bound

$$\left|\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz\right| \leq \frac{4\pi e^R}{e^{2R}-1}. \tag{4.113}$$

The residue theorem shows that

$$\int_{\gamma} f(z)dz = 2\pi i \operatorname{Res}(f, i\pi/2), \tag{4.114}$$

so

$$\left|(1+e^{-\pi\xi})I_R - 2\pi i \operatorname{Res}(f, i\pi/2)\right| \leq \frac{4\pi e^R}{e^{2R}-1}, \tag{4.115}$$

and hence

$$\lim_{R\to\infty} I_R = \frac{2\pi i \operatorname{Res}(f, i\pi/2)}{1+e^{-\pi\xi}}. \tag{4.116}$$

We compute

$$\begin{aligned}\operatorname{Res}(f, i\pi/2) = \lim_{z\to i\pi/2}(z-i\pi/2)f(z) = \lim_{z\to i\pi/2}\left(2e^z e^{i\xi z}\left(\frac{f(z)-f(i\pi/2)}{z-i\pi/2}\right)^{-1}\right)\\ = \frac{2e^{i\pi/2}e^{-\xi\pi/2}}{2e^{2i\pi/2}} = -ie^{-\xi\pi/2},\end{aligned} \tag{4.117}$$

and hence

$$\frac{2\pi i \operatorname{Res}(f, i\pi/2)}{1+e^{-\pi\xi}} = 2\pi\frac{e^{-\pi\xi/2}}{1+e^{-\pi\xi}} = \frac{2\pi}{e^{\pi\xi/2}+e^{-\pi\xi/2}} = \frac{\pi}{\cosh(\pi\xi/2)}. \tag{4.118}$$

Finally, we deduce from the above that

$$\int_{\mathbb{R}} \frac{\cos(\xi x)}{\cosh(x)}dx = \frac{\pi}{\cosh(\pi\xi/2)}. \tag{4.119}$$

Making a change of variables allows us to further compute

$$\int_{\mathbb{R}} \frac{\cos(2\pi\xi x)}{\cosh(\pi x)}dx = \frac{1}{\cosh(\pi\xi)}, \tag{4.120}$$

which is an interesting identity in the branch of math known as Fourier analysis, as it shows that hyperbolic secant, rescaled by π, is its own Fourier transform.

△

The residue theorem also allows us to easily compute integrals involving certain rational functions.

Proposition 4.26. *Let X be a complex Banach space, and suppose that $q : \mathbb{C} \to X$ and $p : \mathbb{C} \to \mathbb{C}$ are polynomials such that $\deg(p) \geq \deg(q) + 2$. If γ is a loop chain in $\mathbb{C}\backslash Z(p)$ homologous to $\partial B(0,R)$, where $Z(p) \subset B(0,R)$, then*

$$\int_\gamma \frac{q}{p} = 0. \tag{4.121}$$

Proof. Since the roots of p lie in $B(0,R)$ we can use Cauchy-Goursat to see that

$$\int_\gamma \frac{q}{p} = \int_{\partial B(0,R)} \frac{q}{p}, \tag{4.122}$$

so it suffices to prove that this latter integral vanishes.

Define $I : [R,\infty) \to \mathbb{C}$ via

$$I(r) = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{q}{p}. \tag{4.123}$$

Since p and q are polynomials, we can pick constants $C_0, C_1, C_2 \in (0,\infty)$ such that

$$|p(z)| \geq C_0 \, |z|^{\deg(p)} - C_1 \text{ and } \|q(z)\|_X \leq C_2(1 + |z|^{\deg(q)}). \tag{4.124}$$

Hence, the condition $\deg(p) \geq \deg(q) + 2$ implies that

$$|I(r)| \leq \frac{2\pi r}{2\pi} \max_{|z|=r} \frac{\|q(z)\|_X}{|p(z)|} \leq \frac{C_2 r(1 + r^{\deg(q)})}{C_0 r^{\deg(p)} - C_1} \to 0 \text{ as } r \to \infty. \tag{4.125}$$

On the other hand, the rational function q/p is meromorphic on $\mathbb{C}$, so the residue theorem implies that

$$I(r) = \sum_{z \in Z(p)} \operatorname{Res}(q/p, z). \tag{4.126}$$

This means that I is a constant function that vanishes at infinity, and hence $I(r) = 0$ for all $r \geq R$. □

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