Closed book and notes; calculators not permitted.

1. An undamped, forced harmonic oscillator may be modelled by the equation $mx'' + kx = f(t)$, where $m$ is the mass of the weight, $k$ is the spring constant, and $f(t)$ is the force applied to the mass. Consider the case where $m = 2$, $k = 8$ and $f(t) = 3 \cos(\omega t)$.

(a) Find a particular solution, $x_p(t)$.

(b) For what value of $\omega$ will resonance occur? How can this be determined from the particular solution found in part (a)?

2. Compute the Laplace transform of the function

$$f(t) = e^{at}g(t), \quad \text{where} \quad g(t) = \begin{cases} 2t & 0 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$

3. Use Laplace transforms to solve the initial value problem

$$x'' + 4x = 1, \quad \text{where} \quad x(0) = 1, \text{ and } x'(0) = 0.$$ 

4. (a) Consider the second order, linear differential equation

$$y'' - 2y' - 8y = 0.$$ 

Find the general solution, and the particular solution satisfying $y(0) = 0$ and $y'(0) = 0$.

(b) The equation

$$x'' + 4x = F \sin(\omega t)$$

models a forced, undamped mass-spring system. Find the general solution for this differential equation. (Assume that $\omega \neq 2$.)

(c) What can you say about the behavior of the solutions to the differential equation in (4b) when $\omega = 2$? (You do not need to find any solutions, merely describe their behavior.)

5. Compute the Laplace transform of the function

$$g(t) = \begin{cases} e^{2t} & 0 \leq t \leq 3 \\ e^{-3t} & 3 < t < 5 \\ 0 & t \geq 5 \end{cases}$$
6. Determine the following inverse Laplace transforms:

(a) \( \mathcal{L}^{-1} \left\{ \frac{s+1}{s^2+9} \right\} \)

(b) \( \mathcal{L}^{-1} \left\{ \frac{e^{-at}}{(s-2)^2+1} \right\} \)

7. Use the Laplace transform method to solve the initial value problem

\[ x'' + 4x = h(t-2); \quad x(0) = 0, x'(0) = 1. \]

8. Use the Laplace transform method to solve the initial value problem

\[ x'' + 2x' + 5x = 0; \quad x(0) = 1, x'(0) = 2. \]
A Short List of Laplace Transforms

<table>
<thead>
<tr>
<th>Function</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$</td>
</tr>
<tr>
<td>$\cos(kt)$</td>
<td>$\frac{s}{s^2 + k^2}$</td>
</tr>
<tr>
<td>$\sin(kt)$</td>
<td>$\frac{k}{s^2 + k^2}$</td>
</tr>
<tr>
<td>$t \cos(kt)$</td>
<td>$\frac{s^2 - k^2}{(s^2 + k^2)^2}$</td>
</tr>
<tr>
<td>$t \sin(kt)$</td>
<td>$\frac{2ks}{(s^2 + k^2)^2}$</td>
</tr>
<tr>
<td>$h(t-a)$</td>
<td>$\frac{e^{-as}}{s}$</td>
</tr>
</tbody>
</table>

Properties of the Laplace Transform

First Shift Theorem: $L\{e^{at} f(t)\} = F(s-a)$.

Second Shift Theorem: $L\{h(t-a) f(t-a)\} = e^{-as} F(s)$.

First Integration Theorem: $L\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$.

Second Integration Theorem: $L\{\frac{f(t)}{t}\} = \int_0^\infty F(\sigma)d\sigma$.

First Differentiation Theorem: $L\{f'(t)\} = sF(s) - f(0)$.

Second Differentiation Theorem: $L\{t^n f(t)\} = (-1)^n \left(\frac{d^n}{ds^n}\right) F(s)$. 