Review #3

1. Consider the linear system of differential equations

\[ \frac{dx}{dt} = Ax, \]

where

\[ A = \begin{bmatrix} 1 & 5 \\ 10 & -4 \end{bmatrix}. \]

(a) Find the eigenvalues of A.
(b) Find the eigenvectors of A.
(c) Solve the initial value problem \( \frac{dx}{dt} = Ax, \ x(0) = \begin{bmatrix} 0 \\ 9 \end{bmatrix}. \)

2. Let

\[ B = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}. \]

The real number 2 is an eigenvalue of B with multiplicity 3, and \( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \) is an eigenvector.
There are no other linearly independent eigenvectors.

(a) Find a chain of generalized eigenvectors of B with length 3.
(b) Find 3 linearly independent solutions to \( \frac{dx}{dt} = Bx. \)
(c) Find the general solution to \( \frac{dx}{dt} = Bx. \)

3. A mass-spring-dashpot system can be modeled by the second order equation

\[ m \frac{d^2x}{dt^2} = -k_s \frac{dx}{dt} - k_d x, \]

where \( m \) is the mass, \( k_s \) is the spring constant and \( k_d \) is the damping coefficient.

(a) A certain system of this type with \( m = 1 \) can also be modeled by the first order system

\[ \begin{bmatrix} x' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}. \]

What is the spring constant in this system? What is the damping coefficient?
(b) The complex number $-2 + i$ is an eigenvalue of the matrix $\begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix}$. An eigenvector corresponding to $-2 + i$ is $\begin{bmatrix} 1 \\ -2 + i \end{bmatrix}$. What is the general solution?

(c) What is the general solution to the second order equation $\frac{d^2x}{dx^2} = -4\frac{dx}{dt} - 5x$?

4. Consider the system of differential equations,

$$\begin{align*}
\frac{dx}{dt} &= 3x - xy \\
\frac{dy}{dt} &= -4y + 2xy.
\end{align*}$$

(a) Find the equilibrium points of this system.

(b) Classify the equilibrium point at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as to type and stability.