Closed book and notes; calculators not permitted.

1. **(20 points)** Let $T$ be a linear transformation so that

$$
T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, T \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

(a) Find a matrix $A$ so that $T(x) = Ax$. (Hint: $A = [T(e_1), T(e_2), T(e_3)]$).

(b) Is the linear transformation $T$ invertible? If yes, what is the matrix of $T^{-1}$, if no, why not?

2. **(20 points)** Let $M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$.

(a) Find a basis for $\text{Col}(M)$.

(b) Find a basis for $\text{Nul}(M)$.

(c) What is the dimension of $\text{Nul}(M^T)$? the dimension of $\text{Col}(M^T)$?

3. **(20 points)** Let

$$
\mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.
$$

(a) Find an orthonormal basis for $\text{span}(\mathbf{v}_1, \mathbf{v}_2)$.

(b) Find an orthonormal basis for $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(c) Find the QR factorization for $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$.

4. **(20 points)** Consider the matrix $A = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix}$.

For what values of $k$ is $A$ invertible?
5. **(20 points)** Let $A$ and $B$ be $n \times n$ matrices, $S$ is an invertible $n \times n$ matrix. $B = S^{-1}AS$.

   (a) Show that $A$ and $B$ have the same eigenvalues.
   (b) What is the relation between the eigenvectors of $A$ and $B$?

6. **(20 points)** Let

   \[ A = \begin{bmatrix} 2 & 0 \\ 6 & -1 \end{bmatrix} \]

   (a) Find the eigenvalues and eigenvectors of $A$.
   (b) Plot several trajectories of the discrete dynamical system

   \[ x_{k+1} = Ax_k \]

   (c) Let $\{x_k\}$ be a solution of the difference equation $x_{k+1} = Ax_k$, $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Compute $\{x_1\}$ and find a formula for $\{x_k\}$.

7. **(20 points)**

   Let $P_4$ denote the linear space of all polynomials of degree $\leq 4$. Which of the following subsets of $P_4$ are subspaces of $P_4$?

   (a) $\{p(t) : \int_0^1 p(t) dt = 0\}$.
   (b) $\{p(t) : p(-t) = p(t), \text{ for all } t\}$.
   (c) $\{p(t) : p'(0) = 4\}$. ( $'$ is the derivative.)

8. **(20 points)**

   Let

   \[ u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}. \]

   Note that $u_1, u_2$, and $u_3$ form an orthonormal basis for $\mathbb{R}^3$. Let

   \[ A = 3u_1u_1^T + 7u_2u_2^T + 9u_3u_3^T. \]

   (a) Show that $A$ is symmetric. (Hint: $(A + B)^T = A^T + B^T$).
   (b) Find $Au_1, Au_2$, and $Au_3$. 

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(c) Diagonalize $A$. That is, write $A$ as

$$A = S \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} S^{-1},$$

for some matrix $S$.

9. (15 points) Let

$$A = \begin{bmatrix} -1 & 1 & -8 \\ 2 & 0 & 6 \\ 2 & -1 & 11 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Is $v$ in $\text{im}(A)$?

10. (15 points) Let

$$A = \begin{bmatrix} 1 & -1 & -2 & -10 \\ 0 & 0 & 1 & 7 \\ -2 & 2 & 2 & 6 \end{bmatrix}$$

(a) Find a basis for the nullspace of $A$.  
(b) Find the dimension of $\text{Nul}(A)$.  
(c) Find the dimension of $\text{Col}(A)$.

11. (20 points) Let

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \quad \text{and} \quad v_3 = \begin{bmatrix} 5 \\ 3 \\ 6 \end{bmatrix}.$$ 

(a) Find the QR factorization of $A = [v_1 \ v_2 \ v_3]$.  
(b) What does $\det(A)$ tell you about the vectors $v_1$, $v_2$ and $v_3$.

12. (20 points) Consider the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of $A$.  
(b) Find a basis for each eigenspace.  
(c) What is $A^n$? What is $A^n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?
(d) Find an explicit formula for \( x_n = A^n x(0) \), where \( x(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \)

13. (20 points) Let \( A = \begin{bmatrix} -3 & 7 \\ 7 & -3 \end{bmatrix} \). Find an orthogonal matrix \( S \) and diagonal matrix \( D \) such that \( A = SDS^{-1} \).

14. (10 points) Let \( u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( v_1 = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \), \( v_2 = \begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix} \), \( \sigma_1 = 3 \), and \( \sigma_2 = 7 \). Note that \( \{u_1, u_2\} \) is an orthonormal set, as is \( \{v_1, v_2\} \). Let \( A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T \).

   (a) Show that \( A^T A = \sigma_1^2 v_1 v_1^T + \sigma_2^2 v_2 v_2^T \).

   (b) Show that \( v_1 \) and \( v_2 \) are eigenvectors of \( A^T A \) with eigenvalues \( \sigma_1^2 \) and \( \sigma_2^2 \) respectively.

   (c) Show that \( u_1 = \frac{1}{\sigma_1} A v_1 \), and \( u_2 = \frac{1}{\sigma_2} A v_2 \).