Exam #3 Review  
April 20, 2005  
Name: ________________________________

Closed book and notes; calculators not permitted.

1. Let  

\[ A = \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}. \]

(a) Find the eigenvalues and eigenvectors of \( A \).
(b) Diagonalize \( A \). That is, write \( A \) as  

\[ A = S \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} S^{-1}, \]

for some matrix \( S \).
(c) Compute \( \lim_{n \to \infty} A^n \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \).

2. Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the reflection in the plane given by the equation \( x_3 = 0 \).

(a) Find the matrix \( B \) of this transformation with respect to the basis \( \mathcal{B} = \{v_1, v_2, v_3\} \) where  

\[ v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \]

(b) \( [u]_\mathcal{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \). Find \([T(u)]_\mathcal{B}\) and \(T(u)\).

3. Let  

\[ A = \begin{bmatrix} -7 & 6 \\ -15 & 12 \end{bmatrix}. \]

(a) Find the eigenvalues of \( A \).
(b) Find an eigenbasis for \( A \).
(c) Let \( T(\mathbf{x}) = A\mathbf{x} \). Find the matrix of \( T \) with respect to the eigenbasis found in (b).

4. Let

\[
A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix}.
\]

(a) Find a basis for \( \text{Nul}(A) \).
(b) Find a basis for \( \text{Col}(A) \).
(c) What is the dimension of \( \text{Nul}(A) \)? \( \text{Col}(A) \)?
   What is \( \text{dim}(\text{Nul}(A)) + \text{dim}(\text{Col}(A)) \)? Why?

5. (20 points) \( \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \) and \( \mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \) are bases for \( \mathbb{R}^2 \).

(a) If \( \mathbf{x} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \), find \( [\mathbf{x}]_{\mathcal{B}} \) and \( [\mathbf{x}]_{\mathcal{C}} \).
(b) If \( [\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \), then what is \( [\mathbf{y}]_{\mathcal{C}} \)?

6. (20 points) Let

\[
A = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 5 & 1 \\ 0 & 0 & 6 \end{bmatrix}
\]

(a) Find the eigenvalues of \( A \).
(b) Find a basis for each eigenspace of \( A \).
(c) Find (if possible) a diagonalization of \( A \), i.e. an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \). If it is not possible, explain why is is not.

7. (20 points) Let \( T : \mathcal{P}_2 \to \mathbb{R}^3 \) be defined by

\[
T(p(x)) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}.
\]

(a) Find the matrix for this linear transformation with respect to the basis of Hermite polynomials \( \mathcal{B} = \{1, 2x, -2 + 4x^2\} \) on \( \mathcal{P}_2 \) and the standard basis \( \mathcal{E} = \{e_1, e_2, e_3\} \) on \( \mathbb{R}^3 \).
(b) What is the rank of this linear transformation? Is this linear transformation invertible?
8. **(20 points)** Let \( M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix} \).

(a) Find a basis for \( \text{Col}(M) \).

(b) Find a basis for \( \text{Nul}(M) \).

(c) What is the dimension of \( \text{Nul}(M^T) \)? the dimension of \( \text{Col}(M^T) \)?

9. **(20 points)** Let \( A \) and \( B \) be \( n \times n \) matrices, \( S \) is an invertible \( n \times n \) matrix. \( B = S^{-1}AS \).

(a) Show that \( A \) and \( B \) have the same eigenvalues.

(b) What is the relation between the eigenvectors of \( A \) and \( B \)?

10. **(20 points)** Let

\[
    A = \begin{bmatrix} 2 & 0 \\ 6 & -1 \end{bmatrix}
\]

(a) Find the eigenvalues and eigenvectors of \( A \).

(b) Plot several trajectories of the discrete dynamical system

\[
    x_{k+1} = Ax_k
\]

(c) Let \( \{x_k\} \) be a solution of the difference equation \( x_{k+1} = Ax_k, \ x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Compute \( \{x_1\} \) and find a formula for \( \{x_k\} \).