Final Exam Review
May 4, 2004

Name:____________________________

Closed book and notes; calculators not permitted.

1. (20 points) Let \( T \) be a linear transformation so that
\[
T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, T \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

(a) Find a matrix \( A \) so that \( T(x) = Ax \). (Hint: \( A = [T(e_1), T(e_2), T(e_3)] \).

(b) Is the linear transformation \( T \) invertible? If yes, what is the matrix of \( T^{-1} \), if no, why not?

2. (20 points) Let \( M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix} \).

(a) Find a basis for \( \text{Col}(M) \).

(b) Find a basis for \( \text{Nul}(M) \).

(c) What is the dimension of \( \text{Nul}(M^T) \)? the dimension of \( \text{Col}(M^T) \)?

3. (20 points) Let
\[
\mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}
\]

(a) Find an orthonormal basis for \( \text{span}(\mathbf{v}_1, \mathbf{v}_2) \).

(b) Find an orthonormal basis for \( \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \).

(c) Find the QR factorization for \( A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \).

4. (20 points) Consider the matrix \( A = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix} \).

For what values of \( k \) is \( A \) invertible?
5. **(20 points)** Let $A$ and $B$ be $n \times n$ matrices, $S$ is an invertible $n \times n$ matrix. $B = S^{-1}AS$.

(a) Show that $A$ and $B$ have the same eigenvalues.
(b) What is the relation between the eigenvectors of $A$ and $B$?

6. **(20 points)**

Sketch the curve defined by

$$Q(x_1, x_2) = 6x_1^2 + 4x_1x_2 + 3x_2^2 = 1$$

by the following steps.

(a) Rewrite the quadratic form $Q(x_1, x_2)$ as $Q(x_1, x_2) = \mathbf{x}^TA\mathbf{x}$.
(b) Find out the principal axes and diagonalize $A$.
(c) Give the formula of the curve in the coordinate system defined by the principal axes and then sketch the curve $Q(x_1, x_2) = 1$.

7. **(20 points)** Let

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}.$$  

(a) Find the singular values of $A$, $\sigma_1$ and $\sigma_2$ with $\sigma_1 \geq \sigma_2$.
(b) Find an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2$ for $\mathbb{R}^2$ so that $\|A\mathbf{v}_1\| = \sigma_1$, $\|A\mathbf{v}_2\| = \sigma_2$ and $A\mathbf{v}_1 \cdot A\mathbf{v}_2 = 0$.
(c) Show that for any unit vector $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, $(c_1^2 + c_2^2) = 1$,

$$\sigma_2 \leq \|A\mathbf{v}\| \leq \sigma_1.$$  

8. **(20 points)** Let

$$A = \begin{bmatrix} 2 & 0 \\ 6 & -1 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of $A$.
(b) Plot several trajectories of the discrete dynamical system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k.$$  

(c) Let $\{\mathbf{x}_k\}$ be a solution of the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Compute $\{\mathbf{x}_1\}$ and find a formula for $\{\mathbf{x}_k\}$.
9. (20 points)
Let \( \mathcal{P}_4 \) denote the linear space of all polynomials of degree \( \leq 4 \). Which of the following subsets of \( \mathcal{P}_4 \) are subspaces of \( \mathcal{P}_4 \)?

(a) \( \{ p(t) : \int_0^1 p(t) \, dt = 0 \} \).
(b) \( \{ p(t) : p(-t) = p(t), \text{ for all } t \} \).
(c) \( \{ p(t) : p(0) = 4 \} \). (‘ is the derivative.)

10. (20 points)
Let
\[
\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.
\]

Note that \( \mathbf{u}_1, \mathbf{u}_2, \) and \( \mathbf{u}_3 \) form an orthonormal basis for \( \mathbb{R}^3 \). Let
\[
A = 3 \mathbf{u}_1 \mathbf{u}_1^T + 7 \mathbf{u}_2 \mathbf{u}_2^T + 9 \mathbf{u}_3 \mathbf{u}_3^T.
\]

(a) Show that \( A \) is symmetric. (Hint: \( (A + B)^T = A^T + B^T \)).
(b) Find \( A \mathbf{u}_1 \), \( A \mathbf{u}_2 \), and \( A \mathbf{u}_3 \).
(c) Diagonalize \( A \). That is, write \( A \) as
\[
A = S \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} S^{-1},
\]
for some matrix \( S \).

11. (15 points) Let
\[
A = \begin{bmatrix} -1 & 1 & -8 \\ 2 & 0 & 6 \\ 2 & -1 & 11 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.
\]

Is \( \mathbf{v} \) in \( \text{im}(A) \)?

12. (15 points) Let
\[
A = \begin{bmatrix} 1 & -1 & -2 & -10 \\ 0 & 0 & 1 & 7 \\ -2 & 2 & 2 & 6 \end{bmatrix}
\]

(a) Find a basis for the nullspace of \( A \).
(b) Find the dimension of \( \text{Nul}(A) \).
(c) Find the dimension of \( \text{Col}(A) \).
13. (20 points) Let
\[
\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 3 \\ 6 \end{bmatrix}.
\]
(a) Find the QR factorization of \( A = [\mathbf{v}_1 \, \mathbf{v}_2 \, \mathbf{v}_3] \).
(b) What does \( \det(A) \) tell you about the vectors \( \mathbf{v}_1, \mathbf{v}_2 \) and \( \mathbf{v}_3 \).
(c) In this case what does \( \det(R) \) tell you about \( \det(A) \)? Why?

14. (20 points) Consider the matrix \( A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \).

(a) Find the eigenvalues of \( A \).
(b) Find a basis for each eigenspace.
(c) What is \( A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \)? What is \( A^n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \)?
(d) Find an explicit formula for \( \mathbf{x}_n = A^n \mathbf{x}(0) \), where \( \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \).

15. (20 points) Let \( A = \begin{bmatrix} -3 & 7 \\ 7 & -3 \end{bmatrix} \). Find an orthogonal matrix \( S \) and diagonal matrix \( D \) such that \( A = SDS^{-1} \).

16. (10 points) Let \( \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} \frac{\sqrt{2}}{4} \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \sigma_1 = 3, \quad \text{and} \quad \sigma_2 = 7. \) Note that \( \{\mathbf{u}_1, \mathbf{u}_2\} \) is an orthonormal set, as is \( \{\mathbf{v}_1, \mathbf{v}_2\} \). Let \( A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T \).

(a) Show that \( A^T A = \sigma_1^2 \mathbf{v}_1 \mathbf{v}_1^T + \sigma_2^2 \mathbf{v}_2 \mathbf{v}_2^T \).
(b) Show that \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are eigenvectors of \( A^T A \) with eigenvalues \( \sigma_1^2 \) and \( \sigma_2^2 \) respectively.
(c) Show that \( \mathbf{u}_1 = \frac{1}{\sigma_1} A \mathbf{v}_1 \), and \( \mathbf{u}_2 = \frac{1}{\sigma_2} A \mathbf{v}_2 \).
(d) Find the singular value decomposition \( A = U \Sigma V^T \).