21-122 Integration, Differential Equations, and Approximation D. Handron

Exam #2 Review Spring 2020

1. (a) Sketch the direction field for $\frac{dy}{dt} = y - t$ in the region where $0 \le x \le 4$ and $0 \le y \le 4$.
   (b) Use Euler's Method with $\Delta t = \frac{1}{2}$ to find an approximate solution to the initial value problem
$$\frac{dy}{dt} = y - t \quad ; \quad y(0) = 1.$$

2. Find the general solution to the differential equation
$$\frac{dy}{dt} = (y^2 + 1)(t + t^2).$$

3. (a) Find the general solution to the differential equation $\frac{dy}{dt} = y\cos(t)$.
   (b) Find the particular solution that satisfies the initial condition $y(0) = 1$. Note that for this solution, $y(\frac{\pi}{2}) = e \approx 2.718$.
   (c) Use Eulers method with initial condition $y(0) = 1$, i.e. $(t_0, y_0) = (0, 1)$, and stepsize $\Delta t = \frac{\pi}{4}$ to find an approximate value for $y(\frac{\pi}{2})$. It may help to know that $1 + \frac{\pi}{4} \approx 1.79$ and $1.79\frac{\pi\sqrt{2}}{8} \approx 0.99$.

4. The size of a population of rabbits on a small island is modeled by the logistic equation
$$\frac{dP}{dt} = 5P\left(1 - \frac{P}{1000}\right).$$
where P denotes the number of rabbits, and t is time measured in years.
   (a) Sketch the phase line and slope field for this differential equation.
   (b) On the slope field above, sketch the solution satisfying $P(0) = 200$. What is $\lim_{t\to\infty} P(t)$?
   (c) Suppose that the island is stocked with 100 additional rabbits each year. Modify the differential equation to account for this additional assumption.
   (d) What are the equilibrium points of the modified system? If $P(0) = 0$, what is $\lim_{t\to\infty} P(t)$? It might help to know that $\sqrt{27} \approx 5.2$ and you might want to consider the phase line for the modified system.

5. (a) Let $a_n = \frac{n^2-1}{n^2}$. What is $\lim_{n\to\infty} a_n$? Does $\sum_{n=1}^{\infty} a_n$ converge? Why or why not?

(b) Let $b_n = \frac{1}{n\ln(n)}$. What is $\lim_{n\to\infty} b_n$? Does $\sum_{n=1}^{\infty} b_n$ converge? Why or why not?

(c) Let $t_n = \frac{(n!)^2}{(2n)!}$. Is the sequence $\{t_n\}$ monotonic (increasing or decreasing)? (Hint: what can you say about t_n/t_{n+1}?)

(d) Is the sequence $\{t_n\}$ bounded? Does the limit $\lim_{n\to\infty} t_n$ exist? Why or why not?

6. The **Sierpinski triangle** is constructed by removing the center one-fourth of an equilateral triangle with area 1, then removing the centers of three smaller remaining triangles, and so on. Show that the sum of the sreas of the removed triangles is 1.

   [EDIT: By the "center one-fourth" of a triangle, I mean the triangle formed by connecting the midpoints of the edges.]

7. Newton's method may be used to solve the equation

$$(x+8)^2 = 0$$

   Begin with $x_1 = 8$, and compute x_2, x_3, x_4 and x_5.

8. (a) Find the limit of the sequence $\left\{\frac{(-1)^{n-1}n}{n^2+1}\right\}_{n=1}^{\infty}$.

   (b) Let

$$a_n = \frac{n!}{n^3}\sqrt{\frac{n}{n+1}}$$

   Find $\lim_{n\to\infty} a_n$.

9. Determine if each of the following series converges. If the series converges find the sum.

   (a) $\sum_{n=1}^{\infty} \ln(\frac{3n}{2n+5})$

   (b) $\sum_{n=1}^{\infty} 3\left(\frac{2}{3}\right)^n \left(\frac{1}{4}\right)^{n-1}$

10. Consider the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right)$$

(a) Find the partial sums s_3, s_4 and s_5.

(b) Find a formula for the nth partial sum, s_n.

(c) What is the limit $\lim_{n\to\infty} s_n$?

(d) What is the sum of the series, $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right)$?

11. Show that the infinite series

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$$

converges. How close is the partial sum s_{10} to the sum of the series?

12. Determine whether the infinite series

$$\sum_{n=1}^{\infty}\frac{n3^n}{\sin^2(n)2^n}$$

converges or diverges. If it converges, how large must n be such that the remainder R_n satisfies $|R_n| < \frac{1}{10}$?

13. **(20 points)** A figure is drawn in the following way. First begin with a 1×1 square. Attach to each corner a $\frac{1}{2} \times \frac{1}{2}$ square. Now on each of the free corners, attach a $\frac{1}{4} \times \frac{1}{4}$ square. Continue in this fashion. The figure is the "limit as $n \to \infty$". What is the area of this figure.

(0 points) If you finish early, think about the following: Will the construction above ever overlap itself? What is the smallest square that can be drawn around the figure? What is the area of this square? How much of this square will the figure fill?