Week #4 Written Assignment: Due on Friday, February 7.

1. The purpose of this exercise is to explain the idea underlying Simpson's rule for approximating the integral $\int_a^b f(x), dx$, and to justify the formula for computing the Simpson's rule approximation of an integral. The fundamental idea is to approximate

$$\int_{x_{i-1}}^{x_{i+1}} f(x)\,dx$$

by the area under a parabola that passes through the points $(x_{i-1}, f(x_{i-1})$, $(x_i, f(x_i)$, $(x_{i+1}, f(x_{i+1})$.

   (a) Setting $\Delta x = \frac{b-a}{n}$ we can write $x_{i-1} = x_i - \Delta x$ and $x_{i+1} = x_i + \Delta x$. Explain why

$$\int_{-\Delta x}^{\Delta x} f(x + x_i)\,dx = \int_{x_i-\Delta x}^{x_i+\Delta x} f(x)\,dx$$

   (b) Note that the graph of $y = f(x + x_i)$ passes through the points $(-\Delta x, f(x_i - \Delta x)$, $(0, f(x_i)$, $(+\Delta x, f(x_i+\Delta x)$. For convenience, let's use the notation $y_- = f(x_i - \Delta x)$, $y_0 = f(x_i)$, $y_+ = f(x_i + \Delta x)$.
   Find the polynomial $p(x) = Ax^2 + Bx + C$ that satisfies $p(-\Delta x) = y_-$, $p(0) = y_0$, $(p(+\Delta x) = y_+$ by solving

$$\begin{aligned} A(-\Delta x)^2 + B(-\Delta x) + C &= y_- \\ A(0)^2 + B(0) + C &= y_0 \\ A(+\Delta x)^2 + B(+\Delta x) + C &= y_+ \end{aligned}$$

   (c) Show that $\int_{-\Delta x}^{\Delta x}(Ax^2 + Bx + C)\,dx = \frac{\Delta x}{3}(y_- + 4y_0 + y_+)$.

   (d) Explain why we say that $\int_{x_i-\Delta x}^{x_i+\Delta x} f(x)\,dx \approx \frac{\Delta x}{3}[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})]$

   (e) Derive the Simpson's rule approximation

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_3) + \cdots + 2f(x_{n-2} + 4f(x_{n-1}) + f(x_n)\right].$$

   for the integral $\int_a^b f(x)\,dx$.

2. (Stewart, 7th edition, #7.8.50) Show that $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

3. (Stewart, 7th edition, #7.8.78) Find the value of C for which the integral

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right) dx$$

converges, and evaluate the integral for that value of C.

4. The Laplace transform of a function $f : [0, \infty) \to \mathbb{R}$ is the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

The domain of F is the set of all $s \in \mathbb{R}$ for which the integral converges.

Find the Laplace transform and it's domain for each of the following functions

(a) $f(t) = 1$

(b) $f(t) = t$