ANTIPLANE SHEAR FLOWS IN VISCO-PLASTIC SOLIDS EXHIBITING ISOTROPIC AND KINEMATIC HARDENING

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Abstract. The authors consider antiplane shearing motions of an incompressible visco-plastic solid. The particular constitutive equation employed assumes that the stress tensor has an "elastic" component and a component which can exhibit hysteresis. The model exhibits both "kinematic" and "isotropic" hardening. Our results consist of a set of energy type estimates for the resulting system, L_2 contractivity estimates for the solution operator, and finally an analysis of the approach of our system to a "rate independent" model as a distinguished parameter describing our flow rule approaches zero. We also include some computational results for simple piecewise constant data.

Key words. antiplane shear flows, visco-plastic material, energy estimates, kinematic and isotropic hardening.

AMS subject classifications. 73E05, 73E50, 73E60, 73E70

1. Introduction. In this note we consider antiplane shear flows of an incompressible, isotropic, visco-plastic solid. This work generalizes and compliments earlier work of Greenberg [1], [2], Greenberg and Nouri [3], and Nouri and Rascle [4]. In particular, our work shows that the energy estimates previously obtained in [1] - [4] have counterparts for a broader class of visco-plastic materials.

The constitutive assumption we use postulates that the Cauchy stress tensor is additively decomposable into a component without hysteresis, i.e., that depends only on the present value of deformation gradient and an elasto-plastic component that exhibits hysteresis, i.e., that can depend on the past history of deformation gradient, and the flow rule we employ for the plastic strain tensor is basically an isotropic hardening rule with yield determined by the elasto-plastic stresses. The presence of a component without hysteresis for the stress tensor yields a model which exhibits both "kinematic" and "isotropic" hardening. Mihăilescu-Suliciu, Suliciu, and Williams [5] analyzed the relationship between solutions of ordinary differential equations that govern the motions of elasto-plastic and visco-plastic oscillators. Their constitutive assumptions also admit both kinematic and isotropic hardening.

The organization of this paper is as follows. In section 2 we develop the equations describing antiplane shear flows in visco-plastic solids. Section 3 focuses on a set of a-priori energy type estimates and also estimates which guarantee that the solution operator is a contraction in L_2. These estimates depend on a small parameter and as that parameter tends to zero they supply information about rate independent limit models. Similar estimates for a purely isotropic hardening model have been obtained by Rascle and Nouri [4]. In section 4 we present some numerical experiments.

2. Model Development. In antiplane shear flows of an incompressible isotropic visco-plastic solid, material points

$$\xi = \xi_1 \mathbf{e}_1 + \xi_2 \mathbf{e}_2 + \xi_3 \mathbf{e}_3 \tag{2.1}$$

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move to [1]

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \tag{2.2}$$

where

$$x_1 = \xi_1,\ x_2 = \xi_2,\ \text{and } x_3 = \xi_3 + \phi\left(\xi_1, \xi_2, t_1\right).$$

For such motions the Eulerian velocity field is given by

$$\mathbf{u}\left(x_1, x_2, x_3, t_1\right) = w\left(x_1, x_2, t_1\right)\mathbf{e}_3 \tag{2.3}$$

where

$$w(x_1, x_2, t_1) = \frac{\partial \phi}{\partial t_1}\left(x_1, x_2, t_1\right)$$

and the shear strains are given by

$$\gamma_1 = \frac{\partial \phi}{\partial x_1} \text{ and } \gamma_2 = \frac{\partial \phi}{\partial x_2}. \tag{2.4}$$

The latter are related to the former by the compatibility relations

$$\frac{\partial \gamma_1}{\partial t_1} = \frac{\partial w}{\partial x_1} \quad \text{and} \quad \frac{\partial \gamma_2}{\partial t_1} = \frac{\partial w}{\partial x_2}. \tag{2.5}$$

The deformation gradient, F, associated with this motion is given by

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma_1 & \gamma_2 & 1 \end{pmatrix}. \tag{2.6}$$

Our basic constitutive assumption postulates a plastic deformation tensor, P, of the same form as F; that is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_1 & p_2 & 1 \end{pmatrix} \tag{2.7}$$

where p_1 and p_2 are functions of $x_1,\ x_2$, and t_1, and postulates that the Cauchy Stress tensor take the form

$$T = -\pi I + \mathcal{S}_1 + \mathcal{S}_2. \tag{2.8}$$

Here π is the hydrostatic pressure,

$$\mathcal{S}_1 = \beta\left(FF^T - \frac{1}{3}\,\text{trace}\left(FF^T\right) I\right) \tag{2.9}$$

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[1] $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are the standard basis for elements for R^3 and $\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i\mathbf{e}_j^T$ are the standard basis elements for linear operators from R^3 to R^3

or equivalently

$$\mathcal{S}_1 = \beta \begin{pmatrix} -\frac{1}{3}\left(\gamma_1^2 + \gamma_2^2\right) & 0 & \gamma_1 \\ 0 & -\frac{1}{3}\left(\gamma_1^2 + \gamma_2^2\right) & \gamma_2 \\ \gamma_1 & \gamma_2 & \frac{2}{3}\left(\gamma_1^2 + \gamma_2^2\right) \end{pmatrix}, \tag{2.10}$$

$$\mathcal{S}_2 = \mu \left(RR^T - \frac{1}{3} \text{ trace } \left(RR^T\right) I \right) \tag{2.11}$$

or equivalently

$$\mathcal{S}_2 = \mu \begin{pmatrix} -\frac{1}{3}\left(r_1^2 + r_2^2\right) & 0 & r_1 \\ 0 & -\frac{1}{3}\left(r_1^2 + r_2^2\right) & r_2 \\ r_1 & r_2 & \frac{2}{3}\left(r_1^2 + r_2^2\right) \end{pmatrix}, \tag{2.12}$$

$$R = FP^{-1} = P^{-1}F \tag{2.13}$$

or equivalently

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_1 & r_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma_1 - p_1 & \gamma_2 - p_2 & 1 \end{pmatrix} \tag{2.14}$$

and β and μ are positive constants. Equations (2.9) - (2.14) imply that $\mathcal{S}_1$ is an isotropic trace free function of F that has the interpretation of a stress without hysteresis, whereas $\mathcal{S}_2$ is an isotropic trace free function of R that is interpreted as the "elasto-plastic" stress and exhibits hysteresis for appropriate deformation histories.

In what follows it will be convenient to let

$$s_1 := \mu\left(\gamma_1 - p_1\right) \quad \text{and} \quad s_2 := \mu\left(\gamma_2 - p_2\right) \tag{2.15}$$

and to use $s_1(x_1,\ x_2,\ t_1)$, $s_2(x_1,\ x_2,\ t_1)$, $p_1(x_1,\ x_2,\ t_1)$ and $p_2(x_1,\ x_2, t_1)$ as the basic descriptors of our system. With this choice we have

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{s_1}{\mu} + p_1 & \frac{s_2}{\mu} + p_2 & 1 \end{pmatrix}, \tag{2.16}$$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{s_1}{\mu} & \frac{s_2}{\mu} & 1 \end{pmatrix}, \tag{2.17}$$

$$\mathcal{S}_2 = \begin{pmatrix} -\frac{1}{3\mu}\left(s_1^2 + s_2^2\right) & 0 & s_1 \\ 0 & -\frac{1}{3\mu}\left(s_1^2 + s_2^2\right) & s_2 \\ s_1 & s_2 & \frac{2}{3\mu}\left(s_1^2 + s_2^2\right) \end{pmatrix}, \tag{2.18}$$

(2.19) and

$$\mathcal{S}_1 = \beta \begin{pmatrix} -\frac{1}{3}\left(\left(\frac{s_1}{\mu}+p_1\right)^2+\left(\frac{s_2}{\mu}+p_2\right)^2\right) & 0 & \frac{s_1}{\mu}+p_1 \\ 0 & -\frac{1}{3}\left(\left(\frac{s_1}{\mu}+p_1\right)^2+\left(\frac{s_2}{\mu}+p_2\right)^2\right) & \frac{s_2}{\mu}+p_2 \\ \frac{s_1}{\mu}+p_1 & \frac{s_2}{\mu}+p_2 & \frac{2}{3}\left(\left(\frac{s_1}{\mu}+p_1\right)^2+\left(\frac{s_2}{\mu}+p_2\right)^2\right) \end{pmatrix}.$$

Balance of momentum in the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions implies that

$$\frac{\partial}{\partial x_1}\left(\pi + \frac{\left(s_1^2+s_2^2\right)}{3\mu} + \frac{\beta}{3}\left(\left(\frac{s_1}{\mu}+p_1\right)^2+\left(\frac{s_2}{\mu}+p_2\right)^2\right)\right) = 0, \tag{2.20}$$

$$\frac{\partial}{\partial x_2}\left(\pi + \frac{\left(s_1^2+s_2^2\right)}{3\mu} + \frac{\beta}{3}\left(\left(\frac{s_1}{\mu}+p_1\right)^2+\left(\frac{s_2}{\mu}+p_2\right)^2\right)\right) = 0, \tag{2.21}$$

and thus that the pressure, π, is given by

$$\pi = \pi_0\left(x_3,\ t\right) - \frac{\left(s_1^2+s_2^2\right)}{3\mu} - \frac{\beta}{3}\left(\left(\frac{s_1}{\mu}+p_1\right)^2+\left(\frac{s_2}{\mu}+p_2\right)^2\right). \tag{2.22}$$

Balance of momentum in the $\mathbf{e}_3$ direction yields

$$\rho_0\frac{\partial w}{\partial t_1} - \frac{\partial}{\partial x_1}\left(s_1+\beta\left(\frac{s_1}{\mu}+p_1\right)\right) - \frac{\partial}{\partial x_2}\left(s_2+\beta\left(\frac{s_2}{\mu}+p_2\right)\right) = -\frac{\partial \pi_0}{\partial x_3}. \tag{2.23}$$

Here, ρ_0 is the constant mass density of the material. Since $\frac{\partial \pi_0}{\partial x_3}$ depends on x_3 and t_1, whereas all quantities in the left-hand side of (2.23) depend on x_1, x_2 and t_1, we conclude that for antiplane shear flows $\frac{\partial \pi_0}{\partial x_3}$ is independent of x_3. **In what follows we shall assume this term is zero**.

We now turn our attention to the flow rule for the nonconstant components p_1 and p_2 of P. We assume the isotropic hardening rule

$$\frac{\partial p_1}{\partial t_1} = \frac{s_1}{\sqrt{s_1^2+s_2^2}}\frac{\left(\sqrt{s_1^2+s_2^2}-s_y(d)\right)_+}{\mu T_0 \epsilon} \tag{2.24}$$

and

$$\frac{\partial p_2}{\partial t_1} = \frac{s_2}{\sqrt{s_1^2+s_2^2}}\frac{\left(\sqrt{s_1^2+s_2^2}-s_y(d)\right)_+}{\mu T_0 \epsilon}. \tag{2.25}$$

Here,

$$(x)_+ = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}, \tag{2.26}$$

d is the accumulated plastic strain and satisfies

$$\frac{\partial d}{\partial t_1} = \sqrt{\left(\frac{\partial p_1}{\partial t_1}\right)^2 + \left(\frac{\partial p_2}{\partial t_1}\right)^2} = \frac{\left(\sqrt{s_1^2+s_2^2} - s_y(d)\right)_+}{\mu T_0 \epsilon} \tag{2.27}$$

and finally $d \to s_y(d)$ is the yield stress which we assume satisfies

$$0 < s_y(0) \le s_y(d), \quad 0 < m \le s_y'(d) \quad \text{and} \quad -M \le s_y''(d) \le 0 \tag{2.28}$$

for $d \ge 0$. The parameter T_0 has the interpretation of a relaxation time, μ is the "shear" modulus, and $\epsilon > 0$ is a dimensionless small parameter.

Equations (2.5), (2.15), and (2.23) -(2.27) combine to yield the following system for $s_1,\ s_2,\ p_1,\ p_2,\ d$, and w:

$$\frac{1}{\mu}\frac{\partial s_1}{\partial t_1} - \frac{\partial w}{\partial x_1} = -\frac{\partial p_1}{\partial t_1}, \tag{2.29}$$

$$\frac{1}{\mu}\frac{\partial s_2}{\partial t_1} - \frac{\partial w}{\partial x_2} = -\frac{\partial p_2}{\partial t_1}, \tag{2.30}$$

$$\rho_0\frac{\partial w}{\partial t_1} - \frac{\partial}{\partial x_1}\left(\frac{(\beta+\mu)}{\mu}s_1 + \beta p_1\right) - \frac{\partial}{\partial x_2}\left(\frac{(\beta+\mu)}{\mu}s_2 + \beta p_2\right) = 0, \tag{2.31}$$

$$\frac{\partial p_1}{\partial t_1} = \frac{s_1}{\sqrt{s_1^2+s_2^2}}\frac{\left(\sqrt{s_1^2+s_2^2} - s_y(d)\right)_+}{\mu T_0 \epsilon}, \tag{2.32}$$

$$\frac{\partial p_2}{\partial t_1} = \frac{s_2}{\sqrt{s_1^2+s_2^2}}\frac{\left(\sqrt{s_1^2+s_2^2} - s_y(d)\right)_+}{\mu T_0 \epsilon}, \tag{2.33}$$

and

$$\frac{\partial d}{\partial t_1} = \frac{\left(\sqrt{s_1^2+s_2^2} - s_y(d)\right)_+}{\mu T_0 \epsilon}. \tag{2.34}$$

For convenience, we put (2.29) - (2.34) into dimensionless form. We let

$$\begin{cases} x = \sqrt{\dfrac{\rho_0}{\mu+\beta}}\dfrac{x_1}{T_0}, \quad y = \sqrt{\dfrac{\rho_0}{\mu+\beta}}\dfrac{y_1}{T_0}, \quad t = \dfrac{t_1}{T_0}, \\ u = \sqrt{\dfrac{\rho_0}{\mu+\eta}}w, \quad \tau_1 = \dfrac{s_1}{\mu}, \quad \tau_2 = \dfrac{s_2}{\mu} \quad \text{and} \quad \tau_y(d) = \dfrac{s_y(d)}{\mu}. \end{cases} \tag{2.35}$$

Then, (2.29) - (2.34) transforms to

$$\frac{\partial \tau_1}{\partial t} - \frac{\partial u}{\partial x} = -\frac{\partial p_1}{\partial t} \tag{2.36}$$

(2.37)
$$\frac{\partial \tau_2}{\partial t} - \frac{\partial u}{\partial y} = -\frac{\partial p_2}{\partial t}$$

(2.38)
$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\tau_1 + \frac{\beta}{\mu+\beta}p_1\right) - \frac{\partial}{\partial y}\left(\tau_2 + \frac{\beta}{\mu+\beta}p_2\right) = 0$$

(2.39)
$$\frac{\partial p_1}{\partial t} = \frac{\tau_1\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)_+}{\epsilon\sqrt{\tau_1^2+\tau_2^2}}$$

(2.40)
$$\frac{\partial p_2}{\partial t} = \frac{\tau_2\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)_+}{\epsilon\sqrt{\tau_1^2+\tau_2^2}}$$

and

(2.41)
$$\frac{\partial d}{\partial t} = \frac{\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)_+}{\epsilon} \geq 0.$$

In section 3 we shall present a set of energy type estimates for (2.36) - (2.41).

3. Energy Estimates. In this section we shall focus on a series of energy type estimates for solutions of the Cauchy problem for (2.36) - (2.41). We shall continually use an integral identity, presented in (3.2) below, for solutions of the divergence identity:

(3.1)
$$\frac{\partial f}{\partial t} - \frac{\partial q_1}{\partial x} - \frac{\partial q_2}{\partial y} = g.$$

The basic identities are

$$\frac{d}{ds} \iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2} \leq R+t-s\}} f(x,y,s)\,dxdy =$$

$$(R+t-s)\int_0^{2\pi} \left[q_1\cos\theta + q_2\sin\theta - f\right](x_0 + (R+t-s)\cos\theta, y_0 + (R+t-s)\sin\theta, s)\,d\theta$$

$$+ \iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2} \leq R+t-s\}} g(x,y,s)\,dxdy$$

and

$$\iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2}\le R\}} f\left(x,y,s\right)dxdy =$$

$$\iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2}\le R+t\}} f\left(x,y,0\right)dxdy$$

$$+\int_0^t (R+t-s)\left(\int_0^{2\pi}[q_1\cos\theta+q_2\sin\theta-f](x_0+(R+t+s)\cos\theta, y_0+(R+t-s)\sin\theta, s)d\theta\right)ds$$

$$+\int_0^t\left(\iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2}\le R+t-s\}} g\left(x,y,s\right)dxdy\right)ds. \tag{3.2}$$

Our first identity is obtained by multiplying (2.36) by τ_1, (2.37) by τ_2, (2.38) by u and (2.41) by $\frac{\mu}{\beta+\mu}\tau_y(d)$ and adding the equations. The result is that (3.1) holds with

$$f=\frac{\mu\left(\tau_1^2+\tau_2^2\right)}{2\left(\mu+\beta\right)}+\frac{\beta\left(\left(\tau_1+p_1\right)^2+\left(\tau_2+p_2\right)^2\right)}{2\left(\mu+\beta\right)}+\frac{u^2}{2}+\frac{\mu}{\mu+\beta}\int_0^d\tau_y\left(\eta\right)d\eta \tag{3.3}$$

$$q_1=u\left(\frac{\mu}{\mu+\beta}\tau_1+\frac{\beta}{\mu+\beta}\left(\tau_1+p_1\right)\right) \tag{3.4}$$

$$q_2=u\left(\frac{\mu}{\mu+\beta}\tau_2+\frac{\beta}{\mu+\beta}\left(\tau_2+p_2\right)\right) \tag{3.5}$$

and

$$g=\frac{-\mu}{\epsilon\left(\mu+\beta\right)}\left(\left(\sqrt{\tau_1^2+\tau_2^2}-\tau_y\left(d\right)\right)_+\right)^2. \tag{3.6}$$

The observation that

$$q_1\cos\theta+q_2\sin\theta\le|u|\left(\frac{\mu}{\mu+\beta}\sqrt{\tau_1^2+\tau_2^2}+\frac{\beta}{\mu+\beta}\sqrt{\left(\tau_1+p_1\right)^2+\left(\tau_2+p_2\right)^2}\right) \tag{3.7}$$

implies that

$$q_1\cos\theta+q_2\sin\theta-f\le$$

$$-\frac{\mu}{2\left(\mu+\beta\right)}\left(|u|-\sqrt{\tau_1^2+\tau_2^2}\right)^2-\frac{\beta}{2\left(\mu+\beta\right)}\left(|u|-\sqrt{\left(\tau_1+p_1\right)^2+\left(\tau_2+p_2\right)^2}\right)^2 \tag{3.8}$$

$$-\frac{\mu}{\mu+\beta}\int_0^d\tau_y(\eta)d\eta\le 0$$

and thus the last two integrals on the right-hand side of (3.2) are less than or equal to zero.[2]

To obtain the contractivity of the solution operator for (2.36) - (2.41) we look at two solutions to (2.36) - (2.41) which we label a and b, respectively. Their differences satisfy

$$\frac{\partial}{\partial t}\left(\tau_1^a-\tau_1^b\right)-\frac{\partial}{\partial x}\left(u^a-u^b\right)=-\frac{\partial}{\partial t}\left(p_1^a-p_1^b\right), \tag{3.9}$$

$$\frac{\partial}{\partial t}\left(\tau_2^a-\tau_2^b\right)-\frac{\partial}{\partial y}\left(u^a-u^b\right)=-\frac{\partial}{\partial t}\left(p_2^a-p_2^b\right), \tag{3.10}$$

$$\frac{\partial}{\partial t}\left(u^a-u^b\right)-\frac{\partial}{\partial x}\left(\tau_1^a-\tau_1^b+\frac{\beta\left(p_1^a-p_1^b\right)}{\mu+\beta}\right)-\frac{\partial}{\partial y}\left(\tau_2^a-\tau_2^b+\frac{\beta\left(p_2^a-p_2^b\right)}{\mu+\beta}\right)=0, \tag{3.11}$$

$$\frac{\partial}{\partial t}\left(p_1^a-p_1^b\right)=\frac{\tau_1^a\left(\sqrt{\left(\tau_1^a\right)^2+\left(\tau_2^a\right)^2}-\tau_y\left(d^a\right)\right)_+}{\epsilon\sqrt{\left(\tau_1^a\right)^2+\left(\tau_2^a\right)^2}}-\frac{\tau_1^b\left(\sqrt{\left(\tau_1^b\right)^2+\left(\tau_2^b\right)^2}-\tau_y\left(d^b\right)\right)_+}{\epsilon\sqrt{\left(\tau_1^b\right)^2+\left(\tau_2^b\right)^2}}, \tag{3.12}$$

$$\frac{\partial}{\partial t}\left(p_2^a-p_2^b\right)=\frac{\tau_2^a\left(\sqrt{\left(\tau_1^a\right)^2+\left(\tau_2^a\right)^2}-\tau_y\left(d^a\right)\right)_+}{\epsilon\sqrt{\left(\tau_1^a\right)^2+\left(\tau_2^a\right)^2}}-\frac{\tau_2^b\left(\sqrt{\left(\tau_1^b\right)^2+\left(\tau_2^b\right)^2}-\tau_y\left(d^b\right)\right)_+}{\epsilon\sqrt{\left(\tau_1^b\right)^2+\left(\tau_2^b\right)^2}}, \tag{3.13}$$

and

$$\frac{\partial}{\partial t}\left(d^a-d^b\right)=\frac{\left(\sqrt{\left(\tau_1^a\right)^2+\left(\tau_2^a\right)^2}-\tau_y\left(d^a\right)\right)_+}{\epsilon}-\frac{\left(\sqrt{\left(\tau_1^b\right)^2+\left(\tau_2^b\right)^2}-\tau_y\left(d^b\right)\right)_+}{\epsilon}. \tag{3.14}$$

We next multiply (3.9) by $\left(\tau_1^a-\tau_1^b\right)$, (3.10) by $\left(\tau_2^a-\tau_2^b\right)$, (3.11) by $\left(u^a-u^b\right)$, and (3.14) by

$$\frac{\mu}{\mu+\beta}\left(\tau_y\left(d^a\right)-\tau_y\left(d^b\right)\right)=\frac{\mu}{\mu+\beta}\left(\int_0^1\tau_y'\left((1-\eta)\,d^b+\eta d^a\right)d\eta\right)\left(d^a-d^b\right)$$

and add the resulting equations. The result is that (3.1) holds where now

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[2]We are, of course, using the fact that $d\geq 0, d_t\geq 0$, and that $0\leq\tau_y(0)d\leq\int_0^d\tau_y(\eta)dy$.

$$f = \frac{1}{2}\left(\frac{\mu\left(\left(\tau_1^a - \tau_1^b\right)^2 + \left(\tau_2^a - \tau_2^b\right)^2\right)}{\mu+\beta} + \frac{\beta\left(\left(\tau_1^a - \tau_1^b + p_1^a - p_1^b\right)^2 + \left(\tau_2^a - \tau_2^b + p_2^a - p_2^b\right)^2\right)}{\mu+\beta}\right.$$
$$\left. + \left(u^a - u^b\right)^2 + \frac{\mu}{\mu+\beta}\left(\int_0^1 \tau_y'\left((1-\eta)\,d^b + \eta d^a\right) d\eta\right)\left(d^a - d^b\right)^2\right), \tag{3.15}$$

$$q_1 = \left(u^a - u^b\right)\left(\frac{\mu}{\mu+\beta}\left(\tau_1^a - \tau_1^b\right) + \frac{\beta}{\mu+\beta}\left(\tau_1^a - \tau_1^b + p_2^a - p_2^b\right)\right), \tag{3.16}$$

$$q_2 = \left(u^a - u^b\right)\left(\frac{\mu}{\mu+\beta}\left(\tau_2^a - \tau_2^b\right) + \frac{\beta}{\mu+\beta}\left(\tau_2^a - \tau_2^b + p_2^a - p_2^b\right)\right), \tag{3.17}$$

and

$$g = \frac{\mu}{\mu+\beta}\left(g_1 + \frac{g_2}{\epsilon}\right) \tag{3.18}$$

where

$$g_1 = \frac{1}{2}\left(\int_0^1 \tau_y''\left((1-\eta)\,d^b + \eta d^a\right)\left((1-\eta)\,d_t^b + \eta d_t^a\right) d\eta\right)\left(d^a - d^b\right)^2 \tag{3.19}$$

and

$$g_2 = \left(\tau_y\left(d^a\right) - \tau_y\left(d^b\right)\right)\left(\left(\sqrt{\left(\tau_1^a\right)^2 + \left(\tau_2^a\right)^2} - \tau_y\left(d^a\right)\right)_+ - \left(\sqrt{\left(\tau_1^b\right)^2 + \left(\tau_2^b\right)^2} - \tau_y\left(d^b\right)\right)_+\right)$$
$$- \left(\tau_1^a - \tau_1^b\right)\left(\frac{\tau_1^a\left(\sqrt{\left(\tau_1^a\right)^2 + \left(\tau_2^a\right)^2} - \tau_y\left(d^a\right)\right)_+}{\sqrt{\left(\tau_1^a\right)^2 + \left(\tau_2^a\right)^2}} - \frac{\tau_1^b\left(\sqrt{\left(\tau_1^b\right)^2 + \left(\tau_2^b\right)^2} - \tau_y\left(d^b\right)\right)_+}{\sqrt{\left(\tau_1^b\right)^2 + \left(\tau_2^b\right)^2}}\right)$$
$$- \left(\tau_2^a - \tau_2^b\right)\left(\frac{\tau_2^a\left(\sqrt{\left(\tau_1^a\right)^2 + \left(\tau_2^a\right)^2} - \tau_y\left(d^a\right)\right)_+}{\sqrt{\left(\tau_1^a\right)^2 + \left(\tau_2^a\right)^2}} - \frac{\tau_2^b\left(\sqrt{\left(\tau_1^b\right)^2 + \left(\tau_2^b\right)^2} - \tau_y\left(d^b\right)\right)_+}{\sqrt{\left(\tau_1^b\right)^2 + \left(\tau_2^b\right)^2}}\right) \tag{3.20}$$

The arguments leading to (3.8) yield the inequality

$$q_1\cos\theta + q_2\sin\theta - f \leq -\frac{\mu}{2(\mu+\beta)}\left(\left|u^a - u^b\right| - \sqrt{\left(\tau_1^a - \tau_1^b\right)^2 + \left(\tau_2^a - \tau_2^b\right)^2}\right)^2$$
$$-\frac{\beta}{2(\mu+\beta)}\left(\left|u^a - u^b\right| - \sqrt{\left(\tau_1^a - \tau_1^b + p_1^a - p_1^b\right)^2 + \left(\tau_2^a - \tau_2^b + p_2^a - p_2^b\right)^2}\right)^2$$
$$-\frac{\mu}{2(\mu+\beta)}\left(\int_0^1 \tau_y'\left((1-\eta)\,d^b + \eta d^a\right) d\eta\right)\left(d^a - d^b\right)^2 \leq 0. \tag{3.21}$$

We further note that the hypotheses that $0 < \tau_y(0) \le \tau_y(d)$, $0 < \frac{m}{\mu} \le \tau_y'(d)$, and $-\frac{M}{\mu} \le \tau_y''(d) \le 0$ for $d \ge 0$ guarantees that $g_1 \le 0$. Thus, to show that

$$\iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2} \le R\}} f(x,y,t)\,dxdy \le \iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2} \le R+t\}} f(x,y,0)\,dxdy \tag{3.22}$$

where f is given by (3.15) it suffices to show that $g_2 \le 0$. We first note that g_2 may be written as

$$\begin{aligned} g_2 = &-\left(\sqrt{(\tau_1^a)^2+(\tau_2^a)^2}-\tau_y(d^a)\right)_+\left(\sqrt{(\tau_1^a)^2+(\tau_2^a)^2}-\tau_y(d^a)\right)\\ &-\left(\sqrt{(\tau_1^b)^2+(\tau_2^b)^2}-\tau_y(d^b)\right)_+\left(\sqrt{(\tau_1^b)^2+(\tau_2^b)^2}-\tau_y(d^b)\right)\\ &+\left(\sqrt{(\tau_1^a)^2+(\tau_2^a)^2}-\tau_y(d^a)\right)_+\left(\frac{\tau_1^a\tau_1^b+\tau_2^a\tau_2^b}{\sqrt{(\tau_1^a)^2+(\tau_2^a)^2}}-\tau_y(d^b)\right)\\ &+\left(\sqrt{(\tau_1^b)^2+(\tau_2^b)^2}-\tau_y(d^b)\right)_+\left(\frac{\tau_1^a\tau_1^b+\tau_2^a\tau_2^b}{\sqrt{(\tau_1^b)^2+(\tau_2^b)^2}}-\tau_y(d^a)\right) \end{aligned} \tag{3.23}$$

and (3.23), together with

$$\tau_1^a\tau_1^b+\tau_2^a\tau_2^b \le \sqrt{(\tau_1^a)^2+(\tau_1^b)^2}\sqrt{(\tau_2^a)^2+(\tau_2^b)^2}, \tag{3.24}$$

implies that

$$\begin{aligned} g_2 \le &-\left(\sqrt{(\tau_1^a)^2-(\tau_2^a)^2}-\tau_y(d^a)\right)_+\left(\sqrt{(\tau_1^a)^2+(\tau_2^a)^2}-\tau_y(d^a)\right)\\ &-\left(\sqrt{(\tau_1^b)^2+(\tau_2^b)^2}-\tau_y(d^b)\right)_+\left(\sqrt{(\tau_1^b)^2+(\tau_2^b)^2}-\tau_y(d^b)\right)\\ &+\left(\sqrt{(\tau_1^a)^2+(\tau_2^a)^2}-\tau_y(d^a)\right)_+\left(\sqrt{(\tau_1^b)^2+(\tau_2^b)^2}-\tau_y(d^b)\right)\\ &+\left(\sqrt{(\tau_1^b)^2+(\tau_2^b)^2}-\tau_y(d^b)\right)_+\left(\sqrt{(\tau_1^a)^2+(\tau_2^a)^2}-\tau_y(d^a)\right) \end{aligned} \tag{3.25}$$

or that

$$g_2 = -\left(\left(\sqrt{\left(\tau_1^a\right)^2 + \left(\tau_2^a\right)^2} - \tau_y\left(d^a\right)\right)_+ - \left(\sqrt{\left(\tau_1^b\right)^2 + \left(\tau_2^b\right)^2} - \tau_y\left(d^b\right)\right)_+\right)^2$$

$$+\left(\sqrt{\left(\tau_1^a\right)^2 + \left(\tau_2^a\right)^2} - \tau_y\left(d^a\right)\right)_+ \left(\sqrt{\left(\tau_1^b\right)^2 + \left(\tau_2^b\right)^2} - \tau_y\left(d^b\right) - \left(\sqrt{\left(\tau_1^b\right)^2 + \left(\tau_2^b\right)^2} - \tau_y\left(d^b\right)\right)_+\right)$$

$$+\left(\sqrt{\left(\tau_1^b\right)^2 + \left(\tau_2^b\right)^2} - \tau_y\left(d^b\right)\right)_+ \left(\sqrt{\left(\tau_1^a\right)^2 + \left(\tau_2^a\right)^2} - \tau_y\left(d^a\right) - \left(\sqrt{\left(\tau_1^a\right)^2 + \left(\tau_2^a\right)^2} - \tau_y\left(d^a\right)\right)_+\right). \tag{3.26}$$

Since each of the terms on the right-hand side of (3.26) is nonpositive we conclude that $g_2 \le 0$ and thus (3.22) holds, which is the desired contractivity of the solution operator associated with (2.36) - (2.41). Similar arguments have been used previously by Greenberg and Nouri [3] and Nouri and Rascle [4].

Our next objective is a set of derivative estimates for solutions of the Cauchy problem for (2.36) - (2.41). In what follows D will be one of the following operators: $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, or $\frac{\partial}{\partial y}$. Differentiation of the original system yields the following equations for $D\tau_1$, $D\tau_2$, Du, Dp_1, Dp_2, and Dd:

$$\frac{\partial}{\partial t}\left(D\tau_1\right) - \frac{\partial}{\partial x}\left(Du\right) = -\frac{\partial}{\partial t}\left(Dp_1\right), \tag{3.27}$$

$$\frac{\partial}{\partial t}\left(D\tau_2\right) - \frac{\partial}{\partial y}\left(Du\right) = -\frac{\partial}{\partial t}\left(Dp_2\right), \tag{3.28}$$

$$\frac{\partial}{\partial t}\left(Du\right) - \frac{\partial}{\partial x}\left(D\tau_1 + \frac{\beta}{\mu+\beta}Dp_1\right) - \frac{\partial}{\partial y}\left(D\tau_2 + \frac{\beta}{\mu+\beta}Dp_2\right) = 0, \tag{3.29}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(Dp_1\right) &= \frac{\tau_1}{\epsilon\sqrt{\tau_1^2+\tau_2^2}}H\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)\left(\frac{\tau_1 D\tau_1 + \tau_2 D\tau_2}{\sqrt{\tau_1^2+\tau_2^2}} - \tau_y'(d)\, Dd\right)\\ &+ \frac{\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)_+}{\epsilon\left(\tau_1^2+\tau_2^2\right)^{3/2}}\left(\tau_2^2 D\tau_1 - \tau_1\tau_2 D\tau_2\right),\end{aligned} \tag{3.30}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(Dp_2\right) &= \frac{\tau_2}{\epsilon\sqrt{\tau_1^2+\tau_2^2}}H\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)\left(\frac{\tau_1 D\tau_1 + \tau_2 D\tau_2}{\sqrt{\tau_1^2+\tau_2^2}} - \tau_y'(d)\, Dd\right)\\ &+ \frac{\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)_+}{\epsilon\left(\tau_1^2+\tau_2^2\right)^{3/2}}\left(\tau_1^2 D\tau_2 - \tau_1\tau_2 D\tau_1\right),\end{aligned} \tag{3.31}$$

and

$$\frac{\partial}{\partial t}(Dd) = \frac{H\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)}{\epsilon}\left(\frac{\tau_1 D\tau_1 + \tau_2 D\tau_2}{\sqrt{(\tau_1^2)+(\tau_2)^2}} - \tau_y'(d)\, Dd\right). \tag{3.32}$$

In equations (3.30) - (3.32)

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}. \tag{3.33}$$

If we now multiply (3.27) by $D\tau_1$, (3.28) by $D\tau_2$, (3.29) by Du, (3.32) by $\frac{\mu}{\mu+\beta}\tau_y'(d)\, Dd$ and add the resulting expressions we find that (3.1) and (3.2) hold where now

$$f = \frac{1}{2}\left(\frac{\mu\left((D\tau_1)^2+(D\tau_2)^2\right)}{\mu+\beta} + \frac{\beta\left((D(\tau_1+p_1))^2+(D(\tau_2+p_2))^2\right)}{\mu+\beta} + (Du)^2 + \frac{\mu}{\mu+\beta}\tau_y'(d)(Dd)^2\right), \tag{3.34}$$

$$q_1 = Du\left(\frac{\mu}{\mu+\beta}D\tau_1 + \frac{\beta}{\mu+\beta}D(\tau_1+p_1)\right), \tag{3.35}$$

$$q_2 = Du\left(\frac{\mu}{\mu+\beta}D\tau_2 + \frac{\beta}{\mu+\beta}D(\tau_2+p_2)\right), \tag{3.36}$$

and

$$\begin{aligned} g &= \frac{\mu\tau_y''(d)}{2(\mu+\beta)}d_t(Dd)^2 \\ &- \frac{\mu}{\epsilon(\mu+\beta)}H\left(\sqrt{\tau_1^2+\tau_2^2}-\tau_y(d)\right)\left(\frac{\tau_1 D\tau_1+\tau_2 D\tau_2}{\sqrt{\tau_1^2+\tau_2^2}} - \tau_y'(d)\, Dd\right)^2 \\ &- \frac{\mu}{\epsilon(\mu+\beta)}\frac{\left(\sqrt{\tau_1^2+\tau_2^2}-\tau_y(d)\right)_+}{(\tau_1^2+\tau_2^2)^{3/2}}(\tau_2 D\tau_1 - \tau_1 D\tau_2)^2. \end{aligned} \tag{3.37}$$

The hypothesis that $\tau_y'' \leq 0$ and $d_t \geq 0$ then guarantee that $g \leq 0$. Moreover, the boundary density satisfies

$$\begin{aligned} q_1\cos\theta + q_2\sin\theta - f \leq &-\frac{\mu\left(|Du| - \sqrt{(D\tau_1)^2+(D\tau_2)^2}\right)^2}{2(\mu+\beta)} \\ &-\frac{\beta\left(|Du| - \sqrt{(D(\tau_1+p_1))^2+(D(\tau_2+p_2))^2}\right)^2}{2(\mu+\beta)} - \frac{\tau_y'(d)(Dd)^2}{2} \leq 0 \end{aligned} \tag{3.38}$$

and the latter two inequalities along with (3.2) yield the desired derivative bounds.

We conclude this section with an examination of the behavior of our system as the small parameter ϵ approaches zero from above. In what follows we let

$$\Omega\left(x_0, y_0, R, t-s\right) = \left\{(x,\ y) \,|\, \sqrt{\left(x-x_0\right)^2 + \left(y-y_0\right)^2} \leq R+t-s\right\} \tag{3.39}$$

when $0 \leq s \leq t$ and

$$B\left(x_0,\ y_0,\ R,\ t\right) = \left\{(x,y,s) \,|\, \sqrt{\left(x-x_0\right)^2 + \left(y-y_0\right)^2} \leq R+t-s, 0 \leq s \leq t\right\}. \tag{3.40}$$

The a-priori estimates associated with (3.2) when f is given by (3.3) or (3.34) and D is one of the operators $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, or $\frac{\partial}{\partial y}$ guarantee that if the initial values for a family $\left(\tau_1^\epsilon,\ \tau_2^\epsilon,\ p_1^\epsilon,\ p_2^\epsilon,\ d^\epsilon, u^\epsilon\right)$ of distributional solutions are in L_2^{loc} independently of ϵ, then we may choose a sequence ϵ_i, $i = 1,\ 2, \ldots$, with the ϵ_i's decreasing to zero and limit functions $\left(\tau_1^0,\ \tau_2^0,\ p_1^0,\ p_2^0,\ d^0,\ u^0\right)$ with the following properties:

(i) For any (x_0, y_0), $R > 0$, and $t > 0$ the sequence $\left(\tau_1^{\epsilon_i},\ \tau_2^{\epsilon_i},\ p_1^{\epsilon_i},\ p_2^{\epsilon_i},\ d^{\epsilon_i},\ u^{\epsilon_i}\right)$ converge strongly in $L_2\left(B\left(x_0, y_0, R, t\right)\right)$ to $\left(\tau_1^0,\ \tau_2^0,\ p_1^0,\ p_2^0,\ d^0,\ u^0\right)$. Moreover, the limit functions have weak t, x, and y derivatives and the sequences $D\left(\tau_1^{\epsilon_i}\tau_2^{\epsilon_i}, p_1^{\epsilon_i},\ p_2^{\epsilon_i},\ d^{\epsilon_i},\ u^{\epsilon_i}\right)$ converge weakly in $L_2\left(B\left(x_0, y_0, R, t\right)\right)$ to $D\left(\tau_1^0,\ \tau_2^0,\ p_1^0,\ p_2^0,\ d^0,\ u^0\right)$ where again $D = \frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$.

(ii) The hypotheses (2.28) and (2.35) on the yield stress further guarantee that $\tau_y\left(d^{\epsilon_i}\right)$ converges strongly in $L_2\left(B\left(x_0,\ y_0, R,\ t\right)\right)$ to $\tau_y\left(d^0\right)$ and that $D\tau_y\left(d^{\epsilon_i}\right) = \tau_y'\left(d^{\epsilon_i}\right)Dd^{\epsilon_i}$ converges weakly in $L_1\left(B\left(x_0,\ y_0, R,\ t\right)\right)$ to $D\tau_y\left(d^0\right) = \tau_y'\left(d^0\right)Dd^0$ where once again $D = \frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$. Moreover, equations (2.30) -(2.41) and the convergence results (i) above imply that $\tau_m^{\epsilon_i}\frac{\partial p_n^{\epsilon_i}}{\partial t}$, m and $n = 1$ and 2, and $\sqrt{\left(\tau_1^{\epsilon_i}\right)^2 + \left(\tau_2^{\epsilon_i}\right)^2}\frac{\partial d}{\partial t}^{\epsilon_i}$ converge weakly in $L_1\left(B\left(x_0,\ y_0,\ R,\ t\right)\right)$ to $\tau_m^0\frac{\partial p_n^0}{\partial t}$, m and $n = 1$ and 2, and $\sqrt{\left(\tau_1^0\right)^2 + \left(\tau_2^0\right)^2}\frac{\partial d^0}{\partial t}$ respectively.

Finally, the identities

$$\tau_1^{\epsilon_i}\frac{\partial p_1^{\epsilon_i}}{\partial t} + \tau_2^{\epsilon_i}\frac{\partial p_2^{\epsilon_i}}{\partial t} = \sqrt{\left(\tau_1^{\epsilon_i}\right)^2 + \left(\tau_2^{\epsilon_i}\right)^2}\frac{\partial d}{\partial t}^{\epsilon_i} \tag{3.41}$$

$$\tau_2^{\epsilon_i}\frac{\partial p_1^{\epsilon_i}}{\partial t} - \tau_1^{\epsilon_i}\frac{\partial p_2^{\epsilon_i}}{\partial t} = 0 \tag{3.42}$$

guarantee that the limit functions satisfy

$$\tau_1^0\frac{\partial p_1^0}{\partial t} + \tau_2^0\frac{\partial p_1^0}{\partial t} = \sqrt{\left(\tau_1^0\right)^2 + \left(\tau_2^0\right)^2}\frac{\partial d^0}{\partial t} \tag{3.43}$$

and

$$\tau_2^0\frac{\partial p_1^0}{\partial t} - \tau_1^0\frac{\partial p_2^0}{\partial t} = 0. \tag{3.44}$$

(iii) For any $\delta > 0$, the measure

$$m\left(\left\{(x,y,s)\ \in\ B\left(x_0, y_0, R, t\right) | \sqrt{\left(\tau_1^0\right)^2 + \left(\tau_2^0\right)^2} - \tau_y\left(d^0\right) \geq \delta\right\}\right) = 0. \tag{3.45}$$

The last identity implies that

$$(3.46)\quad \begin{aligned} & m\left(\left\{(x,y,s)\ \in\ B\left(x_0,y_0,R,t\right) | \sqrt{\left(\tau_1^0\right)^2+\left(\tau_2^0\right)^2}-\tau_y\left(d^0\right)\geq 0\right\}\right) \\ = \ & m\left(\left\{(x,y,s)\ \in\ B\left(x_0,y_0,R,t\right) | \sqrt{\left(\tau_1^0\right)^2+\left(\tau_2^0\right)^2}-\tau_y\left(d^0\right)= 0\right\}\right). \end{aligned}$$

In what follows we shall refer to

$$\left\{(x,y,s)\ \in\ B\left(x_0,y_0,R,t\right) | \sqrt{\left(\tau_1^0\right)^2+\left(\tau_2^0\right)^2}-\tau_y\left(d^0\right)= 0\right\}$$

as the yield set. On open subsets $\mathcal{U}$ of the yield set we introduce Θ by

$$(3.47)\quad \tau_1^0=\tau_y\left(d^0\right)\cos\Theta \text{ and } \tau_2^0=\tau_y\left(d^0\right)\sin\Theta.$$

The relationships (3.41) and (3.42) then imply that

$$(3.48)\quad \cos\Theta\frac{\partial p_1^0}{\partial t}+\sin\Theta\frac{\partial p_2^0}{\partial t}=\frac{\partial d^0}{\partial t}$$

and

$$(3.49)\quad \sin\Theta\frac{\partial p_1^0}{\partial t}-\cos\Theta\frac{\partial p_2^0}{\partial t}=0.$$

Since the weak limits $\left(\tau_1^0,\ \tau_2^0,\ p_1^0,\ p_2^0, u^0\right)$ also satisfy the conservation laws (2.36) - (2.38) we find that in $\mathcal{U}$ the following equations are satisfied in the distributional sense

$$(3.50)\quad \cos\Theta\tau_y'\left(d^0\right)\frac{\partial d^0}{\partial t}-\tau_y\left(d^0\right)\sin\Theta\frac{\partial\Theta}{\partial t}-\frac{\partial u^0}{\partial x}=-\frac{\partial p_1^0}{\partial t}$$

$$(3.51)\quad \sin\Theta\tau_y'\left(d^0\right)\frac{\partial d^0}{\partial t}+\tau_y\left(d^0\right)\cos\Theta\frac{\partial\Theta}{\partial t}-\frac{\partial u^0}{\partial y}=-\frac{\partial p_2^0}{\partial t}$$

and

$$(3.52)\quad \frac{\partial u^0}{\partial t}-\frac{\partial}{\partial x}\left(\tau_y\left(d^0\right)\cos\Theta+\frac{\beta}{\mu+\beta}p_1^0\right)-\frac{\partial}{\partial y}\left(\tau_y\left(d^0\right)\sin\Theta+\frac{\beta}{\mu+\beta}p_2^0\right)=0.$$

These equations represent a closed system for $\left(p_1^0,\ p_2^0,\ d^0,\ u^0,\ \Theta\right)$ on the yield surface. They imply that (3.52) holds and that

$$(3.53)\quad \left(1+\tau_y'\left(d^0\right)\right)\frac{\partial d^0}{\partial t}=\cos\Theta\frac{\partial u^0}{\partial x}+\sin\Theta\frac{\partial u^0}{\partial y},$$

$$(3.54)\quad \frac{\partial\Theta}{\partial t}=\frac{1}{\tau_y\left(d^0\right)}\left(-\sin\Theta\frac{\partial u^0}{\partial x}+\cos\Theta\frac{\partial u^0}{\partial y}\right),$$

$$(3.55)\quad \frac{\partial p_1^0}{\partial t}=\frac{\cos\Theta}{1+\tau_y'\left(d^0\right)}\left(\cos\Theta\frac{\partial u^0}{\partial x}+\sin\Theta\frac{\partial u^0}{\partial y}\right),$$

and

(3.56) $$\frac{\partial p_2^0}{\partial t} = \frac{\sin\Theta}{1+\tau_y'(d^0)}\left(\cos\Theta\frac{\partial u^0}{\partial x} + \sin\Theta\frac{\partial u^0}{\partial y}\right).$$

Not surprisingly, we find that in open sets $\mathcal{E}$ of $\left\{(x,\ y,s)\ \in\ B(x_0,y_0,R,t)\,|0<\sqrt{(\tau_1^0)^2+(\tau_2^0)^2}<\tau_y\left(d^0\right)\right\}$ the weak limits satisfy the elasticity equations

(3.57) $$\frac{\partial \tau_1^0}{\partial t} - \frac{\partial u^0}{\partial x} = 0,$$

(3.58) $$\frac{\partial \tau_2^0}{\partial t} - \frac{\partial u^0}{\partial y} = 0.$$

(3.59) $$\frac{\partial u^0}{\partial t} - \frac{\partial \tau_1^0}{\partial x} - \frac{\partial \tau_2^0}{\partial y} = \frac{\beta}{\mu+\beta}\left(\frac{\partial p_1^0}{\partial x} + \frac{\partial p_2^0}{\partial y}\right)$$

and

(3.60) $$\frac{\partial p_1^0}{\partial t} = \frac{\partial p_2^0}{\partial t} = \frac{\partial d^0}{\partial t} = 0.$$

The assertions in part (i) which pertain to $(\tau_1^{\epsilon_i}, \tau_2^{\epsilon_i}, p_1^{\epsilon_i}, p_2^{\epsilon_i}, u^{\epsilon_i})$ follow from (3.2), (3.3), and (3.34). Equations (2.28), (2.35), (3.2), and $\frac{\partial d}{\partial t}^{\epsilon_i} \geq 0$ imply that the d^{ϵ_i}'s are bounded in $L_1\left(B\left(x_0,y_0,R,t\right)\right)$. Their L_2 boundedness follows from the inequality

(3.61) $$\left(d^{\epsilon_i}\left(x\ y,s\right)\right)^2 \leq 2\left(d\left(x,y,0\right)\right)^2 + 2s\int_0^s \left(\frac{\partial d}{\partial \eta}^{\epsilon_i}\right)^2 (x,y,\eta)\,d\eta$$

which in turn implies that

$$\int_0^t \left(\iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2}\,\leq\, R+t-s\}} \left(d^{\epsilon_i}\left(x,y,s\right)\right)^2 dxdy \right) ds$$

$$\leq 2t \iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2}\,\leq\, R+t\}} d^2\left(x,y,0\right) dxdy$$

$$+2\int_0^t s\left(\int_0^s \left(\iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2}\,\leq\, R+t-\eta\}} \left(\frac{\partial d}{\partial \eta}^{\epsilon_i}\right)^2 (x,y,\eta)\,dxdy \right) d\eta \right) ds$$

$$\leq 2t \iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2}\,\leq\, R+t\}} d^2\left(x,y,0\right) dxdy$$

$$+2t^2\int_0^t \left(\iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2}\,\leq\, R+t-\eta\}} \left(\frac{\partial d}{\partial \eta}^{\epsilon_i}\right)^2 (x,y,\eta)\,dxdy \right) d\eta$$

(3.62)

As noted previously, the assertions of (ii) follow directly from those of (i) and the governing equations (2.39) - (2.41).

The veracity of (iii) follows from the inequality

$$
\begin{aligned}
0 &\leq \delta m\{(x,y,s) \in B(x_0,y_0,R,t) | \sqrt{(\tau_1^0)^2+(\tau_2^0)^2} - \tau_y(d^o) \geq \delta\} \\
&\leq \iiint_{\{(x,y,s)\in B(x_0,y_0,R,t) | \sqrt{(\tau_1^0)^2+(\tau_2^0)^2}-\tau_y(d^0)\geq\delta\}} \left(\sqrt{(\tau_1^0)^2+(\tau_2^0)^2} - \tau_y(d^0)\right) dxdyds \\
&= \iiint_{\{(x,y,s)\in B(x_0,y_0,R,t) | \sqrt{(\tau_1^0)^2+(\tau_2^0)^2}-\tau_y(d^0)\geq\delta\}} \left(\sqrt{(\tau_1^0)^2+(\tau_2^0)^2} - \sqrt{(\tau_1^{\epsilon_i})^2+(\tau_2^{\epsilon_i})^2}\right) dxdyds \\
&+ \iiint_{\{(x,y,s)\in B(x_0,y_0,R,t) | \sqrt{(\tau_1^0)^2+(\tau_2^0)^2}-\tau_y(d^0)\geq\delta\}} \left(\tau_y(d^{\epsilon_i}) - \tau_y(d^0)\right) dxdyds \\
&+ \iiint_{\{(x,y,s)\in B(x_0,y_0,R,t) | \sqrt{(\tau_1^0)^2+(\tau_2^0)^2}-\tau_y(d^0)\geq\delta\}} \left(\sqrt{(\tau_1^{\epsilon_i})^2+(\tau_2^{\epsilon_i})^2} - \tau_y(d^{\epsilon_i})\right) dxdyds,
\end{aligned}
\tag{3.63}
$$

the strong convergence results of part (i) which guarantee that the first two integrals on the right-hand side of (3.63) converge to zero as the ϵ_i's tend to zero, and from the observation that the third integral is bounded from above by

$$
\iiint_{B\ (x_0,y_0,R,t)} \left(\sqrt{(\tau_1^{\epsilon_i})^2+(\tau_2^{\epsilon_i})^2} - \tau_y\left(d^{\epsilon_i}\right)\right)_+ dxdyds
$$

which in turn is bounded by

$$
(m\left(B\left(x_0,y_0,R,t\right)\right))^{1/2} \left(\iiint_{B\ (x_0,y_0,R,t)} \left(\left(\sqrt{(\tau_1^{\epsilon_i})^2+(\tau_2^{\epsilon_i})^2} - \tau_y\left(d^{\epsilon_i}\right)\right)_+\right)^2 dxdyds\right)^{1/2}.
\tag{3.64}
$$

The identity (3.2) with f given by (3.3) and g by (3.6) guarantees that

$$
\begin{aligned}
&\iiint_{B\ (x_0,y_0,R,t)} \left(\left(\sqrt{(\tau_1^{\epsilon_i})^2+(\tau_2^{\epsilon_i})^2} - \tau_y\left(d^{\epsilon_i}\right)\right)_+\right)^2 dxdyds \leq \\
&\frac{\epsilon_i(\mu+\beta)}{\mu} \iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2}\leq R+t\}} \left(\frac{\mu\left(\tau_1^2+\tau_2^2\right)}{2(\mu+\beta)} + \frac{\beta\left((\tau_1+p_1)^2+(\tau_2+p_2)^2\right)^2}{2(\mu+\beta)} + \frac{u^2}{2} + \frac{\mu}{\mu+\beta}\int_0^d \tau_y\left(\eta\right)d\eta\right)(x,y,0)\,dxdy
\end{aligned}
\tag{3.65}
$$

and (3.64) and (3.65) imply that the third integral on the right-hand side of (3.63) tends to zero as ϵ_i tends to zero.

To establish (3.57)-(3.60) in open subsets $\mathcal{E}$ of $\left\{(x,y,s)\in B\left(x_0,y_0,R\ t\right)|0<\sqrt{\left(\tau_1^0\right)^2+\left(\tau_2^0\right)^2}<\tau_y\left(d^0\right)\right\}$ it suffices to show that $\frac{\partial d^0}{\partial t}=0$ on $\mathcal{E}$. Equations (3.43) and (3.44) will then guarantee that $\frac{\partial p_1^0}{\partial t}=\frac{\partial p_2^0}{\partial t}=0$ and these identities, along with (2.36)-(2.38) will guarantee that (3.57)-(3.60) hold.

In what follows we let $\delta>0$,

$$E_\delta=\left\{(x,\ y,\ s)\in B\left(x_0,\ y_0,R,\ t\right)|\sqrt{\left(\tau_1^0\right)^2+\left(\tau_2^0\right)^2}-\tau_y\left(d^0\right)\le-\delta<0\right\}. \tag{3.66}$$

and observe that

$$\left|\iint_{E_\delta}\frac{\partial d^0}{\partial s}dxdyds\right|\le\left|\iiint_{E_\delta}\left(\frac{\partial d^0}{\partial s}-\frac{\partial d^{\epsilon_i}}{\partial s}\right)dxdyds\right|+\left|\iiint_{E_\delta\cap\left\{\frac{\partial d^{\epsilon_i}}{\partial s}>0\right\}}\frac{\partial d^{\epsilon_i}}{\partial s}dxdyds\right| \tag{3.67}$$

The weak convergence of $\frac{\partial d^{\epsilon_i}}{\partial s}$ to $\frac{\partial d^0}{\partial s}$ guarantees that the first integral on the right-hand side of (3.67) may be made arbitrarily small. We estimate the second integral by

$$\left(\iiint_{B\left(x_0,y_0,R,t\right)}\left(\frac{\partial d^{\epsilon_i}}{\partial s}\right)^2dxdyds\right)^{1/2}m\left(E_\delta\cap\left\{\frac{\partial d^{\epsilon_i}}{\partial s}>0\right\}\right)^{1/2}.$$

That the first factor is bounded follows from (3.2) with f given by (3.34) and $D=\frac{\partial}{\partial t}$. Thus, it suffices to show that $\lim_{i\to\infty}m\left(E_\delta\cap\left\{\frac{\partial d^{\epsilon_i}}{\partial s}>0\right\}\right)=0$. To establish this assertion we observe that $\left\{\frac{\partial d^{\epsilon_i}}{\partial s}>0\right\}=\left\{\sqrt{\left(\tau_1^{\epsilon_i}\right)^2+\left(\tau_2^{\epsilon_i}\right)^2}-\tau_y\left(d^{\epsilon_i}\right)>0\right\}$, and that

$$\begin{aligned}\delta m\left(E_\delta\cap\left\{\tfrac{\partial d^{\epsilon_i}}{\partial s}>0\right\}\right)&\le\iiint_{E_\delta\cap\left\{\frac{\partial d^{\epsilon_i}}{\partial s}>0\right\}}\left(\tau_y\left(d^0\right)-\sqrt{\left(\tau_1^0\right)^2+\left(\tau_2^0\right)^2}\right)dxdyds\\&\le\iiint_{E_\delta\cap\left\{\frac{\partial d^{\epsilon_i}}{\partial s}>0\right\}}\left(\tau_y\left(d^0\right)-\tau_y\left(d^{\epsilon_i}\right)\right)dxdyds\\&+\iiint_{E_\delta\cap\left\{\frac{\partial d^{\epsilon_i}}{\partial s}>0\right\}}\left(\sqrt{\left(\tau_1^{\epsilon_i}\right)^2+\left(\tau_2^{\epsilon_i}\right)^2}-\sqrt{\left(\tau_1^0\right)^2+\left(\tau_2^0\right)^2}\right)dxdyds.\end{aligned} \tag{3.68}$$

The strong convergence results of (i) imply that the latter two integrals tend to zero as the ϵ_i's tend to zero thereby yielding $\lim_{i\to\infty} m\left(E_\delta \cap \left\{\frac{\partial d}{\partial s}^{\epsilon_i} > 0\right\}\right) = 0$.

4. Computational Experiments. In this section we present some computational experiments for the dimensionless system (2.36) - (2.41) when the normalized yield stress is given by

$$\tau_y = 1 + c_1 + (c_1 - c_2)\, d - \frac{c_1}{1+d} \tag{4.1}$$

and

$$0 < c_2 < c_1. \tag{4.2}$$

Since the flows associated with this system may be quite complicated we restrict our attention to problems with Riemann type data where

$$(\tau_1, \tau_2, p_1, p_2, d)\left(x, y, 0^+\right) \equiv (0,0,0,0,0) \tag{4.3}$$

and

$$u\left(x, y, 0^+\right) = \begin{cases} u_0, & \text{if} \quad xy > 0 \\ -u_0, & \text{if} \quad xy < 0 \end{cases} \tag{4.4}$$

where u_0 is a constant. The solutions generated by this data exhibit a high degree of symmetry and thus when visualizing them we may confine our attention to one of the four quadrants $(k-1)\frac{\pi}{2} \le \theta \le \frac{k\pi}{2}$,
$k = 1, \ \ldots, 4$. The data for $u(x, y, 0^+)$ is not H_1^{loc} but the functions

$$u^h(x,y,0^+) = \begin{cases} u_0, \text{ if } x > \frac{h}{2} \text{ and } y > \frac{h}{2} \text{ or } x < -\frac{h}{2} \text{ and } y < -\frac{h}{2}, \\ -\quad u_0, \text{ if } x < -\frac{h}{2} \text{ and } y > \frac{h}{2} \text{ or } x > \frac{h}{2} \text{ and } y < -\frac{h}{2}, \\ -\quad u_0 + \frac{2u_0}{h}\left(x + \frac{h}{2}\right), \text{ if } -\frac{h}{2} \le x \le h/2 \text{ and } y \ge \frac{h}{2}, \\ u_0 - \frac{2u_0}{h}\left(x + \frac{h}{2}\right), \text{ if } -\frac{h}{2} \le x \le \frac{h}{2} \text{ and } y \le -\frac{h}{2}, \\ u_0 - \frac{2u_0}{h}\left(y + \frac{h}{2}\right), \text{ if } -\frac{h}{2} \le y \le \frac{h}{2} \text{ and } x \le -\frac{h}{2}, \\ -\quad u_0 + \frac{2u_0}{h}\left(y + \frac{h}{2}\right), \text{ if } -\frac{h}{2} \le y \le \frac{h}{2} \text{ and } x \ge \frac{h}{2}, \\ u_0 - \frac{2u_0}{h}\left(x + \frac{h}{2}\right) - \frac{2u_0}{h}\left(y + \frac{h}{2}\right) + \frac{3u_0}{h^2}\left(x + \frac{h}{2}\right)\left(y + \frac{h}{2}\right), \\ \quad \text{if } -\frac{h}{2} \le x \le \frac{h}{2} \text{ and } -\frac{h}{2} \le y \le \frac{h}{2}. \end{cases} \tag{4.5}$$

are H_1^{loc} and this, together with our L_2^{loc} contractivity estimate of the previous section, is sufficient to guarantee that the solution to (2.36) - (2.41) taking on the data (4.3) and (4.5) has a strong L_2^{loc} limit as $h \to 0^+$ which satisfies (2.36) - (2.41), (4.3), and (4.4). This limiting behavior is true when $\epsilon > 0$ is fixed and also in the $\epsilon = 0^+$ limit when the rate independent equations (3.52) - (3.60) govern.

Our updating algorithm is as follows. We assume we are given $(\tau_1, \tau_2, p_1, p_2, d, u)^N (x, y)$ on the $x - y$ plane. These represent the approximate solution at time $t = (N - 1/2)\,\delta$ where δ is our time step and $N \geq 1$. To advance these data we successively solve the following systems:

$$(4.6) \qquad \begin{cases} \dfrac{\partial \tau_1}{\partial t} - \dfrac{\partial u}{\partial x} = 0, \quad \dfrac{\partial \tau_2}{\partial t} = 0, \quad \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}\left(\tau_1 + \dfrac{\beta}{\mu + \beta} p_1\right) = 0, \\ \text{and} \quad \dfrac{\partial p_1}{\partial t} = \dfrac{\partial p_2}{\partial t} = \dfrac{\partial d}{\partial t} = 0, \quad 0 \leq t \leq \delta, \end{cases}$$

$$(4.7) \qquad \begin{cases} \dfrac{\partial \tau_1}{\partial t} = 0, \quad \dfrac{\partial \tau_2}{\partial t} - \dfrac{\partial u}{\partial y} = 0, \quad \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial y}\left(\tau_2 + \dfrac{\beta}{\mu + \beta} p_2\right) = 0, \\ \text{and} \quad \dfrac{\partial p_1}{\partial t} = \dfrac{\partial p_2}{\partial t} = \dfrac{\partial d}{\partial t} = 0, \quad 0 \leq t \leq \delta, \end{cases}$$

and

$$(4.8) \quad \begin{cases} \dfrac{\partial}{\partial t}(\tau_1 + p_1) = \dfrac{\partial}{\partial t}(\tau_2 + p_2) = \dfrac{\partial u}{\partial t} = 0, \\ \dfrac{\partial p_1}{\partial t} = \dfrac{\tau_1\left(\sqrt{\tau_1^2 + \tau_2^2} - \tau_y(d)\right)_+}{\epsilon\sqrt{\tau_1^2 + \tau_2^2}}, \qquad \dfrac{\partial p_2}{\partial t} = \dfrac{\tau_2\left(\sqrt{\tau_1^2 + \tau_2^2} - \tau_y(d)\right)_+}{\epsilon\sqrt{\tau_1^2 + \tau_2^2}} \\ \text{and} \quad \dfrac{\partial d}{\partial t} = \dfrac{\left(\sqrt{\tau_1^2 + \tau_2^2} - \tau_y(d)\right)_+}{\epsilon}, \quad 0 \leq t \leq \delta. \end{cases}$$

Our principal reason for this splitting is that the systems (4.6) and (4.7) may be updated exactly by elementary characteristic methods and (4.8) may be easily integrated to any desired order of accuracy via Runge Kutta methods.

For (4.6) we use $(\tau_1, \tau_2, p_1, p_2, d, u)^N$ as initial data and let $\left(\tau_1^1, \tau_2^1, p_1^1, p_2^1, d^1, u^1\right)$ denote the solution to (4.6) with these data at time $t = \delta$. We then solve (4.7) using $\left(\tau_1^1, \tau_2^1, p_1^1, p_2^1, d, u^1\right)$ as initial data and let $\left(\tau_1^2, \tau_2^2, p_1^2, p_2^2, d^2, u^2\right)$ denote the solution at $t = \delta$. We next repeat the process but first solve (4.7) with the data $(\tau_1, \tau_2, p_1, p_2, d, u)^N$ and let $\left(\tau_1^3, \tau_2^3, p_1^3, p_2^3, d^3, u^3\right)$ denote the solution at $t = \delta$. We then use $\left(\tau_1^3, \tau_2^3, p_1^3, p_2^3, d^3, u^3\right)$ as data for (4.6) and let $\left(\tau_1^4, \tau_2^4, p_1^4, p_2^4, d^4, u^4\right)$ denote the solution at $t = \delta$. Finally we average the approximate solutions indexed by (2) and (4) and denote the result as $\left(\tau_1^5, \tau_2^5, p_1^5, p_2^5, d^5, u^5\right)$; that is

$$(4.9) \ \left(\tau_1^5, \tau_2^5, p_1^5, p_2^5, d^5, u^5\right) = \frac{1}{2}\left(\tau_1^2 + \tau_1^4, \tau_2^2 + \tau_2^4, p_1^2 + p_1^4, p_2^2 + p_2^4, d^2 + d^4, u^2 + u^4\right).$$

We note that this particular approximation represents a second order update to the "elastic" wave equation:

$$
\begin{cases}
\frac{\partial \tau_1}{\partial t} - \frac{\partial u}{\partial x} = 0, \quad \frac{\partial \tau_2}{\partial t} - \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\tau_1 + \frac{\beta\, p_1}{\mu+\beta}\right) - \frac{\partial}{\partial y}\left(\tau_2 + \frac{\beta\, p_2}{\mu+\beta}\right) = 0, \\
\frac{\partial p_1}{\partial t} = \frac{\partial p_2}{\partial t} = \frac{\partial d}{\partial t} = 0, \quad 0 \le t \le \delta
\end{cases}
\tag{4.10}
$$

taking on the data $(\tau_1, \tau_2, p_1, p_2, d, u)^N$ at $t = 0$ and does better than either of the approximates labeled 2 or 4; in particular solution symmetries are preserved via the averaging algorithm.

The final step in our algorithm involves solving (4.8) with the data $(\tau_1^5, \tau_2^5, p_1^5, p_2^5, d^5, u^5)$. Over the interval $0 \le t \le \delta$ we have

$$
\tau_1 + p_1 \equiv \tau_1^5 + p_1^5, \ \tau_2 + p_2 \equiv \tau_2^5 + p_2^5, \text{ and } u \equiv u^5 \tag{4.11}
$$

and

$$
\begin{cases}
\frac{\partial \tau_1}{\partial t} = \frac{\tau_1\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)_+}{\epsilon\sqrt{\tau_1^2+\tau_2^2}}, \qquad \frac{\partial \tau_2}{\partial t} = \frac{\tau_2\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)_+}{\epsilon\sqrt{\tau_1^2+\tau_2^2}} \\
\text{and} \quad \frac{\partial d}{\partial t} = \frac{\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)_+}{\epsilon}.
\end{cases}
\tag{4.12}
$$

If we let

$$
\tau_1 = J\cos\Theta \text{ and } \tau_2 = J\sin\Theta, \tag{4.13}
$$

then equation (4.12) implies

$$
J + d \equiv J^5 + d^5 \quad \text{and} \quad \Theta \equiv \Theta^5, \quad 0 \le t \le \delta \tag{4.14}
$$

and

$$
\frac{\partial d}{\partial t} = \frac{\left(J^5 + d^5 - d - \tau_y(d)\right)_+}{\epsilon}, \quad 0 \le t \le \delta\ . \tag{4.15}
$$

In (4.14), $J^5 = \sqrt{(\tau_1^5)^2 + (\tau_2^5)^2}$ and $0 \le \Theta^5 < 2\pi$ satisfies

$$
\cos\Theta^5 = \frac{\tau_1^5}{J^5} \text{ and } \sin\Theta^5 = \frac{\tau_2^5}{J^5}. \tag{4.16}
$$

In what follows we let d^6 denote our update of (4.15) taking on the data d^5 at $t = 0$. Equation (4.14) then implies that

$$
\begin{cases}
J^6 = J^5 + d^5 - d^6, \quad \tau_1^6 = J^6\frac{\tau_1^5}{J^5}, \quad \tau_2^6 = J^6\frac{\tau_2^5}{J^5}, \\
u^6 = u^5, \quad p_1^6 = p_1^5 + \tau_1^5 - \tau_1^6, \quad \text{and} \quad p_2^6 = p_2^5 + \tau_2^5 - \tau_2^6.
\end{cases}
\tag{4.17}
$$

Our approximate solution at $t = (N + 1/2)\,\delta$ is given by the update labeled 6. To obtain the approximate solution at $t = \delta/2$ we merely solve (4.8) over the interval $0 \leq t \leq \delta/2$ with the prescribed initial data and take the value of their update at $t = \delta/2$ to be $(\tau_1, \tau_2, p_1, p_2, d, u)^1$.

The snapshots shown in Figure 1-18 were run with the normalized yield stress given by (4.1) when $c_1 = 1$ and $c_2 = .5$. The parameter u_0 defining the initial data was set to 1.5 and we chose $\delta = h = .01$. The parameter ϵ was set to 0.1. Surface renderings of $J = \sqrt{\tau_1^2 + \tau_2^2}, d$, and u are shown at times .3, .4, and .5.

The purely one dimensional nature of the solutions away from the corner where strong interactions take place is evident from these simulations and it is clear from these calculations that our algorithm captures the sharp contact discontinuities in J and u correctly. Our algorithm is easy to implement and avoids a number of thorny issues we would have to contend with if we tried to integrate the reduced $\epsilon = 0^+$ equations directly.

REFERENCES

[1] J. M. GREENBERG, *Models of elastic-perfectly plastic materials*, Euro. J. Appl. Math., 1(1990), pp. 131-150.

[2] J. M. GREENBERG, *The longtime behavior of elastic-perfectly plastic materials*, in Free Boundary Problems Involving Solids—Proceedings of the International Colloquium Free Boundary Problems: Theory and Applications, J. M. Chadam and H. K. Rasmussen, eds., Longman Scientific and Technical, Harlow, UK, 1993, pp. 28-34.

[3] J. M. GREENBERG AND ANNE NOURI *Antiplane Shearing Motions of a Visco-plastic Solid*, Siam J. Math. Anal., Vol. 24, No. 4, 1993, pp. 943-967.

[4] A. NOURI AND M. RASCLE, *A Global Existence and Uniqueness Theorem for a Model Problem in Dynamic Elasto-Plasticity with Isotropic Strain-Hardening*, Siam J. Math. Anal., Vol. 26, No. 4, 1995, pp. 850-868.

[5] M. MIHĂILESCU-SULICIU, I. SULICIU, AND W. O. WILLIAMS, *On Visco-plastic and Elastic-Plastic Oscillators*, Q. Applied Math., Vol. 47, No. 1, 1989, pp. 105-116.