21-131 Analysis I – PRACTICE FOR TEST 2
Test 1– 11:30 October 20, 2004
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ON ALL PROBLEMS, EXPLAIN HOW YOU REACH YOUR CONCLUSIONS

MATERIAL COVERED: CHAPTER I UP TO – AND EXCLUDING – Section 2.5

1 Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that

$$\bar{I}(f)[a, b] = \bar{I}(f)[a, c] + \bar{I}(f)[c, b]$$

for all $c \in (a, b)$.

Note: Here we are using the notation

$$\bar{I}(f)([a, b]) := \inf \left\{ \int_a^b t : t \geq f \text{ on } [a, b], \quad t \text{ is a step function} \right\}.$$

2 Let $g : [-2, 0] \to \mathbb{R}$ be a bounded function such that

$$g(x) = -g(-x - 2)$$

for all $x \in (-2, 1)$. Prove that $\bar{I}(g)[-2, 0] = 0$.

Note: The translation $f(x) := g(x - 1)$ defined on $[-1, 1]$ is *odd*, i.e. $f(x) = -f(-x)$ for all $x \in (-1, 0)$.

3 Prove that if f is a Lipschitz function on $[0, b]$ satisfying $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in [0, b]$ and for some $L > 0$ then

$$\int_0^b f(x)\, dx \leq f(0)b + \frac{b^2}{2}L.$$

4 Let $f : [-1, 4] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x^2 + 1 & -1 \leq x \leq 0, \\ 3 & 0 < x < 1, \\ x^3 & 1 \leq x \leq 4. \end{cases}$$

Prove that f is integrable and find $\int_{-1}^{4} f(x)\,dx$.

5 Let $f : [a, b] \to [0, +\infty)$ be integrable. Prove that f^2 is also integrable.

6 Assume that for all $n \in \mathbb{N}$ the following hold:

$$\sum_{i=1}^{n} \sqrt{i} \leq \frac{2}{3} n\sqrt{n} + C\sqrt{n}, \quad \sum_{i=1}^{n-1} \sqrt{i} \geq \frac{2}{3} n\sqrt{n} - C\sqrt{2n}$$

for some constant $c \in \mathbb{R}$. Show that $x \mapsto \sqrt{x}$ is integrable on $[0, 2]$ and that $\int_0^2 \sqrt{x}\,dx = \frac{4\sqrt{2}}{3}$.

7 Assume that for all $n \in \mathbb{N}$ the following hold:

$$\sum_{i=n}^{2n-1} i^{-2} \leq \frac{1}{2n} + \frac{3}{n^2}, \quad \sum_{i=n+1}^{2n} i^{-2} \geq \frac{1}{2n} - \frac{1}{n^2}.$$

Prove that $x \mapsto x^{-2}$ is integrable on $[1, 2]$ and that $\int_1^2 x^{-2}\,dx = \frac{1}{2}$.

8 Prove that

$$\frac{1}{4} \leq \int_0^1 \frac{x}{1 + x^4}\,dx \leq \frac{1}{2}$$

9 Let $f : [0, 2] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x & x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Is f integrable on $[0, 2]$?

10 Find the area of the region between the curves $y = x^3$ and $y = \sqrt{x}$ on $[0, 2]$.