21-131 Analysis I – PRACTICE FOR TEST 1
Test 1– 11:30 September 24, 2004
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ON ALL PROBLEMS, EXPLAIN HOW YOU REACH YOUR CONCLUSIONS

MATERIAL COVERED: ALL UP TO – AND EXCLUDING – CHAPTER 1 on Integral Calculus

1Justify your answers invoking the appropriate Axioms and Theorems proved in class.

(i) Let $a > 0$ and $b > 0$ with $a^3 < b^3$. Prove that $a < b$.

(ii) Given four positive real numbers a, b, c, d such that

$$\frac{a}{b} < \frac{c}{d}$$

show that

$$\frac{a}{b} < \frac{a+c}{b+d} \quad \text{and} \quad \frac{c}{d} > \frac{a+c}{b+d}.$$

2 Find the supremum and infimum (if they exist) of the sets

i) $\{\frac{1}{n}, \quad n \in \mathbb{N}\}$;

ii) $\{\frac{1}{n}, \quad n \in \mathbb{Z}, \quad n \neq 0\}$;

iii) $\{x \in \mathbb{R} : \quad x = 0 \quad \text{or } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}\}$;

iv) $\{(1 + \frac{1}{n})^n, \quad n \in \mathbb{N}\}$.

Which of the above sets have maximum and/or minimum?

3 Let S be a nonempty set of reals that has an upper bound. Define

$$T = \{2s + 1 : s \in S\}.$$

(i) Prove that $2\sup(S) + 1$ is an upper bound for T.

(ii) Prove that $2\sup up(S) + 1$ is the least upper bound for T.

4 If x and y are two real numbers, we define $\max(x, y)$ to be x if $x \geq y$ and y otherwise. Similarly, we define $\min(x, y)$ to be the smallest of x and y. Show that

$$\min(\min(x, y), z) = \min(x, \min(y, z))$$

5 Let A and B be two nonempty bounded subsets of $\mathbb{R}$.

(i) Show that if $\sup A < \inf B$ then A and B are disjoint sets.

(ii) Assume that $\inf A < \sup B$. Show that there exist $a \in A$ and $b \in B$ such that $a < b$.

6 Let $x \geq 1$. Show that there exists a positive integer $n \in \mathbb{N}$ such that

$$n^5 \leq x < (n+1)^5.$$

7 Show by mathematical induction that

$$\sum_{k=1}^{n} (4k - 1) = 2n^2 + n$$

holds for all positive integers n.