Review of numerical methods for derivative pricing.

1. Choice of a state process
2. Conditional expectation for Gaussian distribution
3. Discontinuous functions
4. Interpolation and approximation

1 Choice of a state process

Criteria:

a) Easy to implement

b) "General" form (should appear in a number of models)

Examples:

Black & Scholes:

$$dS_t = S_t\,(r\,dt + \sigma\,dW_t)$$

"Extended" Black model:

$$dS_t = S_t\,\big((\theta_t - \lambda_t \ln S_t)\,dt + \sigma\,dW_t\big)$$

Hull & White:

$$dr_t = (\theta_t - \lambda r_t)dt + \sigma dW_t$$

Black & Karasinski:

$$d\ln r_t = (\theta_t - \lambda \ln r_t)dt + \sigma dW_t$$

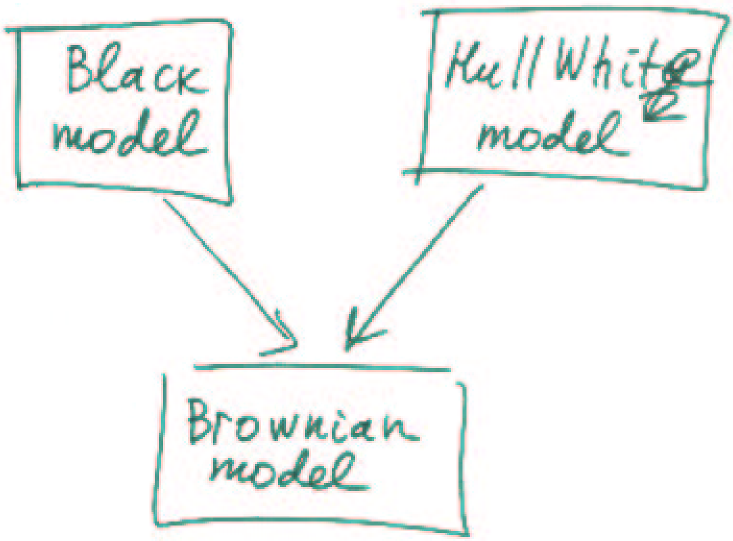
Black & Derman & Toy:

$$dr_t = r_t(\theta_t dt + \varkappa_t dW_t)$$

Common state process:

$$X_t = \int_0^t \sigma_u \, dW_u$$

Architecture of cfl library.



Remark on multi-factor models.

Example (FX model)

Sources of randomness:

1. FX rate.
2. Domestic interest rates.
3. Foreign interest rates.

<u>Remark 1</u>: Choose the # of factors according to the type of transaction.

- Example: (FX swap) 5

A: pays interest (fixed or float) in USD

B: pays interest (fixed or float) in Euro.

If we want to price FX swaption, then # of factors depends on the type of underlying swap:

16

Remark 2 Assume that we have fixed-float case. We need 2 factors or 2 Brownian motions.

Typical solution:

interest in foreign currency

$$dS_t = S_t\left((r_t - q)\,dt + \sigma\, dW_t\right)$$

$$dr_t = a(\theta_t - \lambda r_t)\,dt + \varkappa\, dB_t$$

Hull-White

Black

$$\rho = \frac{dW_t\, dB_t}{dt} \quad \text{: correlation coefficient}$$

However state processes are $\boxed{4}$

$$X_t^1 = W_t$$

$$X_t^2 = \int_0^t e^{\lambda t}\, dB_t$$

Hence,

$$dX_t^1\, dX_t^2 = \rho e^{\lambda t}\, dt$$

Correlation coëfficient becomes time dependent!

$\Longrightarrow$ difficult numerical implementation

Better construction: 8

$$\rho(t) = \frac{dW_t \, dB_t}{dt} = \rho e^{-\lambda t}$$

Then

$$dX_t^1 dX_t^2 = \rho \, dt$$

constant

$\Longrightarrow$ easy numerical scheme. We have model ~~with~~ where state process can be chosen as (Y_t^1, Y_t^2) two independent Brownian motions.

Conditional expectation 9

w.r.t. Gaussian distribution.

State process:

$$X_t = \int_0^t \sigma_u \, dW_u$$

deterministic function

Wiener process (Brownian motion)

To mplement the roll back operator we need to compute

the operator of
conditional expectation:

$$g(x_s) = \mathbb{E}_s\left[f(x_t)\right]$$

Given: $f = f(x)$

Compute: $g = g(x)$

If
$$T = \int_s^t \sigma^2(u)\,du$$
then
$$g(x) = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{2T}} f(y)\,dy$$

In numerical implementation the values of both functions: $f = f(x)$ (input) and $g = g(x)$ (output) are given on a grid:

h — grid step

Popular methods:

(a) Finite differences

(b) FFT (spectral)

Finite differences: 12

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = f(x) \end{cases}$$

Then $u(x,T) = g(x)$

δt : time step

δx : space step

Approximation for $\dfrac{\partial u}{\partial t}$:

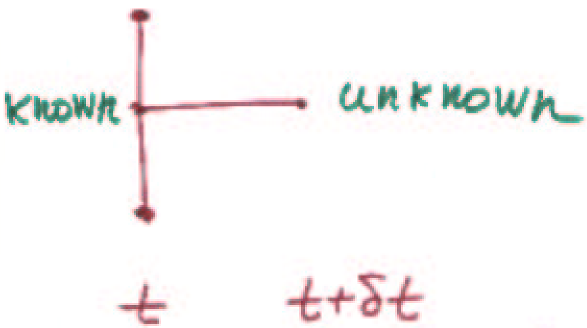
$$\frac{\partial u}{\partial t} \approx \frac{u(x, t+\delta t) - u(x,t)}{\delta t}$$

$$\text{known } t \longrightarrow t+\delta t \text{ unknown}$$

13

Explicit scheme :

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+\delta x, t) - 2u(x,t) + u(x-\delta x, t)}{(\delta x)^2}$$



$$u(x, t+\delta t) = (1-2p)\,u(x,t) + p\,\{u(x-\delta x, t) + u(x+\delta x, t)\}$$

where $p = \dfrac{\delta t}{2(\delta x)^2}$

14

Error analysis:

$$\text{Error} \asymp \delta t + (\delta x)^2$$

Stability analysis:

Definition:

Bounded input $\Longrightarrow$ Bounded output

Method: Take input:

$$f(x) = e^{ikx}$$

Then

$$u(x,t) = e^{ikx}\, g_k^{(t/\delta t)}$$

where

15

$$q_k = (1-2p) + p\left\{e^{-ik\delta x} + e^{ik\delta x}\right\} = 1 - 2p \sin^2 \frac{k\delta x}{2}$$

Bounded output

$$\Updownarrow$$

$$|q_k| \le 1, \quad \forall k$$

$$\Updownarrow$$

$$p = \frac{\delta t}{2(\delta x)^2} \le \frac{1}{2}$$

$$\Updownarrow$$

$$\delta t \le (\delta x)^2$$

16

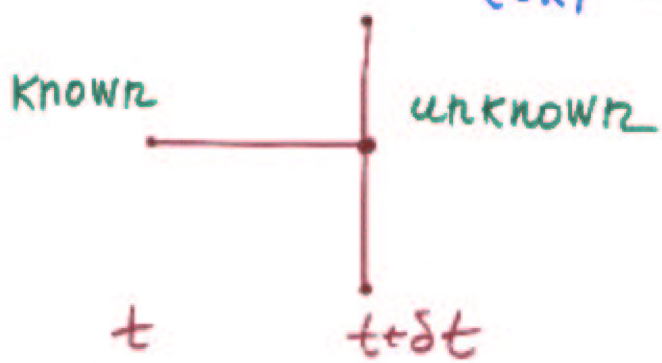
Consequence :
to double # of x
we have to quadriple
# of time layers.

Advantages:
- Great for smoothness
- All coefficients are positive

Disadvantages:
- Very slow

Implicit scheme:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+\delta x, t+\delta t) - 2u(x, t+\delta t) + u(x-\delta x, t+\delta t)}{(\delta x)^2}$$



$$u(x,t) = (1+2p)\,u(x, t+\delta t) - p\big(u(x+\delta x, t+\delta t) + u(x-\delta x, t+\delta t)\big)$$

$$p = \frac{\delta t}{2(\delta x)^2}$$

Tridiagonal system:

$$\begin{bmatrix} 1+2p & -p & & 0 \\ -p & \ddots & \ddots & \\ & \ddots & \ddots & -p \\ 0 & & -p & 1+2p \end{bmatrix} \begin{pmatrix} \\ u(t+\delta t) \\ \\ \end{pmatrix} =$$

$$= \begin{pmatrix} u(t) \\ \\ \\ \end{pmatrix}$$

19

Error analysis:

$$\text{Error} \asymp \delta t + (\delta x)^2$$

Stability analysis:

Input is $f(x) = e^{ikx}$

We have

$$u(x,t) = e^{ikx}\, q_k^{(t/\delta t)}$$

where

$$1 = (1+2p)\, q_k - p\, q_k \ast (e^{ik\delta x} + e^{-ik\delta x})$$

$$q_k = \frac{1}{1 + p \sin^2 \frac{k\delta x}{2}} \leq 1$$

20

Hence scheme is stable for any

$$p = \frac{\delta t}{2(\delta x)^2}$$

Advantages:

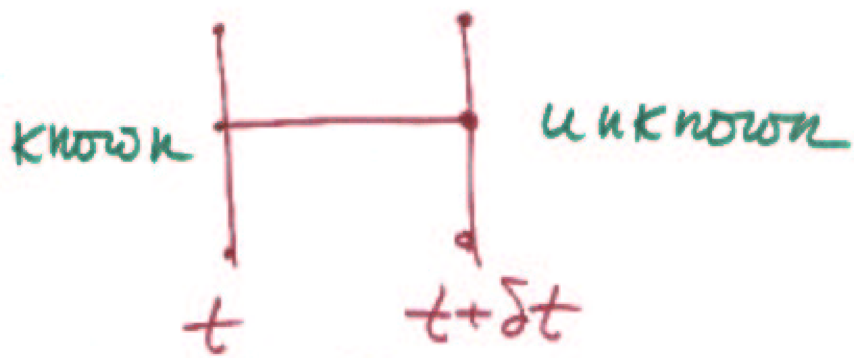
- Good stability
- Can be quite fast

Disadvantages:

- Poor speed w.r.t. δt

Crank-Nicolson:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left(\text{Explicit} + \text{Implicit}\right)$$



stable? YES

Error $\sim (\delta t)^2 + (\delta x)^2$

$\uparrow$!

22

My favorite scheme:
3 layers

Layer 1: start with explicit scheme ← 1/3
(good to smooth discontinuities)

Layer 2: continue with Crank-Nicolson (fast)

Layer 3: end with implicit scheme (stable)

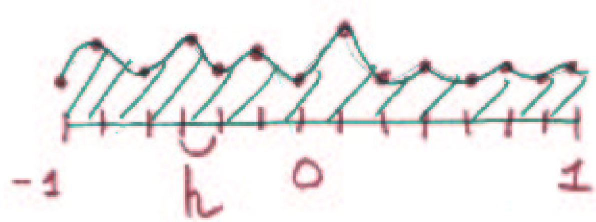
23

3. Numerical integration of discontinuous functions

Principle of Numerical Analysis:

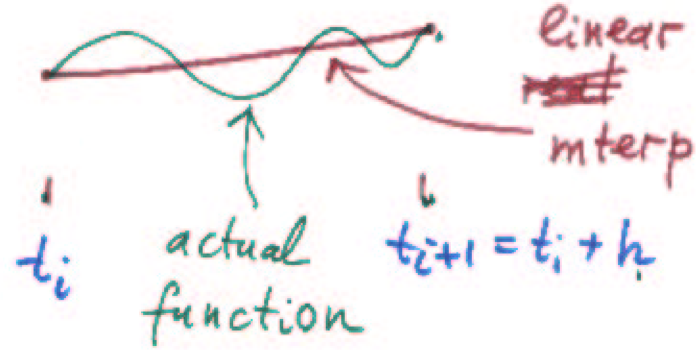
speed $\asymp$ smoothness

Consider the problem of numerical integration on $[-1, 1]$.



Trapezoidal method:

$$\int_{t_i}^{t_i+h} f(x)\,dx \approx \frac{h}{2}\left(f(t_i)+f(t_{i+1})\right)$$



Question: What is the error of this scheme?

Case 1: $f = f(x)$ is smooth (f' is continuous)

Error on $[t_i, t_i + h]$ =

$$\int_{t_i}^{t_i+h} \left(f(x) - \frac{1}{2}\left(f(t_i) + f(t_{i+1})\right)\right) dx$$

$$\asymp \int_{t_i}^{t_i+h} (x - t_i)^2 \, dx \asymp$$

$$\asymp h^3$$

\# of intervals $= \frac{1}{h}$

$$\text{Total error} \asymp \frac{1}{h} \cdot h^3$$

$$\asymp h^2 \quad \left((\delta x)^2\right)$$

Case 2: f is continuous

f' has finite # of discontinuities

25

Example

$$f(x) = \max(g(x), h(x))$$

smooth functions

Error analysis:

(a) If f' is continuous on $[t_i, t_i+h]$, then

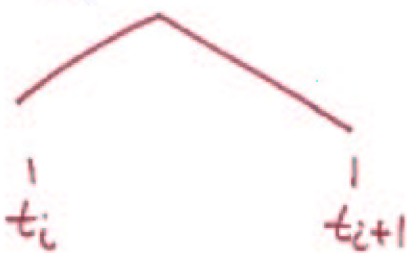
Error on $[t_i, t_i+h] \asymp h^3$

(b) If f' has discontinuity on $[t_i, t_i+h]$, then

Error on $[t_i, t_i+h] \asymp$

$$\int_{t_i}^{t_i+h} \left(f(x) - \frac{1}{2}\left[f(t_i)+f(t_i+h)\right]\right) dx$$

$$\asymp \int_{t_i}^{t_{i+1}} (x-t_i)\, dx \asymp h^2$$



Total error $\asymp$

$$\underbrace{\text{\# of "good" intervals}}_{1/h} * h^3$$

$$+$$

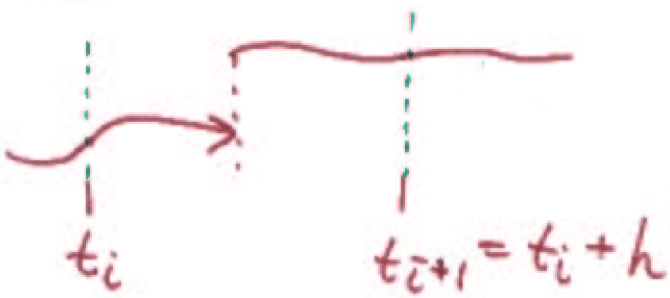
$$\text{\#}\ \underbrace{\text{of "bad" intervas}}_{1} * h^2$$

$\asymp h^2$ (Same as for smooth functions)

Case 3: f has finite # of jumps (f is discontinuous)

a) "Good" interval (h^3)

b) "Bad" interval



30

Error on "bad" $[t_i, t_{i+1}]$ $\asymp$

$$\int_{t_i}^{t_{i+1}} \left(f(t) - \frac{1}{2}(f(t_i) + f(t_{i+1})\right) dt$$

$$\asymp h$$

Total error =

$$\# \text{ of "good" } \underbrace{\text{intervals}}_{\asymp 1/h} * h^3$$

$$+$$

$$\# \text{ of } \underbrace{\text{"bad intervals"}}_{1} * h$$

$$\asymp h \quad (\gg h^2)$$

31

Question: How to increase the speed of convergence?

Idea: "smart" representation of indicator functions

Consider indicator function:

$$I(x>a)$$

We want to create a representation of this indicator function on grid

~~so that~~

$$I(x>a) \longrightarrow \mathbb{I}^{smart}(x_i>a)$$

so that

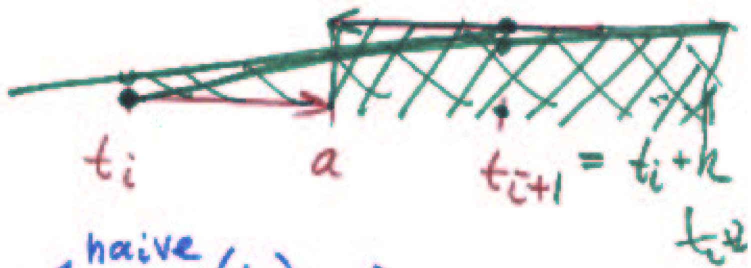
$\forall$ smooth function

$f = f(x)$

$$\int_{-1}^{1} f(x)\, I(x>a)\, dx = \int_{a}^{1} f(x)\, dx$$

$$\approx \frac{h}{2} \sum_{x_i}{}' \Big(I^{\text{smart}}(x_i > a)\, f(x_i) + I^{\text{smart}}(x_{i+1} > a)\, f(x_{i+1}) \Big) + O(h^2)$$

Remark. We represent discontinuous functions using ~~[illegible]~~ "smart" indicators.



$$\Pi^{naive}(t_i) = 0$$

$$\Pi^{naive}(t_{i+1}) = 1$$

$$\Pi^{smart}(t_i) = \frac{t_{i+1} - a}{2h}$$

$$\Pi^{smart}(t_{i+1}) = \frac{t_{i+1} - a}{2h} + \frac{1}{2}$$

4. Interpolation & approximation.

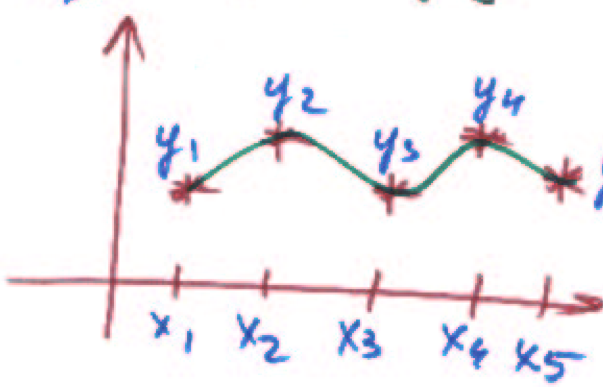
34

Basic idea: given

arguments $(x_1 \dots x_N)$

values of a function $(y_1 \dots y_N)$

$\Rightarrow$ restore $f = f(x)$



Distinction between interpolation & approximation:

a) Interpolation: both arguments and values are inputs

b) Approximation: can select arguments

36

Interpolation methods:

a) linear

b) cubic spline

Approximation methods:

a) Chebyshev polinomials

b) Trigonometric polynomials