21

Hull & White model for interest rates

Standard form:

$$dr_t = (\theta_t - \lambda r_t)\,dt + \varkappa\, dW_t$$

r_t: short-term rate

λ: mean-reversion rate

$\varkappa$: volatility of short-term rate

(θ_t): some deterministic function (to calibrate any discount curve)

2.2

Time-dependent version:

$$dr_t = (\theta_t - \lambda_t r_t)\,dt + \varkappa_t\,dW_t$$

Calibration:

(θ_t): calibrates discount curve

(λ_t) & $(\varkappa_t)$: implied volatility curve.

2.3

Notations:

$B(s,t)$: discount factor computed at s for maturity t

$r(s,t)$: continuously compounded yield

$$B(s,t) = e^{-(t-s)\, r(s,t)}$$

$$r(s,t) = -\frac{1}{t-s} \ln B(s,t)$$

Hull & White model in [2H] Black (HJM) methodology.

Input parameters:

$\left(B(0,t)\right)_{t\geq 0}$: initial discount curve

initial time — maturity of zero-coupon bond

$\left(A(t)\right)_{t\geq 0}$: initial shape curve for changes in discount curve

Convention:

$A(0) = 0 \qquad A'(0) = 1$

$$
\begin{array}{c}
r(0) \uparrow 1 \text{ b.p. (basic point)} \\
\Updownarrow \\
r(0,t) \uparrow \dfrac{A(t)}{t} \text{ b.p.} \\
\Updownarrow \\
\dfrac{\delta B(0,t)}{B(0,t)} \downarrow A(t) \text{ b.p.}
\end{array}
$$

$\boxed{2.5}$

26

In practice

1. discount factors with longer maturities move "faster" $\Longleftrightarrow$ A is increasing ($A' > 0$)

2. Yields with longer maturities move "slower" $\Longleftrightarrow$ $\left(\frac{A(t)}{t}\right)_{t \geq 0}$ is decreasing

$$\left(A'(t) \leq \frac{A(t)}{t} \right)$$

27

It is reasonable to assume that

A' is decreasing

$(A'' < 0)$

Hence,

$$A'(t) = \exp\left(-\int_0^t \lambda_u \, du\right)$$

where

(λ_t) : mean-reversion rate

$(\lambda_t \geq 0)$

$\left(\Sigma(t)\right)_{t\geq 0}$: normalized volatility curve [28]

$$\underbrace{\Psi(0,s,t)}_{\text{implied volatility}} = \left(A(t)-A(s)\right)\Sigma(s)$$

0: ~~xx~~ initial time

s: maturity of option

t: maturity of underlying zero-coupon bond

$$\Sigma(t) = \sqrt{\frac{1}{t}\int_0^t \sigma^2(u)\,du}$$

$\boxed{2.9}$

$\varkappa(t) = A'(t)\,\sigma(t)$:
volatility of short-term
rate

12.10

$F(s,t,u)$: forward price

s : current time

t : delivery time for forward

u : maturity of underlying zero-coupon bond

Hull & White model:

$$dF(s,t,u) = F(s,t,u) * (A(u) - A(t))\, \tilde{\sigma}_s \, dW_s^t$$

W^t : Brownian motion for forward measure $\mathbb{P}^t$.

Output for Hull & White model. 2.11

We compute the value of the option as the function of

$$x = r(0) - r(x)$$

initial short-term rate

perturbed short-term rate

Output:

$$V_0 = \left(V_0(x) \right)_{-\frac{\Delta}{2} \le x \le \frac{\Delta}{2}}$$

Δ : interval for changes in r