5.1

Plan

1. Pricing of path dependent derivatives: theory & implementation.
2. Classes IResetValues, PathDependent, IExtend, Extended
3. Examples of evaluation of path dependent derivatives.

15.2

Example

Assume that we have the "standard" implementation of Black model. That means that at any time t we can manipulate the random variables in the form

$$f\,(S_t)$$

deterministic function

stock price

6.3

Suppose that we have to price "forward start" call option

0	t_1	t_2
issue time	start time	maturity

which payoff at maturity is given by

$$\max\left(S_{t_2} - \underbrace{S_{t_1}}_{\text{strike determined at } t_1}, 0\right)$$

However, this random variable is not "supported" by our standard implementation. 5.4

Solution: extend the dimension of the model

$$S \longrightarrow (S, Y)$$

where Y satisfies

(a) (S, Y) is a state process

(b) $Y_{t_2} = S_{t_1}$

Then the payoff of the forward start ~~call~~ call has the "right" form: 5.5

$$\max(S_{t_2} - Y_{t_2}, 0) =$$
$$= f(S_{t_2}, Y_{t_2})$$

5.6

General framework

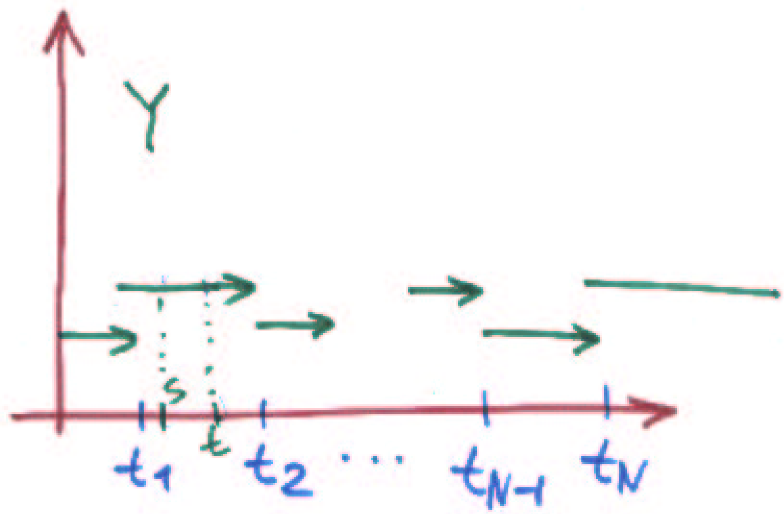
We are given a model with state process X. Assume that the rollback operator is implemented for payoffs determined by X:

$$V_t = f(X_t)$$

$$\underset{s}{g(X_s)} \xleftarrow{\text{rollback}} \underset{t}{f(X_t)}$$

$$g(X_s) = \mathcal{R}(f(X_t), s)$$

Consider also a stochastic process Y which values change at reset times $t_1, \dots, t_N$



Question: when (X, Y) is a state process?

In other words, when 15.8

$$\forall \; s < t, \; \cancel{g}\, f = f(x,y)$$

$$\exists \; g = g(x,y)$$

$$g(X_s, Y_s) = \mathcal{R}\left(f(X_t, Y_t), s\right)$$

$$\underset{s}{g(X_s, Y_s)} \xleftarrow{\text{rollback}} \underset{t}{f(X_t, Y_t)}$$

?

Theorem Assume that $\forall$ reset time t_{i+1} there is a determenistic function $G_{i+1} = G_{i+1}(x, y)$ (reset function) such that 15.9

$$Y_{t_{i+1}} = G_{i+1}\left(\underset{\substack{\nearrow \\ \text{value at } t_{i+1}}}{X_{t_{i+1}}}, \underset{\substack{\nearrow \\ \text{value before } t_{i+1}}}{\underbrace{Y_{t_i}}_{=\,y}}\right)$$

Then (X, Y) is a state process.

5.10

Proof We need to show that $\forall\ s<t$

$f=f(x,y)$, $\exists\ g=g(x,y)$

s.t.

$$g(X_s,Y_s)=\mathcal{R}\big(f(X_t,Y_t),s\big)$$

Using chain rule

$$\begin{tikzcd} s & t \arrow[l, bend right] & u \arrow[l, bend right] \arrow[ll, bend left] \end{tikzcd}$$

$$\mathcal{R}_s(\mathcal{V}_u)=\mathcal{R}_s\big(\mathcal{R}_t(\mathcal{V}_u)\big)$$

we can assume that

$$t_i\le s<t\le t_{i+1}$$

Case 1 : 15.1

$$t_i \le s < t < t_{i+1}$$

We have

$$Y_t = Y_s = Y_{t_i}$$

Hence

$$f(X_t, Y_t) = f(X_t, Y_s)$$

As Y_s is "known" at s we have

$$g(x,y) = R_s\big(f(X_t, y)\big)_{(X_s = x)}$$

Case 2: 15.12

$$t_i \le s < t = t_{i+1}$$

We have

$$Y_t = Y_{t_{i+1}} = G_{i+1}(X_{t_{i+1}}, Y_{t_i})$$

$$= G_{i+1}(X_t, Y_s)$$

Hence

$$f(X_t, Y_t) = h(X_t, Y_s)$$

$$h(x,y) = f(x, G_{i+1}(x,y))$$

and as before

$$g(x,y) = R_s(h(X_t, y))(X_s = x)$$

Example (Hist. value) 15.13

$$Y_t = 0 \qquad t < t_1$$

$$Y_t = S_{t_1} \quad t \geq t_1$$

Example (Hist. max)

$$Y_t = 0 \qquad t < t_1$$

$$Y_t = \max_{t_i \leq t} S_{t_i}$$

$$Y_{t_{i+1}} = \max(Y_{t_i}, S_{t_{i+1}})$$

Example (Hist. average)

$$Y_t = 0 \qquad t < t_1$$

$$Y_t = \frac{1}{n(t)} \sum_{i=1}^{n(t)} S(t_i)$$

$$n(t) = \max \{ i : t_i \leq t \}$$

$$Y_{t_{i+1}} = \frac{1}{i+1} \left(i\, Y_{t_i} + S_{t_{i+1}} \right)$$

15.15

Numerical implementation

Inputs:

- X : "old" state process
- $\mathcal{R}^X$: "old" rollback operator supporting X
- Y : "new" state process
  - $t_1 \dots t_N$: reset times
  - $(G_i)_{1 \le i \le N}$: reset functions

$$Y_{t_{i+1}} = G_{i+1}\left(X_{t_{i+1}}, Y_{t_i}\right)$$

Goal: implement $\mathcal{R}^{(X,Y)}$: rollback operator that "supports" (X,Y). 15.18

Let

$$t_i \le s < t \le t_{i+1}$$

(event times)

$$f = f(x,y)$$

$$g(X_s, Y_s) \leftarrow f(X_t, Y_t)$$

$$\underset{s}{g} = g(x,y) \; - ?^{t}$$

Case 1: 5.17

$$t_i \leq s < t < t_{i+1}$$

Then

$$Y_t = Y_s = Y_{t_i}$$

$$f(X_t, Y_t) = f(X_t, Y_s)$$

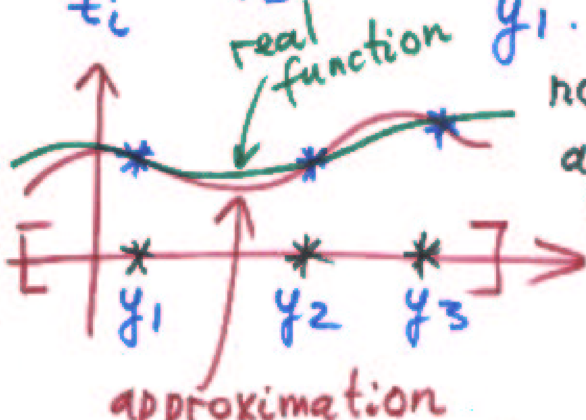
"Naive" scheme: $\forall y$ compute

$$\mathcal{R}_s\big(f(X_t, y)\big)(X_s) = g(X_s, y)$$

"Practical" scheme: 15.18

choose approximation scheme for the values of $Y_{t_i} = Y_s$.

$y_1 \ldots y_m$: nodes of approximation scheme



We then compute

$\mathcal{R}_s\big(f(X_t, y_i)\big)(X_s) = g(X_s, y_i$

for the nodes and

get $g = g(x, y)$ 15.19

using recovery operator

~~for~~ w.r.t. $(y_1 \dots y_m)$.

Case 2:

$$t_i \le s < t = t_{i+1}$$

Then

$$Y_t = Y_{t_{i+1}} =$$

$$= G_{i+1}\left(X_{t_{i+1}}, Y_{t_i}\right)$$

$$= G_{i+1}\left(X_t, Y_s\right)$$

$$f(X_t, Y_t) = h(X_t, Y_s)$$

$$h(x, y) = f(x, G_{i+1}(x, y))$$

we then follow the same technique based on approximation from Case 1. $\boxed{5.50}$