The topological Tverberg problem beyond prime powers

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A 1959 result of Bryan Birch [3] asserts that for any straight-line drawing of the complete graph K_{3q} with $3q$ vertices in the plane, there is a partition into q vertex-disjoint 3-cycles that all surround a common point in the plane. It is natural to wonder whether this results holds more generally if the edges are not assumed to be straight-line segments, but only continuous arcs. This question and its natural generalizations to higher dimensions have turned out to be surprisingly resistant.

The natural generalization of Birch's result to dimension d holds: Any $q(d+1)$ points in $\mathbb{R}^d$ may be partitioned into q sets $X_1, \dots, X_q$ of size $d+1$ such that the simplices spanned by the X_i all intersect in a common point. In fact, generically this intersection of simplices will be full-dimensional, and one can save d points; Helge Tverberg [11] proved that any $(q-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into q sets whose convex hulls all share a common point.

The continuous generalization of Birch's result, and more generally Tverberg's result, has been proved for q a power of a prime [4, 10, 12]. More precisely, any continuous map $f\colon \Delta_{(q-1)(d+1)} \to \mathbb{R}^d$ from the $(q-1)(d+1)$-dimensional simplex to $\mathbb{R}^d$ identifies q points from pairwise disjoint faces, provided that q is a power of a prime. For a linear map f this is precisely Tverberg's theorem. Perhaps surprisingly, the condition that q be a prime power is indeed crucial for this continuous generalization: For any q with at least two distinct prime divisors and d sufficiently large there is a continuous map $f\colon \Delta_{(q-1)(d+1)} \to \mathbb{R}^d$ that never maps q points from pairwise disjoint faces to the same point; see [2, 6, 8, 9]. In fact, Avvakumov, Karasev, and Skopenkov [1] showed that there is such a map $f\colon \Delta_n \to \mathbb{R}^d$ for $n = q(d+1) - q\lceil \frac{d+2}{q+1} \rceil - 2$, provided that q is not a power of prime and $d \geq 2q$.

However, this leaves open the question whether there is a continuous generalization of Birch's original result. Here we prove this generalization and its higher-dimensional versions beyond prime powers; see [7] for details:

Theorem 1. *Let $q \geq 2$ and $d \geq 1$ be integers. Let $n = q(d+1) - 1$. For any continuous map $f\colon \Delta_n \to \mathbb{R}^d$ there are points $x_1, \dots, x_q$ in q pairwise disjoint faces of Δ_n with $f(x_1) = f(x_2) = \cdots = f(x_q)$.*

As a simple consequence of this we obtain a continuous generalization of Birch's theorem:

Corollary 2. *For any continuous drawing of K_{3q} in the plane, where each 3-cycle is embedded, there is a partition of the vertex set into q triples such that the induced 3-cycles all surround a common point.*

Here we require 3-cycles to be embedded since then, by the Jordan curve theorem, each 3-cycle surrounds a well-defined interior region.

Let p be a prime. The p-fold join of a continuous map $f\colon \Delta_n \to \mathbb{R}^d$ is a $\mathbb{Z}/p$-equivariant map $F\colon (\Delta_n)^{*p} \to (\mathbb{R}^{d+1})^p$. Let

$$D = \{(y_1, \dots, y_p) \in (\mathbb{R}^{d+1})^p \ : \ y_1 = y_2 = \cdots = y_p\}$$

denote the diagonal in $(\mathbb{R}^{d+1})^p$. The preimage $F^{-1}(D)$ consists of all ordered p-tuples of (not necessarily distinct) points that f maps to the same point, that is, $F(\lambda_1 x_1 + \cdots + \lambda_p x_p) \in D$ if and only if $\lambda_i = \frac{1}{p}$ for all i and $f(x_1) = f(x_2) = \cdots = f(x_p)$. Since for p a prime the $\mathbb{Z}/p$-action shifting coordinates of $(\mathbb{R}^{d+1})^p$ is free away from the diagonal D, a result of Dold [5] now implies that $F^{-1}(D)$ intersects any $\mathbb{Z}/p$-invariant subcomplex $\Sigma \subset (\Delta_n)^{*p}$ that is homotopically $[(p-1)(d+1)-1]$-connected. The subcomplex $\Sigma \subset (\Delta_n)^{*p}$ that consists only of p-fold joins of pairwise disjoint faces is $(n-1)$-connected, so for $n = (p-1)(d+1)$ this proves the continuous generalization of Tverberg's theorem, provided that $q = p$ is a prime.

The key idea for the proof of Theorem 1 now is to construct for a given integer $q \geq 2$ and a large prime of the form $p = kq + 1$ a $\mathbb{Z}/p$-invariant subcomplex $\Sigma \subset (\Delta_{q(d+1)})^{*p}$ that is $[(p-1)(d+1)-1]$-connected and such that the $\mathbb{Z}/p$-orbit of any vertex contains q consecutive vertices that are pairwise not adjacent. Since Σ is highly connected it follows as before that there are $x_1, \dots, x_p \in \Delta_{q(d+1)}$ with $f(x_1) = f(x_2) = \cdots = f(x_p)$ and such that $\frac{1}{p}x_1 + \cdots + \frac{1}{p}x_p \in \Sigma$. By construction of Σ the points $x_1, x_2, \dots, x_q$ are in pairwise disjoint faces.

This can be used to prove a weaker variant of Theorem 1 for $n = q(d+1)$. To prove the stronger version for $n = q(d+1) - 1$, one can add a dummy vertex to instead argue for $\Delta_{q(d+1)}$ as above. Then observe that for any set $I \subset \mathbb{Z}/p$ of q consecutive numbers modulo p, the points $x_i, i \in I$, are in pairwise disjoint faces of $\Delta_{q(d+1)}$, and the dummy vertex cannot obstruct all of these collections of points, since otherwise q would divide p.

The technical core of the proof of Theorem 1 consists of the construction of suitable complexes Σ, which are highly connected (thus dense) while having large independent sets in each $\mathbb{Z}/p$-orbit (and thus are locally sparse). The construction given in [7] is optimal in the sense that in any $\mathbb{Z}/p$-symmetric $[(p-1)(d+1)-1]$-connected subcomplex of $(\Delta_{q(d+1)})^{*p}$ the largest independent set in some $\mathbb{Z}/p$-orbit has size at most q.

References

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