Sets and Proofs

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Covering Properties of Core Models

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Theorem (Jensen's covering lemma) Assume $0^\#$ does not exist. Let $A$ be any uncountable set of ordinals. Then there is a $B \in L$ such that $B \supseteq A$ and $\text{card}(B) = \text{card}(A)$.

In this paper, we outline Jensen's proof from a modern perspective. We isolate certain key elements of the proof which have become important both within and outside of inner model theory. This leads into an intuitive discussion of what core models are and the difficulties involved in generalizing Jensen's theorem to higher core models. Our hope is to give the reader some insight into these generalizations by concentrating on the simplest core model, $L$.

Jensen's theorem has striking consequences for cardinal arithmetic. Its conclusion implies that if $\omega_2 \leq \beta$ and $\beta$ is a successor cardinal of $L$, then $\text{cf}(\beta) = \text{card}(\beta)$. In particular, if $0^\#$ does not exist, then $L$ computes successors of singular cardinals correctly. The covering lemma also implies that some of the combinatorial principles, which Jensen proved in $L$, really hold. (I.e., they hold in $V$. ) For example, if $0^\#$ does not exist and $\kappa$ is any singular cardinal, then $\square_\kappa$ holds.

By an inner model, we mean a transitive proper class model of ZFC. If $M$ is an inner model, then $M$ has the covering property if for every uncountable set of ordinals $A$, there exists $B \in M$ such that $B \supseteq A$ and $\text{card}(B) = \text{card}(A)$. Core models are certain kinds of inner models which we do not define here, except to say that $L$ is a core model. (See [MiSt] for the precise definition and our Section 2 for a general description.)

If $0^\#$ exists, then $L$ does not have the covering property. Dodd and Jensen found a substitute core model, which they called $K$, and proved that $K$ has the covering property if there is no core model with a measurable cardinal. In fact, if $0^\#$ does not exist, then $K = L$, but not in general.

Expectations must be limited for extensions of the covering lemma to core models with measurable cardinals, because of the example given by Prikry forcing. At this juncture, there are at least three possibilities:
Cardinal arithmetical consequences of the covering property are known as *weak covering properties*. For example, the property that $\text{cf}(\beta) = \text{card}(\beta)$ whenever $\beta$ is a successor cardinal of $M$ is a weak covering property of $M$. The correct computation of successors of singular cardinals is another weak covering property, as is the correct computation of successors of weakly compact cardinals. A first approach is to prove that certain core models have weak covering properties under more relaxed anti-large cardinal hypotheses. Mitchell defined a core model, which he also called $K$, and proved that $K$ has these weak covering properties under the hypothesis that there is no core model with a measurable cardinal $\kappa$ of order $\kappa^{++}$. (See [Mi].) The Mitchell core model and the Dodd-Jensen core model are the same if there is no core model with a measurable cardinal, so there is no ambiguity in the meaning of $K$. Steel extended the definition of $K$ further, by weakening the anti-large cardinal hypothesis to the non-existence of a core model with a Woodin cardinal. More recently, Mitchell, Steel and the author proved the corresponding weak covering theorems. (See [St2], [MiSt], [MiSch], and [SchSt].)

In a second approach, instead of weakening the covering property, one skirts the problem presented by Prikry forcing by considering only core models without measurable cardinals. For example, the minimal inner model closed under the operation $X \mapsto X^+$ has no measurable cardinals, but is "beyond" $\mathcal{L}[U]$. The author and Woodin have proved that this and other core models satisfy the covering property if their "sharps" do not exist. (The precise statements can be found in the forthcoming [SchWo]. Related results were obtained in [Mi].)

Here is a typical example of a core model without measurable cardinals which plays an important role in the theory of projective sets of reals. If $X$ is a set of ordinals and $n < \omega$, then let $\mathcal{M}_n(X)$ be the minimal inner model with $n$ Woodin cardinals which has $X$ as an element. (In particular, $\mathcal{M}_0(X) = \mathcal{L}(X)$.) Under an appropriate large cardinal hypothesis, by [St3], $\mathcal{M}_n = \mathcal{M}_n(\emptyset)$ exists and is a core model. Let $W_n$ be the minimal inner model closed under the operation

$$X \mapsto \mathcal{M}_n(X) \cap \mathcal{P}(\text{sup}(X)).$$

Then $W_n$ is a core model which has no measurable cardinals and so, by [SchWo], $W_n$ has the "full" covering property, like Jensen's theorem for $\mathcal{L}$. ($W_n$ is also characterized as the minimal inner model which is closed under $X \mapsto C_{n+1}(X)$ if $n$ is even, and $X \mapsto Q_{n+1}(X)$ if $n$ is odd. See [St3].)

A third kind of extension to the covering lemma has us deal directly with the problematic Prikry sequences. One may weaken the covering property for a core model $W$ to just

$$A \subseteq f^*(\rho \cup \mathcal{O})$$

for some function $f \in W$, ordinal $\rho < \text{card}(A)^+$, and system of indiscernibles $\mathcal{O}$ for $W$. The first such result is the Dodd-Jensen theorem for $\mathcal{L}[U]$. (See [Do3] and [Do].) Mitchell and Gitik's lower bounds on the consistency strength of failure of the Singular Cardinal Hypothesis require deep analysis in this direction. (See [Gi] and [GiMi].)

These advanced covering results only came about after the elements of Jensen's proof were compartmentalized. The modern perspective which proved suitable for generalizations is known to many researchers in the area, but has not appeared in introductory form. In Sections 1, 3, and 4, we outline Jensen's proof from this modern point of view. We focus on some weak covering properties in Sections 1 and 3, before tackling the full result in Section 4. Having seen an extender in Section 1, we are able to say more about what core models are and to make some comments regarding covering properties of higher core models in Section 2. Section 3 offers an introduction to a common kind of chain argument.

There are many known simplifications to the proof of covering in the case of $\mathcal{L}$, which we do not incorporate here, some of which the reader may notice. Again, the proof we give reflects our underlying interest in generalizations to higher core models.

It must be emphasized that none of the ideas in this paper are due to the author. There are several alternative approaches to Jensen's theorem, each useful and important for different reasons. The reader certainly will want to compare the proof we sketch with the proofs presented in [DeJe], [De], and [Ma].

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## 1. First pass at Jensen's proof: weak covering at countably closed cardinals

We begin by sketching a proof of the following weak covering property. Assume that $\kappa$ is a countably closed cardinal. This means that $\mu^{\kappa_0} < \kappa$ whenever $\mu < \kappa$. For example, if $2^{\kappa_0} < \aleph_\omega$, then $\aleph_\omega$ is countably closed, since

$$(\aleph_n)^{\kappa_0} = \max(\aleph_n, 2^{\kappa_0})$$

whenever $1 \leq n < \omega$. Let $\lambda = (\kappa^+)^+$, and assume that $\text{cf}(\lambda) < \kappa$. In particular, $\lambda < \kappa^+$. We shall show that $0^\#$ exists.

Let $X \prec V_{\kappa^+}$ with

$$\text{sup}(X \cap \lambda) = \lambda,$$

$$\text{card}(X) < \kappa,$$


\[ \kappa \in X, \]

and

\[ \omega^X \subseteq X \]

Such an \( X \) can be realized as the union of a continuous \( \omega_1 \) length chain of elementary submodels of \( V_{\kappa^+} \) where the first submodel contains a witness to \( \text{cf}(\lambda) < \kappa \), and successive submodels contain all the \( \omega \)-sequences from the earlier models. Of course, this approach uses the countable closure of \( \kappa \).

Let \( \pi : N \simeq X \) be elementary with \( N \) transitive. Say \( \pi(\kappa) = \kappa, \pi(\lambda) = \lambda, \delta = \text{crit}(\pi), \) and \( \alpha = \text{OR} \cap N \). Then \( \delta \leq \pi < \lambda < \alpha \) and \( \pi \) is continuous at \( \lambda \) in the sense that

\[ \sup(\pi(\delta)) = \lambda = \pi(\lambda). \]

By the condensation principle for \( L \), \( L^N = L_\alpha \). (And, in fact, \( L_\alpha = J_\alpha \). Our convention here will be to use the \( L \)-hierarchy at levels where it coincides with the \( J \)-hierarchy.)

**Easy Case.** \( \mathcal{P}(\delta) \cap L \subseteq L_\alpha \).

Let \( U \) be the ultrafilter on \( N \cap \mathcal{P}(\delta) \) derived from \( \pi \), namely

\[ U = \{ A \subseteq \delta \mid A \in N \text{ and } \delta \in \pi(A) \} \]

Then, by the case hypothesis, \( U \cap L \) is an ultrafilter on \( L \cap \mathcal{P}(\delta) \). The ultrapower of \( L \) by \( U \) consists of equivalence classes of the form \( [f]_U^L \) where \( f \in L \) and \( f \) is a function from \( \delta \) into \( L \).

**Claim 1.** \( \text{ult}(L, U) \) is wellfounded.

It follows from Claim 1 that \( \text{ult}(L, U) \simeq L \), so the ultrapower map gives a non-trivial elementary embedding of \( L \) into \( L \), hence \( 0^\# \).

Claim 1 is proved using the countable completeness of \( U \). Namely, if the claim fails, then there is a sequence of functions \( \{ f_n \mid n < \omega \} \) such that \( f_n \in L \) and a sequence \( \langle A_n \mid n < \omega \rangle \) such that \( A_n = U \cap L \) with the property that \( f_{n+1}(\xi) \in f_n(\xi) \) for every \( \xi \in A_n \). But \( X \) is closed under \( \omega \)-sequences, so

\[ \pi(A_n \cap n < \omega) = (\pi(A_n) \cap n < \omega) \in X. \]

Therefore, if we set \( A = \bigcap \{ A_n \mid n < \omega \} \), then \( \delta \in \bigcap \pi(A) \), so \( A \in U \). In particular, \( A \neq \emptyset \). But \( \xi \in A \) and \( n < \omega \) implies that \( f_{n+1}(\xi) \in f_n(\xi) \), which is impossible.

**Hard Case.** *Otherwise.*

In this case, we shall get a contradiction by finding an ordinal \( \gamma \) such that

\[ J_{\gamma+1} \models \lambda < \kappa^+. \]

This truly is a contradiction since \( \lambda = (\kappa^+)^L \).

Our first step is to derive an extender from \( \pi \). Consider an arbitrary finite subset \( a \) of \( \lambda \). Say \( \text{card}(a) = n < \omega \). Let \( \mu_a \) be the least ordinal \( \mu \) such that \( \pi(\mu) > \text{max}(a) \). By analogy with how \( U \) was defined in the Easy Case, we define a countably complete ultrafilter \( E_a \) on \( N \cap \mathcal{P}(\mu_a) \) by

\[ E_a = \{ A \subseteq [\mu_a]^n \mid A \in N \text{ and } a \in \pi(Y) \}. \]

To orient the reader, we note that the two ultrafilters \( E(\delta) \) and \( U \) differ in a trivial way:

\[ \mu(\delta) = \delta \]

and for all \( A \subseteq \delta \),

\[ A \in U \iff \{ \xi \mid \xi \in A \} \in E(\delta). \]

The \( (\delta, \lambda) \)-extender derived from \( \pi \) is defined to be the system of ultrafilters

\[ E = \langle E_a \mid a \in [\lambda]^\omega \rangle. \]

We digress to discuss ultrapowers by extenders. Suppose that \( M \) is a transitive set model of a reasonable fragment of set theory with

\[ V^M_X \subseteq N. \]

Then, for each \( a \in [\lambda]^\omega \), \( E_a \cap M \) is an ultrafilter over \( M \). The ultrapower \( \text{ult}(M, E_a) \) consists of equivalence classes \( [f]_{E_a}^M \) where, if \( n = \text{card}(a) \), then \( f \in M \) and \( f \) is a function from \( [\mu_a]^n \) into \( M \). Moreover, the ultrapowers of \( M \) by these ultrafilters form a direct limit system as indicated by the following commutative diagram.

\[ \text{ult}(M, E_a) \rightarrow \text{ult}(M, \pi, \lambda) \]

We have illustrated the case \( a \subseteq b \in [\lambda]^\omega \) and set \( \text{ult}(M, \pi, \lambda) \) equal to the direct limit of the structures \( \text{ult}(M, E_a) \) under the maps \( k_{a,b} \). Then \( k_a \) is the limit of the embeddings \( k_{a,b} \) and the elements of \( \text{ult}(M, \pi, \lambda) \) have the form

\[ [a, f]_{E_a}^M = k_a([f]_{E_a}^M). \]
Admittedly, it takes some abstract nonsense to sort out the definition of $k_{a,b}$, which we leave to the reader. We have also set $i_{E}^{\ast}$ equal to the limit of the ultrapower maps $i_{E}^{M}$.

We shall also be interested in restrictions of $E$, especially

$$E \upharpoonright \kappa = \langle E_{a} \mid a \in [\kappa]^{\omega} \rangle,$$

and their associated ultrapowers. The ultrapower of $M$ by $E \upharpoonright \kappa$,

$$\text{ult}(M, \pi, \kappa),$$

is the direct limit of $\text{ult}(M, E_{a})$ for $a \in [\kappa]^{\omega}$. [It would be reasonable to write $\text{ult}(M, E \upharpoonright \kappa)$ for $\text{ult}(M, \pi, \kappa)$, as is done elsewhere, although we shall not do so here.] There is an embedding of $\text{ult}(M, \pi, \kappa)$ into $\text{ult}(M, \pi, \lambda)$ with critical point at least $\kappa$. In many cases of interest, the two ultrapowers will be equal.

Here are a few remarks which are tie our notion of extender with some commonly used terminology and jargon. (These remarks could be skipped without loss of continuity.)

1. $\delta$ is the **critical point** of $E$. We write $\text{crit}(E) = \delta$.

2. $\lambda$ is the **length** of $E$. We write $\text{lh}(E) = \lambda$.

3. Note that $\pi(\delta) < \lambda$. Equivalently, $\mu_{\delta} > \delta$ for some $a \in [\text{lh}(E)]^{\omega}$. This property of $E$ makes $E$ a long extender.

4. The **superstrong extender derived from** $\pi$ is $E \upharpoonright \pi(\delta)$.

5. We shall see situations in which $E$ is applied to $M$ and $\text{OR} \cap M < \lambda$.

6. In certain contexts, the term “extender” is reserved for extenders which are total, that is, extenders which measure all subsets of their $[\mu_{\delta}]^{\omega}$ for every $a \in [\Delta]^{\omega}$ and $n < \omega$. In cases such as ours, $E$ would be called an **extender fragment**, since it only measures sets in $N$. In the terminology of [St2], $(N, E)$ would be called a **background certificate**.

7. It is possible to express large cardinal axioms, such as the existence of measurable, strong, Woodin, superstrong, supercompact, and huge cardinals, by reference to long extenders. (See [St1].)

Returning to our outline, we next move towards applying the extender $E$ to the longest initial segment of $L$ possible. By the case hypothesis, $a < (\delta^{+})^{L}$. So we may define $\beta$ to be the least ordinal such that for some $\mu < \lambda$,

$$\mathcal{P}(\mu) \cap J_{\beta+1} \not\subseteq L_{\alpha}.$$

Because $\bar{\lambda}$ is the successor cardinal of $\kappa$ in $L_{\alpha}$, we have the some useful equivalent definitions of $\beta$. Namely, $\beta$ is the least ordinal such that

$$\mathcal{P}(\bar{\lambda}) \cap J_{\beta+1} \not\subseteq L_{\alpha}.$$

And, also, $\beta$ is the unique ordinal such that

$$J_{\beta} \models \bar{\lambda} = \kappa^{+},$$

and

$$J_{\beta+1} \models \bar{\lambda} < \kappa^{+}.$$

The proofs of these equivalences use basic facts from [Je].

Clearly,

$$\delta < \alpha < \beta < (\delta^{+})^{L}.$$

Now let $n < \omega$ be least such that for some $\mu < \bar{\lambda}$,

$$\mathcal{P}(\mu) \cap \Sigma_{n+1}^{(J_{\beta}, \bar{\lambda})} \not\subseteq J_{\beta}.$$

Again, using the fact that $\bar{\lambda}$ is the successor cardinal of $\kappa$ in $J_{\beta}$, $n$ is least such that

$$\mathcal{P}(\bar{\lambda}) \cap \Sigma_{n+1}^{(J_{\beta}, \bar{\lambda})} \not\subseteq J_{\beta}.$$

In terms of the projection of $J_{\beta}$, this means that

$$\rho_{n}(J_{\beta}) > \bar{\lambda},$$

and

$$\rho_{n+1}(J_{\beta}) < \bar{\lambda}.$$

We now have that for every $\mu < \bar{\lambda}$,

$$\mathcal{P}(\mu) \cap L_{\bar{\lambda}} = \mathcal{P}(\mu) \cap L_{\alpha} = \mathcal{P}(\mu) \cap J_{\beta} = \mathcal{P}(\mu) \cap \Sigma_{n}^{(J_{\beta}, \bar{\lambda})} \subseteq \mathcal{P}(\mu) \cap \Sigma_{n+1}^{(J_{\beta}, \bar{\lambda})}.$$

In particular, this holds for $\mu = \bar{\lambda}$.

At this point, we make another simplifying assumption, that $n = 0$. To complete the proof in the case $n \geq 1$, instead of working with $J_{\beta}$ as is done below, one works with Jensen's $\Sigma_{n}$-coding structure $(J_{\rho}, \bar{\epsilon}, A)$, where $\rho$ is the $\Sigma_{n}$-projectum of $J_{\beta}$ and $A$ is the $\Sigma_{n}$mastercode for $J_{\beta}$. (See [Je].)

For any structure $\mathfrak{A}$ and any set $S \subseteq |\mathfrak{A}|$, define

$$H^{\mathfrak{A}}(S) = \{ r^{\mathfrak{A}}[g] \mid g \in S \text{ and } r \text{ is a } \Sigma_{1} \text{ Skolem term} \}.$$

Using the fact from [Je] that $J_{\beta}$ is sound, we conclude that for some $x \in J_{\beta}$,

$$J_{\beta} = H^{(J_{\beta}, \bar{\lambda})}(\kappa \cup \{ \bar{\lambda} \}).$$
For example, the first standard parameter of \( J_\beta \) satisfies this equation. The specific choice of \( x \) is not relevant here however. (In this sense, less fine structure is used in the proof of the covering lemma than in the proof of Square.)

**Claim 2.** \( \text{ult}(J_\beta, \pi, \lambda) \) is wellfounded.

The proof of Claim 2 uses countable closure as did that of Claim 1. (In Section 3, we shall sketch how to avoid countable closure altogether.) It follows from Claim 2 that \( \text{ult}(J_\beta, \pi, \kappa) \) is wellfounded.

By Claim 2, we may identify \( \text{ult}(J_\beta, \pi, \lambda) \) with \( J_\gamma \) for some ordinal \( \gamma \). Let

\[
\pi : \beta \rightarrow J_\gamma
\]

be the ultrapower map. By Loe's theorem adapted to long extenders, \( \pi \) is a \( \Sigma_0 \)-elementary embedding of \( J_\beta \) into \( J_\gamma \). Since the ultrapower is formed using only functions in \( J_\beta \), \( \pi \) is a cofinal embedding in the sense that

\[
\sup(\pi(\beta)) = \gamma.
\]

So, in fact, \( \pi \) is \( \Sigma_1 \)-elementary. If \( \beta \) is a limit ordinal, \( x \in J_\beta \), and \( \varphi \) is a \( \Sigma_1 \) formula, then \( J_\beta \models \varphi(x) \) iff

\[
\exists \beta_0 < \beta \ \forall \beta_1 > \beta_0 \ J_\beta \models \varphi(x),
\]

and cofinal \( \Sigma_0 \) embeddings preserve formulas of this form. If \( \beta \) is a successor ordinal, then one uses the \( S \)-hierarchy instead of the \( J \)-hierarchy.

The relationship between \( \pi : L_\alpha \rightarrow L_\alpha^+ \) and \( \tilde{\pi} : J_\beta \rightarrow J_\gamma \) is obscured by the fact that \( \alpha < \beta \) while \( \kappa^+ > \gamma \). However, a little work shows that \( \pi \) and \( \tilde{\pi} \) agree below \( \lambda \). The main point is that for every \( a \in \lambda \),

\[
a \models \pi(\text{id})(a) = [a, \text{id}]^J_\beta_{|B_\pi},
\]

where \( \text{id} \) is the identity map \( u \mapsto u \).

We leave it as an exercise to verify that

\[
\text{ult}(J_\beta, \pi, \lambda) = \text{ult}(J_\beta, \pi, \kappa)
\]

and that \( \tilde{\pi} \) equals the ultrapower map from \( J_\beta \) into \( \text{ult}(J_\beta, \pi, \kappa) \). The main point is that for every \( \mu \in \text{t-\lambda} \), there is a well-order \( W \) of \( \kappa \) in \( J_\beta \) and an ordinal \( \iota < \kappa \) such that \( \pi(\mu) \) is the rank in \( \pi(W) \) of \( \iota \), and so

\[
\pi(\mu) = [\{\iota\}, \xi \mapsto \text{the rank of } \xi \text{ in } W]^J_\beta_{|B_\pi}.
\]

The \( \Sigma_1 \) elementarity of \( \tilde{\pi} \) translates the soundness of \( J_\beta \) into the following properties of \( J_\gamma \):

\[
\tilde{\pi}^\kappa J_\beta = H_1^{(J_\gamma, \kappa)}(\tilde{\pi}(\kappa) \cup \{\tilde{\pi}(x)\})
\]

and

\[
J_\gamma = H_1^{(J_\gamma, \kappa)}(\text{lh}(E) \cup (\tilde{\pi}^\kappa J_\beta)) = H_1^{(J_\gamma, \kappa)}(\lambda \cup \{\tilde{\pi}(x)\}).
\]

Using the fact that \( E \) and \( E \upharpoonright \kappa \) give the same ultrapower,

\[
J_\gamma = H_1^{(J_\gamma, \kappa)}(\lambda \cup \{\tilde{\pi}(x)\}).
\]

Therefore

\[
J_{\gamma+1} \models \tilde{\pi}(\lambda) < \kappa^+.
\]

Since \( \tilde{\lambda} \) is a regular in \( J_\beta \) and \( \tilde{\lambda} > \mu_\alpha \) for all \( \alpha \in [\kappa]^{<\omega} \), it follows that \( \tilde{\pi} \) continuous at \( \lambda \). [Every function of the form

\[
f : [\mu_\alpha]^{<\omega} \rightarrow \tilde{\lambda}
\]

is bounded in \( \tilde{\lambda} \).] Recall that \( \pi \) is also continuous at \( \lambda \). Consequently,

\[
\tilde{\pi} \upharpoonright J_{\lambda+1} = \pi \upharpoonright J_{\lambda+1}
\]

In particular, since \( \lambda = \tilde{\pi}(\lambda) \),

\[
J_{\lambda+1} \models \lambda < \kappa^+,
\]

which is the desired contradiction.

We have shown that if \( 0^\# \) does not exist, \( \kappa \) is a countably complete cardinal, and \( \lambda = (\kappa^+)^L \), then either \( \lambda = \kappa^+ \) or \( \text{cf}(\lambda) = \kappa \). The proof easily adapts to show that if \( 0^\# \) does not exist, \( \lambda = (\kappa^+)^L \), and \( \text{card}(\kappa) \) is countably closed, then either \( \lambda = \text{card}(\kappa)^+ \) or \( \text{cf}(\lambda) = \text{card}(\kappa) \).

2 Intermission: a few words about higher core models

We have seen how embeddings between transitive sets give rise to extenders and vice-versa. In this way, extenders can be seen as witnesses to large cardinal properties. For many reasons, it is desirable to construct inner models with large cardinals for which a Jensen style fine structural analysis is possible. That being the case, it is natural to consider inner models of the general form \( L[B] \), where \( B \) is a sequence of extenders which is intended to witness various large cardinal properties in \( L[B] \). Core models are certain inner models which have this form. We shall not give the precise definition, which can
be found in [MiSt]. Rather, in order to highlight some of the key points, we shall tell a few small lies.

Initial segments of core models are structures of the form

$J^E_\alpha = \langle J^E_\alpha, \in, E, \alpha, E_\alpha \rangle$

and are known as premice. 1 Part of the definition is that either $E_\alpha = \emptyset$, or there is an ordinal $\mu < \alpha$ such that $E_\alpha$ is a $(\mu, \alpha)$-extender over $J^E_\alpha$. In other words, there is a premouse $J^E_\alpha$ and an embedding $\pi : J^E_\alpha \rightarrow J^E_\alpha$ such that $\text{crit}(\pi) = \mu$ and $E_\alpha$ is the extender of length $\alpha$ derived from $\pi$. There is also a coherence condition, which says that

$F_\alpha \upharpoonright \alpha = E_\alpha \upharpoonright \alpha$

and

$F_\alpha = \emptyset$.

The first part of the coherence condition says that $E_\alpha$ is “strong” in the sense of large cardinals. That is, to say, $J^E_\alpha$ and its ultrapower by $E_\alpha$ agree below $\alpha$. The second condition is used to compare premice: if two mice disagree at level $\alpha$, then applying the extenders with index $\alpha$ improves the agreement to levels up to and including $\alpha$ for the ultrapowers. In the right context and with the technology of iteration trees, this naive approach to comparison can be made to work. Roughly speaking, premice which can be compared are known as mice.

We would like to say something about the difficulties in obtaining the covering results mentioned in the Introduction. So let us imagine that we are attempting to prove that a core model, $L[\tilde{E}]$, has the weak covering property. Proceeding along the lines of Section 1, we have a cardinal $\kappa$, its $L[\tilde{E}]$-successor cardinal $\lambda$,

$N \rightarrow X \prec V_{\kappa^+}$

where $N$ is transitive of cardinality $< \kappa$ and ordinal height $\alpha$, and the critical point of $\pi$ is $\delta$. But there is an immediate difficulty with condensation. Namely, the pre-image of $L[\tilde{E}]$ under $\pi$ need not be an initial segment of $L[\tilde{E}]$. In other words,

$\pi^{-1}(L[\tilde{E}]) = L_\alpha[\tilde{E}]$

for some extender sequence $\tilde{E}$, but $E \upharpoonright \alpha$ and $\tilde{E}$ may be different. This difficulty actually arises, and is just one reason for the comparison process using iteration trees mentioned above. (The next step would be to compare $L[\tilde{E}]$ and $L[\tilde{F}]$, but there is no reason to think that this comparison is trivial.)

Suppose that we manage to avoid the first difficulty altogether; there is still a second and more serious problem. Let us consider the case in which, like the Hard Case in Section 1, not every subset of $\delta$ from $L[\tilde{E}]$ is in $L_\alpha[\tilde{E}]$. We may proceed as before, letting $J^E_\gamma$ be the first level of $L[\tilde{E}]$ over which a subset of $\kappa = \pi^{-1}(\delta)$ which is missing from $L_\alpha[\tilde{E}]$ is definable. Assume that $\Sigma_1$ is the minimal complexity for such a definition and proceed assuming that the fine structure of $L[\tilde{E}]$ generalizes that of $L$ in a straightforward way. Countable completeness allows us to conclude that

$\text{ult}(J^E_\alpha, \pi, \lambda)$ is wellfounded.

Say

$\text{ult}(J^E_\alpha, \pi, \lambda) = J^E_\gamma$.

As in the Hard Case of Section 1, $J^E_\gamma$ is a premouse over which $\lambda$ is seen to have cardinality $\leq \kappa$. But, for this to be a contradiction, we would want that $J^E_\gamma \in L[\tilde{E}]$. The difficulty is that $E \upharpoonright \gamma$ and $\tilde{G}$ may be different. Again, the idea that leads to a solution (when there is a solution) is to compare $L[\tilde{E}]$ and $L[\tilde{G}]$ using iteration trees.

3 Second pass at Jensen’s proof: weak covering without countable closure

We now describe how to replace the assumption of countable closure with $\aleph_2 \leq \kappa$ in Section 1. Once again, for simplicity of presentation, let $\kappa$ be a cardinal, not just an $L$-cardinal. We assume that $\text{cf}(\kappa) < \kappa$ and show that $0^+$ exists. In Section 1, countable closure was used to find an $X \prec V_{\kappa^+}$ with $\omega \cdot X \subseteq X$, which, in turn, was used to prove the two claims of wellfoundedness. Even without countable closure, we can show that “many” $X$ satisfy Claims 1 and 2 of Section 1. We shall find such $X$ along an internally approachable chain.

Let $\varepsilon$ be a regular cardinal with $\text{cf}(\kappa) < \varepsilon$ and $\aleph_2 \leq \varepsilon \leq \kappa$. Depending on whether or not $\lambda$ has countable cofinality, either $\varepsilon = \aleph_2$ or $\varepsilon = (\text{cf}(\kappa))^+$ would do.

Let $\langle X_i \mid i < \varepsilon \rangle$ be a continuous chain of elementary substructures of $V_{\kappa^+}$ such that for all $j < \varepsilon$, $\langle X_i \mid i \leq j \rangle \subseteq X_{j+1}$.
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\[ X_j \cap \varepsilon \subseteq \varepsilon, \]

and

\[ \text{card}(X_j) = \text{card}(X_j \cap \varepsilon). \]

Assume also that \( \kappa \in X_0 \). For \( i < \varepsilon \), let \( \varepsilon_i = X_i \cap \varepsilon \). Note that \( \{ \varepsilon_i \mid i < \varepsilon \} \)

is a normal sequence converging to \( \varepsilon \). For \( i < \varepsilon \), let \( \pi_i : N_i \rightarrow V_{\varepsilon_i} \) be the inverse of the transitive collapse of \( X_i \) and \( \alpha_i \) be the ordinal height of \( X_i \).

So \( \text{crit}(\pi_i) = \varepsilon_i \) and \( \pi_i(\varepsilon_i) = \varepsilon \). Say \( \pi_i(\kappa_i) = \kappa \) and \( \pi_i(\lambda_i) = \lambda \). Let \( E_i \) be the extender of length \( \lambda \) derived from \( \pi_i \).

**Easy Case.** There is a stationary set

\[ S \subseteq \{ i < \varepsilon \mid \text{cf}(i) \geq \aleph_1 \text{ and } i = \varepsilon_i \} \]

such that \( i \in S \) implies

\[ \mathcal{P}(\varepsilon_i) \cap L \subseteq L_{\alpha_i}. \]

For \( i \in S \), let \( U_i \) be the ultrafilter on \( N_i \) derived from \( \pi_i \).

**Claim 1.** There is an \( i \in S \) such that \( \text{ult}(L, U_i) \) is wellfounded.

Claim 1 implies that \( 0^\# \) exists. We leave the proof of Claim 1 as an exercise since its proof is similar to that of Claim 2 below.

**Hard Case.** There is a stationary set

\[ S \subseteq \{ i < \varepsilon \mid \text{cf}(i) \geq \aleph_1 \text{ and } i = \varepsilon_i \} \]

such that \( i \in S \) implies

\[ \mathcal{P}(\varepsilon_i) \cap L \nsubseteq L_{\alpha_i}. \]

We call a partial function \( F \) on \( \varepsilon \) a choice function iff \( F(i) \in X_i \) for all \( i \in \text{dom}(F) \).

Fodor’s lemma is used in the following general form in proofs of extensions of the covering lemma (cf. [MiSch]), although here we only need to consider choice functions into the integers.

**Fodor’s lemma.** Suppose that \( F \) is a choice function and that \( \text{dom}(F) \) is stationary in \( \varepsilon \). Then there is a stationary \( S \subseteq \text{dom}(F) \) on which \( F \) is constant.

**Proof.** Let \( \{ G_i \mid i < \varepsilon \} \) be a sequence, which is strictly increasing and continuous with respect to inclusion, such that for all \( i < \varepsilon \), \( G_i \) is a function

from \( \varepsilon_i \) onto \( X_i \). Let \( C = \{ i < \varepsilon \mid \varepsilon_i = i \} \). Then \( C \) is club in \( \varepsilon \) and if \( i \in C \), then \( \text{crit}(\pi_i) = \varepsilon_i = i \). Define \( H \) on \( \text{dom}(F) \cap C \) by \( H(i) = (G_i)^{-1}(F(i)) \).

Then \( H(i) < \varepsilon_i = i \) for all \( i \in \text{dom}(H) \). By the usual Fodor’s lemma, there is a stationary set \( S \subseteq \text{dom}(H) \) on which \( H \) is constant. Suppose that \( i, j \in S \) and \( i < j \). Then \( F(i) = G(H(i)) = G_j(H(i)) = G_j(H(j)) = F(j) \).

Therefore, \( F \) is constant on \( S \).

Fix \( S \) as in the case hypothesis. For \( i \in S \), let \( \beta_i \) be the least ordinal \( \beta \) such that

\[ \mathcal{P}(\kappa_i) \cap J_{\beta_i+1} \nsubseteq L_{\alpha_i} \]

and \( \eta_i \) be the least \( n < \omega \) such that there is a subset of \( \kappa_i \) which is \( \Sigma_{n+1} \) definable over \( J_{\beta_i} \) but not an element of \( J_{\beta_i} \). Let \( n < \omega \) and \( T \subseteq S \) be a stationary set such that \( \eta_i = n \) for \( i \in T \). For simplicity, let us assume that \( n = 0 \). (The other cases are handled using \( \Sigma_n \) fine structure for \( J_{\beta_i} \) as described earlier.)

**Claim 2.** There is an \( i \in T \) such that \( \text{ult}(J_{\beta_i}, \pi_i, \kappa_i) \) is wellfounded.

Claim 2 leads to a contradiction as did the corresponding claim in Section 1. Suppose, then, that \( \text{ult}(J_{\beta_i}, \pi_i, \kappa_i) \) is illfounded for every \( i \in T \). One says that \( J_{\beta_i} \) lifts badly from \( i \) to \( \varepsilon \). Let

\[ j \in T \cap \text{lim}(T). \]

Let \( Y \) be a countable elementary submodel of \( V_{\varepsilon} \) with

\[ (X_i \mid i < \varepsilon), T, j \in Y. \]

Fix an \( i < j \) such that \( i \in T \) and

\[ Y \cap X_j \subseteq X_i. \]

[Recall that \( j \) has uncountable cofinality.] We shall need to consider the natural map \( \pi_{ij} : N_i \rightarrow N_j \) and the extender \( E_{ij} \) of length \( \lambda_i \) derived from \( \pi_{ij} \). The next subclaim shows that if \( J_{\beta} \) lifts badly from \( i \) to \( \varepsilon \), and \( J_{\beta} \) is definable (so that \( \beta \in X_j \)), then \( \beta \) lifts badly from \( i \) to \( j \).

**Subclaim A.** Suppose that \( \beta \leq \beta_i, \beta \in X_j, \) and

\[ \text{ult}(J_{\beta_i}, \pi_i, \kappa_i) \) is illfounded.

\[ \text{ult}(J_{\beta_i}, \pi_i, \kappa_i) \) is illfounded.

\[ \text{ult}(J_{\beta_i}, \pi_i, \kappa_i) \) is illfounded.
Then
\[ \text{ult}(J_{\beta}, \pi_{i, j}, \kappa_j) \text{ is illfounded.} \]

In particular,
\[ \text{ult}(J_{\delta_1}, \pi_{i, j}, \kappa_j) \text{ is illfounded.} \]

Proof. Applying the elementarity of \( \pi_j : N_j \rightarrow V_{\kappa^+} \) we have that
\[ N_j \models \text{ult}(\pi_j^{-1}(\beta_1), \pi_j^{-1}(\kappa_1), \pi_j^{-1}(\kappa)) \text{ is illfounded.} \]

Hence,
\[ N_j \models \text{ult}(J_{\beta}, \pi_{i, j}, \kappa_j) \text{ is illfounded.} \]

Subclaim A then follows by the absoluteness of illfoundedness.

Let \( \beta_{\text{min}} \) be the least \( \beta \leq \beta_i \) such that
\[ \text{ult}(J_{\beta}, \pi_{i, j}, \kappa) \text{ is illfounded.} \]

So \( J_{\beta_{\text{min}}} \) is the least level of \( L \) which lifts badly from \( i \) to \( e \). This definition puts \( \beta_{\text{min}} \in X_j \). Since \( \beta_{\text{min}} \) satisfies the hypothesis of Subclaim A,
\[ \text{ult}(J_{\beta_{\text{min}}}, \pi_{i, j}, \kappa_j) \text{ is illfounded.} \]

In other words, \( J_{\beta_{\text{min}}} \) lifts badly from \( i \) to \( j \).

Subclaim B. There is a \( \beta^* \leq \beta_i \) such that
\[ \text{ult}(J_{\beta^*}, \pi_{i, j}, \kappa_j) \text{ is wellfounded,} \]

while
\[ \text{ult}(J_{\beta^*}, \pi_i, \kappa) \text{ is illfounded.} \]

Suppose that \( \beta^* \) is as in Subclaim B. By the definition of \( \beta_{\text{min}} \) and the second clause of Subclaim B, \( \beta_{\text{min}} \leq \beta^* \). Therefore, since \( J_{\beta_{\text{min}}} \) lifts badly from \( i \) to \( j \), \( J_{\beta^*} \) also lifts badly from \( i \) to \( j \). But this is in direct contradiction with the first clause of Subclaim B.

It remains to find \( \beta^* \) as in Subclaim B. We shall realize \( J_{\beta^*} \) as a kind of “pullback” of \( J_{\beta_1} \) to \( i \).

The soundness of \( J_{\beta_1} \) implies that there is an \( x \in J_{\beta_1} \) such that
\[ J_{\beta_1} = \bigcup \{ Z_{\sigma, x} \mid \sigma < \beta_1 \}. \]

where \( Z_{\sigma, x} \) be the \( \Sigma_1 \) hull in \( (J_{\sigma}, \in) \) of \( \kappa_1 \cup \{ x \} \). [One may take \( x \) to be the standard parameter of \( J_{\beta_1} \).] Moreover, for each \( \sigma < \beta_1 \), the transitive collapse of \( Z_{\sigma, x} \) is \( J_{\gamma} \) for some \( \gamma \leq \lambda_1 \). Thus, there is a directed system \( D \subseteq J_{\lambda_1} \) whose direct limit is \( J_{\beta_1} \). \( D \) consists of the transitive collapses of \( Z_{\sigma, x} \) for \( \sigma < \beta_1 \) and \( x \in J_{\sigma} \). Let \( D^* \subseteq J_{\lambda_1} \) be the direct limit system consisting of those \( J_\gamma \) such that \( \pi_{i, j}(J_{\gamma}) \) is the transitive collapse of \( Z_{\sigma, x} \) for some \( \sigma < \beta_1 \) and \( x \in J_{\sigma} \). [Part of the point here is that \( Z_{\sigma, x} \) might not be in the range of \( \pi_{i, j} \) even if its transitive collapse is.]

There is a natural way in which \( \pi_{i, j} \) extends to an embedding from the direct limit of \( D^* \) into the direct limit \( J_{\beta_1} \) of \( D \). This has several consequences:

**Facts.**

1. the direct limit of \( D^* \) is wellfounded,

2. there is an ordinal \( \beta^* \) such that \( J_{\beta^*} \) is the transitive collapse of the direct limit of \( D^* \),

3. \( \beta^* \leq \beta_i \) and so \( \text{ult}(J_{\beta_i}, \pi_{i, j}, \kappa_j) \) is defined,

4. there is a commutative system of embeddings as in the diagram:

\[ J_{\beta_i} \rightarrow J_{\beta_1} \rightarrow \text{ult}(J_{\beta_1}, \pi_j, \kappa) \]

\[ \downarrow \]

\[ \text{ult}(J_{\beta^*}, \pi_{i, j}, \kappa_j) \rightarrow \text{ult}(J_{\beta_i}, \pi_i, \kappa) \]

5. \( \text{ult}(J_{\beta^*}, \pi_{i, j}, \kappa_j) \) is wellfounded since it embeds into \( J_{\beta_1} \),

6. \( \text{ult}(J_{\beta^*}, \pi_i, \kappa) \) is also defined, however

7. \( \text{ult}(J_{\beta^*}, \pi_i, \kappa) \) is illfounded.

The reason for Fact 7 is that there is a witness to the illfoundedness of \( \text{ult}(J_{\beta_1}, \pi_j, \kappa) \) in the range of the embedding from \( \text{ult}(J_{\beta^*}, \pi_i, \kappa) \) to \( \text{ult}(J_{\beta_1}, \pi_j, \kappa) \). Here are a few hints why. Since \( Y \prec V_{\kappa^+}, \)
\[ (Y, \in) \models \text{ult}(J_{\beta_1}, \pi_j, \kappa) \text{ is illfounded.} \]

So, there are functions \( f_k \in J_{\beta_i} \cap Y \), “coordinates” \( a_k \in [\kappa)^{<\omega} \).
and “measure one” sets  
\[ A_k \in (E_j)_{a_k} \cap J_{\beta_j} \]
so that  
\[ \langle [a_k, f_k]_{E_j}^{J_{\beta_j}} | k < \omega \rangle \]
is an infinite descending chain of \( \text{ult}(J_{\beta_j}, \pi_j, \kappa) \) as witnessed by \( \langle A_k | k < \omega \rangle \). Let \( x_\ell = \langle f_k | k < \ell \rangle \) and choose sufficiently large \( \sigma_\ell \in \beta_j \cap Y \) so that  
\[ J_{\sigma_\ell} \models " f_0 \equiv \ldots \equiv f_\ell \text{ on } A_{\ell} " \]
[More precisely, we may assume that \( a_k \subset a_\ell \) whenever \( k < \ell \). Whenever  
\[ a = \{ a_1 < \ldots < a_m \} = \{ b_{p_1} < \ldots < b_{p_m} \} \subseteq b = \{ b_1 < \ldots < b_n \} \]
and  
\[ u = \{ u_1 < \ldots < u_n \}, \]
then we define  
\[ u^{a, b} = \{ u_{p_1}, \ldots, u_{p_m} \}. \]
What we require above is that  
\[ J_{\sigma_\ell} \models \{ u \in [a_k]^n | f_k (u^{a, b}) \equiv f_k (u) \} \subseteq A_{\ell} \]
whenever \( k < \ell, a = a_k, b = a_\ell, \text{ and } n = \text{card}(b) \). Then the transitive collapse of each \( Z_{x_\ell, y_i} \) is in \( J_{\beta_j} \cap Y \). Some routine checking, which we leave as an exercise, shows that for every \( k < \omega, \]
\[ [a_k, f_k]_{E_j}^{J_{\beta_j}} \]
is in the range of the natural embedding from \( \text{ult}(J_{\beta_j}, \pi_i, \kappa) \) to \( \text{ult}(J_{\beta_j}, \pi_i, \kappa) \). So Fact 7 holds.

We have shown that if \( 0^\# \) does not exist, \( \kappa \) is a cardinal, \( \kappa \geq \aleph_2 \), and \( \lambda = (\kappa^+)\aleph_1 \), then either \( \lambda = \kappa^+ \) or \( \text{cf}(\lambda) = \kappa \). The proof easily adapts to show that if \( 0^\# \) does not exist and \( \lambda \) is any successor cardinal of \( L \) such that \( \aleph_2 \leq \lambda \), then \( \text{cf}(\lambda) = \text{card}(\lambda) \).

4 Third pass at Jensen's proof: putting it all together

Let us assume that \( 0^\# \) does not exist. We say that \( Y \) covers \( X \) if \( Y \supseteq X \) and \( \text{card}(Y) = \text{card}(X) \). We prove by induction on ordinals \( \lambda \) that for every uncountable \( X \subseteq \lambda \), there is a \( Y \in L \) which covers \( X \).

So fix \( \lambda \) and \( X \). Clearly, by the induction hypothesis, we may assume that \( \text{sup}(X) = \lambda \). We may also assume that \( \lambda \) is an \( L \)-cardinal. \[ \text{[Otherwise, there is a } \kappa < \lambda \text{ and a constructible bijection } f : \kappa \rightarrow \lambda. \text{ By the induction hypothesis, there is a } Y \in L \text{ which covers } f^{-1}(X). \text{ Then } f''Y \text{ covers } X. \]
And, also, we may assume that \( \lambda \) is not a cardinal, since otherwise \( \lambda \) itself covers \( X \). For the same reason, we may assume that \( \text{card}(X) < \text{card}(\lambda) \) and \( \aleph_2 \leq \lambda \). Of course, \( \text{cf}(\lambda) \leq \text{card}(X) \).

Let \( \varepsilon \) be a regular cardinal with \( \text{card}(X) < \varepsilon < \lambda \) and \( \aleph_2 \leq \varepsilon < \lambda \). For example, \( \varepsilon = \text{card}(X)^+ \) would do.

As in Section 3, select an internally approachable chain \( \langle X_i | i < \varepsilon \rangle \) of substructures of \( V_{\lambda^+} \), but make sure that \( X_0 \supseteq X \). Let us use the same notation as in Section 3. Since \( 0^\# \) does not exist, we are in the Hard Case. We may not assume that \( \lambda \) is an \( L \)-successor cardinal. However, we may still define \( \beta_i \) to be the least ordinal \( \beta \) such that for some \( \mu < \lambda, \]
\[ P(\mu) \cap J_{\beta+1} \subset L_{\alpha_i} \]
and \( n_i \) to be the least \( n < \omega \) such that there is a bounded subset of \( \lambda \) which is \( \Sigma_{n+1} \)-definable over \( \langle J_{\beta_i}, \varepsilon \rangle \) but not an element of \( J_{\beta_i} \). As before, we restrict attention to the case \( n_1 = 0 \).

As in Claim 2 of Section 3, we find \( i < \varepsilon \) such that \( \text{ult}(J_{\beta_i}, \pi_i, \lambda) \) is well-founded. (Again, some minor modifications must be made to allow for the possibility that \( \lambda \) is a limit cardinal of \( L \).) Fix such an \( i \) and put \( \pi = \pi_i, \delta = \varepsilon_i, \lambda = \lambda_i, \alpha = \alpha_i, \) and \( \beta = \beta_i \). And, also, say  
\[ \bar{\pi} : J_\beta \rightarrow J_\gamma = \text{ult}(J_{\beta_i}, \pi_i, \lambda). \]

Then, as in Section 1,
\[ \text{ran}(\bar{\pi}) \cap (\lambda + 1) = \text{ran}(\pi) \cap (\lambda + 1) = X_i \cap (\lambda + 1) \supseteq X. \]

By soundness, there is a \( \mu < \lambda \) and an \( x \in J_\beta \) such that  
\[ J_\beta = H_1^{(x_i, \varepsilon)}(\mu \cup \{ x \}). \]
[Let \( \mu \) be the \( \Sigma_1 \) projection and \( x \) be the first standard parameter of \( J_\beta \).] Then  
\[ X \subseteq \text{ran}(\bar{\pi}) = H_1^{(x_i, \varepsilon)}(\pi(\mu) \cup \{ \pi(x) \}) = H_1^{(x_i, \varepsilon)}(\pi(\mu) \cup \{ \pi(x) \}). \]

Since \( \pi(\mu) < \lambda \), by the induction hypothesis, there is a set \( Y \in L \) which covers \( \pi(\mu) \). Therefore \( X \) is covered by  
\[ Z = \lambda \cap H_1^{(\lambda, \varepsilon)}(Y \cup \{ \pi(x) \}) \]
and \( Z \in L \), as desired.
References


