COVERING AT LIMIT CARDINALS OF K

WILLIAM J. MITCHELL AND ERNEST SCHIMMERLING

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1. Introduction

Mitchell proved that if there is no transitive class model of ZFC with a measurable cardinal of maximal Mitchell order, then the core model, K, exists and, for every cardinal μ of K, if $\mu \geq \omega_2$ and $\nu = (\mu^+)^K$, then the cofinality of ν is at least the cardinality of μ. For example, if $\mu = \aleph_\omega$ then $(\mu^+)^K = \aleph_{\omega+1}$. Under the same anti-large cardinal hypothesis, Mitchell also proved that for every regular cardinal ν of K, if $\nu > \omega_2$ and $\mathrm{cf}(\nu) < |\nu|$, then ν is a measurable cardinal in K. Moreover, if ν has uncountable cofinality, then the Mitchell order of ν in K has rank at least $\mathrm{cf}(\nu)$. For example, if $\nu = \aleph_\omega$ and ν is regular in K, then ν is measurable in K. Both results are described as covering theorems for the Mitchell core model, the first for its successor cardinals, the second for its limit cardinals.

Mitchell's theorems on the existence of K and his two covering theorems for K extended the pioneering body of work by Dodd and Jensen on their core model for one measurable cardinal. These two earlier covering lemmas for limit cardinals of K provided companion theorems for results on changing a regular cardinal into a singular cardinal by forcing. If ν is a measurable cardinal, then Prikry forcing converts ν to a singular cardinal of countable cofinality. By Dodd and Jensen, if ν is a regular cardinal and there exists a poset that forces ν to be a singular cardinal, then there is an inner model in which ν is a measurable cardinal. If $\omega < \mathrm{cf}(\kappa) = \kappa < \nu$ and ν is a measurable cardinal with $o(\nu) \geq \kappa$, then Magidor forcing makes $\kappa = \mathrm{cf}(\nu) < \nu = |\nu|$. By Mitchell, if ν is a regular cardinal and there exists a poset that forces $\omega < \kappa = \mathrm{cf}(\nu) < \nu = |\nu|$, then there is an inner model in which ν is measurable with $o(\nu) \geq \kappa$.

In 1990, Steel [St] proved that K exists assuming that there is no transitive class model of ZFC with a Woodin cardinal. This is a weaker anti-large cardinal hypothesis than Mitchell's. However, Steel made a background assumption about V in addition to ZFC. If Ω is a measurable cardinal, then V_Ω satisfies Steel's background assumption. In 2007, Jensen and Steel [JS] eliminated the additional background assumption from Steel's core model theory and other results about K that had been proved in the intervening years. By 1995, Mitchell, Schimmerling and Steel had extended the covering theorem on successor cardinals to the Steel K. See [MSS] and [MS]. Building on this machinery, we extend the covering theorem on limit cardinals to the Jensen-Steel K.

Theorem 1. *Assume that there is no transitive class model with a Woodin cardinal. Let ν be a singular ordinal such that $\nu > \omega_2$ and $\mathrm{cf}(\nu) < |\nu|$. Suppose ν is a regular cardinal in K. Then ν is a measurable cardinal in K. Moreover, if $\mathrm{cf}(\nu) > \omega$, then $o^K(\nu) \geq \mathrm{cf}(\nu)$. For these results, we assume that K is constructed using Jensen indexing. Should $(2^{\mathrm{cf}(\nu)})^+ < |\nu|$, the results also hold for K constructed using Mitchell-Steel indexing.*

There was an intermediate advance between measurable cardinals of maximal Mitchell order and Woodin cardinals. Cox [C] proved Theorem 1 under the more restrictive hypothesis that there is no transitive class model with a strong cardinal.[1] The ideas and exposition in his paper influenced ours.

Woodin's stationary tower, $\mathbb{P}_{<\delta}$, is another poset that is germane to this area and informs possible extensions of core model theory. If δ is a Woodin cardinal and $\mathrm{cf}(\kappa) = \kappa < \mathrm{cf}(\nu) = \nu < \delta$, then the stationary set $\{X \prec V_\nu \mid \nu \cap X \in \nu \text{ and } \mathrm{cf}(\nu \cap X) = \kappa\} \in \mathbb{P}_{<\delta}$ forces $\kappa = \mathrm{cf}(\nu) < \nu = |\nu|$. Can K exist and have a Woodin cardinal? If so, then either K is not forcing absolute or K does not have the covering property at ν.

To explain the disclaimer about indexing at the end of Theorem 1, we remind the reader that K is a particular extender model, so K is constructed from a class sequence, $E = \langle E_\alpha \mid \alpha \in \mathrm{OR}\rangle$, where each E_α is an extender of length α over the initial segment $J_\alpha^{E\restriction\alpha}$ of $K = L[E]$. Two styles of indexing are in vogue for extender models. That used by Jensen requires

$$\alpha = \big(j(\kappa)^+\big)^{\mathrm{Ult}(J_\alpha^{E\restriction\alpha}, E_\alpha)}$$

where $j : J_\alpha^{E\restriction\alpha} \to \mathrm{Ult}(J_\alpha^{E\restriction\alpha}, E_\alpha)$ is the ultrapower map and $\kappa = \mathrm{crit}(j)$. In the other style, which is used by Mitchell and Steel [MS], each α must be the cardinal successor in $\mathrm{Ult}(J_\alpha^{E\restriction\alpha}, E_\alpha)$ of

$$(\kappa^+)^{J_\alpha^{E\restriction\alpha}} \cup \sup\left(\{\xi + 1 \mid \xi \text{ is a generator of } E_\alpha\}\right).$$

[1]The actual hypothesis for Cox's theorem is that $0^\P$ does not exist. The existence of $0^\P$ is equivalent to that of a proper class transitive model with a strong cardinal for which there are indiscernibles. Cox used Jensen indexing.

We started off proving Theorem 1 using Mitchell-Steel indexing because it is used in the most complete published accounts of core model theory. Should ν be a singular strong limit cardinal, or merely $(2^{\mathrm{cf}(\nu)})^+ < |\nu|$, our proof of Theorem 1 is not sensitive to the choice of Mitchell-Steel or Jensen indexing. Unfortunately, we only found a proof of the full result using Jensen indexing. Therefore, at a certain point in the proof of Theorem 1, we will pivot from Mitchell-Steel to Jensen fine structure.

There is reason to expect that the universe of K is the same regardless of which indexing is used. (Of course, we mean under a hypothesis that implies they both exist such as the non-existence of a model with a Woodin cardinal.) Ideas for how such an argument might go are known but we have not seen the details worked out ourselves.

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2. Covering machinery

Assume that there is no transitive class model of ZFC with a Woodin cardinal and ν is a singular ordinal with $\nu > \omega_2$ and $\mathrm{cf}(\nu) < |\nu|$.

To simplify core model theory, we assume that there is a measurable cardinal Ω such that $\nu < \Omega$ and let U_Ω be a normal measure over Ω. We merely point to [JS] for how to do without this assumption. Let $K = K^{V_\Omega}$ be the Steel core model defined over V_Ω. See [S] and [St] for what this means. Assume that ν is a regular cardinal in K. Our goal is to show that ν is a measurable cardinal in K.

This core model K is an extender model with Mitchell-Steel indexing. The definitions and proofs of facts in this section can be modified in obvious ways so as to pertain to Jensen indexing. Only Lemmas 2.5 and 2.6 in this section are particular to Jensen indexing.

In this context, we use the terminology *set mouse* for a mouse of height $< \Omega$ and *weasel* for a mouse of height Ω. Another important weasel is $K^c = (K^c)^{V_\Omega}$, Steel's background certified core model. By the covering theorem of [MS], we know that

$$\mathrm{cf}\left((\nu^+)^K\right) \geq |\nu| \geq \omega_2.$$

Let Ω_0 be a regular cardinal $\geq |\nu|^{++}$. Using terminology from [St], there is a weasel W such that W is an A_0-soundness witness for $\mathcal{J}^K_{\Omega_0}$

and there is an elementary embedding from W to K^c. Then W is a thick weasel that agrees with K below Ω_0 and has the hull and definability properties at every ordinal less than Ω_0.

In this section, we study an arbitrary elementary substructure

$$X \prec (H_{\Omega^+}, \in, U_\Omega)$$

of cardinality $|X| < |\nu|$ with $\{\nu, W\} \subset X$ and $\sup(\nu \cap X) = \nu$. As we proceed, we will identify two requirements on X. A bit later, we will quote results which say there are X that meet the requirements.

Let $\pi : N \simeq X$ be the inverse of the Mostowski collapse isomorphism, so N is transitive. Put $\overline{\nu} = \pi^{-1}(\nu)$ and $\delta = \mathrm{crit}(\pi)$. Clearly, $\delta < \overline{\nu}$. Let $\overline{W} = \pi^{-1}(W)$. Applying the condensation lemma of [MS], we see that $\overline{W}$ and W agree below $(\delta^+)^{\overline{W}}$. In particular,

$$\mathcal{P}(\delta) \cap \overline{W} \subseteq \mathcal{P}(\delta) \cap W.$$

Let $(\overline{\mathcal{T}}, \mathcal{T})$ be the coiteration of $(\overline{W}, W)$. As W is universal, there are no drops along the main branch of $\overline{\mathcal{T}}$, which we indicate by writing

$$[0, \infty]_{\overline{\mathcal{T}}} \cap \mathcal{D}^{\overline{\mathcal{T}}} = \emptyset,$$

and the last model of $\overline{\mathcal{T}}$ is a proper initial segment of the the last model of $\mathcal{T}$, which we express by writing

$$\mathcal{M}^{\overline{\mathcal{T}}}_\infty \lhd \mathcal{M}^{\mathcal{T}}_\infty.$$

Requirement 1. $(\delta^+)^{\overline{W}} < (\delta^+)^W$ *and* $\overline{W} \lhd \mathcal{M}^{\mathcal{T}}_\infty$. *In other words,*

$$\mathcal{P}(\delta) \cap \overline{W} \subsetneqq \mathcal{P}(\delta) \cap W$$

and $\overline{\mathcal{T}}$ *is the trivial iteration tree on* $\overline{W}$.

This is the first of two requirements on X. Later in the paper, we will need to consider, not just one X, but many, in which case the notation will change to N_X, π_X, δ_X, ν_X, W_X, $\mathcal{T}_X$ etc. Next, we organize several definitions tied to X as a series of subsections.

2.1. **Definitions and properties of $\eta(\mu)$, $\mathcal{P}_\mu$ and $m(\mu)$.** Given a cardinal μ of $\overline{W}$, define $\eta(\mu)$ to be the least $\eta < \mathrm{lh}(\mathcal{T})$ such that the extender sequences of $\overline{W}$ and $\mathcal{M}^{\mathcal{T}}_\eta$ agree below μ. Notice that if $\mathcal{M}^{\mathcal{T}}_{\eta(\mu)}$ is a set mouse and $n = \deg^{\mathcal{T}}(\eta(\mu))$, then $\mathcal{M}^{\mathcal{T}}_{\eta(\mu)}$ is $(n+1)$-sound above μ meaning that it is n-sound and

$$\mathrm{Hull}^{\mathcal{M}^{\mathcal{T}}_{\eta(\mu)}}_{\Sigma_{n+1}} \left(\mu \cup p_{n+1}(\mathcal{M}^{\mathcal{T}}_{\eta(\mu)})\right) = \mathcal{M}^{\mathcal{T}}_{\eta(\mu)}.$$

Alternatively, if $\mathcal{M}^{\mathcal{T}}_{\eta(\mu)}$ is a weasel, then it has the definability property at every ordinal in the interval $[\mu, \Omega_0)$. So, for every thick Γ,

$$\Omega_0 \subseteq \mathrm{Hull}^{\mathcal{M}^{\mathcal{T}}_{\eta(\mu)}}(\mu \cup \Gamma)\,.$$

In the special case $\mu = (\lambda^+)^{\overline{W}}$, either $\mathcal{M}^{\mathcal{T}}_{\eta(\mu)}$ is a set mouse that is $(\deg^{\mathcal{T}}(\eta(\alpha)) + 1)$-sound above λ or $\mathcal{M}^{\mathcal{T}}_{\eta(\mu)}$ is a weasel with the definability property at every ordinal in the interval $[\lambda, \Omega_0)$.

If there exists an initial segment $\mathcal{P} \trianglelefteq \mathcal{M}^{\mathcal{T}}_{\eta(\mu)}$ such that, for some $m < \omega$, the projecta of $\mathcal{P}$ satisfy the inequalities

$$\rho_m(\mathcal{P}) \geq \mu > \rho_{m+1}(\mathcal{P}),$$

then define $\mathcal{P}_\mu$ to be to be the least such mouse $\mathcal{P}$ and $m(\mu)$ to be the corresponding $m < \omega$. In this case, $\mathcal{P}_\mu$ is a set mouse that is $(m(\mu) + 1)$-sound above μ. Implicit here is that the Mitchell-Steel fine structure of a type III mouse $\mathcal{P}$ is based on its squash $\mathcal{P}^{\mathrm{sq}}$ whose ordinal height is $\rho_0(\mathcal{P})$ by definition. Continuing with the definition, if there is no initial segment of $\mathcal{M}^{\mathcal{T}}_{\eta(\mu)}$ that projects strictly below μ, then define $\mathcal{P}_\mu = \mathcal{M}^{\mathcal{T}}_{\eta(\mu)}$ and $m(\mu) = \deg^{\mathcal{T}}(\eta(\mu))$.

We remark that if $\mu = (\aleph_{\alpha+1})^{\overline{W}}$, then what we have defined to be $\eta(\mu)$ and $\mathcal{P}_\mu$ were called $\eta(\alpha)$ and $\mathcal{P}_\alpha$ in [MSS] and [MS]. Those papers were more focused on successor cardinals than we are here.

2.2. **Definitions and properties of $\mathcal{Q}_\mu$ and $n(\mu)$.** By recursion, for every cardinal μ of $\overline{W}$, we define a certain mouse $\mathcal{Q}_\mu$ that agrees with $\mathcal{P}_\mu$ below μ. The definition consists of two mutually exclusive cases that we name the *mouse case* and the *protomouse case*. In the mouse case, we define $\mathcal{Q}_\mu = \mathcal{P}_\mu$. The protomouse case means that there are F, κ and λ such that

- $\mathcal{P}_\mu$ is a type II mouse,
- $m(\mu) = 0$,
- $F = \dot{F}^{\mathcal{P}_\mu}$,
- $\kappa = \mathrm{crit}(F)$,
- $\lambda = (\kappa^+)^{\mathcal{P}_\mu}$,
- $\lambda < \mu$,
- $\sup(\pi[\lambda]) < \pi(\lambda)$,

In the protomouse case, we define $\mathcal{Q}_\mu = \mathrm{Ult}\,(\mathcal{Q}_\lambda, F)$ and $n(\mu) = n(\lambda)$. Because λ is a cardinal $\overline{W}$ strictly less than μ, this is a legitimate recursive definition.[2]

[2]For future reference, in the context of Jensen indexing, the protomouse case is defined the same way except that the expression "type II mouse" is replaced by "active mouse".

The reason for the name "protomouse case" is that, letting $\mathcal{R}$ be the internal ultrapower of $\mathcal{P}_\mu$ by the extender of length $\sup(\pi[\mu])$ derived from π and assuming that $\mathcal{R}$ is wellfounded, we have that $\dot{F}^{\mathcal{R}}$ only measures subsets of $[\pi(\kappa)]^{<\omega}$ that are constructed before stage $\sup(\pi[\lambda])$ in $\mathcal{R}$. In other words, $\dot{F}^{\mathcal{R}}$ is an extender fragment but not a total extender over $\mathcal{R}$. This structure $\mathcal{R}$ is a protomouse but not a premouse.

Whether $\mathcal{Q}_\mu$ is defined according to the protomouse case depends on $\mathcal{P}_\mu$ satisfying the seven criteria listed above. It is easy to see that at least one of the seven must fail for $\mathcal{Q}_\mu$. Moreover, the undesirable property of $\mathcal{P}_\mu$ identified in the previous paragraph is not a problem for $\mathcal{Q}_\mu$. Namely, if $\mathcal{S}$ is the ultrapower of $\mathcal{Q}_\mu$ by the extender of length $\sup(\pi[\mu])$ derived from π, and $\mathcal{S}$ is wellfounded, then it is a premouse. This is the most important difference between $\mathcal{P}_\mu$ and $\mathcal{Q}_\mu$ when they are different. What they have in common is that they are both "mice missing from $\overline{W}$ that are minimal with respect to subsets of μ." What we mean by this is contained in our description of the fine structure of $\mathcal{Q}_\mu$ below.

To better understand the definition of $\mathcal{Q}_\mu$ in the protomouse case, note that there is a unique descending sequence of $\overline{W}$ cardinals

$$\mu = \lambda_0 > \lambda_1 = (\kappa_1^+)^{\overline{W}} > \cdots > \lambda_{k+1} = (\kappa_{k+1}^+)^{\overline{W}}$$

such that $\mathcal{Q}_{\lambda_{k+1}}$ is defined by the mouse case, so

$$\mathcal{Q}_{\lambda_{k+1}} = \mathcal{P}_{\lambda_{k+1}}.$$

and, for every $i \leq k$,

$$\kappa_{i+1} = \operatorname{crit}(\dot{F}^{\mathcal{P}_{\lambda_i}})$$

and

$$\mathcal{Q}_{\lambda_i} = \operatorname{Ult}\left(\mathcal{Q}_{\lambda_{i+1}}, \dot{F}^{\mathcal{P}_{\lambda_i}}\right).$$

This is referred to as the *decomposition* of $\mathcal{Q}_\mu$. For the definition to make sense, we need that each of these ultrapowers is wellfounded. This can be viewed as an enhanced iterability condition on W. It follows from the fact that W elementary embeds into K^c and the corresponding enhanced iterability condition on K^c holds, as was proved by Steel.

In the next few paragraphs, we provide information about the fine structure of $\mathcal{Q}_\mu$ in the protomouse case. Using the coherence properties of the extenders $\dot{F}^{\mathcal{P}_{\lambda_i}}$, one sees that the three mice $\mathcal{Q}_\mu$, $\mathcal{P}_\mu$ and $\overline{W}$ agree below μ. For each $i \leq k+1$, let $\phi_i : \mathcal{Q}_{\lambda_i} \to \mathcal{Q}_\mu$ be the corresponding composition of the ultrapower maps and

$$t = \left(s(\dot{F}^{\mathcal{P}_\mu}) - \mu\right) \cup \bigcup_{1 \leq i \leq k} \phi_i\left(s(\dot{F}^{\mathcal{P}_{\lambda_i}}) - \kappa_i\right)$$

where, in general, $s(F)$ is the Dodd parameter of an extender F. See §4 of [SSZ] for the definition and basic facts about Dodd solidity and Dodd amenability.

First suppose that $\mathcal{P}_{\lambda_{k+1}}$ is a set mouse. Then $\mathcal{Q}_\mu$ is a set premouse and $n(\mu) = m(\lambda_{k+1})$. Call this integer n for the moment. Then, the map

$$\phi_{k+1} : \mathcal{P}_{\lambda_{k+1}} \to \mathcal{Q}_\mu$$

is a cofinal Σ_{n+1} elementary embedding that preserves Σ_{n+1} fine structure in the expected ways. In particular, $\mathcal{Q}_\mu$ is n-sound and

$$p_{n+1}(\mathcal{Q}_\mu) - \mu = \phi_{k+1}\left(p_{n+1}(\mathcal{P}_{\lambda_{k+1}}) - \kappa_{k+1}\right)$$

but $\mathcal{Q}_\mu$ is not $(n+1)$-sound as it and $\mathcal{P}_{\lambda_{k+1}}$ have the same $(n+1)$-core. However, $\mathcal{Q}_\mu$ is $(n+1)$-sound relative to the parameter

$$(p_{n+1}(\mathcal{Q}_\mu) - \mu) \cup t.$$

Moreover, this is the least parameter, r, such that

$$\mathcal{P}(\mu) \cap \mathcal{Q}_\mu \subseteq \text{the transitive collapse of } \mathrm{Hull}^{\mathcal{Q}_\mu}_{\Sigma_{n+1}}(\mu \cup r).$$

To refer to this parameter, let us adopt the notation $q_{n+1}(\mathcal{Q}_\mu, \mu)$. The proof of this characterization uses Theorem 4.2 of [SSZ].[3]

Second, suppose that $\mathcal{P}_{\lambda_{k+1}}$ is a weasel. Then, $\mathcal{Q}_\mu$ is a thick weasel and $\phi_{k+1} : \mathcal{P}_{\lambda_{k+1}} \to \mathcal{Q}_\mu$ is an elementary embedding. Also, $\mathcal{Q}_\mu$ has the the t-definability property at every ordinal in the interval $[\mu, \Omega_0)$. In other words, for every thick class Γ,

$$\Omega_0 \subseteq \mathrm{Hull}^{\mathcal{Q}_\mu}(\mu \cup t \cup \Gamma).$$

Moreover, t is the least parameter r such that $\mathcal{Q}_\mu$ has the r-hull property at μ. That is, for every thick class Γ,

$$\mathcal{P}(\mu) \cap \mathcal{Q}_\mu \subseteq \text{the transitive collapse of } \mathrm{Hull}^{\mathcal{Q}_\mu}(\mu \cup r).$$

The proof of this characterization again uses Theorem 4.2 of [SSZ]. In terms of the class projectum and class parameter used in [MSS], we have $\kappa(\mathcal{Q}_\mu) \leq \mu$ and $c(\mathcal{Q}_\mu) - \mu = t$.

For future reference, we record a comparison of definability over $\mathcal{P}_\mu$ and $\mathcal{Q}_\mu$. The proof is routine.

Lemma 2.1. *Assume that $\mathcal{Q}_\mu$ is defined by the protomouse case. Let $p = p_1(\mathcal{P}_\mu) - \mu$, $F = \dot{F}^{\mathcal{P}_\mu}$, $\kappa = \mathrm{crit}(F)$, $\lambda = (\kappa^+)^{\mathcal{P}_\mu}$ and $s = s(F) - \mu$. Suppose $\alpha < \mu$ is large enough that*

- $\lambda \leq \alpha$,

[3]If $\mathcal{P}_\mu \lhd \mathcal{M}_{\eta(\mu)}$, then $\mathcal{P}_\mu$ is sound hence Dodd solid by Theorem 4.2 of [SSZ]. Otherwise, we track the Dodd solidity witnesses along the branch $[0, \eta(\mu)]_\mathcal{T}$ from the last drop to the final model $\mathcal{P}_\mu = \mathcal{M}_{\eta(\mu)}$.

- *$p_1(\mathcal{P}_\mu) \cap \mu \subseteq \alpha$,*
- *$s(F) \cap \mu \subseteq \alpha$,*
- *p is Σ_1 definable from parameters in $\alpha \cup s$ over $\mathcal{P}_\mu$, and*
- *s is Σ_1 definable from parameters in $\alpha \cup p$ over $\mathcal{P}_\mu$,*

Then:

- *If $\mathcal{Q}_\mu$ is a set mouse, $n = n(\mu)$ and $q = q_{n+1}(\mathcal{Q}_\mu, \mu)$, then*
$$\mu \cap \mathrm{Hull}_{\Sigma_1}^{\mathcal{P}_\mu}(\alpha \cup p) = \mu \cap \mathrm{Hull}_{\Sigma_{n+1}}^{\mathcal{Q}_\mu}(\alpha \cup q).$$
- *If $\mathcal{Q}_\mu$ is a weasel, $q = c(\mathcal{Q}_\mu) - \mu$, Γ is a thick class and every member of Γ is fixed by the embedding from W to $\mathcal{Q}_\mu$, then*
$$\mu \cap \mathrm{Hull}_{\Sigma_1}^{\mathcal{P}_\mu}(\alpha \cup p) = \mu \cap \mathrm{Hull}^{\mathcal{Q}_\mu}(\alpha \cup q \cup \Gamma).$$

2.3. **Definitions and properties of $\mathcal{S}_\mu$ and π_μ.** Let $\mathcal{S}_\mu$ be the degree $n(\mu)$ ultrapower of $\mathcal{Q}_\mu$ by the extender of length $\sup(\pi[\mu])$ derived from π and let π_μ be the ultrapower map. For this, we write

$$\pi_\mu : \mathcal{Q}_\mu \to \mathrm{Ult}(\mathcal{Q}_\mu, \pi, \sup(\pi[\mu])).$$

If $\mathcal{S}_\mu$ is wellfounded, then it is a premouse that agrees with W below $\sup(\pi[\mu])$. This means that $((W, \mathcal{S}_\mu), \sup(\pi[\mu]))$ is a phalanx.

Requirement 2. *For every cardinal μ of $\overline{W}$,*

- *$\mathcal{S}_\mu$ is wellfounded and iterable, therefore $\mathcal{S}_\mu$ is a mouse, and*
- *the phalanx $((W, \mathcal{S}_\mu), \sup(\pi[\mu]))$ is iterable.*

We list the most basic facts about $\mathcal{S}_\mu$ and π_μ.

- $\pi_\mu \restriction \mu = \pi \restriction \mu$.
- If $\mathcal{Q}_\mu$ is a set mouse, then $\mathcal{S}_\mu$ is a mouse that is $(n(\mu)+1)$-sound above $\sup(\pi[\mu])$ with
$$p_{n(\mu)+1}(\mathcal{S}_\mu) - \sup(\pi[\mu]) = \pi_\mu(p_{n(\mu)+1}(\mathcal{Q}_\mu) - \mu),$$
and π_μ is a cofinal $\Sigma_{n(\mu)+1}$ elementary embedding. Here, cofinal means that
$$\sup\left(\pi_\mu[\rho_{n(\mu)}(\mathcal{Q}_\mu)]\right) = \rho_{n(\mu)}(\mathcal{S}_\mu).$$
- If $\mathcal{Q}_\mu$ is a weasel, then $\mathcal{S}_\mu$ is a thick weasel,
$$c(\mathcal{S}_\mu) - \sup(\pi[\mu]) = \pi_\mu(c(\mathcal{Q}_\mu) - \mu),$$
and $\mathcal{S}_\mu$ has the $c(\mathcal{S}_\mu)$-definability property at every ordinal in the interval $[\sup(\pi[\mu]), \Omega_0)$. Moreover, π_μ is an elementary embedding.

2.4. **Comparing $\mathcal{S}_\mu$ and W.**

Lemma 2.2. *Suppose $\mathcal{S}_\mu$ is a set mouse. Then $\mathcal{S}_\mu \lhd W$. Moreover, letting $\rho = \rho_{n(\mu)+1}(\mathcal{S}_\mu)$, either $\rho = \sup(\pi[\mu])$ or $(\rho^+)^{\mathcal{S}_\mu} = \sup(\pi[\mu])$.*

Proof. Coiterate to obtain iteration trees $\mathcal{U}$ on the weasel W and $\mathcal{V}$ on the phalanx $((W, \mathcal{S}_\mu), \sup(\pi[\mu]))$. Using the fact that W is a thick weasel with the hull and definability properties at every ordinal less than Ω_0, standard arguments show that the final model on the phalanx side is a strict initial segment of the final model on the weasel side, i.e., $\mathcal{M}^{\mathcal{V}}_{1+\infty} \lhd \mathcal{M}^{\mathcal{U}}_\infty$, the final model on the phalanx side is above the starting model, i.e., $1 \leq_{\mathcal{V}} 1+\infty$, and there are no drops of any kind along the branch $[1, 1+\infty]_{\mathcal{V}}$. In particular, we have a degree $n(\mu)$ iteration map $i^{\mathcal{V}}_{1,1+\infty} : \mathcal{S}_\mu \to \mathcal{M}^{\mathcal{V}}_{1+\infty}$ with critical point at least $\sup(\pi[\mu])$. Using the fact that $\mathcal{S}_\mu$ is $n(\mu)$-sound and $(n(\mu)+1)$-sound above $\sup(\pi[\mu])$, standard arguments show that $\mathcal{V}$ is trivial, so $\mathcal{S}_\mu \lhd \mathcal{M}^{\mathcal{U}}_\infty$.

For contradiction, suppose $\mathcal{U}$ is non-trivial. Should $\mathrm{lh}(E^{\mathcal{U}}_0) > \sup(\pi[\mu])$, it would follow that

$$\mathcal{S}_\mu \lhd \mathcal{J}^{\mathcal{M}^{\mathcal{U}}_\infty}_{\mathrm{lh}(E^{\mathcal{U}}_0)},$$

hence $\mathcal{S}_\mu \lhd W$, which would contradict that $\mathcal{U}$ is non-trivial. Therefore, $\mathrm{lh}(E^{\mathcal{U}}_0) = \sup(\pi[\mu])$. It follows that $\sup(\pi[\mu])$ is a successor cardinal in $\mathcal{M}^{\mathcal{U}}_1$ and $\mathcal{S}_\mu$. Then there exists a cardinal λ of $\overline{W}$ such that $\mu = (\lambda^+)^{\overline{W}}$ and

$$(\pi(\lambda)^+)^{\mathcal{M}^{\mathcal{U}}_1} = (\pi(\lambda)^+)^{\mathcal{S}_\mu} = \sup(\pi[\mu]).$$

From this, we can show that

$$\mathcal{S}_\mu = \mathrm{Ult}_{n(\mu)}(\mathcal{Q}_\mu, \pi, \pi(\lambda)) = \mathrm{Hull}^{\mathcal{S}_\mu}_{\Sigma_{n(\mu)+1}}\left(\pi(\lambda) \cup p_{n(\mu)+1}(\mathcal{S}_\mu)\right).$$

Because $\mathcal{S}_\mu$ collapses $\sup(\pi[\mu])$, $\mathcal{S}_\mu$ is not a proper initial segment of $\mathcal{M}^{\mathcal{U}}_\xi$ for any $\xi \geq 1$, a contradiction. Therefore, $\mathcal{U}$ is trivial.

We have that $\rho \leq \sup(\pi[\mu])$ because $\mathcal{S}_\mu$ is $(n(\mu)+1)$-sound above $\sup(\pi[\mu])$ and $(\rho^+)^{\mathcal{S}_\mu} \geq \sup(\pi[\mu])$ because W and $\mathcal{S}_\mu$ agree on cardinals below $\sup(\pi[\mu])$. Therefore, either $\rho = \sup(\pi[\mu])$ or $\sup(\pi[\mu])$ is the cardinal successor of ρ in $\mathcal{S}_\mu$. $\square$

Lemma 2.3. *Suppose that $\mathcal{S}_\mu$ is a weasel and $\mathcal{Q}_\mu$ is not defined by the protomouse case. Therefore, $\mathcal{Q}_\mu = \mathcal{P}_\mu = \mathcal{M}^{\mathcal{T}}_{\eta(\mu)}$ and there are no drops along the branch $[0, \eta(\mu)]_{\mathcal{T}}$. Define $j : W \to \mathcal{S}_\mu$ by $j = \pi_\mu \circ i^{\mathcal{T}}_{0,\eta(\mu)}$. Assume that $\mu > \delta$ so that j is not the identity. Let $\gamma = \mathrm{crit}(j)$. Then $\gamma < \delta$ and $j(\gamma) > \sup(\pi[\mu])$. Let*

$$G = E_j \restriction \sup(\pi[\mu]).$$

Then $G \in W$, $\mathcal{S}_\mu = \mathrm{Ult}(W, G)$ and j is the ultrapower map.

Proof. We have that $\mathrm{crit}(i^{\mathcal{T}}_{0,\eta(\mu)}) < \delta$ as otherwise $\mathcal{M}^{\mathcal{T}}_{\eta(\mu)}$ would be a set mouse. Hence $\gamma < \delta$. We have iteration trees $\mathcal{U}$ on the weasel W and $\mathcal{V}$ on the phalanx $((W, \mathcal{S}_\mu), \sup(\pi[\mu]))$ that come from coiteration. This time, we end up with $1 \leq_{\mathcal{V}} 1+\infty$, no drops along the main branches $[0,\infty]_{\mathcal{U}}$ and $[1, 1+\infty]_{\mathcal{V}}$ and a common final model $\mathcal{N} = \mathcal{M}^{\mathcal{U}}_\infty = \mathcal{M}^{\mathcal{V}}_{1+\infty}$. In particular, we have fully elementary iteration maps $i^{\mathcal{U}}_{0,\infty} : W \to \mathcal{N}$ and $i^{\mathcal{V}}_{1,1+\infty} : \mathcal{S}_\mu \to \mathcal{N}$ with $\mathrm{crit}(i^{\mathcal{V}}_{1,1+\infty}) \geq \sup(\pi[\mu])$. Put

$$\ell = i^{\mathcal{V}}_{1,1+\infty} \circ j = i^{\mathcal{V}}_{1,1+\infty} \circ \pi_\mu \circ i^{\mathcal{T}}_{0,\eta(\mu)}.$$

So $\ell : W \to \mathcal{N}$ is an elementary embedding. Let ξ be least such that $0 <_{\mathcal{U}} \xi + 1 \leq_{\mathcal{U}} \infty$.[4] Using the hull and definability properties that hold for W we see that if

$$\rho = \min\left(\ell(\gamma), \mathrm{lh}(E^{\mathcal{U}}_\xi)\right),$$

then

$$E_\ell \restriction \rho = E^{\mathcal{U}}_\xi \restriction \rho$$

where E_ℓ is the extender derived from ℓ. If $j(\gamma) \leq \sup(\pi[\mu])$, then

$$E_j \restriction j(\gamma) = E_\ell \restriction j(\gamma) = E^{\mathcal{U}}_\xi \restriction j(\gamma)$$

and this is a superstrong initial segment of an extender on the sequence of $\mathcal{M}^{\mathcal{U}}_\xi$ (possibly its top extender). This contradicts our assumption that there is no transitive class model with a Woodin cardinal. Therefore, $j(\gamma) > \sup(\pi[\mu])$ and

$$E_j \restriction \sup(\pi[\mu]) = E_\ell \restriction \sup(\pi[\mu]) = E^{\mathcal{U}}_\xi \restriction \sup(\pi[\mu]).$$

If this extender belongs to $\mathcal{M}^{\mathcal{U}}_\xi$, then it belongs to W. Otherwise, its trivial completion is the top extender of $\mathcal{M}^{\mathcal{U}}_\xi$, which tells us that $\xi \leq 1$ and, again, the extender belongs to W.

That $\mathcal{S}_\mu$ is the ultrapower of W by this extender and j is the ultrapower map is because

$$\mathcal{S}_\mu = \mathrm{Hull}^{\mathcal{S}_\mu}(\sup(\pi[\mu]) \cup \Gamma)$$

where Γ is the class of fixed points of $i^{\mathcal{U}}_{0,\infty}$ and $i^{\mathcal{V}}_{1,1+\infty}$. □

Lemma 2.4. *Suppose that $\mathcal{S}_\mu$ is a weasel and $\mathcal{Q}_\mu$ is defined by the protomouse case. From earlier, we have*

$$\mu = \lambda_0 > \lambda_1 = (\kappa_1^+)^{\overline{W}} > \cdots > \lambda_{k+1} = (\kappa_{k+1}^+)^{\overline{W}}$$

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[4]Claim 3 on p. 240 of [MSS] corresponds to the assertion that $\xi = 0$ here. However, that claim has a gap in its proof. It is easy enough to do without that claim, both there and here, as well as in [GSS], where it is cited in a calculation of Schindler.

such that there is no dropping along the branch $[0, \eta(\lambda_{k+1})]_{\mathcal{T}}$,

$$\mathcal{Q}_{\lambda_{k+1}} = \mathcal{P}_{\lambda_{k+1}} = \mathcal{M}^{\mathcal{T}}_{\eta(\lambda_{k+1})}$$

and, for every $i \leq k$,

$$\mathcal{Q}_{\lambda_i} = \mathrm{Ult}\left(\mathcal{Q}_{\lambda i+1}, \dot{F}^{\mathcal{P}_{\lambda_i}}\right).$$

For $i \leq k+1$, *let*

$$\phi_i : \mathcal{Q}_{\lambda_i} \to \mathcal{Q}_\mu$$

be the corresponding finite composition of ultrapower embeddings. Put

$$t = \left(s(\dot{F}^{\mathcal{P}_\mu}) - \mu\right) \cup \bigcup_{1 \leq i \leq k} \phi_i \left(s(\dot{F}^{\mathcal{P}_{\lambda_i}}) - \kappa_i\right)$$

From earlier we have that

$$c(\mathcal{S}_\mu) - \sup(\pi[\mu]) = \pi_\mu(t).$$

Let

$$j = \pi_\mu \circ \phi_{k+1} \circ i^{\mathcal{T}}_{0,\eta(\lambda_{k+1})}$$

and $\gamma = \mathrm{crit}(j)$. *Then* $\gamma < \delta$ *and* $j(\gamma) > \max(t)$. *Let*

$$G = E_j \restriction (\sup(\pi[\mu]) \cup \pi_\mu(t)).$$

Then $G \in W$, $\mathcal{S}_\mu = \mathrm{Ult}(W, G)$ *and* j *is the ultrapower map.*

Proof. Let $\mathcal{U}$ be the tree on W and $\mathcal{V}$ be the tree on $((W, \mathcal{S}_\mu), \sup(\pi[\mu]))$ that result from coiteration. We have that $\mathcal{M}^{\mathcal{U}}_\infty = \mathcal{M}^{\mathcal{V}}_{1+\infty}$ and refer to this common final model as $\mathcal{N}$. We have that $1 \leq_{\mathcal{V}} 1+\infty$ and there is no dropping on the branches $[0, \infty]_{\mathcal{U}}$ and $[1, 1+\infty]_{\mathcal{V}}$. We let

$$\ell = i^{\mathcal{V}}_{1,1+\infty} \circ j = i^{\mathcal{V}}_{1,1+\infty} \circ \pi_\mu \circ \phi_{k+1} \circ i^{\mathcal{T}}_{0,\eta(\lambda_{k+1})}.$$

We let ξ be least such that $0 <_{\mathcal{U}} \xi + 1 \leq_{\mathcal{U}} \infty$. Arguing as in the proof of the previous lemma, we see that $\ell(\gamma) \geq j(\gamma) > \sup(\pi[\mu])$ and

$$E^{\mathcal{U}}_\xi \restriction \sup(\pi[\mu]) = E_\ell \restriction \sup(\pi[\mu]) = E_j \restriction \sup(\pi[\mu]).$$

Hence $\phi_{k+1}(i_{0,\eta(\lambda_{k+1})}(\gamma)) \geq \mu$. Hence $i_{0,\eta(\lambda_{k+1})}(\gamma) \geq \mathrm{crit}(\dot{F}^{\mathcal{P}_{\lambda_{k+1}}})$. Hence $\phi_{k+1}(i_{0,\eta(\lambda_{k+1})}(\gamma)) > \max(t)$. Hence $j(\gamma) > \max(\pi_\mu(t))$.

We will use these facts:

- The least parameter relative to which $\mathcal{Q}_\mu$ has the hull property at μ is t.
- The least parameter relative to which $\mathcal{S}_\mu$ has the hull property at $\sup(\pi[\mu])$ is $\pi_\mu(t)$.
- The least parameter relative to which $\mathcal{N}$ has the hull property at $\sup(\pi[\mu])$ is $i^{\mathcal{V}}_{1,1+\infty}(\pi_\mu(t))$.

These facts trace back to the Dodd solidity of $\dot{F}^{\mathcal{P}_\mu}$ above μ each $\dot{F}^{\mathcal{P}_{\lambda_i}}$ above κ_i.

Let $\xi \leq \infty$ be least such that $0 <_{\mathcal{U}} \xi + 1 \leq_{\mathcal{U}} \infty$. The proof of Claim 1 on p. 239 of [MSS] is easily modified to show that there is a parameter r such that $\mathcal{M}_{\xi+1}$ has the r-hull property at $\sup(\pi[\mu])$. Let r be the least such parameter. Then $\mathcal{N}$ has the $i^{\mathcal{U}}_{\xi+1,\infty}(r)$-hull property at $\sup(\pi[\mu])$, so

$$i^{\mathcal{U}}_{\xi+1,\infty}(r) \geq_{\text{lex}} i^{\mathcal{V}}_{1,1+\infty}(\pi_\mu(t)).$$

We claim that[5]

$$r = i^{\mathcal{U}}_{\xi+1,\infty}(r) = i^{\mathcal{V}}_{1,1+\infty}(\pi_\mu(t)).$$

First suppose that $E^{\mathcal{U}}_\xi \in \mathcal{M}^{\mathcal{U}}_\xi$. Then $E^{\mathcal{U}}_\xi$ is Dodd solid by Steel's Theorem 4.2 of [SSZ]. This implies that

$$r = s(E^{\mathcal{U}}_\xi) - \sup(\pi[\mu])$$

and $r = i^{\mathcal{U}}_{\xi+1,\infty}(r)$ is the least parameter relative to which $\mathcal{N}$ has the hull property at $\sup(\pi[\mu])$. Therefore, $r \leq_{\text{lex}} i^{\mathcal{V}}_{1,1+\infty}(\pi_\mu(t))$. Second, suppose that $E^{\mathcal{U}}_\xi$ is the top extender of $\mathcal{M}^{\mathcal{U}}_\xi$. Let $\alpha + 1$ be the largest drop of $[0, \xi]_{\mathcal{U}}$. By [SSZ], the top extender of $(\mathcal{M}^*_{\alpha+1})^{\mathcal{U}}$ is Dodd solid. Let $i : (\mathcal{M}^*_{\alpha+1})^{\mathcal{U}} \to \mathcal{M}_\xi$ be the iteration map. With a little work one sees that

$$r = i(s(\dot{F}^{(\mathcal{M}^*_{\alpha+1})^{\mathcal{U}}})) - \sup(\pi[\mu])$$

and that this parameter is Dodd solid over $\mathcal{M}_\xi$. This again implies that $r = i^{\mathcal{U}}_{\xi+1,\infty}(r)$ and $r \leq_{\text{lex}} i^{\mathcal{V}}_{1,1+\infty}(\pi_\mu(t))$.

It follows that

$$E_\ell \restriction (\sup(\pi[\mu]) \cup \pi_\mu(t)) = E^{\mathcal{U}}_\xi \restriction (\sup(\pi[\mu]) \cup r)\,.$$

If $E^{\mathcal{U}}_\xi \in \mathcal{M}^{\mathcal{U}}_\xi$, then this restriction of E_ℓ belongs to W as Lemma 2.4 asserts. On the other hand, if $E^{\mathcal{U}}_\xi$ is the top extender of $\mathcal{M}^{\mathcal{U}}_\xi$, then, with notation as in the previous paragraph,

$$\dot{F}^{\left(\mathcal{M}^*_{\alpha+1}\right)^{\mathcal{U}}} \restriction \left(\sup\left(\pi[\mu]\right) \cup s\left(\dot{F}^{\left(\mathcal{M}^*_{\alpha+1}\right)^{\mathcal{U}}}\right)\right) = E^{\mathcal{U}}_\xi \restriction \left(\sup\left(\pi[\mu]\right) \cup r\right)$$

and this extender still belongs to W.[6]

That $\mathcal{S}_\mu$ is the ultrapower of W by this extender and j is the ultrapower map is because

$$\mathcal{S}_\mu = \text{Hull}^{\mathcal{S}_\mu}(\sup(\pi[\mu]) \cup \pi_\mu(t) \cup \Gamma)$$

where Γ is the class of fixed points of $i^{\mathcal{U}}_{0,\infty}$ and $i^{\mathcal{V}}_{1,1+\infty}$. □

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[5]A similar calculation of Schindler appears in the part of [GSS] that builds on [MSS].

[6]In the usual way, we are identifying isomorphic extenders here.

The following two result are not essential to our proof of Theorem 1 but will simplify the last part of the proof after we switch to Jensen indexing.

Lemma 2.5 (Schindler)**.** *Assume that we have been working with the core model constructed using Jensen indexing. Then W is the only weasel on $\mathcal{T}$. I.e., for every ζ, if 0 is the $\mathcal{T}$ predecessor of $\zeta+1$, then $\zeta+1$ is a drop of $\mathcal{T}$.*

Proof. Suppose otherwise. We have $i^{\mathcal{T}}_{0,\zeta+1} : W \to \mathcal{M}^{\mathcal{T}}_{\zeta+1} = \mathrm{Ult}(W, E^{\mathcal{T}}_{\zeta})$. Let $\kappa = \mathrm{crit}(i^{\mathcal{T}}_{0,\zeta+1})$, $\lambda = i^{\mathcal{T}}_{0,\zeta+1}(\kappa)$ and $\mu = (\lambda^+)^{\mathcal{M}^{\mathcal{T}}_{\zeta+1}}$. Then $E^{\mathcal{T}}_{\zeta}$ is the extender of length μ derived from $i^{\mathcal{T}}_{0,\eta+1}$. We have that $\overline{W}$ and W agree below $(\kappa^+)^{\overline{W}} = (\kappa^+)^W < \delta$. Clearly, $\mathcal{P}_\mu = \mathcal{Q}_\mu = \mathcal{M}^{\mathcal{T}}_{\zeta}$ and $\mathcal{S}_\mu = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\zeta}, \pi, \sup(\pi[\mu])) = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\zeta}, \pi, \pi(\lambda))$. By the version of Lemma 2.2 for Jensen indexing, $\mathcal{S}_\mu \lhd W$. But λ is a cardinal of $\overline{W}$, so $\pi(\lambda)$ is a cardinal of W and the top predicate of $\mathcal{S}_\mu$ is a superstrong extender over W that maps κ to $\pi(\lambda)$. This contradicts our assumption that there is no inner model with a Woodin cardinal. □

Lemma 2.6. *Assume that we have been working with the core model constructed using Jensen indexing. Suppose μ is a cardinal of $\overline{W}$ and $\mu \geq \delta$. Then $\mathcal{P}_\mu$ and $\mathcal{Q}_\mu$ are set mice.*

Proof. That $\mathcal{P}_\mu$ is a set mouse is immediate from Lemma 2.5. Let $F = \dot{F}^{\mathcal{P}_\mu}$, $\kappa = \mathrm{crit}(F)$ and $\lambda = (\kappa^+)^{\mathcal{P}_\mu}$. Suppose that $\mathcal{Q}_\mu$ is defined by the protomouse case and is a weasel. By Lemma 2.5 and the definition of $\mathcal{Q}_\mu$, we may assume that $\mathcal{Q}_\lambda = \mathcal{P}_\lambda = W$ and $\mathcal{Q}_\mu = \mathrm{Ult}(W, F)$. Then

$$\mathcal{P}(\kappa) \cap \overline{W} = \mathcal{P}(\kappa) \cap \mathcal{P}_\mu = \mathcal{P}(\kappa) \cap W$$

but, by Requirement 1,

$$\mathcal{P}(\delta) \cap \overline{W} \subsetneqq \mathcal{P}(\delta) \cap W.$$

Thus, $\lambda < \delta = \mathrm{crit}(\pi)$. In particular, π is continuous at λ, so we are not in the protomouse case. □

2.5. Satisfying the requirements.

Theorem 2 ([MSS])**.** *Suppose that $X \prec (H_{\Omega^+}, \in, U)$, $\{\nu, W\} \subseteq X$, $\sup(\nu \cap X) = \nu$, $|X| < \nu$ and ${}^\omega X \subseteq X$. Then*

- *Requirement 1 holds for X and*
- *Requirement 2 holds for every successor cardinal μ of $\overline{W}$.*

Theorem 3 ([MS])**.** *Let ε be a regular cardinal such that*

$$\max(\omega_2, \mathrm{cf}(\nu)^+) \leq \varepsilon < \nu.$$

Suppose $\langle X_i \mid i < \varepsilon \rangle$ *is an internally approachable continuous chain of elementary substructures of* $(H_{\Omega^+}, \in, U)$ *such that* $\{\nu, W\} \subseteq X_j$, X_j *is transitive below* ε *and* $|X_j| = |\varepsilon \cap X_j|$ *for every* $j < \varepsilon$. *Then there exists a club set* $C \subseteq \varepsilon$ *such that, for every* $j \in C$ *with* $\mathrm{cf}(j) > \omega$,

- *Requirement 1 holds for* X_j *and*
- *Requirement 2 holds for every successor cardinal* μ *of* W_j, *the image of* W *under the Mostowski collapse of* X_j.

Note that Theorem 2 assume countable closure, which would be impossible if ν is not a countably closed cardinal, while Theorem 3 does not. Here, we extend both theorems.

Theorem 4. *Theorems 2 and 3 remain true should the word "successor" be removed.*

The proof of Theorem 4 is the proofs of Theorem 2 and 3 with only obvious changes; we do not include it here. Here are four minor tips or remarks that borrow notation from [MSS] and [MS].

- Granted Theorem 3 has already been proved, the machinery that goes into its proof and that of Theorem 4 simplifies because $W^i \trianglelefteq \mathcal{M}_\infty^{\mathcal{T}^i}$ rather than merely W^i embedding into the final models of various iteration trees on W.
- The pull-back construction in §2 of [MS] must be modified when μ^j is a limit cardinal of W^j. Consider the case in which $\mathcal{Q}^j$ is a set mouse. Then $\mathcal{Q}^j$ and W^j agree below μ^j and $\pi_{i,j}(\mu^i) = \sup(\pi_{i,j}[\mu^i])$, and we need a directed system $D \subseteq J_{\mu^j}^{W^j}$ whose limit is $\mathcal{A}_n(\mathcal{Q}^j)$, the Σ_n coding structure of $\mathcal{Q}^j$. For this, we consider various $\kappa < \mu^j$ and $\xi < \rho_n(\mathcal{Q}^j)$, and declare that the Mostowski collapse of $\mathrm{Hull}_{\Sigma_1}^{\mathcal{A}_n(\mathcal{Q}^j)\restriction\xi}(\kappa \cup p_1(\mathcal{A}_n(\mathcal{Q}^j)))$ to be one of the structures of D. Unlike the case in which μ^j is a successor cardinal in W^j, there is not a fixed $\kappa < \mu^j$ that suffices for the definition of D.
- The proof in [MS] is not organized as an induction but it builds on the proof in [MSS], which is an induction. Correspondingly, there is a sense in which Requirement 2 for a limit cardinal μ follows from Requirement 2 for every successor cardinal $\lambda < \mu$. This traces to the fact that, for every iteration tree that extends $\mathcal{T} \restriction (\eta(\mu)+1)$ and every extender F used on such an extension, setting $\kappa = \mathrm{crit}(F)$ and $\lambda = (\kappa^+)^{\overline{W}}$, if $\kappa < \mu$, then the model to which F is applied is $\mathcal{P}_\lambda$.
- In fact, $\overline{\nu} = \pi^{-1}(\nu)$ is the only limit cardinal of $\overline{W}$ for which we need Requirement 2 in order to prove Theorem 1.

2.6. **Definition of $\mathcal{F}$.** Let $\varepsilon = \max(\omega_2, \mathrm{cf}(\nu)^+)$. Define $\mathcal{F}$ to be the stationary family of X such that there exists $\langle X_i \mid i < \varepsilon\rangle$ and C as described in Theorem 4 and $X = X_j$ for some $j \in C$ with $j = \varepsilon \cap X_j = \mathrm{type}(j \cap C)$ and $\mathrm{cf}(j) = \max(\omega_1, \mathrm{cf}(\nu))$.

Observe that if $X \in \mathcal{F}$, then $\mathrm{OR} \cap X$ is $< \mathrm{cf}(\nu)$-closed. This means that if $\gamma \in \mathrm{OR} \cap X$ and $\mathrm{cf}(\mathrm{type}(\gamma \cap X)) < \mathrm{cf}(\nu)$, then $\sup(\gamma \cap X) \in X$. [*Proof*: Say $X = X_j$ as in Theorem 4. Pick an unbounded $S \subseteq \gamma \cap X$ with $\mathrm{type}(S) = \mathrm{cf}(\mathrm{type}(\gamma \cap X))$. Find $i < j$ such that $S \subseteq \gamma \cap X_i$. Then $\sup(\gamma \cap X) = \sup(\gamma \cap X_i) \in X$.] Equivalently, $\pi_X : N_X \simeq X \prec H_{\Omega^+}$ is continuous at every ordinal of cofinality strictly less than $\mathrm{cf}(\nu)$.

In §4, we will shrink $\mathcal{F}$ to stationary subfamilies twice, first to $\mathcal{G}$, then to $\mathcal{H}$.

3. The critical sequence

We continue to use Mitchell-Steel indexing in this section in a way that adapts easily to Jensen indexing. Only Lemma 3.6 distinguishes between the two indexing styles. The part of Theorem 1 that applies to both styles is proved at the end of this section.

Recall that ν is our singular cardinal that is regular in K. From now on in this section, $\mu = \pi^{-1}(\nu)$ and $\theta = \eta(\mu)$. Thus μ is what we were calling $\overline{\nu}$ in the previous section. Then μ is a limit cardinal of $\overline{W}$. Hence μ is not the index of an extender on the $\overline{W}$, nor is it the index of an extender on the $\mathcal{M}^{\mathcal{T}}_{\theta}$ sequence, so these two mice agree up to and including μ.

Lemma 3.1. *Suppose $\mathcal{Q}_\mu$ is a set mouse. Then, for every $\alpha < \mu$*

$$\sup\left(\mu \cap \mathrm{Hull}^{\mathcal{Q}_\mu}_{\Sigma_{n(\mu)+1}}\left(\alpha \cup p_{n(\mu)+1}(\mathcal{Q}_\mu)\right)\right) < \mu.$$

Proof. Deny. Let

$$H = \nu \cap \mathrm{Hull}^{\mathcal{S}_\mu}_{\Sigma_{n(\mu)+1}}\left(\pi(\alpha) \cup p_{n(\mu)+1}(\mathcal{S}_\mu)\right).$$

Then $\sup(H) = \nu$. By Lemma 2.2, $\mathcal{S}_\mu \lhd W$. Hence $H \in W$ and $|H|^W < \nu$, which contradicts that ν is regular in W. □

Lemma 3.2. *Suppose $\mathcal{Q}_\mu$ is a weasel. Then, for every $\alpha < \mu$, there exists a thick class Γ such that*

$$\sup\left(\mu \cap \mathrm{Hull}^{\mathcal{Q}_\mu}\left(\alpha \cup c(\mathcal{Q}_\mu) \cup \Gamma\right)\right) < \mu.$$

Proof. Deny. Let $\beta = \pi(\alpha)$.

First suppose that $\mathcal{Q}_\mu$ is defined by the mouse case. Then $\mathcal{Q}_\mu = \mathcal{P}_\mu = \mathcal{M}^{\mathcal{T}}_\theta$ and, for every thick class Γ,

$$\sup\left(\mu \cap \mathrm{Hull}^{\mathcal{M}^{\mathcal{T}}_\theta}\left(\alpha \cup \Gamma\right)\right) = \mu.$$

We have elementary embeddings $i^{\mathcal{T}}_{0,\theta} : W \to \mathcal{M}^{\mathcal{T}}_\theta$ and $\pi_\mu : \mathcal{M}^{\mathcal{T}}_\theta \to \mathcal{S}_\mu$. Let $j = \pi_\mu \circ i^{\mathcal{T}}_{0,\theta}$ and $\gamma = \mathrm{crit}(j)$. Then $\gamma < \delta = \mathrm{crit}(\pi_\mu)$. By Lemma 2.3, $j(\gamma) > \nu$ and $E_j \restriction \nu \in W$. Factor j through a series of ultrapowers by restrictions of E_j as follows

$$W \to \mathcal{S}' = \mathrm{Ult}(W, E_j \restriction \beta) \to \mathcal{S}'' = \mathrm{Ult}(W, E_j \restriction \nu) \to \mathcal{S}_\mu$$

Let $i : W \to \mathcal{S}'$ and $\sigma : \mathcal{S}' \to \mathcal{S}''$ name the displayed embeddings. Then $\beta \leq i(\gamma) < (|\beta|^+)^W < \nu$. We claim that $\sup(\sigma[i(\gamma)]) = \nu$. Towards seeing this, let Γ be the thick class of fixed points of all the relevant embeddings and consider any $\xi \in \mu \cap \mathrm{Hull}^{\mathcal{M}^{\mathcal{T}}_\theta}(\alpha \cup \Gamma)$. By our assumption about this hull and the fact that $\sup(\pi[\mu]) = \nu$, it suffices to show $\pi(\xi) \in \sigma[i(\gamma)]$. Pick $a \in \alpha^{<\omega}$, $b \in \Gamma^{<\omega}$ and a Skolem term τ_φ such that $\xi = \tau_\varphi^{\mathcal{M}^{\mathcal{T}}_\theta}[a, b]$. Then $\pi(\xi) < \nu < j(\gamma)$, which implies that

$$\tau_\varphi^{\mathcal{S}'}[\pi(a), b] < i(\gamma).$$

Moreover,

$$\sigma(\tau_\varphi^{\mathcal{S}'}[\pi(a), b]) = \tau_\varphi^{\mathcal{S}''}[\pi(a), b] = \tau_\varphi^{\mathcal{S}_\mu}[\pi(a), b] = \pi(\xi).$$

Combining the claim and the fact that $\sigma \restriction i(\gamma)$ belongs to W, we see that $\mathrm{cf}^W(\nu) \leq i(\gamma) < \nu$. But ν was assumed to be regular in W.

Now suppose that $\mathcal{S}_\mu$ is defined by the protomouse case. To get a contradiction, modify the previous paragraph in the obvious way using Lemma 2.4 instead of Lemma 2.3. □

Lemma 3.3. $\mathcal{P}_\mu = \mathcal{M}^{\mathcal{T}}_\theta$.

Proof. Otherwise, $\mathcal{P}_\mu \lhd \mathcal{M}^{\mathcal{T}}_\theta$. We rule out $\mathcal{Q}_\mu = \mathcal{P}_\mu$ by applying Lemma 3.1 with $\alpha = \rho_{m(\mu)+1}(\mathcal{P}_\mu)$. Therefore, $\mathcal{Q}_\mu$ is defined by the protomouse case. Apply Lemma 7.1 from the Appendix to $\mathcal{P}_\mu$ with $\alpha = \max(\lambda, \rho_1(\mathcal{P}_\mu))$, and $q = p_1(\mathcal{P}_\mu)$ to see that $\tau(\dot{F}^{\mathcal{P}_\mu}) \leq \alpha < \mu$. This leads to a contradiction of Lemma 3.1 or Lemma 3.2 depending on whether $\mathcal{Q}_\mu$ is a set mouse or a weasel. □

Lemma 3.4.

$$\sup\left(\{\mathrm{lh}(E^{\mathcal{T}}_\zeta) \mid \zeta + 1 <_{\mathcal{T}} \theta\}\right) = \sup\left(\{\mathrm{crit}(E^{\mathcal{T}}_\zeta) \mid \zeta + 1 <_{\mathcal{T}} \theta\}\right) = \mu.$$

Proof. Otherwise, we contradict Lemma 3.1 or 3.2 with

$$\alpha = \sup\left(\{\mathrm{lh}(E^{\mathcal{T}}_\zeta) \mid \zeta + 1 <_{\mathcal{T}} \theta\}\right).$$

□

Lemma 3.5. *μ is a regular cardinal in $\mathcal{M}^{\mathcal{T}}_\theta$.*

Proof. Follows from Lemmas 3.1 and 3.2. □

Lemma 3.6. *Suppose that $\zeta <_{\mathcal{T}} \theta$, there are no drops of any kind on $[\zeta, \theta)_{\mathcal{T}}$ and μ is in the range of $i^{\mathcal{T}}_{\zeta,\theta}$. Let I be the set of $\eta \in [\zeta, \theta)_{\mathcal{T}}$ such that, if $\kappa = \mathrm{crit}(i^{\mathcal{T}}_{\eta,\theta})$, then $i^{\mathcal{T}}_{\eta,\theta}(\kappa) = \mu$. Then I is a club subset of $[\zeta, \theta)_{\mathcal{T}}$ Moreover, if we consider all we have done so far to be about Jensen fine structure rather than Mitchell-Steel fine structure, then I is a tail of $[\zeta, \theta)_{\mathcal{T}}$. I.e., there exists $\zeta_0 <_{\mathcal{T}} \theta$ such that $I = [\zeta_0, \theta)_{\mathcal{T}}$.*

Proof. We claim the following is impossible:

- $i^{\mathcal{T}}_{\zeta,\theta}(\lambda) = \mu$,
- $\kappa = \mathrm{crit}(i^{\mathcal{T}}_{\zeta,\theta}) < \lambda$ and
- $i^{\mathcal{T}}_{\zeta,\theta}$ is continuous at λ.

Suppose otherwise. If $\mathcal{M}^{\mathcal{T}}_\theta$ is a set premouse, then

$$\lambda \cap \mathrm{Hull}^{\mathcal{M}^{\mathcal{T}}_\zeta}_{\Sigma_{m(\mu)+1}} \left(\kappa \cup p_{m(\mu)+1}(\mathcal{M}^{\mathcal{T}}_\eta)\right) = \lambda,$$

so,

$$\sup\left(\mu \cap \mathrm{Hull}^{\mathcal{M}^{\mathcal{T}}_\theta}_{\Sigma_{m(\mu)+1}} \left(\kappa \cup p_{m(\mu)+1}(\mathcal{M}^{\mathcal{T}}_\theta)\right)\right) = \mu,$$

which contradicts Lemma 3.1. If $\mathcal{M}^{\mathcal{T}}_\theta$ is a weasel, then, for every thick class Γ,

$$\lambda \cap \mathrm{Hull}^{\mathcal{M}^{\mathcal{T}}_\zeta}(\kappa \cup \Gamma) = \lambda$$

so

$$\sup\left(\mu \cap \mathrm{Hull}^{\mathcal{M}^{\mathcal{T}}_\theta}(\kappa \cup \Gamma)\right) = \mu,$$

which contradicts Lemma 3.2.

Notice that if ζ, κ and λ are as in the statement of the claim, then because $i^{\mathcal{T}}_{\zeta,\theta}$ is discontinuous at λ, there must exist $\eta \in (\zeta, \theta)_{\mathcal{T}}$ such that $i^{\mathcal{T}}_{\zeta,\theta}(\lambda) = \mathrm{crit}(i^{\mathcal{T}}_{\eta,\theta})$, hence $\eta \in I$. Therefore, I is an unbounded subset of $[\zeta, \theta)_{\mathcal{T}}$. The rules for forming normal iteration trees imply that I is closed. The rules for forming normal iteration trees on Jensen style mice imply that I is a tail. □

We are not ready to specialize to Jensen indexing yet. Let I be the club subset of $[0, \theta)_{\mathcal{T}}$ that comes from Lemma 3.6. For $\eta \in I$, define $\mu_\eta = \mathrm{crit}(i^{\mathcal{T}}_{\eta,\theta}) = (i^{\mathcal{T}}_{\eta,\theta})^{-1}(\mu)$. The set of critical points

$$C = \{\mu_\eta \mid \eta \in I\}$$

is club in μ and will play an important role in the rest of the paper.

We introduce notation for the order zero measure on the extender sequence of an arbitrary mouse, $\mathcal{M}$. By the extender sequence of $\mathcal{M}$ we mean the sequence $\dot{E}^{\mathcal{M}}$ from which the universe of $\mathcal{M}$ is constructed followed by the top extender, $\dot{F}^{\mathcal{M}}$. The index and the length are the same for extenders on the sequence of $\mathcal{M}$. If there exists a extender with critical point κ on the extender sequence of $\mathcal{M}$ that is total over $\mathcal{M}$, then let $EZ(\mathcal{M},\kappa)$ be the witness with least length. Otherwise, leave it undefined. The corresponding order zero measure over $\mathcal{M}$ is

$$U(\mathcal{M},\kappa) = \left\{ A \in \mathcal{P}(\kappa) \cap \mathcal{M} \mid \{\{\alpha\} \mid \alpha \in A\} \in EZ(\mathcal{M},\kappa)_{\{\kappa\}} \right\}.$$

The extender $EZ(\mathcal{M},\kappa)$ and the measure $U(\mathcal{M},\kappa)$ have exactly the same ultrapowers.

Returning to our specific situation, as μ is measurable in $\mathcal{M}^{\mathcal{T}}_\theta$, both $EZ(\mathcal{M}^{\mathcal{T}}_\theta,\mu)$ and $U(\mathcal{M}^{\mathcal{T}}_\theta,\mu)$ are defined.

Lemma 3.7. *For every $A \in \mathcal{P}(\mu) \cap \mathcal{M}^{\mathcal{T}}_\theta$, the following are equivalent:*

- $A \in U(\mathcal{M}^{\mathcal{T}}_\theta,\mu)$
- $\exists \zeta \in [\zeta_0,\theta)_{\mathcal{T}}\ \forall \eta \in [\zeta,\theta)_{\mathcal{T}}\ A \cap \mu_\eta \in U(\mathcal{M}^{\mathcal{T}}_\eta,\mu_\eta)$
- $\sup(\{\eta \in [\zeta,\theta)_{\mathcal{T}} \mid A \cap \mu_\eta \in U(\mathcal{M}^{\mathcal{T}}_\eta,\mu_\eta)\}) = \theta$

Lemma 3.8. $EZ(\mathcal{Q}_\mu,\mu) = EZ(\mathcal{M}^{\mathcal{T}}_\theta,\mu)$ *and* $U(\mathcal{Q}_\mu,\mu) = U(\mathcal{M}^{\mathcal{T}}_\theta,\mu)$.

Proof. In the protomouse case, observe that $EZ(\mathcal{M}^{\mathcal{T}}_\theta,\mu)$ is not the top extender $\mathcal{M}^{\mathcal{T}}_\theta$ and use the coherence of $\dot{F}^{\mathcal{M}_\theta}$ with the extender sequence of $\mathcal{M}^{\mathcal{T}}_\theta$. □

Lemma 3.9. $U(\mathcal{S}_\mu,\nu)$ *is defined and belongs to* W.

Proof. By Lemma 2.2, if $\mathcal{S}_\mu$ is a set mouse, then $\mathcal{S}_\mu \lhd W$. Should $\mathcal{S}_\mu$ be a weasel, continue the analysis of the coiteration of W and $((W,\mathcal{S}_\mu),\nu)$ that we started in the proofs of Lemmas 2.3 and 2.4 and show that the extenders applied in $\mathcal{U}$ and $\mathcal{V}$ all have length strictly greater than that of $EZ(\mathcal{S}_\mu,\nu)$. □

Lemma 3.10. *In W, $U(\mathcal{S}_\mu,\nu)$ is not an ultrafilter over ν. It is an ultrafilter on the family of subsets of ν that are constructed in W before stage $(\nu^+)^{\mathcal{S}_\mu} = \pi_\mu((\mu^+)^{\mathcal{Q}_\mu}) = \sup(\pi_\mu[(\mu^+)^{\mathcal{Q}_\mu}]) < \nu^+ = (\nu^+)^W$. In particular, if $A \in \mathcal{P}(\nu) \cap X \cap W$, then A is measured by $U(\mathcal{S}_\mu,\nu)$*

Indexing ambiguous part of Theorem 1. We sketch how to finish the proof of Theorem 1 under the added assumption that $(2^{\mathrm{cf}(\nu)})^+ < |\nu|$. We are still using Mitchell-Steel indexing for our presentation but everything here adapts to Jensen indexing.

Before applying the added assumption, consider an arbitrary X like those we have been discussing all along. Suppose $\zeta \in I$ is the $\mathcal{T}$ predecessor of $\xi + 1$ and $E^{\mathcal{T}}_\xi$ is an extender of order zero over μ_ζ. Then $\xi = \zeta$ and $\zeta <_T \zeta + 1 <_T \theta$. Let I_0 be the set of such ζ and $C_0 = \{\mu_\zeta \mid \zeta \in I_0\}$. If I_0 is an unbounded subset of I, then, for every $A \in \mathcal{P}(\nu) \cap \mathcal{S}_\mu$,

$$A \in U(\mathcal{S}_\mu,\nu) \iff \pi[C_0] \subseteq^* A.$$

In this sense, $\pi[C_0]$ would generate $U(\mathcal{S}_\mu, \nu)$. Otherwise, let $I_1 = I - \max(I_0)$ and $C_1 = \{\mu_\zeta \mid \zeta \in I_1\}$. Notice that if $\zeta \in I - I_0$, then μ_ζ is a measurable cardinal in $\overline{W}$, so $\pi(\mu_\zeta)$ is a measurable cardinal in W. It is straightforward to calculate that, for every $A \in \mathcal{P}(\nu) \cap \mathcal{S}_\mu$,

$$A \in U(\mathcal{S}_\mu, \nu) \iff \{\pi(\mu_\zeta) \mid \zeta \in I_1 \text{ and } A \cap \pi(\mu_\zeta) \in U(W, \pi(\mu_\zeta)\} \subseteq^* A.$$

In this sense, $\pi[C_1]$ would generate $U(\mathcal{S}_\mu, \nu)$.

Using $\varepsilon = (2^{\mathrm{cf}(\nu)})^+ < |\nu|$, build a chain to find X as in Theorem 4 with ${}^{\mathrm{cf}(\nu)}X \subseteq X$. Shrink C if necessary to a club subset of $[0, \theta)_\mathcal{T}$ of order type $\mathrm{cf}(\nu)$. Then $\pi[C_i] \in X$ where $i \in \{0, 1\}$ is defined according to the previous paragraph. As $\pi[C_i]$ generates a measure over $\mathcal{P}(\nu) \cap W \cap X$ and $\{\nu, W, \pi[C_i]\} \subseteq X \prec H_{\Omega^+}$, it follows that $\pi[C_i]$ actually generates a total measure on W according to the same recipe. Call this measure U. It is easy to see that $U \in X$ and U is amenable to W. This can be used to show that $\pi^{-1}(U) = U(\mathcal{M}^\mathcal{T}_\theta, \mu)$. In a similar situation later in the paper, we will argue that $\mathrm{Ult}(W, U)$ is wellfounded, the phalanx $((W, \mathrm{Ult}(W, U)), \nu)$ is iterable and $U \in W$. Therefore ν is a measurable cardinal in W. The ideas for getting $o(\nu) \geq \mathrm{cf}(\nu)$ when ν has uncountable cofinality will be discussed later too. In summary, the closure of X under $\mathrm{cf}(\nu)$ sequences yields a club subset $\pi[C_i]$ of ν that belongs to X and generates a measure on $\mathcal{P}(\nu) \cap W$ that belongs to W. It is common to refer to $\pi[C_i]$ as a set of indiscernibles for X. We do not need to relate indiscernibles arising from various elementary substructures in this case.

4. Growing partial measures

Beware that there will be a pivot to Jensen indexing alone in the middle of this section, just before Lemma 4.3.

The family $\mathcal{F}$, which was defined after the statement of Theorem 4, is stationary. By Fodor's Lemma, there is a stationary subfamily $\mathcal{G}$ such that the following questions have the same answers for every $X \in \mathcal{G}$.

- Is $\mathcal{M}^{\mathcal{T}_X}_{\theta_X}$ is a set mouse or a weasel?
- What is the value of $m_X(\mu_X)$?
- Is $\mathcal{Q}^X_{\mu_X}$ defined by the protomouse case?
- Is $\mathcal{Q}^X_{\mu_X}$ is a set mouse or a weasel?
- What is the value of $n_X(\mu_X)$?

We have constant values $m = m_X(\mu_X)$ and $n = n_X(\mu_X)$ for every $X \in \mathcal{G}$. This section is about $X, Y \in \mathcal{G}$ such that $X \subseteq Y$.

As the reader has already noticed, notation from earlier sections will be decorated with X or Y. We will simplify some notation for the sake of readability. The inverse of the Mostowski collapse of X will be

$$\pi_X : N_X \simeq X$$

and the models of the iteration tree $\mathcal{T}_X$ will be $\mathcal{M}^X_\eta$ for $\eta \leq \infty_X$. The particular model $\mathcal{M}^X_{\theta_X}$ will be written $\mathcal{M}_X$. Similarly, instead of $\mathcal{S}^X_{\mu^X}$ we will write $\mathcal{S}_X$. There is a small typographical dilemma regarding how to simplify $\pi^X_{\mu^X}$ since π_X is already taken. With the idea that the ultrapower map "lifts" π_X, we decided to use upper case notation:

$$\Pi_X : \mathcal{Q}_X \to \mathcal{S}_X = \mathrm{Ult}(\mathcal{Q}_X, \pi_X, \nu).$$

Among other things, we know that

$$\Pi_X \upharpoonright \mu_X = \pi_X \upharpoonright \mu_X.$$

Each of $\mathcal{S}_X$ and $\mathcal{S}_Y$ has what it thinks is a total extender with critical point ν on its sequence. The least such extenders correspond to measures of order zero $U_X = U(\mathcal{S}_X, \nu)$ and $U_Y = U(\mathcal{S}_Y, \nu)$ respectively. Neither measure is total on W. These partial measures belong to W. An additional condition on X and Y will imply that $U_X \subseteq U_Y$. Eventually, we will take a union of enough of these partial measures to obtain a total measure over W and prove it too belongs to W.

As $X \subseteq Y$, we may define $\pi_{XY} = (\pi_Y)^{-1} \circ \pi_X$. Then

$$\pi_{XY} : N_X \to N_Y$$

is an elementary embedding. We take the fine structural ultrapower of $\mathcal{Q}_X$ by the extender of length μ_Y derived from π_{XY} to define

$$\Pi_{XY} : \mathcal{Q}_X \to \mathcal{S}_{XY} = \mathrm{Ult}(\mathcal{Q}_X, \pi_{XY}, \mu_Y).$$

We define the factor embedding

$$\Phi_{XY} : \mathcal{S}_{XY} \to \mathcal{S}_X$$

by

$$\Phi_{XY}([a, \Pi_{XY}(f)]^{\mathcal{Q}_X}_{E_{\pi_{XY}} \upharpoonright \mu_Y}) = \Pi_X(f)(\pi_Y(a))$$

for every $a \in [\mu_Y]^{<\omega}$ and function f that either belongs to $\mathcal{Q}_X$ if $m = 0$ or is Σ_{m_X} definable with parameters over $\mathcal{Q}_X$ if $m > 0$. Then

$$\Pi_X = \Phi^{XY} \circ \Pi_{XY}.$$

Because $\mathcal{S}_{XY}$ embeds into $\mathcal{S}_X$, it is also a mouse. So,

$$\Phi_{XY} \upharpoonright \mu_Y = \pi_Y \upharpoonright \mu_Y.$$

We will write

$$((\overrightarrow{\mathcal{P}_Y} \upharpoonright \mu_Y, \mathcal{M}_Y), \mu_Y)$$

for the phalanx

$$(\langle \mathcal{P}^Y_{(\aleph_{\alpha+1})^{W_Y}} \mid \alpha < \beta \rangle^\frown \langle \mathcal{M}_Y \rangle, \langle (\aleph_{\alpha+1})^{W_Y} \mid \alpha < \beta \rangle^\frown \langle \mu_Y \rangle)$$

where β is the ordinal such that

$$(\aleph_\beta)^{W_Y} = \mu_Y.$$

Recall that $\mathcal{M}_Y = \mathcal{M}^{\mathcal{T}_Y}_{\theta_Y} = \mathcal{P}^Y_{\mu_Y}$ and the extender sequences of $\mathcal{M}_Y$ and W_Y agree up to and including their cardinal μ_Y. This phalanx displayed above is only superficially different from the phalanx derived from $\mathcal{T}_Y \upharpoonright \theta_Y + 1$, which is defined in other literature. Iteration trees on the phalanx $((\overrightarrow{\mathcal{P}_Y} \upharpoonright \mu_Y, \mathcal{M}_Y), \mu_Y)$ are only superficially different from iteration trees that extend $\mathcal{T}_Y \upharpoonright \theta_Y + 1$. Therefore, $((\overrightarrow{\mathcal{P}_Y} \upharpoonright \mu_Y, \mathcal{M}_Y), \mu_Y)$ is an iterable phalanx. Similarly, we will write

$$((\overrightarrow{\mathcal{P}_Y} \upharpoonright \mu_Y, \mathcal{S}_{XY}), \mu_Y)$$

for the phalanx

$$(\langle \mathcal{P}^Y_{(\aleph_{\alpha+1})^{W_Y}} \mid \alpha < \beta \rangle ^\frown \langle \mathcal{S}_{XY} \rangle, \langle (\aleph_{\alpha+1})^{W_Y} \mid \alpha < \beta \rangle ^\frown \langle \mu_Y \rangle).$$

All we have done is change the starting model from $\mathcal{M}_Y$ to $\mathcal{S}_{XY}$, whose extender sequence also agrees with that of W_Y up to and including μ_Y.

Lemma 4.1. *$((\overrightarrow{\mathcal{P}_Y} \upharpoonright \mu_Y, \mathcal{S}_{XY}), \mu_Y)$ is an iterable phalanx.*

Sketch. Use the two requirements listed so far and ideas made explicit in [MSS]. First reduce the iterability to that of

$$((\overrightarrow{\mathcal{Q}_Y} \upharpoonright \mu_Y, \mathcal{S}_{XY}), \mu_Y).$$

Second reduce the iterability to that of

$$((\overrightarrow{\mathcal{S}_Y} \upharpoonright \mu_Y, \mathcal{S}_X), \nu).$$

Third reduce the iterability to that of a W based phalanx. (This steps involves consequences of Requirement 2 on $\mathcal{S}_X$ and the various $\mathcal{S}^Y_\lambda$ for various successor $\lambda < \nu$.) Finally, reduce the iterability to that of a K^c based phalanx. By a theorem of Steel, K^c based phalanxes are iterable. $\square$

Lemma 4.2. *Suppose that $\mathcal{S}_{XY}$ is a set mouse. Then $\mathcal{S}_{XY} \trianglelefteq \mathcal{M}_Y$.*

Sketch. Coiterate the phalanxes

$$((\overrightarrow{\mathcal{P}_Y} \upharpoonright \mu_Y, \mathcal{M}_Y), \mu_Y)$$

and

$$((\overrightarrow{\mathcal{P}_Y} \upharpoonright \mu_Y, \mathcal{S}_{XY}), \mu_Y)$$

to produce iteration trees $\mathcal{U}$ and $\mathcal{V}$.[7] The coiteration is superficially different from the coiteration of W and the latter phalanx, so the final model of $\mathcal{V}$ is an initial segment of the final model of $\mathcal{U}$. Using the fact that $\mathcal{S}_{XY}$ is sound above μ_Y, standard arguments show that $\mathcal{U}$ and $\mathcal{V}$ are trivial. □

We remark that there is a version of Lemma 4.2 when $\mathcal{S}_{XY}$ is a weasel. But it is complicated to state and we will not have an opportunity to use it in this paper, so we omitted it.

We are about to shrink our stationary set of elementary substructures again. First, let us consider the case $\mathrm{cf}(\nu) > \omega$. Fix a club $D \subseteq \nu$ such that $\mathrm{type}(D) = \mathrm{cf}(\nu)$. Replace D with $\lim(D)$ so that every member is a limit ordinal with cofinality $< \mathrm{cf}(\nu)$. From §2.6, we have that $\nu \cap X$ is $< \mathrm{cf}(\nu)$ club in ν. Hence $D \cap X$ is $< \mathrm{cf}(\nu)$ club in ν. Then $\pi_X^{-1}(D \cap X)$ is club in μ_X and π_X is continuous at all its members. (We will not take advantage of this continuity until §6.) Earlier, we defined the set of critical points,

$$C_X = \{\mu_\eta^X \mid \eta \in I_X\}.$$

Define $B_X = C_X \cap \pi_X^{-1}[D \cap X]$. Then B_X is a club subset of C_X, $\mathrm{type}(B_X) = \mathrm{cf}(\nu)$ and π_X is continuous at every member of B_X. In the case $\mathrm{cf}(\nu) > \omega$ that we have been discussing, define

$$\mathcal{H} = \{X \in \mathcal{G} \mid D \subseteq X\}.$$

Then $\mathcal{H}$ is a stationary subfamily of $\mathcal{G}$ and, for all $X, Y \in \mathcal{H}$,

$$\pi_Y[B_Y] \subseteq D \subseteq X.$$

Now we turn to the case $\mathrm{cf}(\nu) = \omega$. For each $X \in \mathcal{G}$, choose an unbounded $B_X \subseteq C_X$ such that $\mathrm{type}(B_X) = \omega$. Because X is the union of an internally approachable chain whose length has uncountable cofinality, there exists $D_X \in X$ such that $\pi_X[B_X] \subseteq D_X \subseteq X \cap \nu$. Apply Fodor's lemma on the choice function $X \mapsto D_X$ to find a stationary family $\mathcal{H} \subseteq \mathcal{G}$ and a fixed value D such that $D_X = D$ for every $X \in \mathcal{H}$. For all $X, Y \in \mathcal{H}$, if $X \subseteq Y$, then

$$\pi_Y[B_Y] \subseteq D \subseteq X \cap \nu.$$

In both cases, we have defined a stationary subfamily $\mathcal{H}$ of $\mathcal{G}$ and, for every $X \in \mathcal{H}$, a club subset B_X of C_X such that $\mathrm{type}(B_X) = \mathrm{cf}(\nu)$ and, for every $Y \in \mathcal{H}$, if $X \subseteq Y$, then $\pi_Y[B_Y] \subseteq X$. Moreover, π_X is continuous at every member of B_X in the case $\mathrm{cf}(\nu) > \omega$.

[7] We have been encouraged to remind the reader that §3 of [SSZ] explains how the rules for forming iteration trees on these phalanxes should be modified in the anomalous case, which happens when $\mathcal{P}_\mu$ is a type III mouse and we wish to apply an extender to $\mathcal{P}_\mu$ whose critical point is the largest cardinal of $\mathcal{P}_\mu$.

We have gone as long as we know how without picking between Mitchell-Steel and Jensen indexing but now we switch solely to Jensen indexing for the rest of §4 and §§5-6. By Lemma 3.6, for each $X \in \mathcal{F}$, there exists $\zeta_0^X <_{\mathcal{T}_X} \theta_X$ such that

$$I_X = [\zeta_0^X, \theta_X)_{\mathcal{T}_X}.$$

For the remainder of this section, assume $X, Y \in \mathcal{H}$ and $X \subseteq Y$. The following is the key result that depends on Jensen indexing.

Lemma 4.3. $\pi_Y[B_Y] \subseteq^* \pi_X[C_X]$.

Proof. First suppose that $\mathcal{Q}_X = \mathcal{M}_X$ and $\mathcal{Q}_Y = \mathcal{M}_Y$. By Lemma 2.6, both are set mice. By Lemma 4.2, $\mathcal{S}_{XY} \trianglelefteq \mathcal{M}_Y$. Consider any $\eta \in [\zeta_0^Y, \theta_Y)_{\mathcal{T}_Y}$ such that $\mu_\eta^Y \in B_Y$ and $\pi_Y(\mu_\eta^Y) \geq \min(\pi_X[C_X])$. If $\mathcal{S}_{XY} \triangleleft \mathcal{M}_Y$, then assume that

$$\mathcal{S}_{XY} \in \mathrm{Hull}_{\Sigma_{m+1}}^{\mathcal{M}_Y}\left(\mu_\eta^Y \cup p_{m+1}(\mathcal{M}_Y)\right).$$

Now, by the motivating property in the definition of $\mathcal{H}$,

$$\pi_Y(\mu_\eta^Y) \in X \cap \nu$$

so there exists $\alpha < \mu_X$ such that $\pi_X(\alpha) = \pi_Y(\mu_\eta^Y)$. It is enough to see that $\alpha \in C_X$. Otherwise, we have $\zeta + 1 \in [\zeta_0^X, \theta_X)_{\mathcal{T}_X}$ such that $\mu_{\zeta^*}^X < \alpha < \mu_{\zeta+1}$ where ζ^* is the $\mathcal{T}_X$ predecessor of $\zeta + 1$. Because $E_\zeta^X \restriction \alpha$ is not a superstrong extender over $\mathcal{M}_\zeta^X$,

$$\alpha \neq \mu_{\zeta+1}^X \cap \mathrm{Hull}_{\Sigma_{m+1}}^{\mathcal{M}_{\zeta+1}^X}\left(\alpha \cup p_{m+1}(\mathcal{M}_{\zeta+1}^X)\right).$$

Hence

$$\alpha \neq \mu_X \cap \mathrm{Hull}_{\Sigma_{m+1}}^{\mathcal{M}_X}\left(\alpha \cup p_{m+1}(\mathcal{M}_X)\right).$$

Hence

$$\Pi_{XY}(\alpha) \neq \mu_Y \cap \mathrm{Hull}_{\Sigma_{m+1}}^{\mathcal{S}_{XY}}\left(\Pi_{XY}(\alpha) \cup p_{m+1}(\mathcal{S}_{XY})\right).$$

But

$$\begin{aligned}\Pi_{XY}(\alpha) &= (\Phi_{XY})^{-1}(\Pi_X(\alpha)) \\ &= (\Phi_{XY})^{-1}(\Pi_Y(\mu_\eta^Y)) = (\Pi_Y)^{-1}(\Pi_Y(\mu_\eta^Y)) = \mu_\eta^Y.\end{aligned}$$

Hence

$$\mu_\eta^Y \neq \mu_Y \cap \mathrm{Hull}_{\Sigma_{m+1}}^{\mathcal{S}_{XY}}\left(\mu_\eta^Y \cup p_{m+1}(\mathcal{S}_{XY})\right).$$

Because η was sufficiently large,

$$\mu_\eta^Y \neq \mu_Y \cap \mathrm{Hull}_{\Sigma_{m+1}}^{\mathcal{M}_Y}\left(\mu_\eta^Y \cup p_{m+1}(\mathcal{M}^Y)\right).$$

But this is false.

For the protomouse case, use Lemma 2.1 to see hulls in $\mathcal{M}_X$ versus $\mathcal{Q}_X$ are equal below μ_X and to see hulls in $\mathcal{M}_Y$ and $\mathcal{Q}_Y$ are equal below

μ_Y. The hypotheses of Lemma 2.1 add requirements on what it means for η to be sufficiently large. The additional requirements are:

$$\mu^Y_\eta > (\mathrm{crit}(\dot{F})^+)^{\mathcal{M}_Y},\ \max(s(\dot{F}^{\mathcal{M}_Y}) \cap \mu),\ \max(p_1(\mathcal{M}_Y)),$$

$s(\dot{F}^{\mathcal{M}_Y})$ is Σ_1 definable from parameters in $\mu^Y_\eta \cup p_1(\mathcal{M}_Y)$ in $\mathcal{M}_Y$ and vice-versa,

$$\alpha > (\mathrm{crit}(\dot{F})^+)^{\mathcal{M}_X},\ \max(s(\dot{F}^{\mathcal{M}_X}) \cap \mu),\ \max(p_1(\mathcal{M}_X)),$$

$s(\dot{F}^{\mathcal{M}_X})$ is Σ_1 definable from parameters in $\alpha \cup p_1(\mathcal{M}_X)$ in $\mathcal{M}_X$ and vice-versa. $\square$

We defined $U_X = U(\mathcal{S}_X, \nu)$ and $U_Y = U(\mathcal{S}_Y, \nu)$. Recall these are the order zero measures of $\mathcal{S}_X$ and $\mathcal{S}_Y$ respectively.

Lemma 4.4. *$U_X \subseteq U_Y$.*

Proof. It suffices to show that $U_X \cap \mathrm{ran}(\Pi_X) \subseteq U_Y$. The reason is that the $\mathrm{ran}(\Pi_X)$ is unbounded in $(\nu^+)^{\mathcal{S}_X}$ and, for every

$$\alpha \in (\nu^+)^{\mathcal{S}_X} \cap \mathrm{ran}(\Pi_X),$$

there exists a surjection $f : \nu \to J^{\mathcal{S}_X}_\alpha \cap U_X$ in the range of Π_X, hence the diagonal intersection of f also belongs to the range of Π_X.

Consider any $A \subseteq \mu_X$ with $A \in \mathcal{Q}_X$. Now, we assume that $\Pi_X(A) \in U_Y$ and prove that $\Pi_X(A) \in U_X$. The converse will follow by the "ultra" in ultrafilter and considering $\mu_X - A$.

Note that $\mathcal{M}_X$ and $\mathcal{Q}_X$ have the same subsets of μ_X and

$$U(\mathcal{M}_X, \mu_X) = U(\mathcal{Q}_X, \mu_X).$$

Also, for every $\eta \in [\zeta^X_0, \theta_X)_{\mathcal{T}_X}$,

$$i^{\mathcal{T}_X}_{\eta,\theta_X}(U(\mathcal{M}^X_\eta, \mu^X_\eta)) = U(\mathcal{M}_X, \mu_X).$$

The reader should briefly contemplate the meaning of this equation when the order zero extenders are the top extenders and not actually members of the mice. Our goal of showing that $\Pi_X(A) \in U_X$ is equivalent to each of the following:

- $A \in U(\mathcal{M}_X, \mu_X)$
- $\sup(\{\eta \in [\zeta, \theta_X)_{\mathcal{T}_X} \mid A \cap \mu^X_\eta \in U(\mathcal{M}^X_\eta, \mu^X_\eta)\}) = \theta_X$

We will verify the second condition.

Observe that $\Pi_{XY}(A) \in \mathcal{S}_{XY}$, so $\Pi_{XY}(A) \in \mathcal{Q}_Y$ by Lemma 4.2. As $\mathcal{M}_Y$ and $\mathcal{Q}_Y$ have the same subsets of μ_Y, we have that $\Pi_{XY}(A) \in \mathcal{M}_Y$. Note that $U(\mathcal{M}_Y, \mu_Y) = U(\mathcal{Q}_Y, \mu_Y)$. Our assumption that $\Pi_X(A) \in U_Y$ is equivalent to each of the following:

- $\Pi_{XY}(A) \in U(\mathcal{M}_Y, \mu_Y)$
- $\exists \zeta \in [\zeta^Y_0, \theta_Y)_{\mathcal{T}_Y}\ \forall \eta \in [\zeta, \theta_Y)_{\mathcal{T}_Y}\ \Pi_{XY}(A) \cap \mu^Y_\eta \in U(\mathcal{M}^Y_\eta, \mu^Y_\eta)$

The following statements are equivalent to each other:

- $\exists \zeta \in [\zeta_0^Y, \theta_Y)_{\mathcal{T}_Y}\ \mu_\zeta^Y$ is not measurable in W_Y.
- $\exists \zeta \in [\zeta_0^Y, \theta_Y)_{\mathcal{T}_Y}\ E_\zeta^{\mathcal{T}_Y} = EZ(\mathcal{M}_\zeta^{\mathcal{T}_Y}, \mu_\zeta^Y)$
- $\exists \zeta \in [\zeta_0^Y, \theta_Y)_{\mathcal{T}_Y}\ \forall \eta \in [\zeta, \theta_Y)_{\mathcal{T}_Y}\ E_\zeta^{\mathcal{T}_Y} = EZ(\mathcal{M}_\zeta^{\mathcal{T}_Y}, \mu_\zeta^Y)$
- $\exists \zeta \in [\zeta_0^Y, \theta_Y)_{\mathcal{T}_Y}\ \forall \eta \in [\zeta, \theta_Y)_{\mathcal{T}_Y}\ \mu_\eta^Y$ is not measurable in W_Y

The rest of the proof of the lemma divides into two cases depending on whether the four equivalent statements above are true or false. First suppose they are false. Consider any $\eta \in [\zeta_0^Y, \theta_Y)_{\mathcal{T}_Y}$. As μ_η^Y is measurable in W_Y, it is also measurable in $\mathcal{S}_{XY}$ and $\mathcal{M}_Y$ by the agreement between these mice on bounded subsets of μ_Y. By the agreement between extender sequences,

$$U(\mathcal{M}_\eta^Y, \mu_\eta^Y) = U(\mathcal{M}_Y, \mu_\eta^Y) = U(W_Y, \mu_\eta^Y) = U(\mathcal{S}_{XY}, \mu_\eta^Y)$$

Assume that η is large enough that

$$\Pi_{XY}(A) \cap \mu_\eta^Y \in U(\mathcal{M}_\eta^Y, \mu_\eta^Y).$$

Lemma 4.3 implies that $B_Y \subseteq^* \pi_{XY}[C_X]$. Assume that $\mu_\eta^Y \in B_Y$ and η is large enough to guarantee that $\mu_\eta^Y \in \pi_{XY}[C_X]$. Pick $\zeta \in [\zeta_0^X, \theta_X)_{\mathcal{T}_X}$ such that $\pi_{XY}(\mu_\zeta^X) = \mu_\eta^Y$. Now we have that

$$\Pi_{XY}(A) \cap \Pi_{XY}(\mu_\zeta^X) \in U(\mathcal{S}_{XY}, \mu_\eta^Y) = \Pi_{XY}(U(\mathcal{M}_X, \mu_\zeta^X)).$$

Therefore,

$$A \cap \mu_\zeta^X \in U(\mathcal{M}_X, \mu_\zeta^X).$$

We have seen that this holds for unboundedly many $\zeta \in [\zeta_0^X, \theta_X)_{\mathcal{T}_X}$, so we have met our goal in the first case.

Now for the other case. Assume that $\zeta_1 \in [\zeta_0^Y, \theta_Y)_{\mathcal{T}_Y}$ and μ_η^Y is not measurable in W_Y for every $\eta \in [\zeta_1, \theta_Y)_{\mathcal{T}_Y}$. This means that, from $\mathcal{M}_{\zeta_1}^{\mathcal{T}_Y}$ to $\mathcal{M}_{\theta_Y}^{\mathcal{T}_Y}$, we have the linear iteration by the images of a measure of order zero:

$$E_\eta^{\mathcal{T}_Y} = EZ(\mathcal{M}_\eta^Y, \mu_\eta^Y) = i_{\zeta_1,\eta}^{\mathcal{T}_Y}(EZ(\mathcal{M}_{\zeta_1}^Y, \mu_{\zeta_1}^Y)).$$

Let $\zeta_2 \in [\zeta_1, \theta_Y)_{\mathcal{T}_Y}$ be large enough that

$$\Pi_{XY}(A) \in \operatorname{ran}(i_{\zeta_2,\theta_Y}^{\mathcal{T}_Y}).$$

For the case we are discussing, the fact that $\Pi_{XY}(A) \in U(\mathcal{M}_Y, \mu_Y)$ is equivalent to

$$\{\mu_\eta \mid \zeta_2 \leq \eta < \theta_Y\} \subseteq \Pi_{XY}(A).$$

Again, we use that $B_Y \subseteq^* \pi_{XY}[C_X]$. Consider any $\eta \in [\zeta_2, \theta_Y)_{\mathcal{T}_Y}$ such that $\mu_\eta^Y \in B_Y$ and η is large enough to guarantee that $\mu_\eta^Y \in \pi_{XY}[C_X]$. Pick $\varepsilon \in [\zeta_0^X, \theta_X)_{\mathcal{T}_X}$ such that $\mu_\eta^Y = \pi_{XY}(\mu_\varepsilon^X)$. Then μ_ε^X is not measurable in W_X. Hence $E_\varepsilon^{\mathcal{T}_X} = EZ(\mathcal{M}_\varepsilon^X, \mu_\varepsilon^X)$. In other words, the situation on $\mathcal{T}_X$ is like that of $\mathcal{T}_Y$ in that, eventually, we are iterating linearly by the images of a measure of order zero. Now it is routine to translate $\pi_{XY}(\mu^X_\varepsilon) \in \Pi_{XY}(A)$ to $\mu^X_\varepsilon \in A$ and, then, to $A \in U(\mathcal{M}_X, \mu_X)$ as was our goal. $\square$

5. The total measure

Recall that after Theorem 4 we defined a stationary family $\mathcal{F}$. In §3, we applied Fodor's Lemma to a certain choice function to obtain stationary subfamilies $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$. All the facts proved so far hold for $X \subseteq Y$ that both belong to $\mathcal{H}$. Define

$$U = \bigcup \{U_X \mid X \in \mathcal{H}\}.$$

The results in this section depend on Lemma 4.4, which we know only for Jensen indexing.

Lemma 5.1. *U is an ultrafilter over W and is amenable to W.*

Proof. $\mathcal{H}$ is directed: for $X, Y \in \mathcal{H}$, there exists $Z \in \mathcal{H}$ such that $X \cup Y \subseteq Z$. By Lemma 4.4, both U_X and U_Y are contained in U_Z. Hence U_X and U_Y are compatible filters. For every $A \subseteq \nu$ with $A \in W$, there exists $X \in \mathcal{H}$ with $A \in X$, which implies that A is measured by U_X; see Lemma 3.10. Amenability holds because each U_X belongs to W by Lemma 3.9. $\square$

Lemma 5.2. $\mathrm{Ult}(W, U)$ *is wellfounded and the phalanx*

$$((W, \mathrm{Ult}(W, U)), \nu)$$

is iterable.

Proof. Pick $X \in \mathcal{H}$ with $U \in X$. Recall that $W_X \lhd \mathcal{M}^{\mathcal{T}_X}_\infty$. By Lemma 5.1 and elementarity, $\pi_X^{-1}(U)$ is an ultrafilter over W_X that is amenable to W_X. Let $\beta = (\mu_X^+)^{W_X}$. Then $\Pi_X \restriction J^{W_X}_\beta = \pi_X \restriction J^{W_X}_\beta$. Using this equation and amenability, we conclude that

$$\pi_X^{-1}(U) = \bigcup_{\alpha<\beta} \pi_X^{-1}(U \cap J^W_{j(\alpha)}) = \bigcup_{\alpha<\beta} \Pi_X^{-1}(U_X \cap J^W_{j(\alpha)}) = U(\mathcal{Q}_X, \mu_X).$$

Recall that

$$U(\mathcal{Q}_X, \mu_X) = U(\mathcal{P}_X, \mu_X) = U(\mathcal{M}_X, \mu_X)$$

and that this measure has exactly the same ultrapowers as the extender

$$EZ(\mathcal{Q}_X, \mu_X) = EZ(\mathcal{P}_X, \mu_X) = EZ(\mathcal{M}_X, \mu_X).$$

Now:

- $\mathcal{T}_X$ is an iteration tree on W,

- $\mathcal{M}_X$ is a model of $\mathcal{T}_X$,
- $EZ(\mathcal{M}_X, \mu_X)$ is an extender on the sequence of $\mathcal{M}_X$, and
- W_X is an initial segment of a later model of $\mathcal{T}_X$.

Viewed this way, $\mathrm{Ult}(W_X, EZ(\mathcal{M}_X, \mu_X))$ is an "iterate" of W albeit not technically an iterate of W because an extender from an earlier model was applied to a later model. (This violation occurred only once. Infinitely many violations would lead to illfoundedness.) Nevertheless, W satisfies an enhanced form of iterability that implies $\mathrm{Ult}(W_X, EZ(\mathcal{M}_X, \mu_X))$ is wellfounded and the phalanx

$$((W_X, \mathrm{Ult}(W_X, EZ(\mathcal{M}_X, \mu_X))), \mu_X)$$

is iterable. The proof uses the fact that W embeds into K^c and the corresponding enhanced iterability theorem for K^c, which is due to Steel.[8] Thus, $\mathrm{Ult}(W_X, U(\mathcal{M}_X, \mu_X))$ is wellfounded and the phalanx

$$((W_X, \mathrm{Ult}(W_X, U(\mathcal{M}_X, \mu_X))), \mu_X)$$

is iterable. Wellfoundedness is downward absolute to N_X. Iterability is too. At each limit stage in the construction of an iteration tree of length $\leq \Omega_X$ in N_X, use 1) the existence of a cofinal wellfounded branch in V, 2) the uniqueness of this branch, not just in V but also in $V[G]$ where G is V-generic for $\mathrm{Coll}(\omega, \Omega_X)$, 3) Shoenfield absoluteness from $V[G]$ to $N_X[G]$ and 4) the weak homogeneity of the collapse poset in N_X. Lemma 5.2 follows as $\pi_X : N_X \to H_{\Omega^+}$ is an elementary embedding. (We are only interested in iteration trees of length $\leq \Omega$ here.) $\square$

As an aside, it almost looks like, in the proof of Lemma 5.2, we could instead reduce the iterability of

$$((W_X, \mathrm{Ult}(W_X, EZ(\mathcal{Q}_X, \mu_X))), \mu_X)$$

to that of $((W_X, Q_X), \mu_X)$, which we know is iterable. But not quite since, in a one-step iteration tree on the latter phalanx, $EZ(\mathcal{Q}_X, \mu_X)$ would be applied to $\mathcal{Q}_X$ rather than W_X.

Lemma 5.3. *$U \in W$.*

Proof. Coiterate W and $((W, \mathrm{Ult}(W, U)), \nu)$. Use standard arguments to see that the final model on the phalanx side is above $\mathrm{Ult}(W, U)$, the two final models of this coiteration are equal, and the order zero initial segment of the first extender used along the main branch on the W side is $EZ(W, \nu)$. Hence $U = U(W, \nu) \in W$. $\square$

[8]See [A] for even more general rules about choosing and applying extenders, and corresponding iterability results about countable elementary substructures of rank initial segments of V.

This completes the proof that ν is measurable in K constructed using Jensen indexing without assuming that $(2^{\mathrm{cf}(\nu)})^+ < |\nu|$.

6. Uncountable cofinality and more measures

In this section, we sketch the "moreover" part of Theorem 1 for Jensen indexing. Assuming $\mathrm{cf}(\nu) > \omega$, we deduce $o^K(\nu) > \alpha$ for every $\alpha < \mathrm{cf}(\nu)$ by induction. The base case $\alpha = 1$ was shown in the previous section. Here, we outline the case $\alpha = 2$ in a way that reveals the remaining ideas for the rest of the induction. The way to finish is described at the end of this section.

Lemma 6.1. *Assume* $\mathrm{cf}(\nu) > \omega$. *Then*

$$\{\lambda < \nu \mid \lambda \text{ is a measurable cardinal in } W\}$$

contains a club subset of ν.

Sketch. Deny. Let S be a stationary subset of ν that has no measurable cardinals of W. We may assume that $S \subseteq D$ where D is the club in ν of type $\mathrm{cf}(\nu)$ that was fixed just before we defined $\mathcal{H}$ in §4. Then $S \subseteq D \cap X$ for every $X \in \mathcal{H}$. Put $S_X = \pi_X^{-1}[S]$. Then S_X is stationary in μ_X. Let κ_X be the least limit point of $S_X \cap C_X$ that belongs to S_X. It will be relevant that π_X is continuous at κ_X. Fodor's Lemma gives $\lambda \in S$ and a stationary subfamily $\mathcal{I}$ of $\mathcal{H}$ such that $\pi_X(\kappa_X) = \lambda$ for every $X \in \mathcal{I}$. Now repeat the construction in §4 but replacing

$$\nu,\ D,\ \mathcal{H},\ \mu_X,\ C_X,\ B_X$$

by

$$\lambda,\ S \cap \lambda,\ \mathcal{I},\ \kappa_X,\ C_X \cap \kappa_X,\ B_X \cap S_X$$

to see that λ is measurable in W. Contradiction! $\square$

Lemma 6.2. *Assume* $\mathrm{cf}(\nu) > \omega$. *Then there exists* $U_1 \in W$ *such that, in* W, U_1 *is a normal measure over* ν *and*

$$\{\lambda < \nu \mid \lambda \text{ is a measurable cardinal in } W\} \in U_1.$$

In particular, ν *has Mitchell order at least* 2 *in* W.

Sketch. Change notation to $D_0 = D$, the club subset of ν of order type ν we fixed in §5. From Lemma 6.1, we have a club subset D_1 of ν such that every member of D_1 is a measurable cardinal in W. We may assume that $D_1 \subseteq D_0$. Change notation to $E(\mathcal{M}, \kappa, 0) = EZ(\mathcal{M}, \kappa)$ and $U(\mathcal{M}, \kappa, 0) = U(\mathcal{M}, \kappa)$. Define $E(\mathcal{M}, \kappa, 1)$ be the second total extender with critical point κ on the $\mathcal{M}$ sequence and let $U(\mathcal{M}, \kappa, 1)$ be the normal measure derived from $E(\mathcal{M}, \kappa, 1)$ should they exist. Now repeat the construction in §5 but replacing D_0, $E(\mathcal{M},\kappa,0)$, and $U(\mathcal{M},\kappa,0)$ by D_1, $E(\mathcal{M},\kappa,1)$, and $U(\mathcal{M},\kappa,1)$ to define

$$U_1 = \bigcup_{X\in\mathcal{H}} U_1(\mathcal{S}_\nu,\nu,1)$$

and argue that U_1 has the necessary properties. $\square$

The pattern for finishing the proof of Theorem 1 by induction on $\beta < \mathrm{cf}(\nu)$ is clear. The steps depend on $\mathrm{cf}(\nu) > \omega$ for the combinatorics of club and stationary subsets of ν. Recursively, we have constructed clubs $\nu \supseteq D_0 \supseteq \cdots \supseteq D_\alpha \supseteq \cdots$ such that D_α consists of cardinals that are measurable of order α in W for each $\alpha < \beta$. We let $D_\beta = \bigcap_{\alpha<\beta} D_\alpha$. Then D_β is club in ν; this is where we use $\beta < \mathrm{cf}(\nu)$. Using D_β, $E(\mathcal{M},\kappa,\beta)$, and $U(\mathcal{M},\kappa,\beta)$, we find U_β by following the template of §5 and Lemma 6.2. In particular, U_β concentrates on the set of $\lambda < \nu$ such that λ is measurable of order at least β in W. Continuing as in Lemma 6.1, we prove there exists a club $D_{\beta+1} \subseteq D_\beta$ whose members are measurable cardinals of order at least β in W. And so on.

7. Appendix

Here, we return to Mitchell-Steel indexing. The first result in this section records a relationship between the projectum and the Dodd projectum that was used implicitly and explicitly in the proof of Theorem 1. (E.g., in the proof of Lemma 3.3.)

Lemma 7.1. *Let $\mathcal{M} = \mathcal{J}_\beta^{\mathcal{M}}$ be a type II premouse, $F = \dot{F}^{\mathcal{M}}$, $\kappa = \mathrm{crit}(F)$, $\lambda = (\kappa^+)^{\mathcal{M}}$, $\tau = \tau(F)$, $s = s(F)$, $\lambda \le \alpha < \beta$ and $q \in [\beta]^{<\omega}$. Assume $\mathrm{Hull}_{\Sigma_1}^{\mathcal{M}}(\alpha \cup q) = \mathcal{M}$. Then $\tau \le \alpha$.*

Proof. Let $D = J_\lambda^{\mathcal{M}}$. Pick $e \in D$ and $t \in [\tau]^{<\omega}$ such that $q = [s\cup t, e]_F^D$. It is enough to show that $\mathcal{M} \subseteq \mathrm{Ult}_0(D, F \restriction \alpha \cup t)$. Consider any $x \in \mathcal{M}$. Pick $r \in [\alpha]^{<\omega}$ and $\sigma < \beta$ such that x is Σ_1 definable from q and r over $\mathcal{M} \restriction \sigma$. Pick $\delta < \lambda$ large enough that $e \in J_\delta$, $\kappa < \delta$ and $\mathcal{M} \restriction \sigma$ is coded by $F \cap (J_\delta^{\mathcal{M}} \times [s_0+1]^{<\omega})$. Let $f : \kappa \to J_\delta^{\mathcal{M}}$ be a surjection with $f \in D$. Let $g : [\kappa]^2 \to D$ be the function

$$g : \{\eta,\theta\} \mapsto \{(a, f(\xi) \cap [\eta]^{|a|} \mid a \in [\theta+1]^{<\omega}, \xi < \eta \text{ and } a \in f(\xi)\}.$$

Then

$$[\{\kappa, s_0\}, g]_F^D = F \cap (J_\delta^{\mathcal{M}} \times [s_0+1]^{<\omega}).$$

Using g, we find a function $h \in D$ such that $x = [r \cup s \cup t, h]_F^D$. $\square$

We thought we needed the following lemma for the proof of Theorem 1 but it turned out we did not. Still, the lemma and its proof seem likely to have applications elsewhere. The lemma strengthens the result that the standard parameter is universal. Its proof involves strengthening the result of [MSt] that extenders used on maximal single-rooted iteration trees are close to the models to which they are applied.

Lemma 7.2. *Let $\mathcal{M}$ be a mouse and $\rho = \rho_1(\mathcal{M})$. Then $\mathcal{M}$ and $\mathfrak{C}_1(\mathcal{M})$ have the same subsets of ρ that are Σ_1 definable with parameters.*

Proof. Obviously, for every $A \subseteq \rho$, if A is Σ_1 definable with parameters over $\mathfrak{C}_1(\mathcal{M})$, then A is Σ_1 definable with parameters over $\mathcal{M}$. We must prove the converse. Coiterate to obtain iteration trees $\mathcal{U}$ on $\mathcal{M}$ and $\mathcal{V}$ on $((\mathcal{M}, \mathfrak{C}_1(\mathcal{M})), \rho)$. As in the proof of Theorem 8.1 of [MSt], we get that $1 \leq_{\mathcal{V}} 1{+}\infty$, there is no dropping along $[0, \infty]_{\mathcal{U}}$ and $[1, 1{+}\infty]_{\mathcal{V}}$, and the iteration trees have a common final model $\mathcal{M}^{\mathcal{U}}_{\infty} = \mathcal{M}^{\mathcal{V}}_{1+\infty}$ that we call $\mathcal{N}$. We have cofinal Σ_1 elementary iteration maps $i^{\mathcal{U}}_{0,\infty} : \mathcal{M} \to \mathcal{N}$ and $i^{\mathcal{V}}_{1,1+\infty} : \mathfrak{C}_1(\mathcal{M}) \to \mathcal{N}$ both of which are the identity below ρ. It suffices to see that if $A \subseteq \rho$ and A is Σ_1 definable with parameters over $\mathcal{N}$, then A is Σ_1 definable with parameters over $\mathfrak{C}_1(\mathcal{M})$. It suffices to see that if ζ is the $\mathcal{V}$ predecessor of $\theta + 1$, then $(E^{\mathcal{V}}_{\theta})_a$ is Σ_1 definable with parameters over $(\mathcal{M}^*_{\theta+1})^{\mathcal{V}}$ for every $a \in [\mathrm{lh}(E^{\mathcal{V}}_{\theta})]^{<\omega}$. The corresponding fact about iteration trees with one root is proved by induction in Lemma 6.1.5 of [MSt]. Assume that E_η is close to $\mathcal{M}^*_{\eta+1}$ whenever $1 \leq \eta < \theta$. Let θ^* be the $\mathcal{V}$ predecessor of $\theta + 1$. If $\theta^* \geq 1$, then the proof on pp. 61–63 of [MSt] adapts in a straightforward manner. Assume that $\theta^* = 0$. This means that $\mathrm{crit}(E^{\mathcal{V}}_{\theta}) < \rho$ and $(E^{\mathcal{V}}_{\theta})_a \subseteq J^{\mathcal{M}}_{\rho}$. The difference with the proof in [MSt] is that $E^{\mathcal{V}}_0$ is undefined here. For every $\eta \geq 1$, if $(E^{\mathcal{V}}_{\theta})_a \in \mathcal{M}^{\mathcal{V}}_{\eta}$, then $(E^{\mathcal{V}}_{\theta})_a \in \mathfrak{C}_1(\mathcal{M})$, hence $(E^{\mathcal{V}}_{\theta})_a \in \mathcal{M}$. Therefore, we may assume that E_θ is the top extender of $\mathcal{M}_\theta$. Our accumulated assumptions imply that there are no drops on the branch $b = \{\eta \mid \eta \leq_{\mathcal{V}} \theta\}$. If $1 \leq_{\mathcal{U}} \theta$, then our assumptions imply that $(E_\theta)_a$ is Σ_1 definable with parameters over $\mathfrak{C}_1(\mathcal{M})$, hence over $\mathcal{M}$. If $0 <_{\mathcal{U}} \theta$, then $\rho_1(\mathcal{M}_\theta) > \rho$, so $(E_\theta)_a$ belongs to $\mathcal{M}_\theta$, hence to $\mathfrak{C}_1(\mathcal{M})$, hence to $\mathcal{M}$. □

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Department of Mathematics, University of Florida, Gainesville, FL, 32611-8105, USA

Email address: wjm@ufl.edu

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213-3890, USA

Email address: eschimme@andrew.cmu.edu