Aronszajn Trees and the SCH

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These are notes on results presented by Itay Neeman at the Appalachian Set Theory workshop on February 28, 2009. Spencer Unger was the official note-taker and based these notes closely on Neeman's lectures.

Definition. *A* λ-tree *is a tree of height* λ *with levels of size* $< \lambda$. *A* λ-Aronszajn *tree is a* λ*-tree with no cofinal branch.*

Fact (Specker)**.** *If* κ *is regular and* $2^{<\kappa} = \kappa$*, then there is a* κ^+*-Aronszajn tree.*

In fact, the tree constructed by the above method is *special* in the sense of the following definition.

Definition. *A* κ^+*-tree is* special *if and only if there is a function* $f : T \to \kappa$ *such that for all* $x, y \in T$*, if* xTy *then* $f(x) \neq f(y)$*.*

Definition. SCH*:* κ *is singular and* $2^{<\kappa} = \kappa$ *implies* $2^\kappa = \kappa^+$

Violating SCH is hard. It was first done by starting with κ supercompact and then,

1. Forcing to get κ measurable and $2^\kappa > \kappa^+$
2. Use Prikry forcing to make κ singular of cofinality ω.

The best result of this form is due to Gitik,

Theorem 1. $Con(\neg \text{SCH}) \Leftrightarrow Con(\exists \kappa\ o(\kappa) = \kappa^{++})$

When we force to obtain the first item on the list, there is a κ^+ Aronszajn tree in the extension and it remains an Aronszajn tree after Prikry forcing. In fact, Prikry forcing adds a special Aronszajn tree. This can be found in Work of Ben-David and Magidor which contains a contribution from Grossberg. So the original model for the failure of SCH_κ, has a κ^+-Aronszajn Tree.

This brings up the following question, which Woodin asked in the late 80's.[1] For κ singular of cofinality ω, does the failure of SCH imply that there exists a κ^+-Aronszajn tree? The question was intended to test whether the above (1) + (2) is the only way to get the failure of SCH. Gitik and others showed that the answer to the latter question is no by proving that one can add κ^{++} Prikry sequences. However, Woodin's original question remained open. It turns out, there are other ways to make SCH fail, but they all still gave Aronszajn trees! So a lot of work went into proving that the answer to the above question is "yes".

Definition. *λ has the* Tree Property *if and only if every λ-tree has a cofinal branch. Equivalently, There is no λ Aronszajn tree.*

Given this definition, Woodin's question can be restated. For κ of cofinality ω, does the Tree Property on κ^+ imply SCH_κ?

Many had hope for a positive answer. The line of thought was that the failure of SCH at κ would imply some intermediate combinatorial principle, like Approachability or Weak Square, and then from this one could construct a κ^+ Aronszajn tree.

It turns out that the first step fails. In particular that the failure of SCH is not enough to give an appropriate combinatorial principle. This was shown in work of Gitik and Sharon. Cummings and Foreman showed that futhermore it is possible to have the failure of SCH without any approachability properties.

The purpose of this workshop is to present a proof that the answer to Woodin's question is *no*.

1 The Tree Property

Theorem 2. *(Shelah) Let κ be the limit of supercompact cardinals with* $\mathrm{cof}(\kappa) = \omega$*, then κ^+ has the Tree Property.*

Note that a singular limit, κ, of κ^+ strong compact cardinals is enough to get the above theorem, but the theorem is stated in the form that we will use it.

This was later improved by Magidor and Shelah who showed that one can collapse between the supercompacts to get $\kappa = \tau^{+\omega}$, where τ is the first supercompact, and still have the tree property at κ^+. Moreover, one can make $\kappa = \aleph_\omega$ and still have the tree property at κ^+. For the last step, a little more than a huge cardinal is needed.

In the proof of Shelah's theorem we have Lemma 1 and Lemma 2. In the rest of proof we will have to prove new versions of Lemma 1 and Lemma 2.

Proof of Shelah's Theorem. Let T be a κ^+ tree. Without loss of generality, level α of T is $\{\langle\alpha,\xi\rangle | \xi < \kappa\}$. Let $\kappa_n, n < \omega$ be an increasing sequence of supercompact cardinals cofinal in κ.

Lemma 1 (Version 1)**.** *There is an $n < \omega$ and a cofinal $C \subseteq \kappa^+$ such that for all $\alpha < \beta$ both in C, there are $\xi, \zeta < \kappa_n$ such that $\langle\alpha,\xi\rangle T \langle\beta,\zeta\rangle$.*

Proof. Let $\pi : V \to M$ be a κ^+ supercompactness embedding with critical point κ_0. Let $\gamma^* =_{def} \sup \pi``\kappa^+$. Then $\gamma^* < \pi(\kappa^+)$, as κ^+ is regular. Fix some node on level γ^* of $\pi(T)$, $\langle\gamma^*, \eta^*\rangle$. For each $\alpha < \kappa$, there is some unique, ξ^*_α, such that

$$\langle \pi(\alpha), \xi^*_\alpha \rangle \pi(T) \langle \gamma^*, \eta^* \rangle$$

As $\langle \pi(\kappa_n)\colon n < \omega \rangle$ is cofinal in $\pi(\kappa)$, there is $n_\alpha < \omega$ such that $\xi^*_\alpha < \pi(\kappa_n)$. As κ^+ is regular, we have $n < \omega$ and $C \subseteq \kappa^+$ cofinal such that $n_\alpha = n$ for all $\alpha \in C$.

We check that this n and C are as required for Lemma 1 version 1. Fix $\alpha < \beta$ both in C.
Note: $\langle \pi(\alpha), \xi^*_\alpha \rangle\ \pi(T)\ \langle \gamma^*, \eta^* \rangle$ and similarly for β, $\langle \pi(\beta), \xi^*_\beta \rangle\ \pi(T)\ \langle \gamma^*, \eta^* \rangle$. Since $\pi(T)$ is a tree, $\langle \pi(\alpha), \xi^*_\alpha \rangle\ \pi(T)\ \langle \pi(\beta), \xi^*_\beta \rangle$.
So $M \models \exists \xi^*, \zeta^* < \pi(\kappa_n)\ \langle \pi(\alpha), \xi^* \rangle\ \pi(T)\ \langle \pi(\beta), \zeta^* \rangle$
By elementarity, $V \models \exists \xi, \zeta < \kappa\ \langle \alpha, \xi \rangle\ T\ \langle \beta, \zeta \rangle$. □

Lemma 2 (Version 1)**.** *There is a cofinal $J \subseteq C$, and a map $\alpha \mapsto \xi_\alpha$ such that for all $\alpha < \beta$ both in J, $\langle \alpha, \xi_\alpha \rangle\ T\ \langle \beta, \xi_\beta \rangle$.*

Note. *This lemma is what we want as our map enumerates a cofinal branch.*

Proof. Let $\pi : V \to M$ be a κ^+ supercompactness embedding with critical point κ_{n+1}. The idea is that we have thinned the tree to $\{\langle \alpha, \xi \rangle : \alpha \in C, \xi < \kappa_n\}$. And so we only stretch the leftover tree vertically but not horizontally. Let γ^* be the least element of $\pi(C)$ above $\pi"\kappa^+$. For each $\alpha \in C$ have $\xi^*_\alpha, \eta^*_\alpha < \pi(\kappa_n) = \kappa_n$, such that $\langle \pi(\alpha), \xi^*_\alpha \rangle\ \pi(T)\ \langle \gamma^*, \eta^*_\alpha \rangle$ by an application of Lemma 1 in M!

Fix $\eta^* < \kappa_n$ and $J \subseteq C$ cofinal, such that $\alpha \in J \Rightarrow \eta^*_\alpha = \eta^*$. Note that this is possible as $|C| = \kappa^+$ and there are only κ_n possibilities for a point on level γ^* of $\pi(T)$. For the latter, recall that we have restricted ourselves to looking at the points $\{\langle \gamma^*, \xi \rangle : \xi < \kappa_n\}$. Let $\xi_\alpha = \xi^*_\alpha$ for $\alpha \in J$. So $\pi(\xi_\alpha) = \xi^*_\alpha$.

We claim that J and $\alpha \mapsto \xi_\alpha$ work. Fix $\alpha < \beta$ both in J. Then

$$\langle \pi(\alpha), \pi(\xi_\alpha) \rangle\ \pi(T)\ \langle \gamma^*, \eta^* \rangle$$

$$\langle \pi(\beta), \pi(\xi_\beta) \rangle\ \pi(T)\ \langle \gamma^*, \eta^* \rangle$$

Since $\pi(T)$ is a tree, $\langle \pi(\alpha), \pi(\xi_\alpha) \rangle\ \pi(T)\ \langle \pi(\beta), \pi(\xi_\beta) \rangle$.
By elementarity $\langle \alpha, \xi_\alpha \rangle\ T\ \langle \beta, \xi_\beta \rangle$. □

This completes the proof of Shelah's theorem. □

2 Diagonal Prikry Forcing

In this section, we define the forcing and prove of some of its properties. For the setup we have κ supercompact and $\nu > \kappa$, $cof(\nu) = \omega$. We then force to make $cof(\kappa) = \omega$ and $|\nu| = \kappa$. In the extension, $(\nu^+)^V$ becomes the successor of κ.

To begin, fix $\langle \kappa_n : n < \omega \rangle$ increasing and cofinal in ν with each κ_n regular. Fix $\mathcal{U}_n$ supercompactness measures on $\mathcal{P}_\kappa(\kappa_n)$. Note that $\mathcal{U}_n$ measures sets, A, of the form $A = \{a \subseteq \kappa_n : |a| < \kappa a \cap \kappa \text{ inaccesible } \}$. For $a \in \mathcal{P}_\kappa(\kappa_n)$, define $\text{Cone}(a) = \{b : b \supseteq a, |b \cap \kappa| > |a|\}$.

We are ready to define our poset $\mathbb{P}$. Conditions are of the form $p = \langle g_p, A_p \rangle$, with

$$g_p = \langle g_p(0), \dots\ g_p(k-1) \rangle$$
$$A_p = \langle A_p(k), A_p(k+1), \dots \rangle$$

where $g_p(n) \in \mathcal{P}_\kappa(\kappa_n)$ and $A_p(n) \subseteq \mathcal{P}_\kappa(\kappa_n)$ with $\mathcal{U}_n$ measure 1. Lastly, we require that $g_p = \langle g_p(0), ... g_p(k-1) \rangle$ satisfy

1. For all $n < k$, $g_p(n) \cap \kappa$ is an inaccessible cardinal.
2. For all $n < m < k$, $g_p(m) \in Cone(g_p(n))$

The first condition avoids certain pathologies and the second condition is a kind of increasing property.

The order is defined as follows. $q \leq p$ if and only if

1. g_q extends g_p, ie $g_q \restriction \operatorname{dom} g_p = g_p$.
2. $A_q(n) \subseteq A_p(n)$ for all $n \geq \ell(g_q)$.
3. $g_q(n) \in A_p(n)$ for all n such that $\ell(g_p) \leq n < \ell(g_q)$.

Let G be $\mathbb{P}$-generic over V. Define $g = \bigcup\{g_p : p \in G\}$. Then g is a sequence of length ω, $\langle g_n : n < \omega \rangle$, with $g(n) \in \mathcal{P}_\kappa(\kappa_n)$ for all n.

Remark. *The intuition is that conditions in $\mathbb{P}$ are a description of g. A condition just specifies a finite initial segment of g, plus giving a restriction on later terms of g.*

By genericity, $\bigcup_{n<\omega} g(n) = \nu$. So in $V[G]$, $|\nu| = \sum_{n<\omega} |g(n)| \leq \kappa$, because $|g(n)| < \kappa$ for all n. In fact, for any τ such that $\kappa \leq \tau \leq \nu$, $\tau = \bigcup_{n<\omega} g(n) \cap \tau$ and $|g(n) \cap \tau| < \tau$. Moreover, if τ as above is regular in V, we have $cf(\tau) = \omega$ in $V[G]$.

Note. *If we had κ still srong limit, $2^\kappa > \nu^+$ and cardinals below κ and above ν were not collapsed, then we would have the failure of* SCH_κ *in $V[G]$.*

For $A = \langle A(n) : k \leq n < \omega \rangle$ and $B = \langle B(n) : k \leq n < \omega \rangle$, define

$$A \cap B = \langle A(n) \cap B(n) : k \leq n < \omega \rangle$$

If p, q are conditions and $g_p = g_q$, then p, q are compatible, because $\langle g_p, A_p \cap A_q \rangle$ is stronger than both. This shows that members of an antichain must have different stems. A result of Solovay [4] shows that $|\mathcal{P}_\kappa(\kappa_n)| = \kappa_n$. Therefore we have the following,

$$|\{\text{stems}\}| = \sum_{n<\omega} |\mathcal{P}_\kappa(\kappa_n)| = \nu$$

So $\mathbb{P}$ is ν^+-cc, which tells us that cardinals above ν are preserved.

We would like to see that cardinals below κ are preserved. In fact, we show that no bounded subsets of κ are added. To do this we prove that $\mathbb{P}$ has the 'Prikry property,' a propery we will explain later.

First, we define some notation. Let $\varphi = \varphi(\dot{x}_1, \ldots, \dot{x}_n)$ be a statement in the forcing language. Write $h \Vdash \varphi$ (for a finite stem h) if there is some A such that $\langle h, A\rangle$ is a condition forcing φ.

Note. *$h \Vdash \varphi$ and $h \Vdash \neg\varphi$ is impossible, as we would have $\langle h, A\rangle \Vdash \varphi$ and $\langle h, B\rangle \Vdash \neg\varphi$, but $\langle h, A\rangle$ and $\langle h, B\rangle$ are compatible.*

We write h decides φ, in symbols $h \parallel \varphi$, if $h \Vdash \varphi$ or $h \Vdash \neg\varphi$. Before we prove the Prikry Lemma, we need the definition of diagonal intersection and a few facts about it.

Definition. *Let $\langle \bar{A}_s : s \text{ is a stem}\rangle$ be a sequence such that for each s, $\bar{A}_s$ is a sequence of measure one sets as in the top half of our forcing conditions. Then the diagonal intersection of the above sequence, $\triangle_s \bar{A}_s$, is a sequence of sets whose n^{th} coordinate is the set $\{x \in \mathcal{P}_\kappa(\kappa_n)$: for all h if h is a stem of length n and $h^\frown x$ is a stem, then $x \in A_h\}$.*

Fact. *The n^{th} coordinate of the diagonal intersection is measure one for $\mathcal{U}_n$.*

Fact. *Let $\langle \bar{A}_s : s \text{ is a stem}\rangle$ be a sequence such that $\langle s, \bar{A}_s\rangle \in \mathbb{P}$ and let A^* be their diagonal intersection. If $\langle s, \bar{A}_s\rangle$ decides φ, then $\langle s, A^*(\ell(s)), A^*(\ell(s) + 1)\ldots\rangle$ decides φ in the same way.*

The proof is made easy by noting that $\langle s, A^*(\ell(s)), A^*(\ell(s) + 1)\ldots\rangle$ and $\langle s, \bar{A}_s\rangle$ are compatible.

The Prikry Lemma. *For all stems, h, and for all formulas φ in the forcing language, $h \parallel \varphi$. Equivalently, for every condition p there is a $q \leq p$ such that q decides φ and $g_p = g_q$.*

Proof.

Claim. *$h \nparallel \varphi$ with $\ell(h) = k$ implies for $\mathcal{U}_k$ almost every a, $h^\frown a \nparallel \varphi$*

We prove the contrapositive, that is we prove that if $B =_{def} \{a \in \mathcal{P}_\kappa(\kappa_k) : h^\frown a \text{ decides } \varphi\} \in \mathcal{U}_k$, then h decides φ.

So for each $b \in B$ there is a sequence of measure one sets $\bar{A}_b$ such that $\langle h^\frown b, \bar{A}_b\rangle$ decides φ. We can partition B in to the set of those b such that $\langle h^\frown b, \bar{A}_b\rangle \Vdash \varphi$ and the set of b such that $\langle h^\frown b, \bar{A}_b\rangle \Vdash \neg\varphi$. One of these sets must be measure one for $\mathcal{U}_k$. Without loss of generality, we let $B' \in \mathcal{U}_k$ such that for all $b \in B'$, $\langle h^\frown b, \bar{A}_b\rangle \Vdash \varphi$.

Form the diagonal intersection of the sequence $\langle \bar{A}_b : b \in B'\rangle$ and call it A^*. By the fact above, for all $b \in B'$, we have $\langle h^\frown b, A^*\rangle \Vdash \varphi$. We have a maximal antichain below $\langle h, B', A^*\rangle$ consisting of conditions that force φ. So we have finished the claim.

For the proof of the Prikry Lemma, we assume for a contradiction that there is a stem, h, and a statement φ such that $h \nparallel \varphi$.

Suppose f is a stem of length $n \geq k$ extending h such that f does not decide φ. Then by the claim there is a $\mathcal{U}_n$ measure one set of extensions of f that do not decide φ. Let $A_f(n)$ be this set and let $A_f(m) = \mathcal{P}_\kappa(\kappa_m)$ for all $m > n$. Let $\bar{A}_f = \langle A_f(n), A_f(n+1) \dots \rangle$.

Let $\bar{A}$ be the diagonal intersection of the sequence $\langle \bar{A}_f : f \text{ extends } h \rangle$.

We claim that no extension of $\langle h, \bar{A} \rangle$ decides φ. This will be our contradiction. Suppose $\langle h', \bar{B} \rangle \leq \langle h, \bar{A} \rangle$. Let $\ell(h') = n$. An easy inductive argument using the definition of diagonal intersection shows that for all m with $k \leq m < n$, $h'(m) \in A_{g \upharpoonright m}(m)$. So by the choice of $A_{g \upharpoonright m}(m)$ for each m as above, h' does not decide φ. So no condition extending $\langle h, \bar{A} \rangle$ decides φ. However it is a general forcing fact that we can always extend to decide a statement. This is a contradiction. $\square$

Using the Prikry Lemma it is easy to see that $\mathbb{P}$ adds no bounded subsets of κ. Suppose $\dot{f}$ is a $\mathbb{P}$-name for a function τ to θ, both cardinals such that $\tau < \theta < \kappa$. We want to force values of $\dot{f}$ without extending the stem and then use the closure of the second second coordinate. We can do the first part using the Prikry Lemma.

We assume that $\mathbb{1}_\mathbb{P} \Vdash \dot{f} : \tau \to \theta$. For each $\alpha < \tau$ and $\beta < \theta$, use the Prikry property to find $A_{\alpha,\beta}$ such that $\langle \emptyset, A_{\alpha,\beta} \rangle \parallel \dot{f}(\check{\alpha}) = \check{\beta}$. Let $A = \bigcap_{\substack{\alpha<\tau \\ \beta<\theta}} A_{\alpha,\beta}$

Then $\langle \emptyset, A \rangle$ is a condition by the κ-completeness of each $\mathcal{U}_n$. Moreover, $\langle \emptyset, A \rangle$ decides $\dot{f}$ completely. So $\dot{f}[G] \in V$ and cannot collapse θ. So $\mathbb{P}$ preserves cardinals less than κ and greater than ν.

Gitik-Sharon

What follows is a short summary of Gitik and Sharon's [2] use of Diagonal Prikry forcing. To begin we start with the statement of a theorem of Laver [3], which is used in their work.

Theorem 3. *Assuming there is a supercompact cardinal κ, then there is a forcing extension in which κ is still supercompact and remains supercompact under any κ-directed closed forcing. We say κ is* indestructibly supercompact.

Start from κ indestructibly supercompact, with GCH holding above κ. Take $\nu = \kappa^{+\omega}$, $\kappa_n = \kappa^{+n}$.

First, let $\mathbb{A} = Add(\kappa, \nu^{++})$, and take E to be $\mathbb{A}$-generic over V. Note that κ is still supercompact in $V[E]$.

Now, take $\mathbb{P}$ to be diagonal Prikry forcing for κ, ν as defined above, G $\mathbb{P}$-generic over $V[E]$. Then κ is singular in $V[E][G]$, of cofinality ω, and SCH fails at κ.

Cummings, Foreman showed that there is a bad scale and (with some extra assumptions) a very good scale on κ in $V[E][G]$.

3 The final proof

Recall the question, 'Does the tree property at κ^+ imply SCH_κ? We will show that the answer is no.

We will start with ν as a limit of ω supercompacts, κ_n for $n < \omega$. Let $\kappa = \kappa_0$ be indestructibly supercompact. Let $\mathbb{A}$ be $\mathrm{Add}(\kappa, \nu^{++})$. Let E be $\mathbb{A}$ generic over V. Let G be $\mathbb{P}$ generic over $V[E]$. We will start by showing that ν^+ has the tree property in $V[E]$ and then we will show that it still has the tree property after forcing with $\mathbb{P}$.

We proceed in two steps. The facts at the beginning imply that ν^+ has the tree property at the start, so we just have to prove that this is preserved in each extension. Our first step will be to show that it is preserved in the first extension.

3.1 First Extension

As a warm up we will show

Theorem 4. *ν^+ has the tree property in $V[E]$*

Let $T \in V[E]$ be a ν^+-tree. Without loss of generality level α of T is $\{\langle \alpha, \xi \rangle : \xi < \nu\}$.

Lemma 1 (Version 2)**.** *There is a $C \subseteq \nu^+$ cofinal and $n < \omega$, such that for all $\alpha < \beta$ both in C, there are $\xi, \zeta < \kappa_n$ such that $\langle \alpha, \xi \rangle T \langle \beta, \zeta \rangle$*

The proof is as before. κ is still supercompact in $V[E]$, using the indestructibility of κ.

What about Lemma 2? Before we used a supercompactness embedding $\pi : V \to M$ with critical point point κ_{n+1}. This time we need to lift π to $V[E]$. Let F be $\mathrm{Add}(\kappa, \pi(\nu^{++})$ generic over $V[E]$. In $V[E][F]$, we can extend π to $\pi^* : V[E] \to M[E^*]$. By work of Silver, it suffices to arrange that $\pi\text{''}E \subseteq E^*$. We can do this by interleaving the generic F with the point wise image of E under π to create the generic E^*.

Now we repeat the proof of Lemma 2 using π^*. So in $V[E][F]$ we get that there is a $J \subseteq C$ cofinal and $\alpha \mapsto \xi_\alpha$ for $\alpha \in J$ such that for $\alpha < \beta$ both in J, $\langle \alpha, \xi_\alpha \rangle T \langle \beta, \xi_\beta \rangle$. So we got a branch through T in $V[E][F]$, call it b. We want to show that $b \in V[E]$.

This will follow from

Lemma S. *Let S be tree of height θ in $M \models ZFC$. Let $\mathbb{B} \in M$ be a poset. Assume that $\mathbb{B} \times \mathbb{B}$ is θ-cc and that $\mathbb{B}^{|S|^+}$ does not collapse $|S|^+$. (Note: it doesn't matter which support is used in the power of $\mathbb{B}$ as long as we have the hypothesis.) Then $\mathbb{B}$ does not add cofinal branches through S. More precisely, if F is $\mathbb{B}$ generic over M and $b \in M[F]$ is a branch through S, then $b \in M$.*

Note. *In the above formulation S need not be a θ-tree.*

We will use this lemma with $M = V[E]$, $S = T$, $\theta = \nu^+$ and $\mathbb{B} = Add(\kappa, \pi(\nu^{++}))$. We need to check that the hypotheses hold. $\mathbb{B} \times \mathbb{B}$ is κ^+-cc, because it is $Add(\kappa, \pi(\nu^{++}) + \pi(\nu^{++}))$. And $\mathbb{B}^{\nu^{++}}$ is $Add(\kappa, \nu^{++} \cdot \pi(\nu^{++}))$ if we use supports of size $< \kappa$.

Proof of Lemma S. Work in M. We can assume that θ is regular. If θ were not regular, then we could replace it with $cof(\theta)$ and S by its restriction to $cof(\theta)$ levels cofinal in θ. Let $b \in M[F]$ be a cofinal branch through S. Fix a name $\dot{b}$ such that $\dot{b}[F] = b$. Suppose for a contradiction that $\Vdash_{\mathbb{B}} \dot{b} \notin \check{M}$

The idea is to force with $\mathbb{B}^* = \mathbb{B}^{|S|^+}$. Letting F^* be $\mathbb{B}^*$ generic over M, we see that $F^* = \prod_{\delta < |S|^+} F_\delta$, a product of generics. Then let $b_\delta = \dot{b}[F_\delta]$. By mutual genericity, the b_δs are all different.

Working with $\mathbb{B} \times \mathbb{B}$, use $\dot{b}_{left}$ and $\dot{b}_{right}$ to name the interpretation of $\dot{b}$ by the left and right generics. Note that the assumption $\Vdash_{\mathbb{B}} \dot{b} \notin \check{M}$ implies $\Vdash_{\mathbb{B}\times\mathbb{B}} \dot{b}_{left} \neq \dot{b}_{right}$. We are going to find δ_1, δ_2 such that $\Vdash \dot{b}_{\delta_1} = \dot{b}_{\delta_2}$, to get a contradiction to the fact that the interpretation of the $\dot{b}_\delta$s are distinct.

Let $H \prec M$. We are really taking a substructure of a rank initial segment of M. And arrange that H has the following properties, $\{\theta, S, \dot{b}, \mathbb{B}, \mathbb{B}^*\} \subseteq H$, $H \cap \theta$ is an ordinal and $|H| < \theta$. Since $\mathbb{B}$ is θ-cc, each antichain of $\mathbb{B}$ in H, is contained in H. So for all δ F_δ is $\mathbb{B}$ generic over H. Furthermore, $H[F_\delta] \prec M[F_\delta]$ and $H[F_\delta] \cap M = H$.

Similarly for $\mathbb{B} \times \mathbb{B}$ and $F_{\delta_1} \times F_{\delta_2}$. Let $\eta = H \cap \theta$. For each δ, let β_δ be the node of b_δ on level η. This is done in $M[F^*]$, as $M[F^*]$ has the map $\delta \mapsto \beta_\delta$ for $\delta \in |S|^+$. There are $|S|$ possibilities for β_δ and by assumption $|S|^+$ is a cardinal in $M[F^*]$. So there are δ_1 and δ_2 such that $\beta_{\delta_1} = \beta_{\delta_2}$.

Recall $\Vdash_{\mathbb{B}\times\mathbb{B}} \dot{b}_{left} \neq \dot{b}_{right}$ So there is a condition $\langle p_1, p_2 \rangle \in (F_{\delta_1} \times F_{\delta_2}) \cap H$ forcing this. Extending $\langle p_1, p_2 \rangle$ is needed we can get a level $\gamma \in H$ such that

$$\langle p_1, p_2 \rangle \Vdash \dot{b}_{left}(\gamma) \neq \dot{b}_{right}(\gamma)$$

This implies that $b_{\delta_1}(\gamma) \neq b_{\delta_2}(\gamma)$. As S is a tree and $\gamma < \eta$ we get $b_{\delta_1}(\eta) \neq b_{\delta_2}(\eta)$, a contradiction. □

This finishes the warm up. By Lemma S, the tree property holds in $V[E]$ at ν^+.

3.2 Second Extension

In this section the proof becomes more difficult. We show that the tree property holds at $\nu^+ = \kappa^+$ in $V[E][G]$. Let $T \in V[E][G]$. Without lost of generality, level α of T is $\{\langle \alpha, \xi \rangle : \xi < \kappa\}$. Let $\dot{T} \in V[E]$ such that $\dot{T}[G] = T$. Suppose $\mathbb{1}_{\mathbb{P}} \Vdash \dot{T}$ is a tree with the above form. Working in V[E], we formulate Lemma 1.

Lemma 1 (Version 3)**.** *There is $n < \omega$, $C \subseteq \nu^+$ cofinal so that for all $\alpha < \beta$ both in C, there are $\xi, \zeta < \kappa$ and a stem h of length n such that $h \Vdash \langle \alpha, \xi \rangle T \langle \beta, \zeta \rangle$.*

Proof. As before, fix $\pi : V[E] \to M$, a ν^+ supercompactness embedding with critical point κ. Let G^* be $\pi(\mathbb{P})$ generic over M. Let $T^* = \pi(\dot{T})[G^*]$. Let $\gamma^* = \sup \pi"\nu^+$ and fix η^* such that $\langle \gamma^*, \eta^* \rangle$ is a node of T^* on level γ^*.

For each $\alpha < \nu^+$, fix ξ^*_α such that $\langle \pi(\alpha), \xi^*_\alpha \rangle T^* \langle \gamma^*, \eta^* \rangle$. We have a condition forcing this. Let n_α be the length of its stem. Since $\pi \restriction \nu^+$ belongs to M, all of this is done in $M[G^*]$. In particular, the map $\alpha \mapsto n_\alpha$ belongs to $M[G^*]$. Note that ν^+ is a cardinal in $M[G^*]$ and $\nu^+ < \pi(\kappa)$. So we can find $C \subseteq \nu^+$ cofinal and $n < \omega$, such that $\alpha \in C \Rightarrow n_\alpha = n$.

Claim. *This C, n satisfy the requirements of Lemma 1 version 3*

Let $g^* = \bigcup\{g_p : p \in G^*\}$. Fix $\alpha < \beta$ both in C. Then $g^* \restriction n \Vdash \langle \pi(\alpha), \xi^*_\alpha \rangle \pi(\dot{T}) \langle \gamma^*, \eta^* \rangle$ and $\langle \pi(\beta), \xi^*_\beta \rangle \pi(\dot{T}) \langle \gamma^*, \eta^* \rangle$. As $\Vdash \pi(\dot{T})$ is tree, $g^* \restriction n \Vdash \langle \pi(\alpha), \xi^*_\alpha \rangle \pi(\dot{T}) \langle \pi(\beta), \xi^*_\beta \rangle$.

So $M \models$ "There is a stem h of length n, and $\xi, \zeta < \kappa$ such that $h \Vdash \langle \pi(\alpha), \xi \rangle \pi(\dot{T}) \langle \pi(\beta), \zeta \rangle$." By elementarity, in $V[E]$, There is a stem, h, of length n such that $h \Vdash \langle \alpha, \zeta \rangle \dot{T} \langle \beta, \zeta \rangle$ ◻

Lemma 2 (Version 3)**.** *There is, in V[E], $J \subseteq C$ cofinal and a map $\alpha \mapsto \xi_\alpha (\alpha \in J)$ and a stem $\bar{h}$ such that for all $\alpha < \beta$ both in J, $\bar{h} \Vdash \langle \alpha, \xi_\alpha \rangle \dot{T} \langle \beta, \xi_\beta \rangle$.*

Remark. *Our notation is deceptive. This will not finish the proof. We could have $\bar{h}^\frown a \Vdash \neg \langle \alpha, \xi_\alpha \rangle \dot{T} \langle \beta, \xi_\beta \rangle$!*

Proof. Let $\pi : V \to M$ be a ν^+ supercompactness embedding with critical point κ_{n+1}, where n is given by Lemma 1. Let F be $Add(\kappa, \pi(\nu^{++})$-generic over $V[E]$. In $V[E][F]$, π extends to $\pi^* : V[E] \to M[E^*]$. As before let γ^* be least in $\pi^*(C)$ greater than $\sup \pi^*"\nu^+$. For each $\alpha \in C$, we have $\xi^*_\alpha, \eta^*_\alpha$ and h^*_α such that

$$h^*_\alpha \Vdash \langle \pi^*(\alpha), \xi^*_\alpha \rangle \pi^*(\dot{T}) \langle \gamma^*, \eta^*_\alpha \rangle$$

with $\ell(h^*_\alpha) = n$. This is by Lemma 1 in $M[E^*]$.

We are going to stabilize h^*_α and η^*_α. As before, in $V[E][F]$, there is $J \subseteq C$ cofinal and a fixed stem $\bar{h}$ of length n and an $\eta < \kappa$ such that

$$\alpha \in J \Rightarrow \eta^*_\alpha = \eta \wedge h^*_\alpha = \bar{h}$$

In the above we used that $\mathrm{crit}(\pi^*) = \kappa_{n+1}$, to obtain the fact that stems of length n are the same in $\pi^*(\mathbb{P})$ and $\mathbb{P}$. So $\alpha < \beta$ both in J implies

$$\bar{h} \Vdash \langle \pi^*(\alpha), \xi^*_\alpha \rangle \pi^*(\dot{T}) \langle \gamma^*, \eta \rangle$$
$$\bar{h} \Vdash \langle \pi^*(\beta), \xi^*_\beta \rangle \pi^*(\dot{T}) \langle \gamma^*, \eta \rangle$$

So $\bar{h} \Vdash \langle \pi^*(\alpha), \xi^*_\alpha \rangle \pi^*(\dot{T}) \langle \pi^*(\beta), \xi^*_\beta \rangle$

Set $\xi_\alpha = \xi^*_\alpha$. Note $\pi^*(\xi_\alpha) = \xi^*_\alpha$ and $\pi^*(\bar{h}) = \bar{h}$. So $\bar{h} \Vdash \langle \alpha, \xi_\alpha \rangle T \langle \beta, \xi_\beta \rangle$ by elementarity. This almost finishes the proof. In $V[E][F]$ we have the map $\alpha \mapsto \xi_\alpha$ for $\alpha \in J$. We need to pull this back to $V[E]$.

We apply Lemma S. We do this by viewing the above map as a branch through a particular tree, a tree of attempts to create such a map.

Without loss of generality, we may assume that J is maximal, by which we mean if $\beta \in J$ and $\alpha < \beta$ such that there is a ξ with $\bar{h} \Vdash \langle \alpha, \xi \rangle T \langle \beta, \xi_\beta \rangle$ then $\alpha \in J$ and $\xi = \xi_\alpha$. Again we are using that $\Vdash \dot{T}$ is a tree.

Let f be a function $i \mapsto (\alpha_i, \xi_{\alpha_i})$ where $i \mapsto \alpha_i$ enumerates J in increasing order. Note that for $i < \nu^+$, $f \restriction i \in V[E]$, because $f \restriction i$ is determined in $V[E]$ from $(\alpha_i, \xi_{\alpha_i})$ and $\bar{h}$. So f is a branch through a tree in V[E] of length ν^+. The tree of attempts to construct such a function in $V[E]$. By Lemma S, $f \in V[E]$ as required. □

Are we done? *No!* We can assume that $g = \bigcup \{g_p : p \in G\}$ extends $\bar{h}$. But is $\{\langle \alpha, \xi_\alpha \rangle : \alpha \in J\}$ a branch? Not necessarily. Let $\ell(\bar{h}) = \bar{k}$. Though $\bar{h} \Vdash \langle \alpha, \xi_\alpha \rangle T \langle \beta, \xi_\beta \rangle$, there might be an $a \in \mathcal{P}_\kappa(\kappa_{\bar{k}})$ such that $\bar{h}^\frown a \Vdash \neg \langle \alpha, \xi_\alpha \rangle T \langle \beta, \xi_\beta \rangle$

The set of such a has measure zero, but need not be empty. For all we know $g(\bar{k})$ is such an a, then we would have $\neg \langle \alpha, \xi_\alpha \rangle \dot{T}[G] \langle \beta, \xi_\beta \rangle$ in the extension.

3.2.1 Next Step

The final proof shows the following: There are sequences of measure one sets A_α in $V[E]$, for α in some final of J, such that for all $\alpha < \beta$ both in the final segment of J, $\langle \bar{h}, A_\alpha \cap A_\beta \rangle \Vdash \langle \alpha, \xi_\alpha \rangle \dot{T} \langle \beta, \xi_\beta \rangle$

Recall that A_α and A_β are sequences of measure one sets and that we intersect them point wise. When we have this it will finish the proof. It can be shown that G meets cofinally many of the conditions $\langle \bar{h}, A_\alpha \rangle$ for α in the tail end of J. This gives a branch.

So it remains to find A_α. The idea is to construct $A_\alpha(n)$ by 'induction' on $n \geq \bar{k}$. We will show how to do the construction for $n = \bar{k}$.

Lemma 3. *In V[E], there are sets Z_α for α in some final segment of J, such that each Z_α has $\mathcal{U}_{\bar{k}}$ measure one and for all $\alpha < \beta$ both in the final segment of J, for all $a \in Z_\alpha \cap Z_\beta$, $\bar{h}^\frown a \Vdash \langle \alpha, \xi_\alpha \rangle \dot{T} \langle \beta, \xi_\beta \rangle$.*

Before proceeding with the proof, we will explain a failed attempt which is quite instructive. Fix $\pi : V \to M$ a supercompactness embedding with critical point $\kappa_{\bar{k}+1}$. As before we can lift π to the universe $V[E]$ to get an elementary embedding $\pi^* : V[E] \to M[E^*]$ where E^* is generic for $\pi(Add(\kappa, \nu^{++}))$. Again, as before we can take $\gamma^* > \sup \pi^{*\prime\prime} \nu^+$, with $\gamma^* \in \pi^*(J)$. Then for each α, we get A^*_α such that for all $a \in A^*_\alpha$, $\bar{h}^\frown a \Vdash \langle \pi^*(\alpha), \xi_\alpha \rangle \pi(\dot{T}) \langle \gamma^*, \xi^*_\gamma \rangle$. A^*_α has $\pi^*(\mathcal{U}_{\bar{k}})$ measure one, but the problem is that A^*_α need not be in $V[E]$ let alone measure one for $\mathcal{U}_{\bar{k}}$. So this will not work.

Proof of Lemma 3. The key idea is to work "vertically" instead of "horizontally". A vertical segment will use a version of J for $h^\frown a$. So fix π^* and γ^* as in the last paragraph. Let $\eta^* = \pi^*(\alpha \mapsto \xi_\alpha)(\gamma^*)$. For each $x \in \mathcal{P}_\kappa(\kappa_{\bar{k}})$, let $J_x = \{\alpha \in J : \bar{h}^\frown x \Vdash \langle \pi(\alpha), \xi_\alpha \rangle \pi(\dot{T}) \langle \gamma^*, \eta^* \rangle\}$. The "horizontal" sets are $\{x : \alpha \in J_x\}$. They have $\pi(\mathcal{U}_{\bar{k}})$ measure one, but they need not be in $V[E]$. So we are going to look at the "vertical" segments J_x.

Claim. *If J_x is unbounded in ν^+, then $J_x \in V[E]$*

We apply Lemma S. Let $\dot{J}_x \in V[E]$ be a name for J_x in $Add(\kappa, \pi(\nu^{++}))$. (Recall that F was the generic for this poset.) Since $Add(\kappa, \pi(\nu^{++}))$ is κ^+-cc, there is a set $K_x \in V[E]$ of size $\leq \kappa$ such that $\Vdash_{Add(\kappa,\pi(\nu^{++}))} \dot{J}_x \in K_x$, if $\dot{J}_x$ is unbounded in ν^+.

By shrinking K_x, we may assume that for each $I \in K_x$

1. I is unbounded in ν^+
2. $(\beta \in I \wedge \alpha < \beta \wedge \alpha \in J) \Rightarrow (\alpha \in I \Leftrightarrow h^\frown x \Vdash \langle \alpha, \xi_\alpha \rangle \dot{T} \langle \beta, \xi_\beta \rangle)$

We call condition 2, maximality. We do all of this in V[E], in particular $x \mapsto K_x$ is in $V[E]$. K_x is a set of possible vertical segments that J_x could be.

Claim. *If $I, I' \in K_x$ are distinct, then they are disjoint on a tail.*

Proof. If there is a place, β, where I, I' agree then by maximality they agree below β. So after the first place where I, I' differ, they are disjoint. □

Corollary 5. *For each x, there is $\rho_x < \nu^+$ such that if $I, I' \in K_x$ are distinct, then they are disjoint above ρ_x.*

Proof. Recall, $|K_x| \leq \kappa$. So to find ρ_x we take a supremum over the least place where any pair from K_x differ. This is a supremum over just $|K_x|^2$ many ordinals less than ν^+. So it is less than ν^+. □

Let $\rho = \sup_{x \in \mathcal{P}_\kappa(\kappa_{\bar{k}})} \rho_x < \nu^+$. Recall, for all n, $|\mathcal{P}_\kappa(\kappa_n)| = \kappa_n$. So for any x and for any $I, I' \in K_x$ which are distinct, I, I' are disjoint above ρ.

We are going to work above ρ. Define a function f on $\mathcal{P}_\kappa(\kappa_{\bar{k}}) \times (J \setminus \rho)$ by $f(x, \alpha) =$ the unique $I \in K_x$ such that $\alpha \in I$, if such I exists and undefined otherwise.

Claim. *For $\alpha \in J \setminus \rho$, $\{x : f(x, \alpha)$ is defined$\}$ has $\mathcal{U}_{\bar{k}}$ measure one.*

Proof. Fix $\alpha \in J \setminus \rho$. First note that the set we defined is in $V[E]$. Let Y be its complement. So Y is the set of x where $f(x, \alpha)$ is not defined. Suppose for a contradiction that Y has $\mathcal{U}_{\bar{k}}$ measure one.

Here we actually need the sets from our failed attempt at a proof of Lemma 3. Recall, A^*_α was measure one for $\pi^*(\mathcal{U}_{\bar{k}})$. As $\text{crit}(\pi^*) = \kappa_{\bar{k}+1}$, $\pi^*(Y) = Y$. By elementarity Y has $\pi^*(\mathcal{U}_{\bar{k}})$ measure one. For every $\beta \in J$, the intersection of measure one sets $A^*_\alpha \cap A^*_\beta \cap Y$ is nonempty. For each β, let $x_\beta \in A^*_\alpha \cap A^*_\beta \cap Y$.

As J is unbounded in ν^+ and $|\mathcal{P}_\kappa(\kappa_{\bar{k}})| = \kappa_{\bar{k}}$, there is a fixed x and a $U \subseteq J$ unbounded, such that for all $\beta \in U$, $x = x_\beta$.

By the construction of the A^*_βs, we have $\bar{h}^\frown x \Vdash \langle \pi^*(\alpha), \xi_\alpha \rangle \pi(\dot{T}) \langle \gamma^*, \xi^*_\gamma \rangle$. and similarly for all $\beta \in U$. So by the definition of J_x, $\alpha \in J_x$ and $U \subseteq J_x$. But this means that $f(x, \alpha)$ was defined and equal to J_x, a contradiction. □

Claim. *For α, α' both in $J \setminus \rho$, the set $\{x : f(x, \alpha) = f(x, \alpha')\}$ has $\mathcal{U}_{\bar{k}}$-measure one.*

Proof. By the previous claim there is a measure one set where both are defined. Fix x and suppose that $f(x,\alpha)$ and $f(x,\alpha')$ are both defined. Without loss of generality $\alpha < \alpha'$. We apply maximality to see that $\alpha \in f(x,\alpha')$. It suffices to check that $\alpha' \in f(x,\alpha')$, $\alpha < \alpha'$, $\alpha \in f(x,\alpha)$ and $\bar{h}^\frown x \Vdash \langle\alpha,\xi_\alpha\rangle\dot{T}\langle\beta,\xi_\beta\rangle$. The first three are obvious. And the last one follows from the fact that $f(x,\alpha), f(x,\alpha')$ are candidates for J_x and that $\Vdash \dot{T}$ is a tree. This finishes the proof as we have shown that everywhere $f(x,\alpha), f(x,\alpha')$ are defined they are equal. □

Let α_0 be the least element of $J \setminus \rho$. Define $Z_\alpha = \{x : f(x,\alpha) = f(x,\alpha_0)$ where both are defined$\}$. By the previous claims Z_α has $\mathcal{U}_{\bar{k}}$ measure one. If $x \in Z_\alpha \cap Z_\beta$ then let $I = f(x,\alpha) = f(x,\beta) = f(x,\alpha_0)$ Recall, I is maximal so $\bar{h}^\frown x \Vdash \langle\alpha,\xi_\alpha\rangle\dot{T}\langle\beta,\xi_\beta\rangle$, as required. □

Open Problems

We proved that the tree property at κ does not imply SCH_κ.

1. Does ν^+ still have the tree property after cardinal preserving forcing?
2. Can we make κ of the result into $\aleph_\omega$ or some other small cardinal?

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